

**ZBORNIK RADOVA
1. KONGRESA HRVATSKOG DRUŠTVA ZA
MEHANIKU**

**PROCEEDINGS
OF THE 1st CONGRESS OF CROATIAN
SOCIETY OF MECHANICS**

Pula, Croatia, 14-17. rujna / September 1994

ZBORNİK RADOVA / PROCEEDINGS
1. KONGRESA HRVATSKOG DRUŠTVA ZA MEHANIKU
OF THE 1st CONGRESS OF CROATIAN SOCIETY OF MECHANICS

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PREDGOVOR

Prvi kongres Hrvatskog društva za mehaniku prvo je organizirano druženje znanstvenika iz područja mehanike krutih i deformabilnih tijela i mehanike fluida u neovisnoj Hrvatskoj nakon osamostaljivanja i međunarodnog priznanja. To je prilika da se uz prikaz rezultata istraživanja razmjene znanja i iskustva iz područja teorijske i primijenjene mehanike, gdje su obuhvaćene analitičke, numeričke i eksperimentalne metode.

Ovaj Zbornik sadrži radove koji su izloženi u vremenu od 14. do 16. rujna 1994. godine u Puli. Uvjereni smo da će prikazani rezultati pridonijeti upoznavanju čitatelja s novim metodama koje omogućuju efikasno rješavanje problema u raznim znanstvenim područjima kao što su strojarstvo, građevinarstvo, brodogradnja, rudarstvo i ostala područja. Najveći broj radova predstavlja doprinos numeričkim metodama analize konstrukcija. Nagli razvoj računala omogućio je uspješno rješavanje statičkih i dinamičkih nelinearnih problema. Pokušaji su što točnijeg opisivanja fenomena fizikalne i geometrijske nelinearnosti uz poboljšanje efikasnosti numeričkih algoritama. Nešto manje su zastupljeni radovi koji spadaju u područje analitičkih i eksperimentalnih metoda.

Budući da radovi na Kongresu nisu bili podijeljeni na sekcije, ni u Zborniku nisu grupirani po strogo odvojenim područjima i znanstvenim disciplinama. Tiskani su direktno u obliku koji je dobiven od autora bez prethodne lekture, stoga Organizacijski i znanstveni odbor ne preuzima odgovornost za eventualne pogreške i nejasnoće.

Veliku zahvalnost dugujemo svim sudionicima iz zemlje i inozemstva koji su svojim radovima pridonijeli realizaciji našega Prvog kongresa, koji će, vjerujemo, biti poticaj za daljnje druženje znanstvenika iz područja mehaničkih znanosti a predstavlja početak organiziranja skupova Hrvatskog društva za mehaniku.

Zahvaljujemo sponzorima koji su pružili financijsku potporu kao i svima koji su na bilo koji način pomogli organiziranje Kongresa.

Organizacijski i znanstveni odbor

PREFACE

The first Congress of the Croatian Society of Mechanics is the first organized gathering of scientists in all fields of theoretical and applied mechanics. It provides an opportunity to present research results in the fields of general mechanics, mechanics of solids and mechanics of fluids, using analytical, numerical and experimental methods.

The Proceedings contain papers that will be presented in Pula between September 14th and 16th. We are convinced that the results presented will enable readers to get acquainted with modern methods of analysis that can serve as an efficient tool for solving problems in different fields of science and engineering such as Mechanical Engineering, Civil Engineering, Naval Architecture, Mining and Petroleum Engineering, etc. The overwhelming majority of papers deals with numerical analysis of structures. The rapid development of computers has significantly contributed to solving not only static but also dynamic nonlinear problems. Efforts have been made to describe both physical and geometrical nonlinearities. At the same time the efficiency of numerical algorithms has substantially improved. Nevertheless, papers dealing with analytical and experimental methods are included too.

Since the papers have not been grouped in sections for presentation at the Congress, they have not been assigned to definite sections in the Proceedings either. The authors were asked to prepare camera-ready copies and they were printed as such without any corrections. Therefore, the Organizing and Scientific Committee does not assume liability for errors or inconsistencies.

We are grateful to all participants from Croatia and abroad for their contribution to the success of our first Congress. We hope that this will be the first in a series of equally successful congresses to be held in our beautiful country.

Finally, we wish to express our great gratitude to our sponsors for their financial support and to all those who helped in any way in organizing this Congress

Organizing and Scientific Committee

THE CROATIAN SOCIETY OF MECHANICS

Brief Historical Review

1969 - 1994

This year we celebrate three important jubilees: the 75th anniversary of teaching engineering mechanics at the newly established School of Engineering in Zagreb, the 40th anniversary of the foundation of the Zagreb Section of the former Yugoslav Society of Mechanics and the 25th anniversary of the Croatian Society of Mechanics.

School of Engineering 1919-1994. Since the end of XIX century there had been demands for a school of engineering in Croatia. The first initiative for founding such a school came from the Society of Engineers and Architects in Zagreb. The Society stated formally their demand on September 19, 1881. Due to many obstacles, political and financial, these demands were not met until December 10, 1918, following the fall of Austria-Hungary as a result of the efforts of Dr. Milan Rojc, the energetic Secretary of Education of the Croatian Government. The school was named VISOKA TEHNIČKA ŠKOLA (High Technical School, Technische Hochschule). Regular studies began in 1919. Ing. Milan Čalogović, Ing. Jaroslav Havliček and Architect Ing. Martin Pilar were elected as the first professors at the School. The first Chancellor of the School was architect Edo Šen. In 1926 the school was incorporated into the University of Zagreb under the name of TECHNICAL FACULTY.

Dr. Milivoj Prejac was the first Professor of Engineering Mechanics and Dr. Mirko Roš of Statics. They were elected in 1919. However, Professor Roš left for ETH Zürich in 1924. Meanwhile, Stepan Prokofievich Timoshenko, former professor at the Institute of Engineers of Ways of Communication in St. Petersburg arrived as a refugee in the Kingdom of Serbs, Croats and Slovenes in March 1920. The Professor's Council of the School of Engineering in Zagreb invited him as a Professor of Mechanics and Strength of Materials. S. P. Timoshenko accepted the invitation and arrived in Zagreb in May 1920.

After several years of interruption Timoshenko resumed his scientific work in Zagreb and prepared several papers. These he wrote in English and with the help of R. V. Southwell and E. H. Love they were published in the Philosophical Magazine and the Proceedings of the London Mathematical Society. At the end of his second academic year he left Zagreb, leaving Dr. Konstantin Čališev as his successor. In fact K. Čališev was the first scientist to receive a doctors degree at the newly founded School of Engineering on March 10, 1921. Dr. Rajko Kušević was elected as Assistant Professor of Engineering Statics in 1928 and of Metal Structures in 1929. Professor Prejac had been the head of the Chair for Engineering Mechanics since its foundation in 1919 until the end of the Second World War in 1945.

After the war new professors were elected: Davorin Bazjanac for Engineering Mechanics for students of Mechanical Engineering (ME), Electrical Engineering (EE) and Naval Architecture (NA), Vasilij Andrejev for Engineering Mechanics for students of Civil Engineering (CE), Architecture (A) and Geodesy (G), Dragutin Horvat for Fluid Mechanics for students of ME, EE and NA, Josip Grčić for Fluid Mechanics for the students of CE, A and G, and Mladen Hudec for Engineering Mechanics for students of Mining, Petroleum Engineering and Geology.

Zagreb Section of Mechanics 1954-1994. The first Yugoslav Congress of Applied and Theoretical Mechanics was held at Bled in the Republic of Slovenia in February 1952. At the Congress an initiative was advanced to establish the Yugoslav Society of Mechanics. In 1954 The Yugoslav Society of Mechanics (YSM) was founded. The first President of the Society was Jakov Hlitičijev and the first Secretary General, Vlatko Brčić. That same year the Zagreb Section of the YSM was founded in Zagreb.

The Croatian Society of Mechanics 1969-1994. In 1969 the Zagreb Section of YSM was transformed into the Croatian Society of Mechanics (CSM). The founding Assembly of the Croatian Society of Mechanics was held on April 18, 1969. The first President and the first Secretary of the Society were Davorin Bazjanac and Ivo Alfirević. All the presidents and secretaries of the Society and the dates of their elections are listed in Table 1.

Nineteen Yugoslav Congresses of Theoretical and Applied Mechanics were held between 1952 and 1990. These are: Bled 1952, 1954, 1956 and 1960, Opatija 1958, Split 1962, 1966 and 1968, Vrnjačka Banja 1964 and 1988, Baško Polje 1970 and 1970, Ohrid 1974 and 1990, Sarajevo 1976, Portorož 1978, Kupari 1981, Bečići 1984, Zadar 1986. Members of CSM at all Congresses participated with a number of papers and CSM helped substantially in organizing many of them.

After the recognition of the Republic of Croatia as an independent and sovereign country, CSM ceased to be a part of YSM. The same happened to the Slovenian Society of Mechanics as well as to other societies of mechanics in former Yugoslav Republics. In this way YSM ceased to exist.

Two members of CSM were presidents of YSM: Davorin Bazjanac in two terms 1970-72 and 1972-74, and Zlatko Kostrenčić 1978-81. In 1976 D. Bazjanac was appointed first honorary member of YSM. Another two members of CSM were honorary members of YSM: V. Andrejev 1978 and Z. Kostrenčić 1986.

Table 1 List of presidents and secretaries of the Croatian Society of Mechanics

Date of Assembly	President of the Society	Secretary of the Society
28.4.1969	Davorin Bazjanac	Ivo Alfirević
14.6.1971	Antun Vučetić	Ivan Heidl
25.1.1973	Mladen Hudec	Vilim Korošec
23.12.1974	Zlatko Kostrenčić	Vice Šimić
13.1.1977	Zlatko Kostrenčić	Vice Šimić
16.5.1979	Ivo Alfirević	Nikola Vranković
21.10.1981	Ivo Alfirević	Nikola Vranković
24.11.1983	Stjepan Jecić	Dragan Pustaić
21.11.1985	Stjepan Jecić	Željko Goja
16.12.1987	Aleksandar Kiričenko	Marijan Staneković
13.12.1989	Vilim Korošec	Zorislav Despot
28.11.1991	Osman Muftić	Zorislav Despot
17.12.1992	Ivo Alfirević	Jurica Sorić

The Croatian Society of Mechanics organized seven Yugoslav Symposia of the Theory of Plasticity: Plitvička Jezera 1981, 1982 and 1983, Tuheljske Toplice 1985, Lipik 1987, Opatija 1989 and Pula 1991.

In cooperation with the Austrian Society of Experimental Strain Analysis (ASESA) and the Hungarian Scientific Society of Mechanical Engineering, CSM founded the Danubia-Adria Committee (DAC) in 1982. The first members of the Committee were I. Alfirević, R. Beer, I. Huszar, S. Jecić, H.P. Rosmanith, F. Thamm. The Italian Association of Stress Analysis (AIAS) joined the DAC in 1986 and the Czechoslovakian Society of Mechanics in 1992. Later the Czechoslovakian Society split into two parts: the Czech Society of Mechanics and the Slovak Society of Mechanics. Since May this year both societies have been members of the DAC. Eleven Danubia-Adria Symposia on Experimental Methods in Solid Mechanics have been held to date: Stubičke Toplice 1984, Graz 1985, Budapest 1986, Plzen 1987, Udine 1988, Mösern 1989, Pula 1990, Gödöllo 1991, Trieste 1992, Meftin 1993, Baden 1994. CSM also organized three round table discussions about some important current issues: Stubičke Toplice 1976, Modern Methods in Teaching Mechanics; Split 1979, Application of SI Units in Engineering; Slavonski Brod 1989, Problems and Improvements in Teaching Mechanics. CSM was coorganizer of the third International Symposium on Numerical Methods for Non-Linear Problems, Dubrovnik 1986.

XIV

Inicijativni odbor za osnivanje
Društva za mehaniku SRH

Zagreb, 26.V 1969.

SEKRETARIJATU JAVNE SIGURNOSTI

Z a g r e b

Dvadeset i osmog travnja ove godine održana je osnivačka skupština Društva za mehaniku Socijalističke republike Hrvatske. Molimo Vas da izvršite upis Društva za mehaniku SRH u Registar udruženja građana i da odobrite njegov rad. Održavanje osnivačke skupštine Društva odobreno je potvrdom broj 14 - 3/2 - 7 od 18.

IV 1969. Molbi prilažemo:

- a) dva primjerka Zapisnika osnivačke skupštine
- b) četiri primjerka Statuta društva
- c) dva primjerka spiska osnivača Društva
- d) spisak članova Upravnog i Nadzornog odbora.

- 1. Prof.dr D. Bazjanac D. Bazjanac
- 2. Prof.dr Vasilij Andrejev V. Andrejev
- 3. Prof.dr Antun Vučetić A. Vučetić
- 4. Prof. Mladen Hudec Mladen Hudec
- 5. Mr Ivo AlfIREVIĆ Ivo AlfIREVIĆ
- 6. Inž. Krsto Draža Krsto Draža
- 7. Inž. Ivo Heidl Ivo Heidl
- 8. Inž. Stjepan Jecić Stjepan Jecić
- 9. Inž. Vilim Knoch Vilim Knoch
- 10. Inž. Osman Muftić Osman Muftić

Figure 1 Facsimile of the founding document of the Croatian Society of Mechanics

Since 1971 CSM has been supporting member of the International Centre for Mechanical Sciences (CISM) Udine, Italy. Croatian representatives in the Administrative and Scientific Council were Antun Vučetić, and Vasilij Andrejev and Zlatko Kostrenčić, respectively. On the proposal of and in cooperation with CSM, a course "Photoelasticity of Anisotropic Media" was organized at CISM Udine in 1983.

This is only a brief review of the activities and members of CSM. It was difficult to collect all the relevant information in such a short time and to present it in such a brief review. I can only hope that all relevant events and deserving members have been duly acknowledged. To all those whom I may have inadvertently omitted my most sincere apologies.

Ivo AlfIREVIĆ

SADRŽAJ / CONTENTS

Predgovor	VII
Preface	IX
I. Alfirević: <i>"The Croatian Society of Mechanics - Brief Historical Review"</i>	XI
Popis autora / Author index	XIX
D. R. J. Owen, Đ. Perić, E. A. de Souza Neto: <i>"Developments in Computational Procedures for the Simulation of Industrial Forming Problems"</i>	1
I. Kožar, A. Ibrahimbegović: <i>"New Finite-Rotation Solid Element"</i>	7
B. Gotovac, V. Kozulić, F. B. Damjanić: <i>"On Shell Finite Elements with an Additional Drilling Degree of Freedom"</i>	15
V. Herak Marović, P. Marović, A. Mihanović: <i>"Development of New Quadrilateral Plane Element with Three d.o.f. per Node"</i>	23
A. Mihanović, V. Herak Marović, P. Marović: <i>"Accuracy of the New Fully Compatible Quadrilateral Shell FE"</i>	31
Z. Tonković, J. Sorić, I. Heidl: <i>"On Mixed Interpolation of Eight-Node Shear-Bending Plate Element"</i>	39
I. Smojver, J. Sorić, I. Alfirević: <i>"On Nine Noded Element for Analysis of Laminated Composite Shells"</i>	47
G. Jelenić, M. A. Crisfield: <i>"Geometrically Nonlinear Space Beams with End Releases"</i>	55
B. Brank, Đ. Perić, F. B. Damjanić: <i>"A Non-Linear Finite Element Model for Thin Elastic Shells"</i>	63
J. Sorić, W. Zahlten: <i>"On Geometrically Nonlinear Elastic-Plastic Computation of Compound Shell Structures"</i>	71
Ž. Nikolić, A. Mihanović, P. Marović: <i>"Non-Linear Concrete Model with Curved Reinforcement"</i>	79
Ž. Nikolić, A. Mihanović: <i>"Finite Element Analysis of Postensioned Concrete Structures"</i>	87
I. Demirdžić, A. Ivanković, D. Martinović, S. Muzaferija: <i>"Numerical Method for Solving Linear and Non-Linear Solid Body Problems"</i>	95
M. Stanek, F. Damjanić: <i>"Numerical Modelling of Two-Dimensional Reinforced Concrete Structures"</i>	103
G. Schickhofer, W. Guggenberger: <i>"Elastic Analysis of Flexibly Jointed Laminated Timber Plates"</i>	111
B. Vidak, P. Marović: <i>"Some Parametric Analysis of Contact Elements"</i>	123
V. Kozulić, B. Gotovac, F. B. Damjanić: <i>"Numeričko modeliranje spoja ljuske i stupa"</i>	131

Ž. Šomodi: "A Simplified Model for Numerical Elastoplastic Plane Stress Analysis"	139
N. Vranković: "An Analysis of Plastic Shakedown by the Nonratchetting Condition"	145
S. Glodež, S. Pehan, J. Flašker: "Numerical Analysis of Crack Propagation in Gear Tooth Root"	152
G. Hren, A. Ivanković: "Rapid Crack Propagation by FEM"	160
C. J. Pearce, N. Bićanić: "Computational and Modelling Aspects of the Softening Five Parameter Model for Concrete"	168
T. Rodič: "Numerical Modelling of the Microstructural Changes During Hot Working"	176
M. Schönauer: "Engineering Applications of a Geometrically Nonlinear Enhanced Strain Method Based on a Logarithmic Finite Strain Model and Elastic-Inelastic Material Laws"	183
Z. Ren, N. Bićanić: "Viscoplastic Flow in Singular Regions of Mohr-Coulomb Yield Surface"	193
D. Vučina: "Modeliranje rubnog trenja kod analize metodom konačnih elemenata u formulaciji plastičnog toka"	203
B. Kralj, G. N. Pande, J. Middleton: "Application of a Homogenisation Technique to Masonry"	211
I. Dobovšek: "Interaction Between a Point Defect and a Screw Dislocation in Elastic Crystal"	219
V. Grubišić: "Behaviour of Fiber-Reinforced Plastics and Estimation of Components Service Strength"	226
Ž. Domazet: "Upotreba kompozitnih ojačanja za popravak zamorom oštećenih konstrukcija"	235
Č. Primec, J. Legat, N. Gubeljak: "Utjecaj dužine pukotine (a/W) na CTOD krivulje otpornosti i žilavost loma čelika povećane čvrstoće"	243
N. Gubeljak, J. Legat, A. Alujević: "Ocjena utjecaja stanja materijala na lomno ponašanje čelika povećane čvrstoće"	251
D. Kovačić, D. Čorko: "Mehanička svojstva gradskog otpada"	259
M. Krpan: "The Phenomenon of Friction in Present - Day, Theory and Practice"	269
R. Žarnić: "Inelastic Elements for Infilled Frame Modelling"	277
L. Frgić, B. Paris: "Calculation of Primary Support by Boundary Element Method"	285
Z. Žunić, Đ. Žebeljan: "Applications of Iterative Methods in Elastostatics by the Boundary Element Method"	293
P. Koc, B. Štok: "Optimizacija konstrukcija metodom dekompozicije"	301
P. Marović, G. Gusić: "FEME - One Way of Meshing and Generating Within the CAD Environment"	309
D. Markulak, Z. Žagar: "Primjer korištenja ekspertnih sustava za analizu spregnutih konstrukcija čelika i betona"	317

Z. Žagar: <i>"Finite Elements Communication Method"</i>	325
M. Ulbin, Z. Ren, J. Flašker: <i>"Object Oriented Programming and Finite Element Method"</i>	332
I. Senjanović, Y. Fan, J. Parunov: <i>"Transition from Regular to Chaotic Oscillations of Cubic Dynamical Systems"</i>	340
Ž. Lozina: <i>"Dinamička stabilnost greda i okvirnih nosača u linearnom području"</i>	351
H. Wolf: <i>"The Reliability of Analysis of Forced Vibrations of a Mechanical System with Clearances by Using the Method of Harmonic Balance"</i>	358
M. Stegić, B. Mikulić: <i>"Računanje dinamičkih koeficijenata krutosti i prigušenja uljnog filma Galerkinovom metodom"</i>	365
M. Oblak, B. Butinar, I. Ciglarič: <i>"A Dynamic of the Non-Linear Coulomb Damping Oscillator"</i>	373
Đ. Žebeljan: <i>"Evaluation of Vehicle Passive Safety Systems at Low Speeds by the Finite Element Method"</i>	381
V. Jović: <i>"Contribution to the Finite Element Method Based on the Method of Characteristics in Modelling Hydraulic Networks"</i>	389
Z. Doliner: <i>"Eksperimentalno određivanje viskoznog otpora 'potpuno potopljenih tijela'"</i>	399
Z. Virag, R. Korbar: <i>"Application of Corrective Analytical Differencing Scheme to Solving Navier-Stokes Equation"</i>	407
L. Sopta: <i>"Incompressible Potential Flow in Multy-Connected Domain with Periodic Boundary"</i>	415
L. Škerget, M. Delić, I. Žagar: <i>"Boundary-Domain Integral Formulations of Diffusion-Convective Equations"</i>	422
Z. Mrša: <i>"Finite Element Modelling of Laminar Flow of Incompressible Viscous Fluid with Discretization of Directional Derivative"</i>	430
M. Grabašnjak: <i>"The Random Walk Method of Pollutant Transport in Groundwater"</i>	439
G. Turkalj: <i>"Load Carrying Capacity of Hydrostatic Squeeze Film"</i>	447
S. Jecić, D. Semenski, Z. Bilušić: <i>"Principles of Crack-Tip Mixed Mode Caustics in Mechanically Anisotropic Materials"</i>	455
S. Jecić, G. Krpanec: <i>"Application of Thermoelastic Effect in Experimental Stress Analysis"</i>	463
F. de Bona, S. Zelenika: <i>"Mechanical Characterization of High Precision Manipulators Based on Flexible Elements"</i>	471
O. Muftić: <i>"Determination of Intraabdominal Pressure Using Harmonical Method of Human Figure Design"</i>	479
V. Šimić: <i>"Primjena funkcija kompleksne promjenljive u ravninskim problemima termoelastičnosti"</i>	488

D. Pustaić: <i>"The Analysis of the Crack Tip Opening Displacement (CTOD) and the Magnitude of the Plastic Zone Around a Crack Tip Based on Dugdale's Strip Yield Model and the Estimation of Its Reliability"</i>	496
I. Binkevich, I. Mamuzić, L. Medvedeva: <i>"The Investigation of the Metal Forming Processes Using Nonlinear Programming Technique"</i>	505
B. Vrdoljak: <i>"Radially Symmetric Solutions of Some Elliptic Equations"</i>	512
K. W. Tomantschger, H. Wallner: <i>"On a Hypercomplex Analysis"</i>	520
R. Rosman: <i>"Contribution to the Mechanics of Sandwich Bars"</i>	527
I. Stanke: <i>"Metoda redukcije dimenzije pri proračunu cilindričnog tijela"</i>	535
R. Pavazza: <i>"Naprezanja pri rastezanju štapova otvorenog tankostjenog presjeka s utjecajem smicanja"</i>	542
R. Pavazza, A. Matoković, B. Plazibat: <i>"Analiza naprezanja pri uvijanju štapova otvorenog tankostjenog presjeka s utjecajem smicanja"</i>	550
J. Saucha: <i>"Elastic Stability of Compressed Multispan Bars"</i>	558
S. Polić: <i>"Analysis of the Plates of Variable Stiffness by the Finite Difference Method"</i>	564
N. Bjelajac: <i>"Numerički postupak za određivanje sekundarnih ravnotežnih putanja za ploče bez početne imperfekcije"</i>	572
M. Butković, D. Perak, I. Kobasić: <i>"Thermal Stresses in the Utilizer Steam Boiler Tubes"</i>	580
J. Brnić: <i>"Determination of the Stress Concentration Factors by Finite Element Method"</i>	588
B. Grizelj: <i>"Savijanje kružnih ploča"</i>	597
O. Muftić, T. Jurčević: <i>"Konstrukcija biomehaničkog modela za kompjutorsku simulaciju ljudskog gibanja"</i>	605
B. Mijović: <i>"Flow Studies in Elastic Y-Models Using Newtonian and Non-Newtonian Fluids"</i>	613

POPIS AUTORA / AUTHOR INDEX

- Alfirević, I.: XI, 47
 Alujević, A.: 251

 Bićanić, N.: 168, 193
 Bilušić, Z.: 455
 Binkevich, I.: 505
 Bjelajac, N.: 572
 Brank, B.: 63
 Brnić, J.: 588
 Butinar, B.: 373
 Butković, M.: 580

 Ciglarić, I.: 373
 Crisfield, M. A.: 55

 Čorko, D.: 259

 Damjanić, F. B.: 15, 63, 103, 131
 De Bona, F.: 471
 De Souza Neto, A. E.: 1
 Delić, M.: 422
 Demirdžić, I.: 95
 Dobovšek, I.: 219
 Doliner, Z.: 399
 Domazet, Ž.: 235

 Fan, Y.: 340
 Flašker, J.: 152, 332
 Frgić, L.: 285

 Glodež, S.: 152
 Gotovac, B.: 15, 131
 Grabašnjak, M.: 439
 Grizelj, B.: 597
 Grubišić, V.: 226
 Gubeljak, N.: 243, 251
 Guggenberger, W.: 111
 Gusić, G.: 309

 Heidl, I.: 39
 Herak Marović, V.: 23, 31
 Hren, G.: 160

 Ibrahimbegović, A.: 7
 Ivanković, A.: 95, 160

 Jecić, S.: 455, 463
 Jelenić, G.: 55
 Jović, V.: 389
 Jurčević, T.: 605

 Koc, P.: 301
 Kobasić, I.: 580

 Korbar, R.: 407
 Kovačić, D.: 259
 Kozulić, V.: 15, 131
 Kožar, I.: 7
 Kralj, B.: 211
 Krpan, M.: 269
 Krpanec, G.: 463

 Legat, J.: 243, 251
 Lozina, Ž.: 351

 Mamuzić, I.: 505
 Markulak, D.: 317
 Marović, P.: 23, 31, 79, 123, 309
 Martinović, D.: 95
 Matoković, A.: 550
 Medvedeva, L.: 505
 Middleton, J.: 211
 Mihanović, A.: 23, 31, 79, 87
 Mijović, B.: 613
 Mikulić, B.: 365
 Mrša, Z.: 430
 Muftić, O.: 479, 605
 Muzaferija, S.: 95

 Nikolić, Ž.: 79, 87

 Oblak, M.: 373
 Owen, D. R. J.: 1

 Pande, G. N.: 211
 Paris, B.: 275
 Parunov, J.: 340
 Pavazza, R.: 542, 550
 Pearce, C. J.: 168
 Pehan, S.: 152
 Perak, D.: 580
 Perić, Đ.: 1, 63
 Plazibat, B.: 550
 Polić, S.: 564
 Primec, Č.: 243
 Pustaić, D.: 496

 Ren, Z.: 193, 332
 Rodić, T.: 176
 Rosman, R.: 527

 Saucha, J.: 558
 Schickhofer, G.: 111
 Schönauer, M.: 183
 Semenski, D.: 455
 Senjanović, I.: 340

Smojver, I.: 47
Sopta, L.: 415
Sorič, J.: 39, 47, 71
Stanek, M.: 103
Stanke, I.: 535
Stegić, M.: 365

Šimič, V.: 488
Škerget, L.: 422
Šomodi, Ž.: 139
Štok, B.: 301

Tomantschger, K. W.: 520
Tonkovič, Z.: 39
Turkalj, G.: 447
Ulbin, M.: 332

Vidak, B.: 123
Vranković, N.: 145
Virag, Z.: 407
Vrdoljak, B.: 512
Vučina, D.: 203

Wallner, H.: 520
Wolf, H.: 358

Zahlten, W.: 71
Zelenika, S.: 471

Žagar, I.: 422
Žagar, Z.: 317, 325
Žarnić, R.: 277
Žebeljan, Đ.: 293, 381
Žunić, Z.: 293

DEVELOPMENTS IN COMPUTATIONAL PROCEDURES FOR THE SIMULATION OF INDUSTRIAL FORMING PROBLEMS

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1. INTRODUCTION

The role of computational mechanics in the solution of industrial forming problems is becoming increasingly established due to parallel advances in theoretical formulations, together with associated algorithmic implementation, and developments in workstation technology. Such simulation procedures, which are almost exclusively based on the finite element method, are now routinely employed in the modelling of metal forming operations and the acceptance of such approaches for industrial design is continually increasing, due to improved awareness, availability of appropriate software and reducing computational costs.

Probably the most striking example of developments in this field is the advancements recently made in the numerical solution of *finite strain plasticity* problems. The formulation of rigorous solution procedures has been the subject of intense debate over the last decade and only recently has some consensus been reached on an appropriate constitutive theory based on tensorial state variables to provide a theoretical framework for the macroscopic description of a general elasto-plastic material at finite strains. In computational circles, effort has been directed at the formulation of algorithms for integration of the constitutive equations relying on operator split methodology. In particular the concept of consistent linearisation has been introduced to provide quadratically convergent solution procedures. By employing logarithmic stretches as strain measures a particularly simple model for large inelastic deformations at finite strains is recovered. In particular, the effects of finite strains appear only at the kinematic level and the integration algorithms and corresponding consistent tangent operators for small strain situations can be directly employed.

A further class of nonlinear problem for which considerable advances in numerical modelling have been made in recent years is that of *contact-friction* behaviour. Contact-friction phenomena arise in most forming processes and numerical treatment in the past has, of necessity, relied on temperamental and poorly convergent algorithms. This situation has changed markedly with recognition of the complete analogy that exists between contact-friction behaviour and the classical theory of elasto-plasticity. Hence, the operator split algorithms and consistent linearisation procedures developed for the latter case translate directly to contact-friction models to provide robust and rapidly convergent numerical solutions.

Despite such advances, most applications reported in the literature are still restricted to the generalized Amontons-Coulomb law of perfect friction. As pointed out by Curnier [3] such a simplified theory may represent only a limited range of tribological situations and the state conditions of surfaces in contact are influenced by a number of complex phenomena such as wear, internal straining, lubrication, chemical reactions, etc. In situations involving large sliding distances, typical in deep drawing operations, the evolution of surface wear may become particularly important in the definition of the frictional behaviour. Specially for coated sheet metals, appreciable deviations from the standard Amontons-Coulomb model are observed in experiments [6]. When subjected to high normal pressures and large sliding

distances, the surface coating present in these materials may be worn away exposing the bulk metal thereby causing a rapid increase in the friction coefficient. In these cases, numerical predictions with reasonable accuracy may demand consideration of more realistic friction rules.

A general feature encountered in the FE simulation of finite strain plasticity problems is that the optimal *mesh configuration* changes continually throughout the deformation process requiring mesh derefinement as well as mesh refinement during the adaptive process. At the same time, *error estimation* procedures play a crucial role in quality assurance by providing a reliable finite element solution to be used as a basis for the design process.

2. FINITE STRAIN ELASTO-PLASTICITY

2.1 Constitutive equations for large strain elasto-plasticity

It is only recently that the constitutive theory for finite strain plasticity, commonly based on the multiplicative decomposition of the deformation gradient, and the associated numerical integration algorithms, have received a complete rational basis. Notable contributions in this area are presented in [2,5,8-9,11,14,16]. It is generally accepted that a sufficiently general constitutive model with possible wide application, including most of metal plasticity, together with considerable simplifications in the numerical treatment of finite strain elasto-plasticity, may be defined by employing a quadratic strain energy function in the form of a scalar symmetric function of its principal stretches λ_i^e , ($i = 1, 2, 3$)

$$W(\lambda_{(1)}^e, \lambda_{(2)}^e, \lambda_{(3)}^e) = \mu ((\ln \lambda_{(1)}^e)^2 + (\ln \lambda_{(2)}^e)^2 + (\ln \lambda_{(3)}^e)^2) + \frac{1}{2} \lambda (\ln J^e)^2 \quad (2.1)$$

where μ, λ are non-zero material constants and $J^e = \lambda_{(1)}^e \lambda_{(2)}^e \lambda_{(3)}^e$ is the Jacobian. By denoting a logarithmic strain as $E := \ln U$ and its work conjugate stress as $T := R^T \tau R$ the constitutive equations for large deformations of elasto-plastic solids at finite strains, with attention restricted to isotropic elasto-plasticity, can be represented in a form similar to the standard small strain plasticity equations as (see [2,5,8-9,11-12,14,16]. for details)

$$F = F^e F^p \quad (2.2a)$$

$$T = \partial_{E^e} W = 2\mu E^e + \kappa \text{tr}[E^e] 1 \quad (2.2b)$$

$$\text{sym} [\dot{F}^p F^{p-1}] = \dot{\gamma} \partial_T \Phi(T, K) \quad (2.2c)$$

$$\dot{K} = \dot{\gamma} H(E^p, K) \quad (2.2d)$$

Here $\Phi(T, K)$ is the yield surface, K is the scalar variable describing isotropic hardening while function $H(E^p, K)$ defines the hardening law. Finally, loading/unloading conditions may be formulated in the standard Kuhn-Tucker form

$$\Phi \leq 0, \quad \dot{\gamma} \geq 0, \quad \dot{\gamma} \Phi = 0. \quad (2.3)$$

2.2 Numerical integration of the large strain elasto-plastic constitutive model

Procedures for numerical integration of the rate type constitutive equations which are particularly suitable for implementation are connected with general operator split methodology, where the original problem of evolution is solved through composition, applying first the elastic and then the plastic algorithm.

As a crucial point in the derivation of the algorithm, an *exponential approximation* is used in the discretisation of the plastic flow rule in the plastic corrector stage first employed in its original form in the computational literature by Weber and Anand [16] (see also [2,5,11]). This leads to the following incremental evolution equation

$$F_{n+1}^p = \exp [\Delta \gamma \partial_T \Phi_{n+1}] F_n^p \quad (2.4)$$

It can be easily shown that the above approximation results in

$$\mathbf{E}_{n+1}^e = \mathbf{E}_{n+1}^{e \text{ trial}} - \Delta\gamma \partial_T \Phi_{n+1} \quad (2.5)$$

which is valid whenever $\mathbf{C}_{n+1}^{e \text{ trial}} := (\mathbf{F}_{n+1}^{e \text{ trial}})^T \mathbf{F}_{n+1}^{e \text{ trial}}$ and $\partial_T \Phi_{n+1}$ commute. In conjunction with the quadratic strain energy function expressed in terms of logarithmic stretches (2.1), equations (2.2-5) lead to a particularly simple format of the integration algorithms for multiplicative finite strain plasticity which is equivalent to the *small strain plasticity* algorithms.

REMARK 2.1 A crucial task for forming simulation, particularly bulk forming operations, is the development of appropriate elements which can model the *incompressible* nature of plastic flow without exhibiting locking behaviour. In this respect, the geometrically nonlinear *assumed strain* method, proposed by Simo and Armero [13] is particularly relevant.

3. CONTACT-FRICTION MODELLING

3.1 Normal behaviour

In the definition of the normal contact behaviour, it is assumed that penetration between the two bodies is admissible. In addition, a linear relation between the normal reaction and the normal penetration, $g_N < 0$, is postulated resulting in the constitutive function $p_N = -k_N g_N$ whenever $g_N < 0$ and $p_N = 0$ otherwise, where k_N is the *normal stiffness* or *penalty factor*.

3.2 Tangential behaviour

Following standard arguments of the elastoplasticity theory of friction [3,10], the decomposition of the tangential relative velocity $\dot{\mathbf{u}}_T$ into an *adherence* and a *slip* component is adopted, $\dot{\mathbf{u}}_T = \dot{\mathbf{u}}_T^a + \dot{\mathbf{u}}_T^s$ along with the following constitutive law for the tangent reaction on the slave boundary $\mathbf{p}_T = -k_T \mathbf{u}_T^a$, where k_T is the *tangential contact stiffness*. The evolution law for the slip component is defined by

$$\dot{\mathbf{u}}_T^s = -\dot{\gamma} \partial_{\mathbf{p}_T} \psi \quad (3.1)$$

where the *slip potential* ψ defines the direction of frictional sliding and $\dot{\gamma}$ is consistent with the loading/unloading condition:

$$\phi(\mathbf{p}, A) \leq 0 \quad \dot{\gamma} \geq 0 \quad \dot{\gamma} \phi(\mathbf{p}, A) = 0. \quad (3.2)$$

The *slip criterion* ϕ above, is assumed to be a function of the contact reaction $\mathbf{p}$ and (possibly) a set A of internal variables taking into account the history dependence of the friction phenomenon.

For *isotropic* frictional contact, frictional sliding may occur only in the direction opposite to the tangential reaction. Hence, the slip potential ψ is given by $\psi(\mathbf{p}_T) = \|\mathbf{p}_T\|$. As a central hypothesis of the present model for friction with hardening, the following particular form for the slip criterion is postulated:

$$\phi(\mathbf{p}, w) = \|\mathbf{p}_T\| - \mu(w) p_N \quad (3.3)$$

where the friction coefficient, μ , is assumed to be a function of the single *scalar internal variable* w defined on the *slave* boundary [15]. In addition, the internal variable w is chosen as the density of *frictional work* expended at the point considered. Hence, its evolution equation is defined by

$$\dot{w} = -\mathbf{p}_T \cdot \dot{\mathbf{u}}_T^s. \quad (3.4)$$

The resulting constitutive model for frictional contact with hardening is analogous to classical work hardening plasticity.

4. FURTHER ASPECTS OF THE DEFORMATION OF INELASTIC SOLIDS AT FINITE STRAINS

In this section certain aspects of phenomenologically based refinements in constitutive modelling of inelastic solids at finite strains which are particularly relevant for forming simulations are discussed. Refined material characterisation is important in the accurate modelling of material behaviour and, particularly, it is considered essential for failure description.

4.1 Yield surface representation

For thin sheet metal forming operations, due to the processing of the material, the plastic behaviour of textured polycrystalline sheet is predominantly anisotropic. Full *planar anisotropy* is described by the yield function introduced by Barlat and Lian [1] which for the plane stress state is of the superquadric form (computational aspects of this criterion are discussed [4])

$$f(\boldsymbol{\sigma}, \bar{\sigma}) = a|K_1 + K_2|^M + a|K_1 - K_2|^M + (2 - a)|2K_2|^M - 2\bar{\sigma}^M \quad (4.1)$$

in which

$$K_1 = \frac{\sigma_{xx} + h\sigma_{yy}}{2} \quad \text{and} \quad K_2 = \sqrt{\left(\frac{\sigma_{xx} - h\sigma_{yy}}{2}\right)^2 + p^2\sigma_{xy}^2} \quad (4.2)$$

where a, h, p and M are material constants and $\bar{\sigma}$ is the yield stress from a uniaxial tension test. For a given value of the exponent M material constants a, h, p can be evaluated using R values, i.e. plastic strain ratios of the in-plane strains to the thickness strain obtained from uniaxial tension tests in three different directions.

4.2 Elasto-plastic damaging solids

As experimentally verified for many materials (see Lemaitre [7]) the nucleation and growth of voids and microcracks which accompany large plastic flow causes a reduction of the elastic modulus as well as material softening. In this case, the prediction of rupture with reasonable accuracy requires consideration of the coupling between elastoplasticity and damage.

With attention restricted to isotropic solids, a scalar parameter D is introduced, which represents the density of micro-cracks and micro-cavities in the body. The elasticity law is then given by $\boldsymbol{\sigma} = (1 - D)\mathbf{C} : \boldsymbol{\varepsilon}^e$ where $\mathbf{C}$ is the elastic modulus which is assumed constant and isotropic. Stresses are assumed to satisfy the von Mises yield criterion in the form

$$f(\boldsymbol{\sigma}, \bar{\sigma}, D) = \frac{\sqrt{3J_2(\boldsymbol{\sigma})}}{1 - D} - \bar{\sigma} \quad (4.3)$$

where $\bar{\sigma}(\bar{\varepsilon}^p)$ is the uniaxial yield stress. The evolution laws for internal variables are obtained in a standard way employing the second law of thermodynamics in the form of the Clausius-Duhem inequality and adopting the normal dissipation assumption which results in

$$\dot{\varepsilon}^p = \dot{\gamma} \frac{1}{(1 - D)} \frac{\text{dev}[\boldsymbol{\sigma}]}{\sqrt{3J_2(\boldsymbol{\sigma})}} \quad (4.4a)$$

$$\dot{D} = \dot{\gamma} \frac{1}{1 - D} \left(\frac{-Y}{S_0} \right)^{s_0} \quad (4.4b)$$

where S_0 and s_0 are material constants, and Y is the damage energy release rate expressible in terms of the stresses and accumulated damage, D . A computational framework for the fully coupled elasto-plastic damage model at finite strains is provided in [14].

4.3 Elasto-viscoplastic solids

The constitutive model for elasto-viscoplasticity based on a von Mises yield criterion and power law isotropic hardening is represented as

$$f(\boldsymbol{\sigma}, \bar{\sigma}) = F(\boldsymbol{\sigma}) - \bar{\sigma} \quad (4.5a)$$

$$F(\boldsymbol{\sigma}) = \sqrt{3J_2} \quad (4.5b)$$

$$\bar{\sigma} = h_0 \left(\varepsilon_0 + \bar{\varepsilon}^{vp} \right)^n \quad (4.5c)$$

$$\dot{\varepsilon}^{vp} = \gamma \langle \Psi \rangle \partial_{\boldsymbol{\sigma}} f(\boldsymbol{\sigma}, \bar{\sigma}) \quad (4.5d)$$

$$\dot{\bar{\varepsilon}}^{vp} = \dot{\bar{\sigma}} H^{-1} = \gamma \langle \Psi \rangle \quad (4.5e)$$

where h_0 , ε_0 and n are material constants, $\bar{\varepsilon}^{vp}$ is the equivalent viscoplastic strain and $H = d\bar{\sigma}(\bar{\varepsilon}^{vp})/d\bar{\varepsilon}^{vp}$ is the standard hardening modulus. Furthermore, the viscoplastic flow potential is introduced as

$$\Psi(\boldsymbol{\sigma}, \bar{\sigma}) = \left(\frac{F(\boldsymbol{\sigma})}{\bar{\sigma}} \right)^N - 1 \quad (4.6)$$

where N is the material parameter representing the rate-sensitivity of the material. The evolution problem described by (4.5-6) is widely accepted as a description of rate-dependent deformations of solids and has a firm experimental basis. Its detailed study is provided in [8].

5. CONCLUDING REMARKS

Some recent advances in the finite element analysis of forming problems have been reviewed, indicating the progress that has been made both in the theoretical understanding of inelastic material behaviour under finite strains and the associated numerical implementation. Whilst the state of knowledge in some areas is relatively mature, considerable further understanding and development is required in others. For example, issues related to the modelling of complex contact-friction phenomena are far from settled and the phenomenological model for coated sheet presented here represents only a particular class of nonlinear friction models. A more comprehensive treatment of friction may necessitate the integration of micromechanical studies with computational approaches.

Although *adaptive strategies* are, at present, routinely performed for linear elliptic problems, their extension to nonlinear elliptic problems – in particular to metal forming problems where, typically, large inelastic deformations at finite strains are standard working conditions – is by no means trivial. A comprehensive treatment of adaptive mesh refinement for forming problems requires consideration of the three following issues: mesh generation/adaption techniques, error estimation and transfer operators.

Presently available phenomenological plasticity models for metals provide only a first order approximation of the material behaviour under a restricted class of loading conditions. Often, due to the processing of the material, the plastic behaviour of as-received forming material is predominantly anisotropic. The situation is further aggravated by the lack of any information regarding evolution of material behaviour during the forming process, so that standard isotropic hardening laws are usually employed in FE simulations of forming operations.

The complexities regarding the anisotropic plastic behaviour in forming materials make development of phenomenological models increasingly difficult and require an in-depth knowledge of the polycrystalline texture evolution. Although, some initial results on the computational modelling of the forming of polycrystalline materials have been recently reported [17], substantial further research is required in this area.

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New Finite-Rotation Solid Element

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Summary

The paper describes development of an 8 node solid finite element capable of undergoing both large displacements and large rotations. Constitutive law chosen is for elastic materials.

Element is based on a sound variational basis and consistent linearization is performed so that quadratic convergence rate is obtained (as it is visible from examples).

Incompatible modes introduced by Wilson et al. in 1973 are employed to produce 'locking' free performance. Ibrahimbegović and Wilson [1991] modified the method so that the constant strain condition ('patch-test') is satisfied. Further improvement regarding incompatible modes is use of operator split technique that makes possible separate solution of equilibrium equations which in turn drastically reduces demands for secondary storage.

Solid element developed in this article includes treatment of finite rotations and can undergo large displacements as well as large rotations. Rotation vector is chosen as a parameterization of large rotations and as a consequence stiffness matrix is symmetric. Also, rotation vector is expressed in a unique way so that it can be additively composed and the update procedure is the simplest one possible and without need for secondary storage.

Interpolation functions are introduced at the very end of the work so that description is as element type independent as possible.

Correctness of the procedure is proved with a set of numerical examples.

Section 1 describes variational formulation of the problem. Section 2 introduces constitutive law and discretizes the formulation and presents stress and strains mappings. In section 3 we add incompatible modes. Linearization of the virtual work equations is performed in section 4. Finite element interpolations for both displacements and incompatible modes are introduced in section 5. Various numerical examples follow in section 6.

1 Variational Formulation

Detailed description of the formulation that is behind the approach to finite element procedures described in this paper are to be found in Ibrahimbegović [1992] while here will be presented only the main ideas.

The variational formulation used is for the continuum with independent rotation filed as it appears in geometrically nonlinear theory. Consistently applied this leads us to the finite element formulation capable of undertaking large displacements and large rotations. Starting point is classical potential energy principle for finite displacement (e.g. see Washizu [1982])

$$\Pi(\mathbf{u}) = \int_{\Omega} W(\mathbf{H}(\mathbf{u})) d\Omega - \int_{\Omega} \mathbf{u} \cdot \mathbf{f} d\Omega \quad (1.1)$$

The second integral is an external work for the Dirichlet boundary value problem and $W(\mathbf{H}(\mathbf{u}))$ is stored energy as a function of a chosen finite strain measure.

Usually in geometric nonlinear analysis with small strain assumption we use Green-Lagrange strain tensor while in geometrically linear theory one is advised to use infinitesimal strain tensor. Here on the contrary, we use the Biot strain tensor which is defined as

$$\mathbf{H} = \mathbf{U} - \mathbf{I} \quad (1.2)$$

where $\mathbf{U}$ is a symmetric positive-definite tensor as stated by the polar factorization theorem applied to the deformation gradient

$$\mathbf{F} = \mathbf{A}\mathbf{U} \quad (1.3)$$

$\mathbf{A}$ is orthogonal tensor which describes (unrestricted) rotation. Deformation gradient in terms of the displacement gradient $\nabla \mathbf{u}$ is

$$\mathbf{F} = \nabla \varphi = \mathbf{I} + \nabla \mathbf{u} \quad ; \quad \nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \quad ; \quad \mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad (1.4)$$

where $\mathbf{u} = \varphi(\mathbf{x}) - \mathbf{x}$ is the displacement vector field.

Using the symmetric property of $\mathbf{H}$ we can decompose it into its symmetric and skew-symmetric part

$$\begin{aligned} \mathbf{H} &= \mathbf{E} + \mathbf{\Omega} \\ \mathbf{E} &= \text{sym}(\mathbf{A}^T(\mathbf{I} + \nabla \mathbf{u}) - \mathbf{I}) \\ \mathbf{\Omega} &= \text{skew}(\mathbf{A}^T(\mathbf{I} + \nabla \mathbf{u}) - \mathbf{I}) \end{aligned} \quad (1.5)$$

2 Discrete Formulation and Finite Element Interpolation

Using variational principles described above we discretise them in a form suitable for use in finite element computational process. Our basic equation will be variational formulation with only kinematic fields as dependant variables derived on the basis of the Reisner [1965] principles

$$\Pi_{\gamma}(\mathbf{u}, \mathbf{A}) = \int_{\Omega} \left\{ \frac{1}{2} \mathbf{E} : \mathbf{C} : \mathbf{E} + \frac{1}{2} \gamma \mathbf{\Omega} : \mathbf{\Omega} \right\} d\Omega - \int_{\Omega} \mathbf{u} \cdot \mathbf{f} d\Omega \quad (2.1)$$

where γ is a regularization parameter and superscript ' h ' (mesh parameter) denotes finite element approximated fields in order to make a clear distinction from the corresponding values in the infinite-dimensional spaces implied before.

Hughes and Brezzi [1989] have explained for the geometrically linear theory that variational formulation has to be regularized in order to preserve stability of the solution for interpolations conveniently used in finite elements approximations (isoparametric elements). Although γ can be any positive number it is suggested that for linear isotropic case shear modulus is its optimal value

$$\gamma = \frac{E}{2(1 + \mu)} \quad (2.2)$$

For finite element analysis purpose deformation gradient $\mathbf{F}$ which is normally 3 by 3 matrix is mapped into vector 9 by 1

$$\mathbf{y}_1(\mathbf{u}) = \begin{Bmatrix} 1 + u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{Bmatrix} ; \quad \mathbf{y}_2(\mathbf{u}) = \begin{Bmatrix} u_{1,2} \\ 1 + u_{2,2} \\ u_{3,2} \end{Bmatrix} ; \quad \mathbf{y}_3(\mathbf{u}) = \begin{Bmatrix} u_{1,3} \\ u_{2,3} \\ 1 + u_{3,3} \end{Bmatrix} \quad (2.3)$$

For easier dealing with equations we shall divide our strain measure in its symmetric and skew-symmetric parts, both mapped into vectors

$$\mathbf{E} \Rightarrow \mathbf{e}$$

$$\mathbf{\Omega} \Rightarrow \boldsymbol{\omega} \quad (2.4)$$

Symmetric part of strain mapped into vector is

$$\mathbf{e} = \begin{Bmatrix} \epsilon_{1,1} \\ \epsilon_{2,2} \\ \epsilon_{3,3} \\ 2\epsilon_{1,2} \\ 2\epsilon_{1,3} \\ 2\epsilon_{2,3} \end{Bmatrix} = \begin{bmatrix} \mathbf{a}_1^T & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{a}_2^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{a}_3^T \\ \mathbf{a}_2^T & \mathbf{a}_1^T & \mathbf{0}^T \\ \mathbf{a}_3^T & \mathbf{0}^T & \mathbf{a}_1^T \\ \mathbf{0}^T & \mathbf{a}_3^T & \mathbf{a}_2^T \end{bmatrix} \begin{Bmatrix} \mathbf{y}_1(\mathbf{u}) \\ \mathbf{y}_1(\mathbf{u}) \\ \mathbf{y}_1(\mathbf{u}) \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \mathbf{R}\mathbf{y}(\mathbf{u}) - \mathbf{1} \quad (2.5)$$

and skew-symmetric part

$$\boldsymbol{\omega} = \begin{Bmatrix} \omega_{1,2} \\ \omega_{1,3} \\ \omega_{2,3} \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{a}_2^T & -\mathbf{a}_1^T & \mathbf{0}^T \\ -\mathbf{a}_3^T & \mathbf{0}^T & \mathbf{a}_1^T \\ \mathbf{0}^T & \mathbf{a}_3^T & -\mathbf{a}_2^T \end{bmatrix} \begin{Bmatrix} \mathbf{y}_1(\mathbf{u}) \\ \mathbf{y}_1(\mathbf{u}) \\ \mathbf{y}_1(\mathbf{u}) \end{Bmatrix} = \mathbf{P}\mathbf{y}(\mathbf{u}) \quad (2.6)$$

Matrices $\mathbf{R}$ and $\mathbf{P}$ are used to establish connection between deformation gradient components and symmetric and skew-symmetric strain components. As it can be easily shown they consist of parts of large rotation matrix $\mathbf{A}$ when it is written like

$$\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \quad (2.7)$$

For the finite element procedures Euler-Lagrange equations in matrix form are reduced to

$$\mathbf{n} = \mathbf{C}_e \mathbf{e} ; \quad \mathbf{t} = \mathbf{C}_\omega \boldsymbol{\omega} \quad (2.8)$$

Constitutive matrices $\mathbf{C}_e$ and $\mathbf{C}_\omega$ describe material behavior we intend to use in our analysis. For isotropic linear material $\mathbf{C}_e$ can be found in many places (e.g. see Zienkiewicz and Taylor [1989]). For $\mathbf{C}_\omega$ matrix we borrow from geometrically linear theory

$$\mathbf{C}_\omega = \gamma \mathbf{I}_3 \quad (2.9)$$

3 Addition of Incompatible Modes

Assumed displacement gradient is reparametrized in terms of additive decomposition into 'compatible' $\nabla \mathbf{u}$ and enhanced ('incompatible') part

$$\mathbf{D} = \nabla \mathbf{u} + \tilde{\mathbf{D}} \quad (3.1)$$

It can be seen in the variational formulation in Ibrahimbegović and Frey [1992b] that it is sufficient that the enhanced displacement gradient belongs only to the space of square integrable functions over the domain Ω ($L^2(\Omega)$ space). Direct consequence is that in the finite element approximations $\tilde{\mathbf{D}}$ can be discontinuous over element boundaries ('incompatible'), yet variationally consistent.

For the finite element calculation purposes enhanced displacement gradient field $\tilde{\mathbf{D}}$ is mapped into vector $\mathbf{d}$ that is 9 by 1.

Now we have enriched our strain field and the relations between strains and displacements and incompatible modes are

$$\begin{aligned} \mathbf{e} &= \mathbf{R}(\mathbf{y}(\mathbf{u}) + \mathbf{d}) = \mathbf{R}\mathbf{y}(\mathbf{u}) + \tilde{\mathbf{D}}_e \\ \boldsymbol{\omega} &= \mathbf{P}(\mathbf{y}(\mathbf{u}) + \mathbf{d}) = \mathbf{P}\mathbf{y}(\mathbf{u}) + \tilde{\mathbf{D}}_\omega \end{aligned} \quad (3.2)$$

4 Linearization of the Virtual Work Equations

Relations established up to now make us possible to write the potential energy equation in matrix form

$$\Pi(\mathbf{u}, \boldsymbol{\Lambda}) = \frac{1}{2} \int_{\Omega} (\mathbf{e}^T \cdot \mathbf{C}_e \mathbf{e} + \boldsymbol{\omega}^T \cdot \mathbf{C}_\omega \boldsymbol{\omega}) d\Omega - \Pi_{ext} \quad (4.1)$$

The associated virtual work equations are then obtained by applying directional derivatives in the direction of virtual displacements and rotations $\delta \mathbf{u}$ and in the direction of virtual enhanced displacement gradient $\delta \mathbf{d}$ which gives us two nonlinear equations with the two unknown fields to be solved for, displacement field $(\delta \mathbf{u}, \delta \boldsymbol{\omega})$ and incompatible modes field $\delta \mathbf{d}$

$$\delta \Pi(\mathbf{u}, \boldsymbol{\Lambda}(\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta \mathbf{u}, \delta \boldsymbol{\theta}) = \int_{\Omega} (\delta \mathbf{e}^T \cdot \mathbf{C}_e \mathbf{e} + \delta \boldsymbol{\omega}^T \cdot \mathbf{C}_\omega \boldsymbol{\omega}) d\Omega - \delta \Pi_{ext} = 0 \quad (4.2)$$

$$\delta \Pi(\mathbf{u}, \boldsymbol{\Lambda}(\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta \mathbf{d}) = \int_{\Omega} (\delta \tilde{\mathbf{D}}_e^T \cdot \mathbf{C}_e \mathbf{e} + \delta \tilde{\mathbf{D}}_\omega^T \cdot \mathbf{C}_\omega \boldsymbol{\omega}) d\Omega = 0 \quad (4.3)$$

Note that in the integral expression one must express variation $\delta \mathbf{d}$ as $\delta \tilde{\mathbf{D}}_e$ and $\delta \tilde{\mathbf{D}}_\omega$ since incompatible modes and strains are connected through large rotation operators $\mathbf{R}$ and $\mathbf{P}$.

Linearization of those virtual work equations will result in tangent modulus that permits us to employ Newton's algorithm for solution of the two nonlinear equations. If we apply the linearization process consistently (e.g. see Hughes and Pister [1978]) then quadratic convergence rate should be expected during iterative procedure.

Linearization of the first equation gives

$$\begin{aligned} L(\delta \Pi(\mathbf{u}, \boldsymbol{\Lambda}(\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta \mathbf{u}, \delta \boldsymbol{\theta})) &= \delta \Pi(\mathbf{u}, \boldsymbol{\Lambda}(\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta \mathbf{u}, \delta \boldsymbol{\theta}) - \Pi_{ext} + \\ &\frac{d}{d\alpha} (\delta \Pi(\mathbf{u} + \alpha \Delta \mathbf{u}, \boldsymbol{\Lambda}(\boldsymbol{\theta} + \alpha \Delta \boldsymbol{\theta}), \mathbf{d}) \cdot (\delta \mathbf{u}, \delta \boldsymbol{\theta}))|_{\alpha=0} + \\ &\frac{d}{d\alpha} (\delta \Pi(\mathbf{u}, \boldsymbol{\Lambda}(\boldsymbol{\theta}), \mathbf{d} + \alpha \Delta \mathbf{d}) \cdot (\delta \mathbf{u}, \delta \boldsymbol{\theta}))|_{\alpha=0} = 0 \end{aligned} \quad (4.4)$$

and linearization of the second one

$$\begin{aligned} L(\delta\Pi(\mathbf{u}, \mathbf{A}(\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta\mathbf{d})) &= \delta\Pi(\mathbf{u}, \mathbf{A}(\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta\mathbf{d}) + \\ \frac{d}{d\alpha} (\delta\Pi(\mathbf{u} + \alpha\Delta\mathbf{u}, \mathbf{A}(\boldsymbol{\theta} + \alpha\Delta\boldsymbol{\theta}), \mathbf{d}) \cdot (\delta\mathbf{d}))|_{\alpha=0} &+ \\ \frac{d}{d\alpha} (\delta\Pi(\mathbf{u}, \mathbf{A}(\boldsymbol{\theta}), \mathbf{d} + \alpha\Delta\mathbf{d}) \cdot (\delta\mathbf{d}))|_{\alpha=0} &= 0 \end{aligned} \quad (4.5)$$

5 Finite Element Interpolation

After careful linearization of our virtual work equations we get their matrix counterpart

$$\bigcup_{i=1}^{Nel} \left\{ \begin{matrix} \delta\mathbf{u}_i \\ \delta\boldsymbol{\theta}_i \\ \delta\alpha \end{matrix} \right\} \cdot \left\{ \begin{matrix} \mathbf{r}^e - \mathbf{f}^e \\ \mathbf{h}^e \end{matrix} \right\} + \begin{bmatrix} \mathbf{K}^e & \mathbf{F}^e \\ \mathbf{F}^{eT} & \mathbf{H}^e \end{bmatrix} \left\{ \begin{matrix} \Delta\mathbf{u}_i \\ \Delta\boldsymbol{\theta}_i \\ \Delta\alpha \end{matrix} \right\} = 0 \quad (5.1)$$

In our finite element approximation we have decided to use the 8-node isoparametric solid (brick) element with full numerical integration (8 Gauss integration points). The element is simple to use but simple elements have strong tendency toward locking which is in this element avoided through the use of incompatible modes.

As we have seen before our incompatible strain field does not have to be continuous over element boundaries, so we are choosing the simplest possible interpolation functions

$$\mathbf{M}_1 = 1 - \xi^2 \quad ; \quad \mathbf{M}_2 = 1 - \eta^2 \quad ; \quad \mathbf{M}_3 = 1 - \zeta^2 \quad ; \quad (5.2)$$

With all the necessary parts determined matrices comprised in finite element equation (5.1) can be determined

$$\begin{aligned} \mathbf{R}^T \mathbf{n} + \mathbf{P}^T \mathbf{t} &= \mathbf{p} \\ \mathbf{R}^T \mathbf{C}_e \mathbf{R} + \mathbf{P}^T \mathbf{C}_\omega \mathbf{P} &= \mathbf{C} \\ [\mathbf{y}_i \times \mathbf{I}] &= \mathbf{Y} \\ [\mathbf{p} \times \mathbf{I}] &= \mathbf{X} \end{aligned} \quad (5.3)$$

$$\mathbf{r}^e = \int_{\Omega} \begin{bmatrix} \mathbf{N}_{,a} \\ \mathbf{TN} \end{bmatrix}^T \left\{ \begin{matrix} \mathbf{p} \\ \mathbf{Y}^T \mathbf{p} \end{matrix} \right\} d\Omega \quad (5.4)$$

$$\mathbf{K}^e = \mathbf{K}_m^e + \mathbf{K}_g^e =$$

$$\int_{\Omega} \begin{bmatrix} \mathbf{N}_{,a} \\ \mathbf{TN} \end{bmatrix}^T \left[\begin{bmatrix} \mathbf{C} & \mathbf{YC} \\ \mathbf{Y}^T \mathbf{C} & \mathbf{Y}^T \mathbf{C} \mathbf{Y} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{H} + \boldsymbol{\Xi} \end{bmatrix} \right] \begin{bmatrix} \mathbf{N}_{,a} \\ \mathbf{TN} \end{bmatrix} d\Omega \quad (5.5)$$

Since $\mathbf{F}^{eT}$ is only a transpose of incompatible modes coupling matrix $\mathbf{F}^e$, only the latter one is calculated and is resolved in the condensation process of incompatible modes.

$$\mathbf{F}^e = \int_{\Omega} \left\{ \mathbf{G}^T \right\} \left[\begin{bmatrix} \mathbf{C} & \mathbf{CY} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{X} \end{bmatrix} \right] \begin{bmatrix} \mathbf{N}_{,a} \\ \mathbf{TN} \end{bmatrix} d\Omega \quad (5.6)$$

Besides, there are the incompatible modes residual and the incompatible modes tangent matrix in the second equilibrium equation

$$\mathbf{h}^e = \int_{\Omega} \mathbf{G}^T \mathbf{p} d\Omega$$

$$\mathbf{H}^e = \int_{\Omega} \mathbf{G}^T \mathbf{C} \mathbf{G} d\Omega \quad (5.7)$$

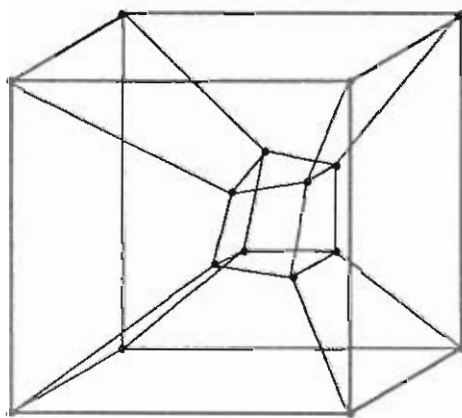
Second equilibrium equation in (5.1) can be eliminated at the element level. Still, if it is done in a usual way one needs to keep matrices $\mathbf{H}^e$, $\mathbf{h}^e$, $\mathbf{F}^e$ and $\boldsymbol{\alpha}$ in a secondary storage because of the update procedure. On the other hand, if one applies the operator split method as in the plasticity analysis (see Ibrahimbegović and Kožar [1994]) equations can be solved independently and only the vector $\boldsymbol{\alpha}$ has to be kept in the storage with the enormous benefit that there are now only 9 parameters per element to be saved instead of 531.

6 Numerical Examples

A series of calculations has been performed to provide us an insight into element's behavior. All examples are solved using FEAP finite element program and Newton incremental-iterative procedure. As it can be seen from the result tables element exhibits quadratic convergence rate.

6.1 Patch test

Standard patch test has been performed to check for constant strain capabilities. There are seven solid elements in a unit cube.



Unit cube
with 7 solid
finite elements.

Minimum
boundary
conditions.

Fig.1. Patch test example.

Boundary conditions are kept to a minimum with only 6 restraints preventing rigid body movements. For a force $P = 1000$, $E = 1000$ and $\mu = 0.1$ large displacement solution of 1.0 and stretching of 0.1 are obtained in a single iteration.

6.2 Real three-dimensional example

Since we are employing vector-like parameterization in our approach it is important to test the element in an example where all three rotations are present. Three-dimensional curved beam first analyzed by Bathe and Bolourchi [1979] has been chosen as a test example for true 3-D large deformation performance test. As it is known beam is curved in plane by a radius of 100 and angle of 45° and loaded normally to that plane (see Fig.2).

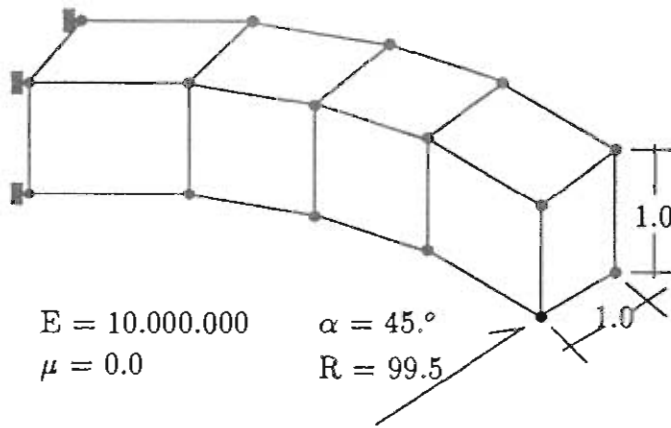


Fig.2. 3-D cantilever beam by Bathe and Bolourchi.

Results are averaged to the axis of the beam and compared with the solution obtained by Bathe and Bolourchi in the following table.

Table 1.- Displacements for the 3-D cantilever beam

Load inc.	u1	u2	u3
$P = 300$			
present	-11.81	7.07	40.02
Bathe,Bolourchi	-11.50	6.80	39.50
$P = 600$			
present	-23.30	13.64	53.21
Bathe,Bolourchi	-23.50	13.40	53.40

Beam has been modeled with 16 solid elements with 8 nodes each (in contrast to original discretization of Bathe and Bolourchi [1979] who used 16 20 node elements). Only 6 load increments with force of 100 each has been applied and quadratic convergence rate has been obtained. Convergence rates for the last load increment are presented in a table (other load steps have very similar convergence).

Table 2.- Convergence rates for the 45-degree bend

Iter. No.	Residual norm	Energy norm
0	5.000×10^1	3.509×10^2
1	6.296×10^5	1.607×10^4
2	6.603×10^1	4.306×10^0
3	7.617×10^3	2.419×10^0
4	5.016×10^{-2}	2.668×10^{-6}
5	7.463×10^{-3}	3.978×10^{-12}
6	4.716×10^{-7}	3.054×10^{-18}

6.3 Stability analysis

Elastic beam column subjected to increasing axial compression is analyzed in this example. In order to perform stability analysis a small perturbation load is applied at the free end of the beam. Model has only 5 solid elements since the buckling shape is simple.

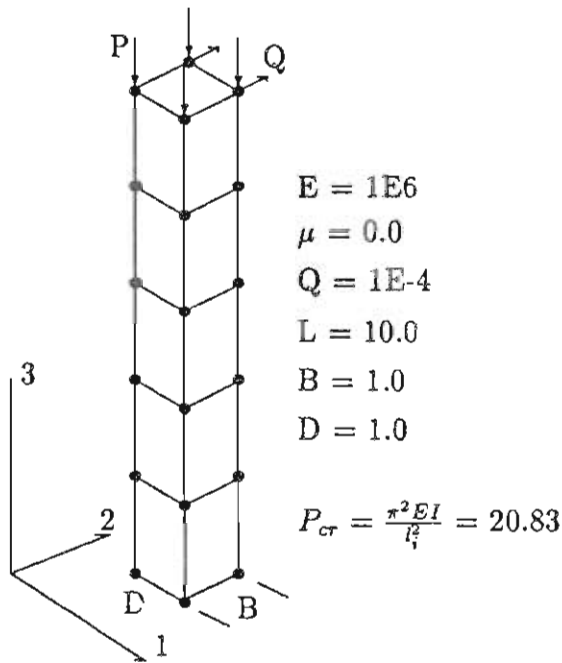


Fig.3. Beam-column buckling.

Very good prediction of beam-column behavior has been obtained.

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ON SHELL FINITE ELEMENTS WITH AN ADDITIONAL DRILLING DEGREE OF FREEDOM

SUMMARY

Inclusion of an additional rotational degree of freedom to the plane stress analysis is presented. According to this formulation, an isoparametric C^0 finite shell element with six degrees of freedom per node is derived. The element exhibits excellent accuracy characteristics. In combination with finite beam element, very efficient numerical model for analysis of complex structures consisting of shells, beams and columns is developed.

O KONAČNIM ELEMENTIMA LJUSKE S ROTACIJSKIM STUPNJEM SLOBODE U SVOJOJ RAVNINI

SAŽETAK

U ovom radu prikazan je postupak uvođenja dodatnog stupnja slobode rotacije u analizu ravninskog stanja naprezanja. Iz takve formulacije je izveden izoparametrijski C^0 konačni element ljuske sa šest stupnjeva slobode u svakom čvoru. Ovaj element pokazuje svojstva visoke preciznosti. U kombinaciji s grednim konačnim elementom dobiva se veoma efikasan numerički model za analizu složenih konstrukcija sastavljenih od ljuski i stupova.

1. INTRODUCTION

The most frequently used shell finite elements are based on the assumption that the rotation around the normal to the shell mid-surface is constrained. In the analysis of a complex structure composed of shells/plates and beams/columns, this constrain prevents the compatibility requirement between adjacent the shell elements and beam finite elements. The same problem arises when the normals to the shell mid-surfaces in a common node of two adjacent shell elements have different directions. In such cases an additional procedure has to be employed to satisfy the compatibility requirements (e.g. "master and slave nodes" at the contact boundary). Introducing a "drilling" degree of freedom at shell element nodes, which is considered here, these difficulties can be easily overcome.

The need to introduce an additional "drilling" or rotational degree of freedom around the normal to shell plane has been present for some time already. There exist different formulations and algorithms to solve this task, but they are generally either too complicated or not accurate enough, therefore they are not convenient to insert into the already developed programmes.

The proposed formulation presented here is based on the analogous concept to the introduction of an additional rotational degree of freedom in the plane stress analysis. This formulation is relatively simple and can ensure accurate results. In the following sections the formulation is briefly described and two numerical examples present its practical merit.

2. EQUILIBRIUM EQUATIONS

The starting point is based on the linear theory for the equilibrium of the isotropic elastic body in a plane stress state.

In general, the equilibrium equation for an infinitesimal volume of the homogeneous isotropic elastic body under body forces and moments, can be written as follows [1]:

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_{\alpha}} + B_{\alpha} = 0 \quad (1)$$

$$\frac{\partial m_{\alpha\beta}}{\partial x_{\alpha}} + e_{\beta\gamma\delta} \cdot \sigma_{\gamma\delta} + C_{\beta} = 0$$

where $\sigma_{\alpha\beta}$ is the asymmetric stress tensor, x_{α} the coordinate direction α , B_{α} the component of a body force, $m_{\alpha\beta}$ is the bending moment tensor, $e_{\beta\gamma\delta}$ a permutation tensor and C_{β} is the component of the moment distributed through the body.

For a plane stress problem (e.g. in the x, y coordinate system), the system of equations (1) can be expressed in the form of the Cosserat equilibrium equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + B_x = 0 \quad (2)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0$$

$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \tau_{xy} - \tau_{yx} + C_z = 0 \quad (2a)$$

In the following, the constant moment will be assumed $\left(\partial m_{xz}/\partial x = \partial m_{yz}/\partial y = 0\right)$, which in general does not crucially affect the procedure. However, it will essentially simplify the procedure. The direct consequence of the simplification is a new shape of the moment equilibrium equation (2a) which can now be written as:

$$\tau_{xy} - \tau_{yx} + C_z = 0 \quad (3)$$

From the equation (3) it is seen that a difference between the shear components is entirely due to the influence of the moment distributed through the body. For example, if $C_z \neq 0$, then the shear tensor is an asymmetric one.

The mechanism of deformation in the case of an asymmetric tensor ($\tau_{xy} \neq \tau_{yx}$) is shown in Figure 1.

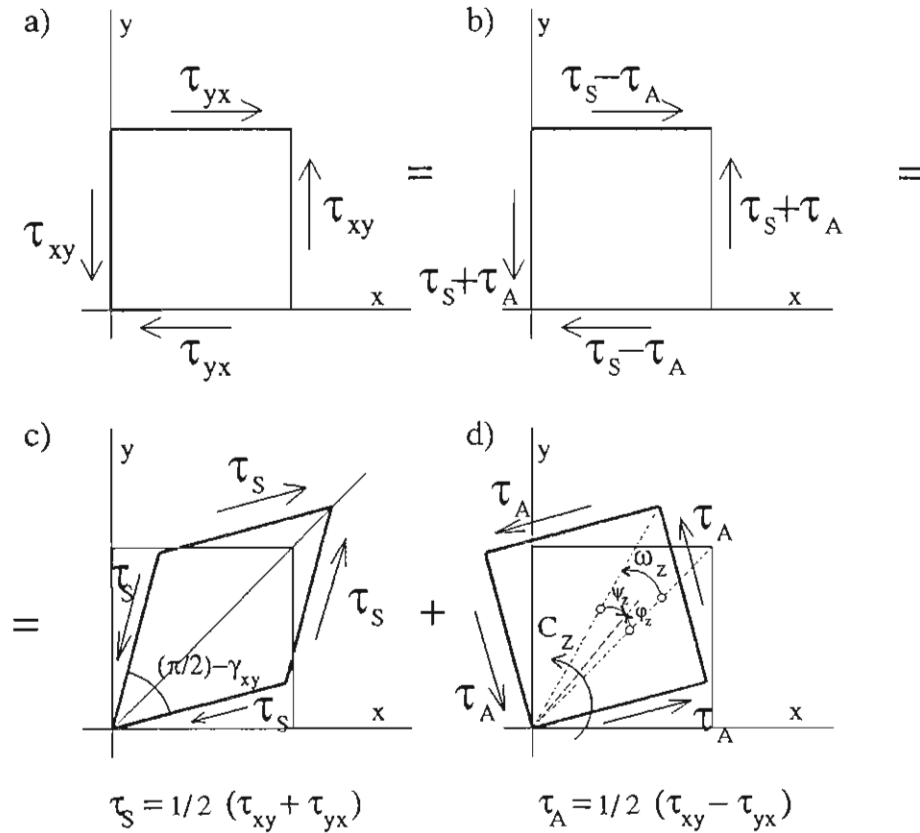


Figure 1. Deformation due to an asymmetric stress tensor

The asymmetric shear tensor in Figure 1.b) can be split into the symmetric τ_S and asymmetric part τ_A according to the Mindlin theory.

The symmetric part of the shear stress causes a shear deformation (Figure 1.c):

$$\gamma_{xy} = \frac{1}{G} \tau_S \quad (4)$$

The skew part of shear stress τ_A , if different from zero, causes a local rotation of the rigid body φ_z , shown in Figure 1.d).

The global rotation of the infinitesimal rectangular body around the z axis is defined by the angle ω_z , expressed with the u, v displacements. The angle φ_z of the local rotation of the infinitesimal volume considered as the rigid body can be defined when a new degree of freedom is introduced in the form of an unknown rotation function ψ_z , so that:

$$\varphi_z = \omega_z - \psi_z \quad (5)$$

The relation between the shear stress τ_A and the local rotation of the rigid body φ_z , can be defined empirically [3],[5],[7], with the parametre γ . According to the results of numerical examples it is shown that this parametre is equivalent to the shear modulus G .

Hence, if the local rotation of an infinitesimal volume is considered separately as a rigid body, it is not possible to establish a unique relationship between shear stress τ_A and the angle of local rotation φ_z . However, if the surrounding continuum of the infinitesimal volume is taken into account, which restrains the local rotation of the considered infinitesimal volume by the shear stress τ_A at the respective boundaries, then, according to Fig. 2, the law of the local "slipping" can be given as:

$$\tau_A = G \cdot \varphi_z \quad (6)$$

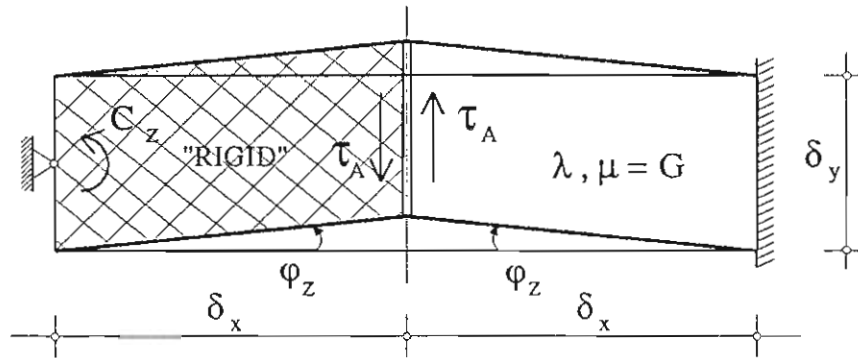


Figure 2. Model of local rotation of the rigid body

3. VARIATIONAL FORMULATION

Using the former equations for the two-dimensional space Ω , the equations (2) can be expressed as:

$$\text{div } \sigma + f = 0 \quad (7)$$

at which point we abandon the usual assumption that the tensor σ_{ij} is symmetric. A general asymmetric tensor is introduced, which, employing additive decomposition, can be written in the following form:

$$\sigma_{ij} = \text{symm } \sigma_{ij} + \text{skew } \sigma_{ij} \quad (8)$$

where:

$$\text{symm } \sigma_{ij} = \frac{1}{2} (\sigma_{ij} + \sigma_{ji}) \rightarrow \text{symm } \sigma = \begin{vmatrix} \sigma_{11} & \tau_s \\ \tau_s & \sigma_{22} \end{vmatrix} \quad (9)$$

and

$$\text{skew } \sigma_{ij} = \frac{1}{2} (\sigma_{ij} - \sigma_{ji}) \rightarrow \text{skew } \sigma = \begin{vmatrix} 0 & -\tau_A \\ \tau_A & 0 \end{vmatrix} \quad (10)$$

In the case of the plane stress state and isotropic material, the material properties are described by the tensor $C = \{C_{ijkl}\}$:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad i, j, k, l \in \{1, 2\} \quad (11)$$

where λ and μ are the Lamé constants, expressed by the Young's modulus and Poisson's ratios as follows:

$$\lambda = \frac{\nu E}{(1 - \nu^2)} \quad (12)$$

$$\mu = \frac{E}{2(1 + \nu)} \quad (13)$$

In the same way as the decomposition of the stress tensor (8) is made, the tensor of the gradient of the displacement field is decomposed into its symmetric and antisymmetric parts. It is well known that the symmetric part has a direct influence on the stress, while the antisymmetric part defines the rotation of the rigid body, and does not create stress. However, it is necessary to notice that the local rotation of the rigid body with respect to the moment stresses leads to a change in the stress state. Introducing a new unknown function of the rotation ψ_z to describe the new degree of freedom, the strain tensor and local rotation have the following form:

$$\nabla \mathbf{u} - \text{skew} \psi_z = \frac{1}{2} \begin{bmatrix} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) & \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \\ \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \\ \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) & 0 \end{bmatrix} - \begin{bmatrix} 0 & \psi_z \\ -\psi_z & 0 \end{bmatrix} \quad (14)$$

$\xleftrightarrow{\hspace{15em}}$
 local rotation of a rigid body

where:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad ; \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \quad ; \quad \gamma_{12} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \quad (15)$$

$$\omega_z = \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \quad ; \quad \varphi_z = \omega_z - \psi_z \quad (16)$$

Using Eqs. (8) - (9) the relationship between generalised displacements and stresses can be finally written in the matrix form:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_s \\ \vdots \\ \tau_A \end{bmatrix} = \begin{bmatrix} \lambda + \mu & \lambda & 0 & 0 \\ \lambda & \lambda + \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ \hline 0 & 0 & 0 & \mu \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \\ \varphi_z \end{bmatrix} \quad (17)$$

or shorter:

$$\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\epsilon} \quad (18)$$

which allows that the variational formulation can be written in a very simple way:

$$\Pi(u, v, \varphi_z) = \frac{1}{2} \int_{\Omega} \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} \, d\Omega - \int_{\Omega} \mathbf{u}^* \mathbf{f}^* \, d\Omega \quad (19)$$

where $\mathbf{u}^*$ and $\mathbf{f}^*$ are the generalised vectors of displacements and forces, respectively.

4. NUMERICAL EXAMPLES

4.1 Bending of a simply supported beam

A simply supported beam under the two types of bending loading conditions, i.e. moments and pair of forces, is analysed. The results are compared with analytical obtained by the Euler-Bernoulli theory, and with numerical results from Ref. [3] where in despite to the similar name different elements were used. The influence of different mesh discretisation is also analysed. Comparison of the results is made in Table 1.

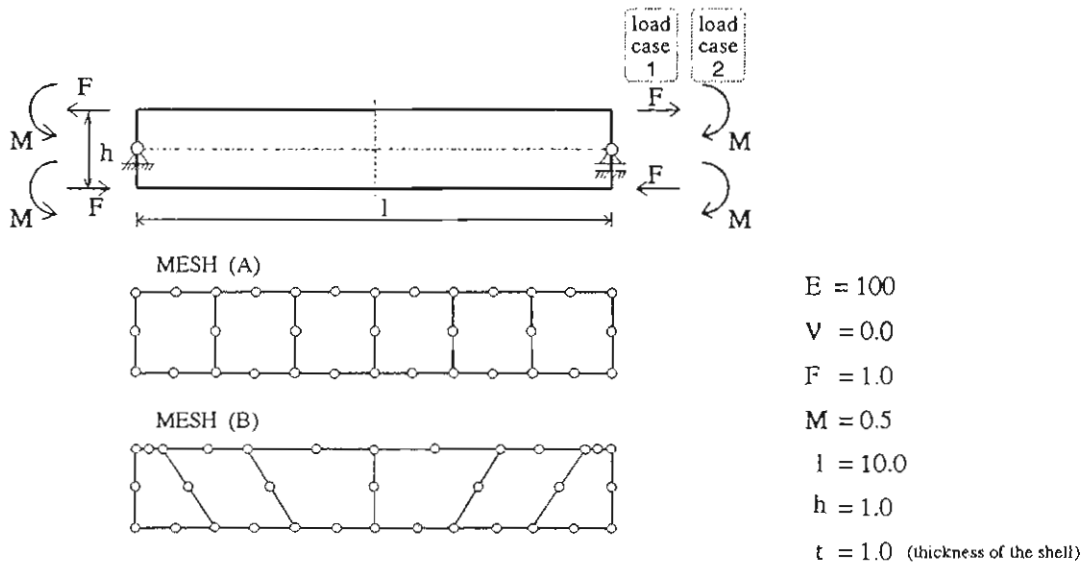


Figure 3. Simply supported beam

Table 1.

MESH	LOAD CASE	Reference [3]		This work	
		DISPLACEMENT	ROTATION	DISPLACEMENT	ROTATION
mesh (A)	load case 1	1.5	0.6	1.5	0.6
mesh (B)	load case 1	1.3227	0.56759	1.48775	0.59776
mesh (A)	load case 2	1.5	0.62983	1.52154	0.61449
mesh (B)	load case 2	1.3276	0.65514	1.49008	0.62037
analytical solution		1.5	0.6	1.5	0.6

4.2 Horizontal square plate supported on one vertical column

A horizontal square plate of 0.15 m thickness supported on a vertical column at one corner is analysed. The ground plane of this structure is shown in Fig. 4. The cross-section of the column is marked with a dotted line. The column compared to the plate represents a rigid body. The plate is loaded with two horizontal forces which create a "drilling" moment at node 1. This example is chosen to represent the problem which can arise when these shell elements are connected with a beam element representing a column.

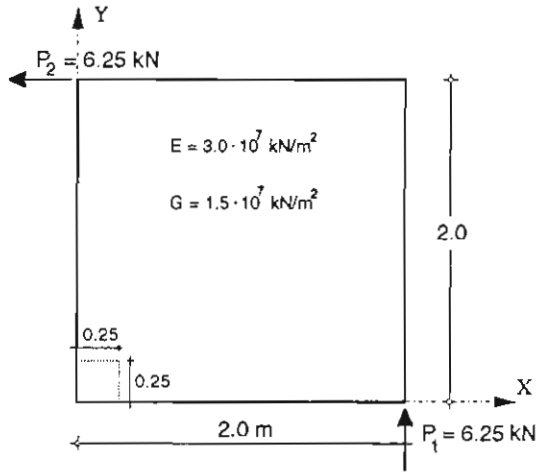


Figure 4. Geometry of a horizontal square plate with loading forces

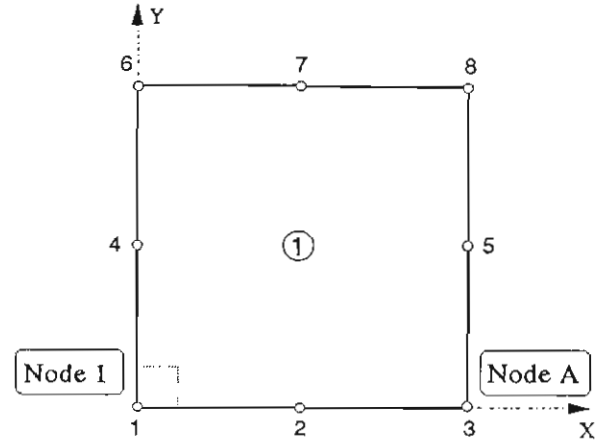


Figure 5. Finite element idealisation of the plate with one 8-node shell element

The plate is discretised with 1 (Fig. 5), 4, 16 and 64 (Fig. 6) 8-node shell elements. The column is idealised with a 2-node beam element. The horizontal displacement of node A and the angle of rotation around the z axis at node 1 for different meshes are listed in Table 2.

Table 2.

MESH	1×1	2×2	4×4	8×8
v_{Ay}	2.202	3.99	11.8	43.73
φ_{1z}	0.7565	0.7565	0.7565	0.7565

It is immediately seen that the angle of rotation is the same for all the four meshes considered. However, the displacement of node A is greater when the element size is smaller, i.e. when a mesh is more refined. The reason for such results, which are unusual for the finite element methods, is due to the way in which the drilling moment is transferred from the plate to the column. Namely, the plate idealised with shell element(s) is connected with the column idealised with a beam element at one single node. When the shell element size is smaller (a refined mesh) the drilling moment is distributed through a smaller volume so that the local effect of concentrated stresses is more accentuated. This is in agreement with the formulation of the drilling rotational degree of freedom as is described in Section 2. To overcome this difficulty it is necessary that the numerical model represents only the real structure which means that a stiff link between plate and column should not only be in one node but all over the area of the column's cross-section. The system is therefore analysed with the mesh of 8×8 shell elements where the size of element 1 is equal to the column's cross section (Fig. 6). The elastic properties of element 1 are multiplied by 10, 100 and 1000. This increase in the material properties can be justified with the real stiffness of the "plate" element 1 which is, in fact, the column. Numerical results obtained are presented in Table 3.

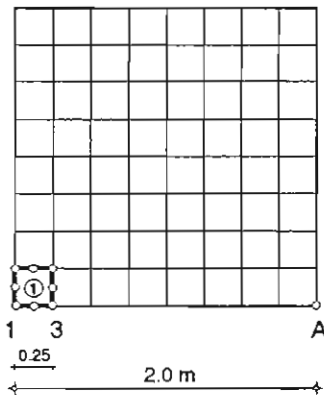


Figure 6. Mesh 8×8

Table 3.

STIFFNESS OF THE ELEMENT ①	(E_1, G_1)	$(E_1, G_1) \times 10$	$(E_1, G_1) \times 10^2$	$(E_1, G_1) \times 10^3$
φ_{1z}	0.7565	0.7565	0.7565	0.7565
φ_{3z}	23.53	3.545	1.054	0.7865
v_{Ay}	43.73	6.756	2.492	2.049

From Table 3. it can be seen that the value of the displacement v_{Ay} is smaller as the element stiffness increases. It can be assumed that the element 1, as a "part" of the column, has a rigid rotation which means that the rotation angle at node 1 and node 3 should be equal. This is true for element stiffness which is magnified 100-1000 times.

The above assumption is tested with a coarse mesh as shown in Fig.7. The plate is discretized with three elements, element 1 being of the column cross section size. The results are shown in Table 4, where once again it can be seen that the analysis performs better when increased stiffness values for element 1 are used. Using such a conclusion and employing a more refined mesh the results will then be more accurate.

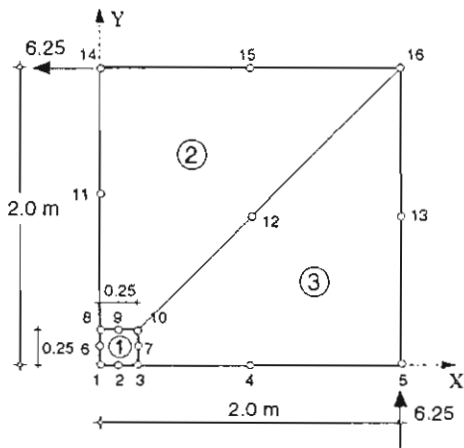


Figure 7.

Table 4.

STIFFNESS OF THE ELEMENT ①	Φ_{1z}	Φ_{3z}	v_{sy}
(E_1, G_1)	0.7565	20.57	39.97
$(E_1, G_1) \times 10$	0.7565	3.106	5.942
$(E_1, G_1) \times 10^2$	0.7565	1.037	2.142
$(E_1, G_1) \times 10^3$	0.7565	0.786	1.719

5. CONCLUSION

The procedure for a simple inclusion of an additional rotational degree of freedom to the plane stress analysis is presented. Employing this concept for the formulation of an additional "drilling" or rotational degree of freedom around the normal to the shell plane, a finite shell element with six degrees of freedom per node is obtained. Such a shell element fully satisfies the displacement and rotation compatibility when linked with a beam finite element. It exhibits high accuracy for regular and distorted meshes. Problems with "drilling" moments, apart of another type of load, can be successfully analysed. The proposed numerical model enables the analysis of complex structures consisting of shells and beams/columns.

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Development of new quadrilateral plane element with three d.o.f. per node

S U M M A R Y

This paper presents a development procedure of new quadrilateral plane element with three degrees of freedom per node, e.g. two displacements and one in-plane rotation per node. In the development procedure different combinations of functions, e.g. linear, parabolic and hermitian, describing displacements and in-plane rotations are tested. Some obtained results of the previously developed plane elements are shown and appropriate conclusions are drawn.

1. INTRODUCTION

The process of developing new four-node quadrilateral finite elements is continued, but is that necessary? Namely, the four-node elements with two translational degrees of freedom per node seem to be very practical, accurate and popular.

The use of drilling degrees of freedom per node to improve performance of plane element has been often tested lately. Developed elements yield quite remarkable results and improvements although they are based on different approaches [1].

This paper presents a development procedure of a new quadrilateral four-node plane element with three degrees of freedom per node, e.g. two translations/displacements and one in-plane rotation per node, often called a drilling degree of freedom. In the development procedure different combinations of functions, e.g. linear, parabolic and hermitian, describing displacements and in-plane rotations are tested.

Developed elements, with different combinations of functions, were tested on a few simple test problems discretized with different meshes. The obtained results were compared with the known analytical results and with themselves. Some relevant conclusions were drawn concerning future investigations due to development of appropriate shell element with six degrees of freedom per node [2-3].

2. INVESTIGATED COMBINATIONS OF FUNCTIONS

The geometry and coordinate systems of the used four-node quadrilateral finite element are presented in Figure 1.

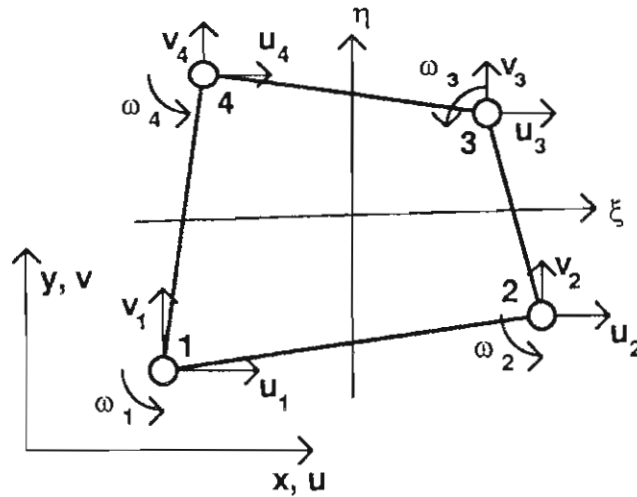


Fig. 1 Four-node quadrilateral element

In order to investigate the influence of different functions for displacements and in-plane rotations, the different combinations of linear, parabolic and hermitian functions were tested.

The geometry mapping for all tested elements was performed with standard bilinear functions in the form:

$$N_g = \frac{1}{4}(1-\xi)(1-\eta) \quad (1)$$

The next combinations of linear (L), parabolic (P) and hermitian (H) functions were tested (the expressions are listed only for the first node of the element):

Combination: 4QL-HL-HL

Displacements: product of hermitian 3rd order and linear functions:

$$\begin{aligned} N_u^{(1)} &= -\frac{1}{8}(\xi-1)\left[3(\eta-1)^2 + (\eta-1)^3\right] \\ N_v^{(1)} &= -\frac{1}{8}(\eta-1)\left[3(\xi-1)^2 + (\xi-1)^3\right] \end{aligned} \quad (2)$$

In-plane rotation: product of hermitian 3rd order and linear functions:

$$\begin{aligned} N_\omega^{(1)} &= \frac{1}{8}(\xi-1)\left[4(\eta+1) - 4(\eta+1)^2 + (\eta+1)^3\right] \\ N_\omega^{(1)} &= -\frac{1}{8}(\eta-1)\left[4(\xi+1) - 4(\xi+1)^2 + (\xi+1)^3\right] \end{aligned} \quad (3)$$

Combination: 4QL-HL-HP

Displacements: product of hermitian 3rd order and linear functions:

$$\begin{aligned} N_u^{(1)} &= -\frac{1}{8}(\xi-1)\left[3(\eta-1)^2 + (\eta-1)^3\right] \\ N_v^{(1)} &= -\frac{1}{8}(\eta-1)\left[3(\xi-1)^2 + (\xi-1)^3\right] \end{aligned} \quad (4)$$

In-plane rotation: product of hermitian 3rd order and parabola 2nd order functions:

$$\begin{aligned}
N_u^{(1)} &= \frac{1}{8}(\xi - \xi^2) \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^3 \right] \\
N_v^{(1)} &= -\frac{1}{8}(\eta - \eta^2) \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right]
\end{aligned} \tag{5}$$

Combination: 4QL-HL-HP1

Displacements: product of hermitian 3rd order and linear functions:

$$\begin{aligned}
N_u^{(1)} &= -\frac{1}{8}(\xi - 1) \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \\
N_v^{(1)} &= -\frac{1}{8}(\eta - 1) \left[3(\xi - 1)^2 + (\xi - 1)^3 \right]
\end{aligned} \tag{6}$$

In-plane rotation: product of hermitian 3rd order and parabola 2nd order functions:

$$\begin{aligned}
N_u^{(1)} &= -\frac{1}{16}(\xi^2 - 2\xi + 1) \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^2 \right] \\
N_v^{(1)} &= \frac{1}{16}(\eta^2 - 2\eta + 1) \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right]
\end{aligned} \tag{7}$$

Combination: 4QL-HL-HP2

Displacements: product of hermitian 3rd order and linear functions:

$$\begin{aligned}
N_u^{(1)} &= -\frac{1}{8}(\xi - 1) \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \\
N_v^{(1)} &= -\frac{1}{8}(\eta - 1) \left[3(\xi - 1)^2 + (\xi - 1)^3 \right]
\end{aligned} \tag{8}$$

In-plane rotation: product of hermitian 3rd order and parabola 2nd order functions:

$$\begin{aligned}
N_u^{(1)} &= -\frac{1}{16}(-\xi^2 - 2\xi + 3) \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^3 \right] \\
N_v^{(1)} &= \frac{1}{16}(-\eta^2 - 2\eta + 3) \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right]
\end{aligned} \tag{9}$$

Combination: 4QL-HL-HH

Displacements: product of hermitian 3rd order and linear functions:

$$\begin{aligned}
N_u^{(1)} &= -\frac{1}{8}(\xi - 1) \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \\
N_v^{(1)} &= -\frac{1}{8}(\eta - 1) \left[3(\xi - 1)^2 + (\xi - 1)^3 \right]
\end{aligned} \tag{10}$$

In-plane rotation: product of hermitian 3rd order and hermitian 3rd order functions:

$$\begin{aligned}
N_u^{(1)} &= -\frac{1}{16} \left[3(\xi - 1)^2 + (\xi - 1)^3 \right] \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^3 \right] \\
N_v^{(1)} &= \frac{1}{16} \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right]
\end{aligned} \tag{11}$$

Combination: 4QL-HH-HL

Displacements: product of hermitian 3rd order and hermitian 3rd order functions:

$$N_u^{(1)} = N_v^{(1)} = \frac{1}{16} \left[3(\xi - 1)^2 + (\xi - 1)^3 \right] \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \tag{12}$$

In-plane rotation: product of hermitian 3rd order and linear functions:

$$\begin{aligned} N_u^{(1)} &= \frac{1}{8}(\xi - 1) \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^3 \right] \\ N_v^{(1)} &= -\frac{1}{8}(\eta - 1) \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right] \end{aligned} \quad (13)$$

Combination: 4QL-HH-HP

Displacements: product of hermitian 3rd order and hermitian 3rd order functions:

$$N_u^{(1)} = N_v^{(1)} = \frac{1}{16} \left[3(\xi - 1)^2 + (\xi - 1)^3 \right] \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \quad (14)$$

In-plane rotation: product of hermitian 3rd order and parabola 2nd order functions:

$$\begin{aligned} N_{ru}^{(1)} &= \frac{1}{8}(\xi - \xi^2) \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^3 \right] \\ N_{rv}^{(1)} &= -\frac{1}{8}(\eta - \eta^2) \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right] \end{aligned} \quad (15)$$

Combination: 4QL-HH-HH

Displacements: product of hermitian 3rd order and hermitian 3rd order functions:

$$N_u^{(1)} = N_v^{(1)} = \frac{1}{16} \left[3(\xi - 1)^2 + (\xi - 1)^3 \right] \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \quad (16)$$

In-plane rotation: product of hermitian 3rd order and hermitian 3rd order functions:

$$\begin{aligned} N_{ru}^{(1)} &= -\frac{1}{16} \left[3(\xi - 1)^2 + (\xi - 1)^3 \right] \left[4(\eta + 1) - 4(\eta + 1)^2 + (\eta + 1)^3 \right] \\ N_{rv}^{(1)} &= \frac{1}{16} \left[3(\eta - 1)^2 + (\eta - 1)^3 \right] \left[4(\xi + 1) - 4(\xi + 1)^2 + (\xi + 1)^3 \right] \end{aligned} \quad (17)$$

3. FINITE ELEMENT FORMULATION

The before mentioned functions were incorporated into an adequate numerical algorithm for the analysis of two-dimensional plane stress and plane strain states based on the standard numerical algorithm [4].

The displacement field can be finally described as:

$$\begin{Bmatrix} u(\xi, \eta) \\ v(\xi, \eta) \end{Bmatrix} = [N_1, N_2, N_3, N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad (18)$$

where: $N_i = \begin{bmatrix} N_u^i & 0 & N_{ru}^i \\ 0 & N_v^i & N_{rv}^i \end{bmatrix}$ and $\{u_i\}^T = [u_i, v_i, \omega_i]$ are shape functions and local displacements.

The deformations are described with the expression:

$$\varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_{xy}]^T \quad (19)$$

where:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (20)$$

The constitutive law has a form:

$$\sigma = D \epsilon \quad (21)$$

with the forces vector:

$$\sigma = [\sigma_x, \sigma_y, \sigma_{xy}]^T \quad (22)$$

The elasticity matrix for linear elastic, isotropic material under plane stress conditions is:

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (23)$$

According to the standard procedure the strain matrix is:

$$\epsilon = B u \quad (24)$$

where strain matrix B has a form:

$$B = [B_1, B_2, B_3, B_4] \quad (25)$$

with submatrix belonging to the i -th node:

$$B_i = \begin{bmatrix} \frac{\partial N_u^i}{\partial x} & 0 & \frac{\partial N_{ru}^i}{\partial x} \\ 0 & \frac{\partial N_v^i}{\partial y} & \frac{\partial N_{rv}^i}{\partial y} \\ \frac{\partial N_u^i}{\partial x} & \frac{\partial N_v^i}{\partial y} & \left(\frac{\partial N_{ru}^i}{\partial y} + \frac{\partial N_{rv}^i}{\partial x} \right) \end{bmatrix} \quad (26)$$

Further procedure follows standard finite element technique [4-7] and will not be repeated herein.

4. SOME PERFORMANCE TESTS OF FINITE ELEMENTS

Two simple and standard problems were chosen for testing developed functions, e.g. : (a) plane 2-D cantilever; (b) plane 2-D simple beam. These investigations, on both problems, were performed with different domain discretizations. The geometrical and material data for the problems are presented in Figures 2 and 3 together with used meshes.

EXAMPLE 1: Plane 2-D cantilever

The cantilever beam, illustrated in Figure 2, is often used to show the ability of plane elements to represent bending deformations. The cantilever beam was analyzed using three different regular meshes, see Figure 2. The material properties were: Young's modulus of elasticity $E = 1000$ MPa, Poisson's ratio $\nu = 0.3$, thickness $t = 1.0$, and continuously distributed shear force $P = 100$ kN.

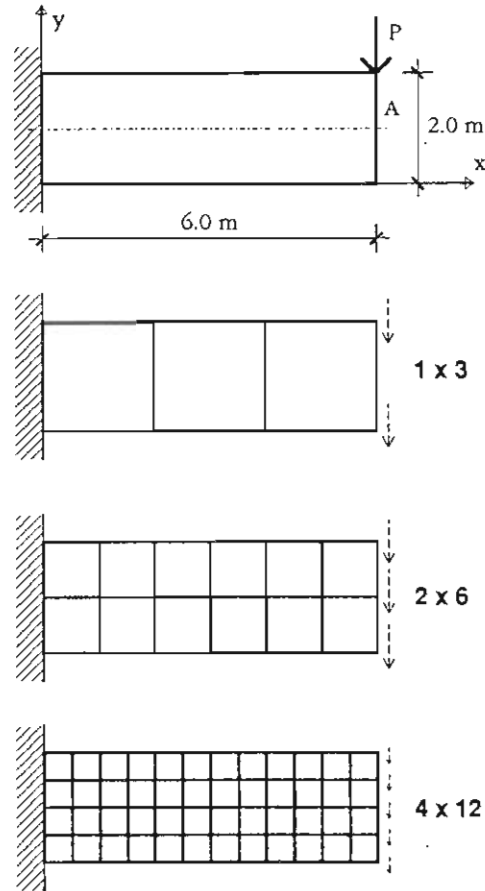


Fig. 2 Plane 2-D cantilever: geometrical data and meshes

Some results of the performed tests on 2-D plane cantilever, vertical displacement of point A at the free end of the cantilever, are presented in Table 1 for different combinations of functions and different meshes.

Table 1. Vertical displacement of point A on the cantilever

Combination	Vertical displacement of point A - δ_A		
	Mesh 1 $\times$ 3	Mesh 2 $\times$ 6	Mesh 4 $\times$ 12
4QL-HL-HL	10.668	11.740	11.668
4QL-HL-HP1	5.614	7.018	7.268
4QL-HL-HP2	8.982	9.842	9.940
4QL-HL-HH	10.446	11.614	11.630
4QL-HH-HL	8.524	9.368	9.515
4QL-HH-HH	9.238	9.810	9.672
Analytical	11.625		

EXAMPLE 2: Plane 2-D simple beam

The presented elements have also been used to solve a simple example, i.e. a simple beam, (Figure 3). The simple beam was also analyzed using three different regular meshes, see Figure 3. The material properties were: Young's modulus of elasticity $E = 1000$ MPa, Poisson's ratio $\nu = 0.3$, thickness $t = 1.0$, and continuously distributed vertical loading $q = 20$ kN/m.

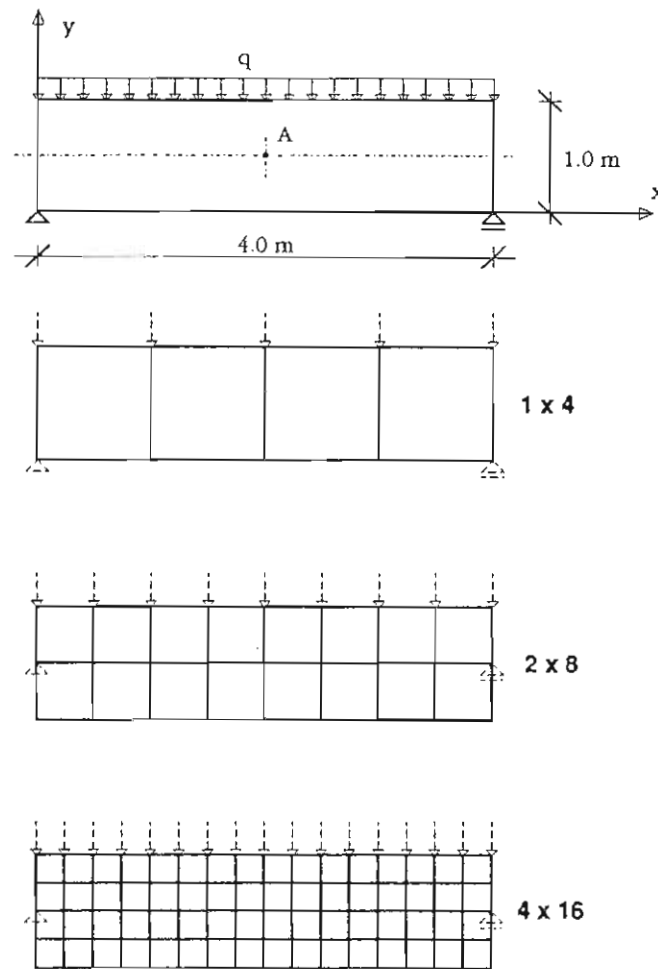


Fig. 3 Plane 2-D simple beam: geometrical data and meshes

Some results of the performed tests on 2-D plane simple beam, vertical displacement of point A at the mid-span of the simple beam, are presented in Table 2 for different combinations of functions and different meshes.

Table 2. Vertical displacement of point A on the simple beam (mid-span)

Combination	Vertical displacement of point A - δ_A		
	Mesh 1 $\times$ 4	Mesh 2 $\times$ 8	Mesh 4 $\times$ 16
4QL-HL-HL	0.845	0.878	0.919
4QL-HL-HP1	0.547	0.654	0.715
4QL-HL-HP2	0.769	0.814	0.855
4QL-HL-HH	0.823	0.873	0.917
4QL-HH-HL	0.663	0.718	0.762
4QL-HH-HH	0.730	0.735	0.768
Analytical	0.914		

The problem of simple beam has been tested with the meshes similar to the one investigated by Ibrahimbegović [8, 9] but due to the shortage of space will not be presented herein. Both, regular and irregular (distorted) meshes were tested there [10].

5. CONCLUSIONS

This paper presents a development procedure of new quadrilateral plane element with three degrees of freedom per node, e.g. two displacements and one in-plane rotation per node. In the development procedure different combinations of functions, e.g. linear, parabolic and hermitian, were tested. The investigations were performed on two simple problems; plane 2-D, cantilever and simple beam.

Summarizing all performed tests and results (only some are presented herein) it was concluded that the most suitable and acceptable combinations of functions for inclusion in 2-D plane numerical algorithm are the combinations 4QL-HL-HL and 4QL-HL-HH.

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Accuracy of the new fully compatible quadrilateral shell FE

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SUMMARY

*This paper presents results of the accuracy research of the new fully compatible quadrilateral (NFCQ) shell finite element method. Introducing the NFCQ shell finite element, authors detected numerical phenomenon named **numerical curvature of the space**. This phenomenon appeared in plane stress systems subjected to concentrated or quasi-concentrated forces applied directly or indirectly. It depends functionally on the mesh size. This paper gives also results of the static and dynamic eigen problems, which show there does not exist lower spectrum limit because the first eigen value tends to zero if mesh density increases. Increasing of degrees of freedom of the system, leads to the conclusion that the vector space of the system is behaving as infinite space. Analytical solutions of the equilibrium problems in infinite space show that under the concentrated or quasi-concentrated loads, displacement is infinite.*

The NFCQ shell finite element method gives the same solution as the analytical one if the mesh size is sufficiently dense. Therefore the phenomenon of the numerical curvature of the space is numerical simulation of the physical reality. While modelling plane stress problems subjected to concentrated forces one must respect this fact. Aforementioned facts are verified on carefully selected examples.

1. INTRODUCTION

The NFCQ shell finite element presents one of the six degrees of freedom shell finite elements per node; three translations and three rotations. According to some authors the sixth degree of freedom is referred to as drilling degree of freedom.

The use of six degrees of freedom has been proposed by few authors since 1988, among which are [1- 8]. In order to improve the performance of membrane elements, corner rotations as third degree of freedom, have been proposed many times since 1965, mostly by Irons and Ahmad [9] and Hughes and Brezzi [10,11].

The NFCQ shell finite element [12,13] is formulation which assures the continuity of membrane displacements and the continuity of their first derivatives in the nodes which defines the full compatibility in the membrane state. The second important segment of compatibility is the continuity of transversal displacements and their first derivatives in the nodes in both directions. The continuity of the rotations means that influences, both flexural and shear, are taken into the account so that the Mindlin formulation can be followed.

Application of this method as shown in Ref. [12,13], leads to governing equilibrium equation

$$\mathbf{Ku} = \mathbf{f} \quad (1)$$

where $\mathbf{K}$ is stiffness matrix, $\mathbf{u}$ is displacement vector and $\mathbf{f}$ is force vector.

The NFCQ shell finite element method is verified on following standard tests.

1.1 Patch tests

A rectangular plate was modelled by using the single element and the skewed mesh (Fig. 1), and it was subjected to constant states of tension (Fig. 1.a), bending (Fig. 1.b) and shearing (Fig. 1.c). The stresses and displacements obtained with both meshes were exactly identical to the analytical answers.

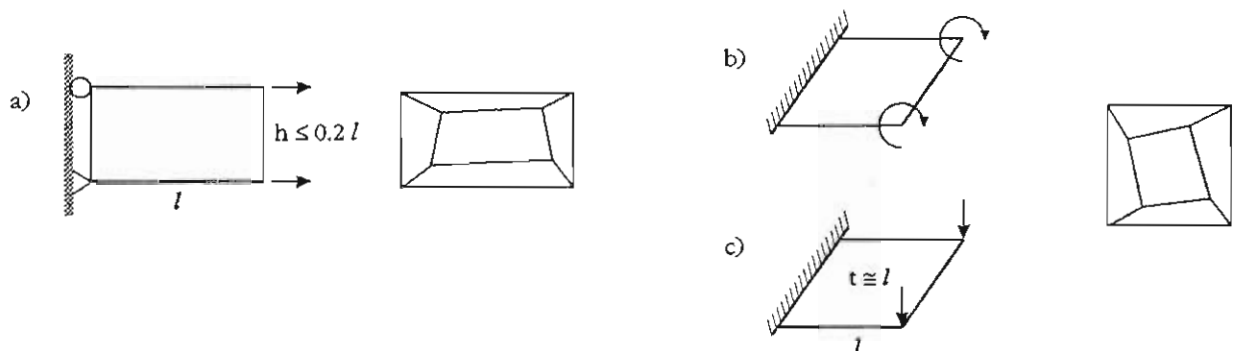


Fig. 1 Patch test meshes

1.2 Cook's membrane

A trapezoidal membrane (Figure 2), suggested by Cook [14], is widely used as a test for the accuracy of quadrilateral elements. In addition to the shear dominated loading, it also displays the effects of mesh distortion. The results for the tip deflection, varying the mesh size, are compared with the results obtained by other authors [1, 4, 5, 15] and are shown in Table 1. Two cases of loading were analyzed: (a) distributed load over free edge; (b) concentrated load in the mid point A.

The obtained results agree closely with those well known results from the literature [1, 4, 5, 14, 15].

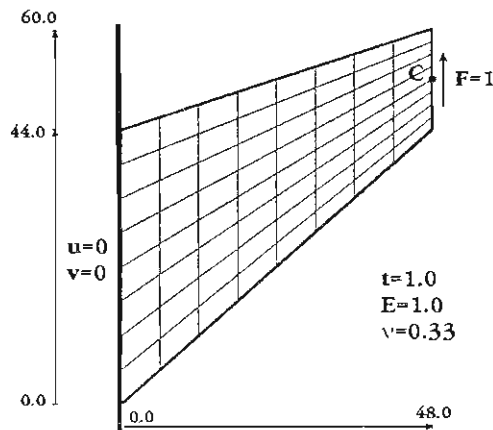


Fig. 2 Cook's membrane

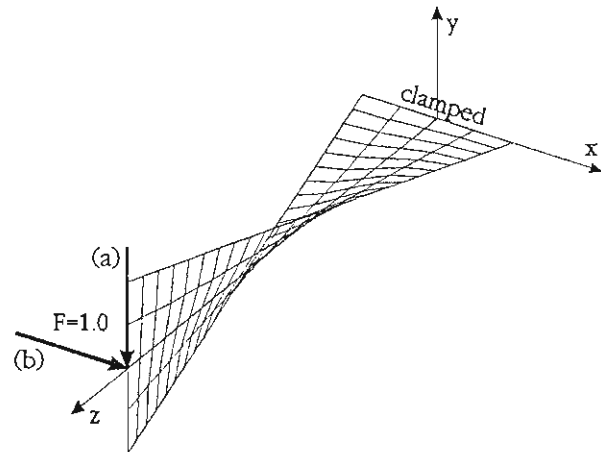


Fig. 3 Twisted thin shell

Table 1. Results of the Cook's membrane

Mesh size	Allman [5]	Simo et al. [1]	Ibrahim-begović [4]	This paper Case (a)	This paper Case (b)
1 x 1	--	16.743	14.065	18.40	18.40
2 x 2	20.27	21.124	20.683	22.30	21.00
4 x 4	22.78	23.018	22.984	23.40	23.50
8 x 8	23.56	23.685	23.626	23.80	25.70
16 x 16	--	--	--	24.00	26.50
32 x 32 [15]	23.91	23.910	23.910	23.91	23.91

1.3 Shearing of a twisted thin shell

This problem, a thin shell twisted 90° (Figure 3), was suggested by Simo et al. [1] as a test to assess the effect of warping on the performance of a shell element. A thin shell of length $l=12$, width $w=1.1$, thickness $t=0.05$, and material properties $E=29.0 \times 10^6$, $\nu=0.22$, twisted by 90° , was cantilevered and subjected to two cases of end loading: (a) unit shear load in the width direction; (b) unit shear load in the thickness direction.

By increasing the number of elements, the obtained results approached the final results from Ref. [1] but from the upper side as a consequence of geometrical description of the problem (plane elements versus twisted elements).

1.4 Hemispherical shell with 18° hole

A standard test problem of a hemispherical shell with a hole known from literature [16], is presented in Figure 4.

The analytical solutions based on asymptotic expansions appear to yield an answer of a 0.093 [1], which is used for the normalization. Table 2 lists the results of this problem and compares the proposed element performance to the mixed and 4-DKQ (discrete Kirchhoff quadrilateral element of Taylor), see Ref. [1].

The performance of the proposed element in this problem was quite remarkable. The 2×2 mesh produced 90% of the numerical answer, and the results with the 4×4 mesh were practically exact. No shear or membrane locking was detected.

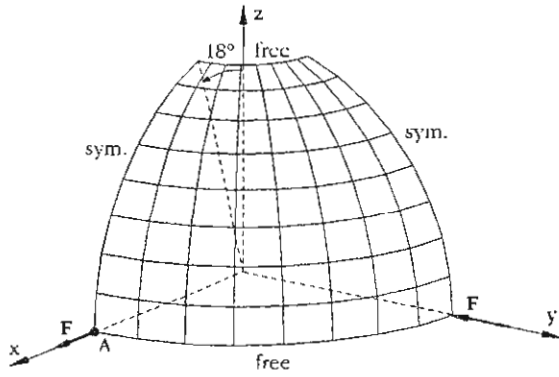


Fig. 4 Hemispherical shell with a hole: radius = 10.0; thickness = 0.04; $E = 6.825 \times 10^7$; $\nu = 0.3$; mesh = $N \times N$ (on quadrant); loading: concentrated forces $F = 1$ as shown

Table 2. Normalized results of the hemispherical shell with 18° hole

Nodes per side	Mixed [1] (%)	4-DKQ [1] (%)	This work (%)
3	91.9	66.3	89.1
5	100.4	92.8	100.9
9	99.8	101.0	-
17	99.9	100.3	100.0

2. NUMERICAL CURVATURE OF THE SPACE

2.1 Discretization sensitivity investigation

The numerical model on which the discretization sensitivity is investigated, i.e. the problem of concentrated force applied on the half plane, is called A-test and is shown in Figure 5.a.

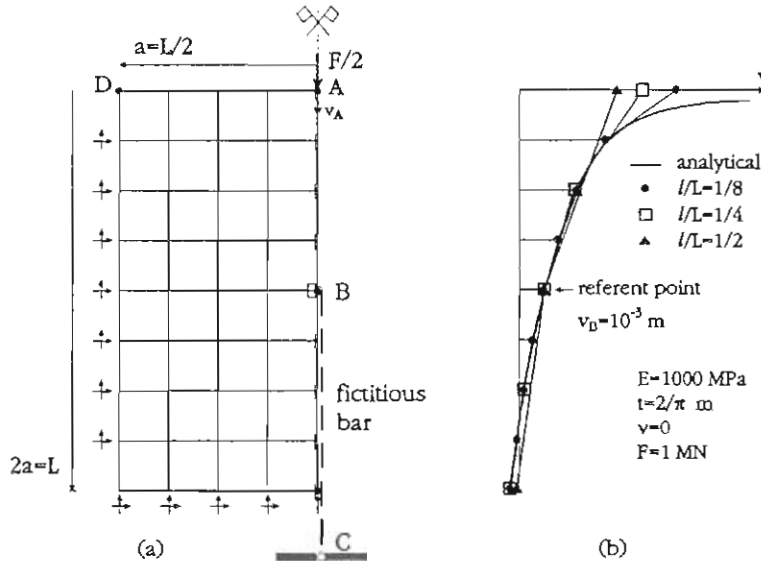


Fig. 5 A-test: (a) numerical model (mesh 1) with geometrical and material properties; (b) vertical displacement under point A for different meshes

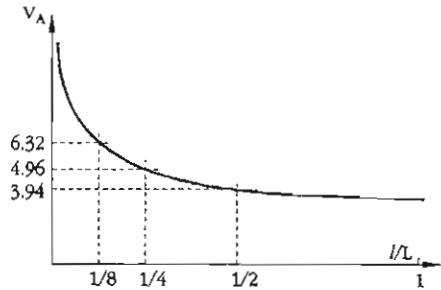


Fig. 6 Diagram: vertical displacement of the point A - mesh discretization

The model is obtained by taking a part of the half plane of size $L \times L = 2a \times 2a$. The influence of the outer part of the half plane is taken into account with stresses at the boundaries taken from the analytical solution [15] and the prescribed displacement of the reference point is achieved with a fictitious bar BC according to the same analytical solution. Three different quadratic meshes were used. Due to the symmetry, the first mesh consisted of $8/2 \times 8 = 32$ elements with a ratio $l/L = 1/8$ where l is the length of finite element. The second mesh consisted of 8 elements with a ratio $l/L = 1/4$, and the third mesh consisted of 2 elements with a ratio $l/L = 1/2$.

The obtained numerical results were compared with the analytical results [17] and some of them are presented in Figure 5.b. As illustrated in Figure 5.b, mesh 1 shows excellent agreement with theoretical results of vertical displacements for all points in the x-axis except in point A where, according to theory [17], the displacement is infinite. Meshes 2 and 3 also show good agreement. The results for every mesh are very similar for both 2×2 and 3×3 Gauss integration rules.

It is very interesting to follow vertical displacement of point A for different meshes. The non dimensional relation of the vertical displacement of point A in correlation to the mesh discretization is shown in Figure 6.

If the numerical model simulates well the real situation, then for $l/L \rightarrow 0$ one can obtain according to theory [17] that $v_A \rightarrow \infty$. Actually, the function:

$$v^0 = 1.165 \left[\ln \left(\frac{l}{L} \right)^2 - \pi \right] \quad (2)$$

presented in Figure 7 is the obtained numerical solution which satisfies analytical conditions.

The displacement of point A in a dimensional form can be obtained from the relation:

$$v_A = v^0 \frac{F}{Et} \quad (3)$$

where F is the applied concentrated force at point A.

According to [17], the vertical displacement of point A in relation to the displacement of point D at the finite distance, with the accepted displacement $v_D=0$, is infinitely large. It follows logically that the displacement v_A in the case of the finite plane will also be infinite. To prove this fact, the so called A1-test (Case 1) is performed which is shown in Figure 7.

The vertical and horizontal displacements resulting from the analysis, obtained by full 3×3 Gauss integration, can be shown with non-dimensional functions:

$$v^0 = 1.20 \left[-\ln\left(\frac{l}{L}\right)^2 + \left(1 - \frac{2}{5}v^2\right)\pi \right], \quad u^0 = 0.63 \left\{ \ln\left(\frac{l}{L}\right)^2 - \left[1 + \frac{v}{2}(1-v)\right]\pi \right\} \quad (4)$$

The general relations, which can be seen from the following examples, can be set to:

$$v^0 = a_i \left[-\ln\left(\frac{l}{L}\right)^2 + \left(1 - \frac{2}{5}v^2\right)\pi \right], \quad u^0 = c_i \left\{ \ln\left(\frac{l}{L}\right)^2 - \left[1 + \frac{v}{2}(1-v)\right]\pi \right\} \quad (5a), (5b)$$

where coefficients a_i and c_i depend on the type of boundary condition.

In standard linear elastic finite systems the next relations are valid:

$$v_i^{(P_i=1)} = \delta_{ij}, \quad k_{ij} = \frac{1}{\delta_{ij}} = \frac{1}{v_i^{(P_i=1)}} \quad (6)$$

where k_{ij} is constant.

In the above solved example for different meshes, the different stiffness are obtained which represent a measure of the numerical sensitivity of the mesh to concentrated loading. The authors have called this phenomenon the **numerical curvature of the space**.

If stiffness is to be kept constant for all meshes, it can be achieved with the condition:

$$E^* = f(E) = \frac{E}{a \left[-\ln\left(\frac{l}{L}\right)^2 + (1+d)\pi \right]} \quad (7)$$

where E^* is the reduced modulus of elasticity. The coefficient a stands for a_i or c_i from Eqs. (5a) and (5b) and the coefficient d stands for the right side of the second member in the brackets from the same equations.

Thus, the numerical curvature of the space presents a numerically non-linear situation. As relation (7) is not the same for both x and y directions, which will be seen later also for the rotation, the numerical non-linearity does not exist only at the material level but also on the space level too.

The same conclusion about variable stiffness can be found in Ref. [18].

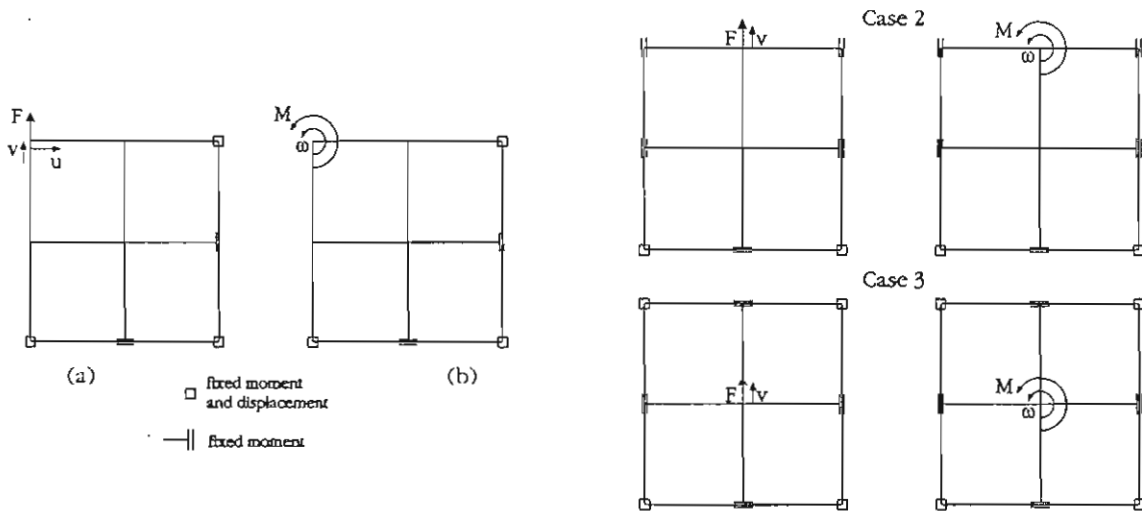


Fig. 7 A1-test: (a) concentrated force; (b) concentrated moment

Fig. 8 Boundary conditions for (Case 2) and (Case 3)

The analysis of the concentrated moment loading on different meshes with 3×3 Gauss in Figure 7.b gives non dimensional value for rotation of point A:

$$\omega^0 = 0.856 \left(\frac{L}{l}\right)^2 \left[1 + \frac{v}{2}(1-v)\right] \quad (8)$$

or generally:

$$\omega^0 = b_i \left(\frac{L}{l}\right)^2 \left[1 + \frac{v}{2}(1-v)\right] \quad (9)$$

The rotation of point A in the dimensional form can be obtained from the relation:

$$\omega_A = \omega^0 \frac{M}{L^2 E I} \quad (10)$$

where M is the applied concentrated moment at point A.

Evidently, the rotation caused by the concentrated moment tends to be quicker towards infinity than the displacement due to the concentrated force. This shows that the finite element mesh is more sensitive for concentrated moments than for concentrated forces.

2.2 Discretization sensitivity with different boundary conditions

The discretization sensitivity is analyzed varying boundary conditions. Three different boundary conditions were analyzed: (Case 1) shown in Figure 8; (Case 2) and (Case 3) shown in Figure 8. The sensitivity functions are given with Eqs. (21a) and (25) with the values of coefficients a_i and b_i as shown in [13]. The results show that coefficients a_i and b_i strongly depend on the boundary conditions.

3. ANALYSIS OF THE STATIC AND DYNAMIC EIGEN SOLUTION

Accuracy of one numerical solution is usually verified by increasing finite element mesh density. The numerical solutions basically are verified by comparing the equivalent vector spaces. Static eigen vectors and corresponding eigen values are defining, in this case, vector space. If the mass of system is uniformly distributed and lumped in nodes, then the dynamic eigen problem is equivalent to the static one. The static eigen problem is defined as

$$(K - \lambda I)u = 0 \quad (11)$$

The eigen problem solution is performed by WYD [19] method. Analysis is carried out on the example shown in Fig. 8 case 3. Three different mesh sizes are modelled, for which first five eigen values are presented in Table 3, together with first three eigen vectors Fig. 9, Fig. 10 and Fig 11.

Table 3: static eigen values for different mesh size

mesh size	λ_1	λ_2	λ_3	λ_4	λ_5
2x2	2224	3130	7735	12110	12990
4x4	84.16	355	4179	4866	8596
8x8	16.99	76.30	125.2	2268	3351

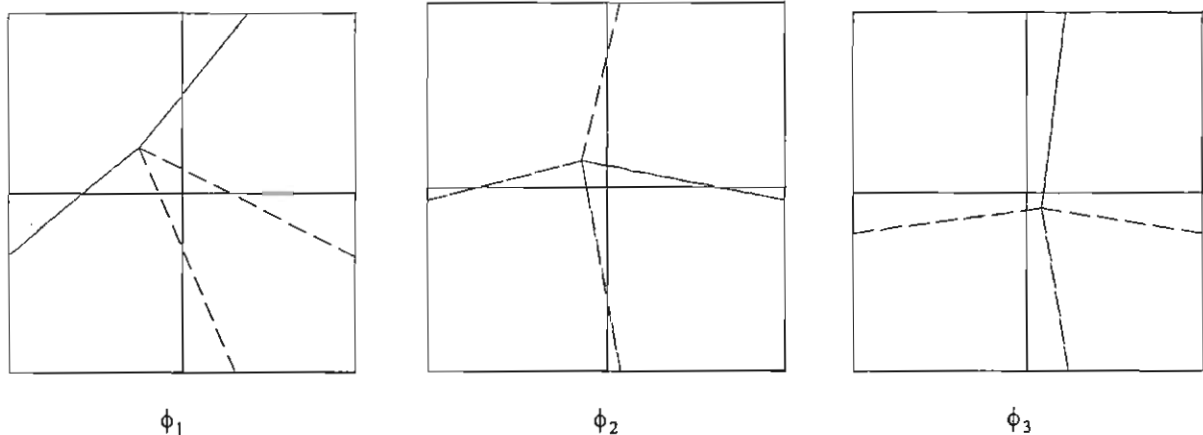


Fig. 9 Static eigen vectors for mesh size 2x2

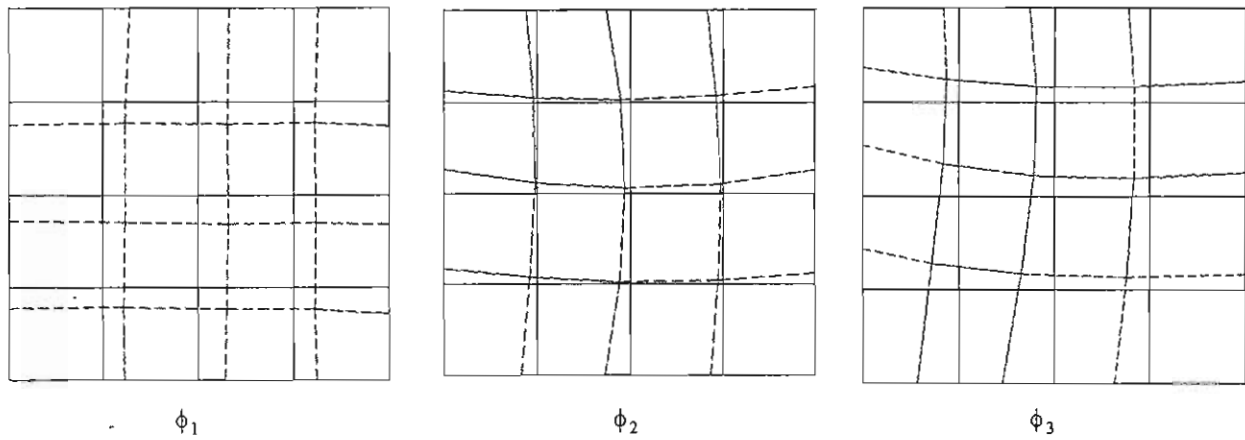


Fig. 10 Static eigen vectors for mesh size 4x4

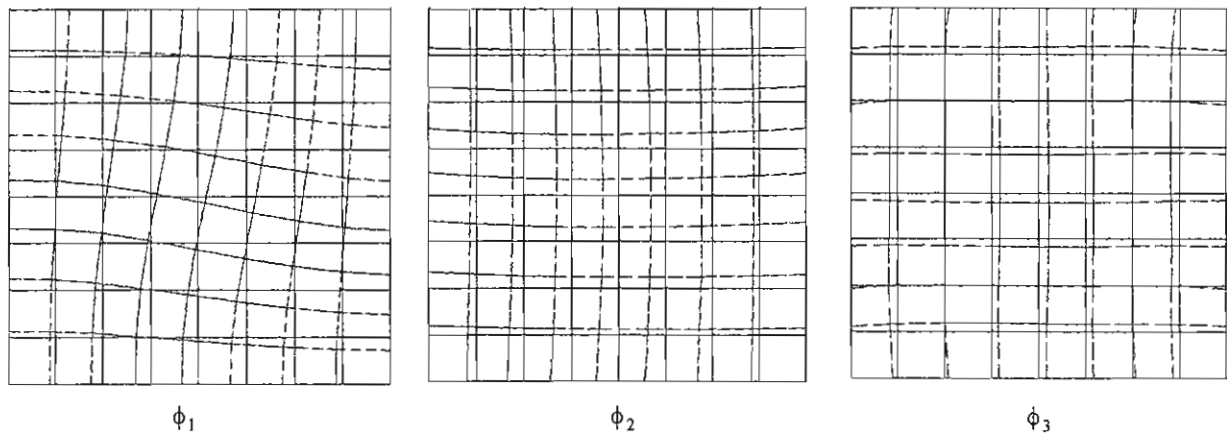


Fig. 11 Static eigen vectors for mesh size 8x8

It is very important to notice that there is no stable lower spectrum limit, although the system formally has finite stiffness and finite mass, and that first eigen value converge to zero as shown in Fig. 12.

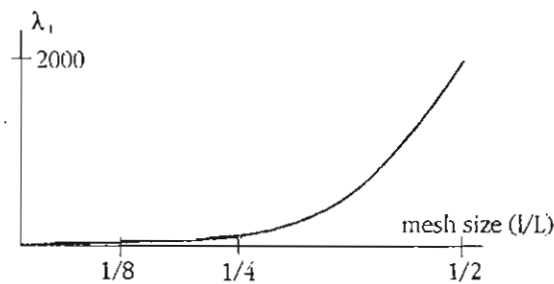


Figure 12. First static eigen values as function of mesh size

Similar conclusions are valuable for the boundary conditions in case 1 Fig. 7, and case 2 Fig. 8. Increasing of degrees of freedom of the system, leads to the conclusion that the vector space of the system is behaving as infinite space. Displacements in cases of the concentrated or quasi-concentrated loads according to analytical solutions given by Girkman in Ref. [17], are infinite. The NFCQ shell finite element method gives same solution as analytical one if mesh size is sufficiently dense. Which means that results of numerical analysis strictly depend on the mesh size. Some authors [5,8,12,18] have been faced with the similar phenomenon.

This paper shows the functional relation between mesh size and its accuracy. It can be concluded, that there exists the mesh size on which the computer program collapses, due to "infinite" or big displacements which have to be calculated.

4. EXAMPLES

Example 4.1

Rectangular deep beam supported by two columns is subjected to concentrated force $F=1.0$ MN, as shown in Fig. 13. Material properties of deep beam are: $E=30000$ MPa, $\nu=0.20$, thickness $t=0.20$ m. Cross section of columns is defined by width 0.30 m and depth 0.20 m, with the same modulus of elasticity and Poisson's ratio.

Analysis is performed on mesh sizes 1x2 and 4x8. Displacements of the former are shown in Fig. 13. Relation between stiffness and mesh size can be noted from moment and shear force values given in Table 4.

Table 4. Moments and shear forces in relation to the mesh size

mesh size	M_A	T_A	M_B	T_B
1x2	0.0188	0.0223	0.0257	-0.0223
4x8	0.0301	0.0382	0.0467	-0.0382

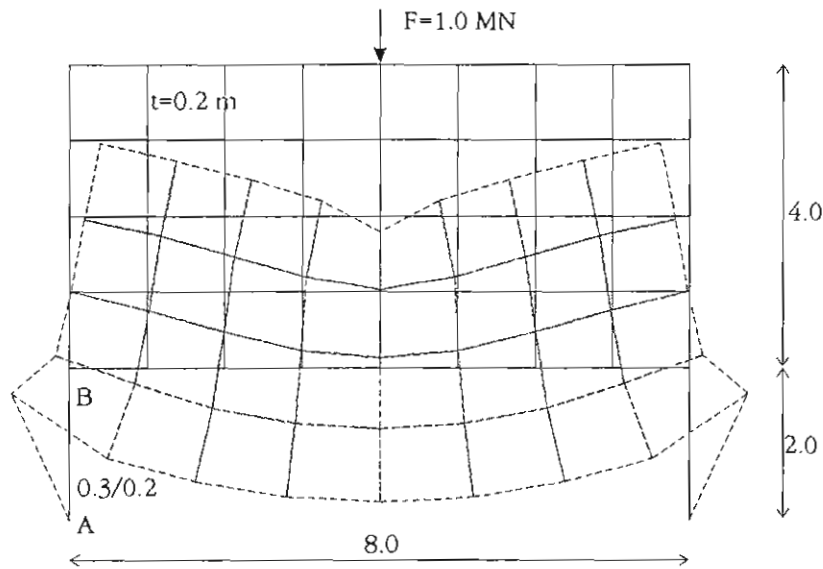


Figure 13. Deep beam under concentrated load

Example 4.2

The same deep beam subjected to distributed load $Q=1.0$ MN, as shown in Fig. 14, is analyzed. In this example concentrated force effect is provoked by column connection. This can be seen from displacements shown on Fig. 14. Relation between stiffness and mesh size can be noted from moment and shear force values given in Table 5.

Table 5. Moments and shear forces in relation to the mesh size

mesh size	M_A	T_A	M_B	T_B
1x2	0.0149	0.018	0.0211	-0.018
4x8	0.0271	0.0351	0.0431	-0.0351

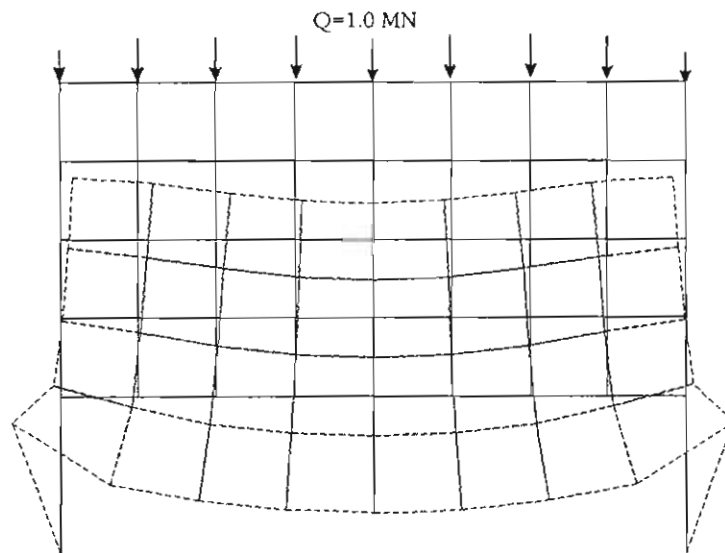


Figure 14. Deep beam under distributed load

5. CONCLUSIONS

According to the presented new fully compatible quadrilateral plate/shell finite element, the computer code has been developed for linear static analyses of complex structures, such as space trusses and frames, plane structures such as shear walls, deep beams, plates, shells and, which is more important, for all their possible combinations.

The presented model closely follows the analytical solutions even in those cases of singularity due to the action of concentrated loads, as it was been illustrated through examples. As a consequence of such high accuracy, discretization sensitivity appears in plane stress problems under concentrated and quasi-concentrated loads. The authors [13] called this phenomenon *numerical curvature of the space*. Due to the analytical solutions given in Ref. [18] any numerical method has to show the same or very similar property.

The analysis of the static eigen problem gives the same conclusion, as presented before, in fact there does not exist lower spectrum limit because the first eigen value tends to zero if mesh density increases. Increasing of degrees of freedom of the system, leads to the conclusion that the vector space of the system is behaving as infinite space. Analytical solutions Ref. [17] show that under the concentrated or quasi-concentrated loads, displacement are infinite. The NFCQ shell finite element method gives same solution as the analytical one if the mesh size is sufficiently dense, which means that the results of numerical analysis strictly depend on the mesh size.

This paper shows the functional relation between mesh size and its accuracy. It can be concluded, that there exists the mesh size on which the computer program collapses, due to the "infinite" or big displacements which have to be calculated.

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ON MIXED INTERPOLATION OF EIGHT-NODE SHEAR-BENDING PLATE ELEMENT

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Summary. This paper presents eight-node, 24-d.o.f. quadrilateral shear-bending Reissner-Mindlin plate element valid for the analysis of thick to thin plates. The formulation of the element is derived from a displacement variational principle using mixed interpolation strategy in order to eliminate undesirable locking effect. The standard mixed interpolation was improved by the new location of the discrete points that leads to the insensitivity to the mesh distortion.

1. Introduction

In recent years the development of efficient plate finite elements has been mostly based on the Reissner-Mindlin thick plate theory, which requires C^0 continuity [8], [9], [11]. By classical formulation the shear locking phenomenon can appear, which may lead to the incorrect solution for thin plates. In order to remove this undesirable effect, several approaches have been proposed [2], [5], [6].

The present paper is concerned with the discrete Kirchhoff approach which involves an initial selection of approximations for the normal rotations within the element and an independent description of the transverse displacement on the boundary. To calculate the bending stiffness matrix, the standard procedures are employed, while the strain-nodal displacement matrix for the shear is derived in an inconsistent non-standard manner. Using mixed interpolation the Kirchhoff hypothesis is constrained to the few discrete points. To improve this approach new locations of discrete points were proposed which leads to the lesser sensitivity to mesh distortions. The eigenvalue analysis and locking test show good efficiency of the element presented for the computation of very thin and very thick plates.

2. Formulation of the element

The plate element presented in this paper was derived from classical eight-node isoparametric quadrilateral finite element based on the Reissner-Mindlin plate theory and represents an extension of the formulation given in [12], where the strain energies due to bending and shear deformation are considered.

The finite element equation is derived from the total potential energy which can be expressed in the following form:

$$\Pi = \frac{1}{2} \int_A (\mathbf{x}^T \mathbf{D}_B \mathbf{x} + \mathbf{\gamma}^T \mathbf{D}_S \mathbf{\gamma}) dA - \int_A q w dA \quad (1)$$

where $\mathbf{x}$ is curvature and $\mathbf{\gamma}$ transverse shear strain which may be written as

$$\mathbf{x} = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -\theta_{xx} \\ -\theta_{yy} \\ -(\theta_{xy} + \theta_{yx}) \end{bmatrix}; \quad \mathbf{\gamma} = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} -\phi_x \\ -\phi_y \end{bmatrix} = \begin{bmatrix} w_{,x} - \theta_x \\ w_{,y} - \theta_y \end{bmatrix} \quad (2)$$

$\mathbf{D}_B$ and $\mathbf{D}_S$ denote matrix of flexural and shear rigidities, respectively

$$\mathbf{D}_B = \frac{E h^3}{12 (1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}; \quad \mathbf{D}_S = \frac{E h}{2 k (1 + \nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3)$$

The coordinates x, y lie in the middle surface, and z is the coordinate through the thickness h . w is the transverse displacement, while θ_x and θ_y represent the rotations of the normal in the $x-z$ and $y-z$ planes. q is the distributed pressure loading and A the area of the middle surface of the plate. k stands for shear correction factor (appropriately set to 1.2). The material data are modulus of elasticity E and Poisson's ratio ν .

In order to derive discrete algorithms the Reissner-Mindlin displacement field is approximated by the serendipity quadratic interpolation functions $N_i(\xi, \eta)$ and the nodal point values $w_i, \theta_{xi}, \theta_{yi}$ in the following way

$$w = \sum_{i=1}^8 N_i(\xi, \eta) w_i, \quad \theta_x = \sum_{i=1}^8 N_i(\xi, \eta) \theta_{xi}, \quad \theta_y = \sum_{i=1}^8 N_i(\xi, \eta) \theta_{yi} \quad (4)$$

where ξ and η are natural coordinates according to isoparametric techniques. After lengthy computations the element stiffness matrix may be derived in the form

$$\mathbf{k} = \mathbf{k}_B + \mathbf{k}_S \quad (5)$$

where first matrix presents bending stiffness

$$\mathbf{k}_B = \int_A \mathbf{B}_B^T \mathbf{D}_B \mathbf{B}_B dA \quad (6)$$

$\mathbf{k}_S$ denotes shear stiffness matrix

$$\mathbf{k}_S = \int_A \mathbf{B}_S^T \mathbf{D}_S \mathbf{B}_S dA \quad (7)$$

In the equation (6) $\mathbf{B}_B$ stands for curvature-displacement matrix and in the expression (7) $\mathbf{B}_S$ represents shear strain-displacement matrix. In the analysis of very thin plates, very stiff results may be obtained when the exact numerical integration is used to the computation element

matrices. This phenomenon is termed shear locking effect. To avoid this deficiency, various remedies based on selective and reduced integration have been proposed [2]. Since those approaches are not effective for very thin plates, other methods are used such as mixed interpolation and new finite element formulations.

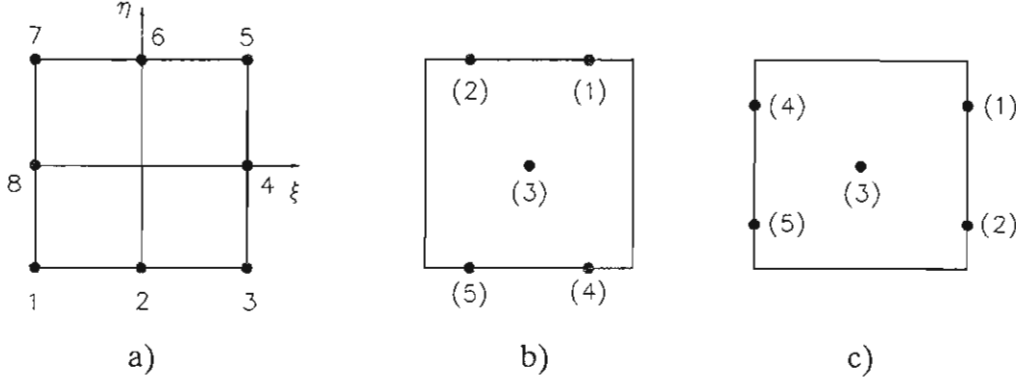


Figure 1. a) The reference finite element
 b) Location points for the transverse shear $\gamma_{\xi\xi}$
 c) Location points for the transverse shear $\gamma_{\eta\eta}$

Mixed interpolation

Mixed interpolation approach is based on the evolution of shear-displacement matrix $\mathbf{B}_s$ using the constrained shear strain fields. It begins with the expression of the transverse shear strain, which can be written by means of (4) and using natural coordinate in the following form

$$\begin{aligned} \gamma_{\xi\xi} = w_{,\xi} - \theta_{\xi} = & \alpha_1(w_i, \theta_{\xi i}) + \alpha_2(w_i, \theta_{\xi i}) \xi + \alpha_3(w_i, \theta_{\xi i}) \eta + \alpha_4(\theta_{\xi i}) \xi^2 + \\ & + \alpha_5(w_i, \theta_{\xi i}) \xi \eta + \alpha_6(w_i, \theta_{\xi i}) \eta^2 + \alpha_7(\theta_{\xi i}) \xi^2 \eta + \alpha_8(\theta_{\xi i}) \xi \eta^2 \end{aligned} \quad (8)$$

A similar expression for $\gamma_{\eta\eta}$ may be obtained by simply interchanging ξ by η . The limiting Kirchhoff conditions ($\gamma_{\xi\xi} = \gamma_{\eta\eta} = 0$) imply $\alpha_1 = \alpha_2 = \dots = \alpha_8 = 0$. The conditions on the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_5$, and α_6 impose a linear relationship between the nodal deflections and the average θ_{ξ} rotation (θ_{η} for $\gamma_{\eta\eta}$) on the element. This physically means that the average element rotation equals the element slope, which is clearly satisfied for thin-plate. On the other hand, $\alpha_4 = \alpha_7 = \alpha_8 = 0$ imposes a uniform or zero rotation on the element, which invariably leads to vanishing of the bending energy and to undesirable locking effect [10]. In order to avoid the locking phenomenon the bending and the shear effect were included in the stiffness matrix using different interpolations. According to classical isoparametric concept the bending stiffness matrix is derived by means of the equations (2) and (4).

The shear strains must be evaluated only in points where the spurious $\alpha_k(\theta_{\xi i})$ and $\alpha_k(\theta_{\eta i})$ vanish. According to this approach, on the edges of the quadrilateral element in $\xi - \eta$ natural

coordinates are imposed as follows [12]:

the tangential transverse shear strain $\gamma_{\xi\zeta}$ ($\gamma_{\eta\zeta}$) is such that the interpolation of the functions $w_{,\xi}$ and θ_{ξ} ($w_{,\eta}$ and θ_{η}) are polynomials of the same order with respect to the natural coordinate ξ (η) at $\eta = \pm 1$ ($\xi = \pm 1$) and for all $\xi \in [-1, 1]$ ($\eta \in [-1, 1]$).

The polynomial of w must be one order higher than that assumed for the θ_{ξ} and θ_{η} . A classic quadratic serendipity interpolation for the rotations θ_{ξ} and θ_{η} is used, while the cubic interpolation for the transverse deflection is expressed as follows

$$w(\xi, \eta) = a_1 + a_2 \xi + a_3 \eta + a_4 \xi^2 + a_5 \xi \eta + a_6 \eta^2 + a_7 \xi^3 + a_8 \xi^2 \eta + a_9 \xi \eta^2 + a_{10} \eta^3 + a_{11} \xi^3 \eta + a_{12} \xi \eta^3 \quad (9)$$

Besides the conventional degrees of freedom w_i , $\theta_{\xi i}$, $\theta_{\eta i}$, the new degrees of freedom representing the tangential derivative of w with respect to the natural coordinates ξ or η , such as $(w_{,\xi})_2$, $(w_{,\eta})_4$, $(w_{,\xi})_6$ and $(w_{,\eta})_8$ in the mid-side nodes 2, 4, 6, and 8 are introduced. In this way the undesirable degrees of freedom appear only in the constant and quadratic terms of the tangential transverse shear strains $\gamma_{\xi\zeta}$ and $\gamma_{\eta\zeta}$ along the edge $\eta = \pm 1$ and $\xi = \pm 1$, respectively. To eliminate that effect, the linear distribution of shear strain along the edges is assumed, which leads to the following expressions

$$(w_{,\xi})_2 = -\frac{1}{2} w_1 + \frac{1}{2} w_3 - \frac{1}{6} \theta_{\xi 1} - \frac{1}{6} \theta_{\xi 3} + \frac{1}{3} \theta_{\xi 2} \quad (10)$$

$$(w_{,\eta})_4 = -\frac{1}{2} w_3 + \frac{1}{2} w_5 - \frac{1}{6} \theta_{\eta 3} - \frac{1}{6} \theta_{\eta 5} + \frac{1}{3} \theta_{\eta 4} \quad (11)$$

$$(w_{,\xi})_6 = \frac{1}{2} w_5 - \frac{1}{2} w_7 - \frac{1}{6} \theta_{\xi 5} - \frac{1}{6} \theta_{\xi 7} + \frac{1}{3} \theta_{\xi 6} \quad (12)$$

$$(w_{,\eta})_8 = \frac{1}{2} w_7 - \frac{1}{2} w_1 - \frac{1}{6} \theta_{\eta 7} - \frac{1}{6} \theta_{\eta 1} + \frac{1}{3} \theta_{\eta 8} \quad (13)$$

Inserting equations (10) to (13) into (9), the sets of monomial terms $P(w_{,\xi})$ and $P(w_{,\eta})$ are determined

$$P(w_{,\xi}) = \{1, \xi, \eta, \xi^2, \xi \eta, \eta^2, \xi^2 \eta, \eta^3\} \quad (14)$$

$$P(w_{,\eta}) = \{1, \xi, \eta, \xi^2, \xi \eta, \eta^2, \xi \eta^2, \xi^3\} \quad (15)$$

The rotations θ_{ξ} and θ_{η} are interpolated from the classic quadratic serendipity functions

$$P(\theta_{\xi}) = P(\theta_{\eta}) = \{1, \xi, \eta, \xi^2, \xi \eta, \eta^2, \xi^2 \eta, \xi \eta^2\} \quad (16)$$

According to definition (8) the monomial terms of tangential transverse shear strains may be expressed as

$$P(\gamma_{\xi\zeta}) = P(w_{,\xi}) \cap P(\theta_{\xi}) \quad (17)$$

$$P(\gamma_{\eta\zeta}) = P(w_{,\eta}) \cap P(\theta_{\eta}) \quad (18)$$

By means of (8), (14), (15), (16) and according to the assumption of linear distribution of $\gamma_{\xi\zeta}$ and $\gamma_{\eta\zeta}$ along the corresponding edges, the expressions (17) and (18) may be written as

$$P(\gamma_{\xi\zeta}) = \{1, \xi, \eta, \xi\eta, \eta^2\} \quad (19)$$

$$P(\gamma_{\eta\zeta}) = \{1, \xi, \eta, \xi^2, \xi\eta\} \quad (20)$$

It is evident from these equations that five points should be defined for determining the transverse shear strain components. To obtain linear variation of the natural transverse shear strains in the tangential direction along the boundary, four points must be located at the edge of the element, while one inner point is introduced in order to ensure the quadratic variation in the normal directions. The shape functions $R_i(\xi, \eta)$ of the $\gamma_{\xi\zeta}$ for the points presented in Figure 1b are derived in the form

$$\begin{aligned} R_1(\xi, \eta) &= 1/4(\eta + \xi/a)(1 + \eta) & ; & & R_2(\xi, \eta) &= 1/4(\eta - \xi/a)(1 + \eta) \\ R_3(\xi, \eta) &= (1 - \eta^2) & ; & & R_4(\xi, \eta) &= 1/4(-\eta + \xi/a)(1 - \eta) \\ R_5(\xi, \eta) &= 1/4(-\eta - \xi/a)(1 - \eta) \end{aligned} \quad (21)$$

Interchanging ξ by η the shape functions $S_i(\xi, \eta)$ of the $\gamma_{\eta\zeta}$ for the points presented in Figure 1c are described by analogous expression

$$S_i(\xi, \eta) = R_i(\eta, \xi) \quad (22)$$

After lengthy derivations, the final expression for the transverse shear strains may be given in the following form

$$\begin{bmatrix} \gamma_{\xi\zeta}(\xi, \eta) \\ \gamma_{\eta\zeta}(\xi, \eta) \end{bmatrix} = \mathbf{N}_{\xi-\eta} \boldsymbol{\gamma}_{\xi\zeta-\eta\zeta} \quad (23)$$

where $\mathbf{N}_{\xi-\eta}$ is the interpolation function matrix

$$\mathbf{N}_{\xi-\eta} = \begin{bmatrix} \mathbf{N}_{\xi} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{\eta} \end{bmatrix}, \quad \begin{cases} \mathbf{N}_{\xi} = [R_1(\xi, \eta) \dots R_5(\xi, \eta)] \\ \mathbf{N}_{\eta} = [S_1(\xi, \eta) \dots S_5(\xi, \eta)] \end{cases} \quad (24)$$

and $\boldsymbol{\gamma}_{\xi\zeta-\eta\zeta}$ is the shear strains matrix which can be written as function of the element degrees of freedom

$$\boldsymbol{\gamma}_{\xi\zeta-\eta\zeta} = \mathbf{A} \mathbf{v}, \quad \begin{cases} \boldsymbol{\gamma}_{\xi\zeta-\eta\zeta}^T = [\gamma_{\xi\zeta 1} \dots \gamma_{\xi\zeta 5} \gamma_{\eta\zeta 1} \dots \gamma_{\eta\zeta 5}] \\ \mathbf{v}_i^T = [w_i \theta_{\xi i} \theta_{\eta i}] \end{cases} \quad (25)$$

In this expression, the matrix $\mathbf{A}$ is deduced from the components of $\boldsymbol{\gamma}$ computed at the discrete points. Finally, the shear-displacement matrix can be obtained as

$$\mathbf{B}_s = \mathbf{J}^{-1} \mathbf{N}_{\xi-\eta} \mathbf{A} \quad (26)$$

where $\mathbf{J}$ denotes Jacobian matrix.

The new location of discrete points

New locations of the discrete points for the element presented are proposed in this paper. The positions of the four points are identical with the locations of the four Gauss integration points

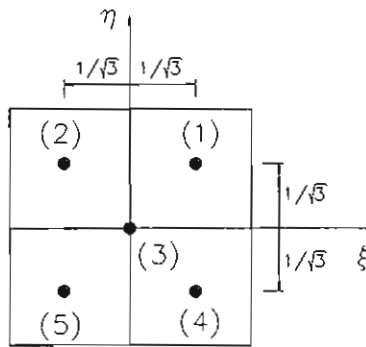


Figure 2. Locations of discrete points

shown in Figure 2, while one discrete point is located at the centre of the element. In contrast to the approach described above, the point locations for the shear strains $\gamma_{\xi\xi}$ don't differ from that for $\gamma_{\eta\eta}$ and the derivation of the shape functions is analogous. The linear distribution of the shear strain along the lines $\eta = \pm 1/\sqrt{3}$ and $\xi = \pm 1/\sqrt{3}$ is imposed in order to eliminate the undesirable degrees of freedom which are expressed by the equations (10)-(13). Since other discrete points are used, the expression of $\mathbf{N}_{\xi-\eta}$ differs from that presented in (24). By computing the stiffness matrices, a reduced Gauss integration scheme (2x2) is used.

3. Element characteristics

To assess the performance characteristics of the element with new discrete points several problems were solved and the results obtained are presented in this section in comparison with the element CL8 from [12] and with the element based on the mixed formulation MIX8 presented in [14] which is implemented in FEMAS program [15].

The eigenvalue analysis shows that the one spurious energy mode occurs when a reduced integration is applied. This mode disappears in a mesh, as well as at the CL8 element. The element also passes the patch tests in bending, twisting and shear.

The effects of mesh distortions were also analyzed. The sensitivity of the numerical results to the element shapes were investigated by computing one quarter of a clamped thick square plate, as shown in Figure 3. The exact three-dimensional solution for the point C, w_{3D} , is used from [13] as reference value and the numerical results are presented in Table I under the following form: $((w_c - w_{c3D})/w_{c3D}) \cdot 100$.

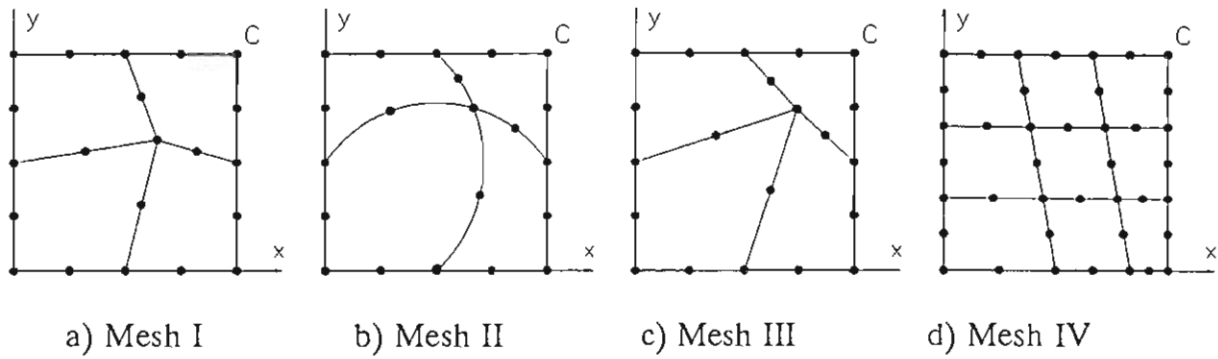


Figure 3. Mesh distortion problems

The Table also shows the results obtained by the CL8 element using full and reduced integration and the MIX8 finite element. As may be seen, the elements with new discrete points show the minimum deviation from the exact three-dimensional solution.

Table I. Test on mesh sensitivity for a clamped square plate ($a/h=10$, $E=7200 \times 10^6$, $\nu=0.3$, uniform load $q=1$) for different mesh distortion

Mesh	CL8	CL8-red. int.	MIX8	New discrete points
Regular 2x2	1.3	2.0	-2.1	0.6
Mesh I	2.4	3.4	-1.1	1.9
Mesh II	-4.8	2.2	-9.2	0.2
Mesh III	2.8	4.5	-1.0	2.5
Regular 3x3	0.7	0.8	-1.9	-1.0
Mesh IV	0.8	0.9	-1.8	0.7

The Figure 5 shows the results obtained by locking test, where the numerical values are compared with the exact Kirchhoff solution. A symmetric quadrant, simply supported thin plate ($a=10$, $E=10.92$, $\nu=0.3$) under unit uniform load $q=1$ is analyzed (see Figure 4).

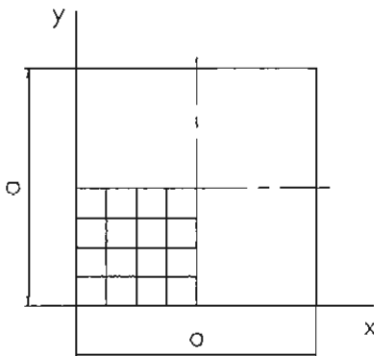


Figure 4. Square plate under uniform loading

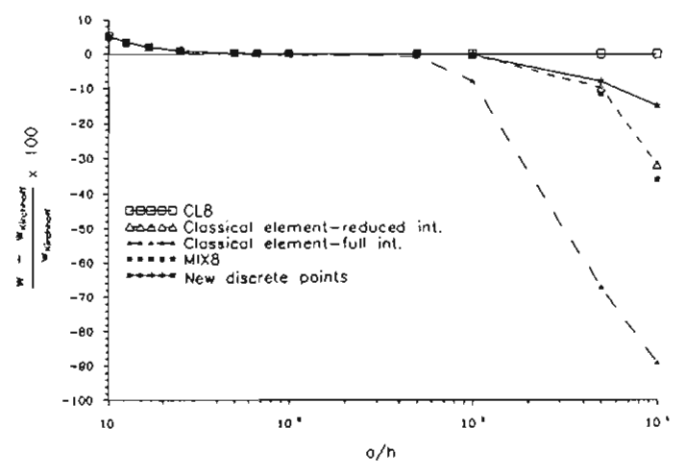


Figure 5. Locking test

4. Conclusions

An eight-node and 24-DOF quadrilateral shear-bending C^0 plate finite element with the discrete points located in the Gauss integration points for the analysis of thick to thin plates has been evaluated and implemented in computer code. This element represents an extension of the element CL8 given in [12].

The element derived is insensitive to mesh distortions, which is a significant advantage over the element CL8. The locking tests show that the transverse shear effects for thin plates are not important. The results of the eigenvalues analysis show that one spurious energy mode for the reduced integration occurs which disappears in a mesh.

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ON NINE NODED ELEMENT FOR ANALYSIS OF LAMINATED COMPOSITE SHELLS

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SUMMARY

The paper presents 9 noded Lagrangian element with 45 D.O.F. used for analysis of laminated composite shells. The element is based on the principle of degeneration, enabling analysis of thick and thin shells. Full and selective integration was applied over 2D domain with numerical integration over thickness of the shell. Developed routines were written in MATLAB with intention of translation such a code in some of the higher programming languages like C or FORTRAN 90. Results for orthotropic layered material models were presented and have shown good agreement with available references.

1. INTRODUCTION

The last twenty years have witnessed a large scale spread of composite materials in various forms and areas of application like aerospace, automotive industry, pressure vessels [1,2]. This is due to their very high specific stiffness and strength. On the other side, their use is very demanding, requiring good understanding of closely related phenomena in structural response calculation, manufacturing and exploitation behaviour. Numerical methods, like finite element method, have proven as a very suitable computation procedure. In many of these applications unidirectional multilayered fiber composites are the primary material consisting of a large number of layers with fibers firmly bonded in matrix. In this analysis the 3D degenerated element has been used. The degeneration principle has the advantage that no special shell theory has to be utilized, the geometry and displacements are directly interpolated using standard shape functions.

2. DESCRIPTION OF THE ELEMENT

The element formulation is based on the well known principle of degeneration [3], taking into

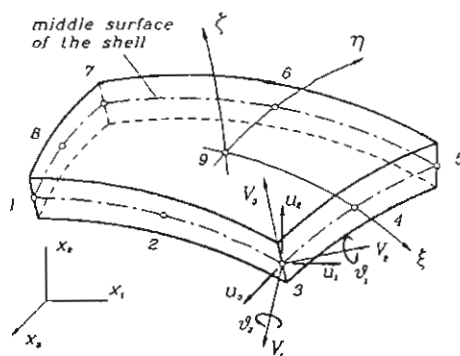


Fig.1 Geometry and kinematics of the shell

account isoparametric formulation of three-dimensional continuum. Lagrangian element with 9 nodes and 5 degrees of freedom per node is considered: three displacements u_i in the direction of global Cartesian coordinate axes x_i and two rotations of the normal ϑ_2 and ϑ_1 around local Cartesian axes V_i described with their unit vectors V_1 and V_2 , Fig 1. The position of the point in the global coordinate system x_i is described using standard shape functions N_k

$$x_i = \sum_{k=1}^9 N_k(\xi, \eta) x_i^k + \frac{\xi h}{2} \sum_{k=1}^9 N_k(\xi, \eta) V_{3,i}^k \quad i=1,3 \quad (1)$$

where x_i^k stands for global coordinates of node k , h for thickness of the shell, and $V_{3,i}^k$ for components of normal vector in the global coordinate system referring to the node k . These components could be easily expressed using nodal rotations by

$$V_{3,i}^k = -V_{2,i}^k \vartheta_2^k + V_{1,i}^k \vartheta_1^k \quad i = 1, 3 \quad (2)$$

Cartesian coordinate system V_i used in defining rotations of the normal in nodes is defined as follows

$$V_1 = \frac{\mathbf{e}_2 \times \mathbf{V}_3}{\|\mathbf{e}_2 \times \mathbf{V}_3\|} \quad V_2 = \mathbf{V}_3 \times \mathbf{V}_1 \quad V_3 = \frac{\frac{\partial \mathbf{x}_1}{\partial \xi} \times \frac{\partial \mathbf{x}_1}{\partial \eta}}{\left\| \frac{\partial \mathbf{x}_1}{\partial \xi} \times \frac{\partial \mathbf{x}_1}{\partial \eta} \right\|} \quad (3)$$

where $\mathbf{e}_2$ is unit vector in global x_2 direction, and derivatives corresponding to the natural coordinates relate those of the global Cartesian coordinates by

$$\mathbf{e}_\xi = \frac{\partial \mathbf{x}_i}{\partial \xi} \quad \mathbf{e}_\eta = \frac{\partial \mathbf{x}_i}{\partial \eta} \quad (4)$$

Expressions (4) are elements of Jacobian transformation matrix (ζ dependence of the Jacobian is not neglected what better describes very curved and thick shells).

$$\mathbf{J}(\xi, \eta, \zeta) = \begin{bmatrix} \frac{\partial \mathbf{x}_1}{\partial \xi} & \frac{\partial \mathbf{x}_1}{\partial \eta} & \frac{\partial \mathbf{x}_1}{\partial \zeta} \end{bmatrix}^T \quad (5)$$

Displacement field is represented through

$$u_i(\xi, \eta, \zeta) = \sum_{k=1}^9 N_k(\xi, \eta) u_i^k + \frac{\zeta h}{2} \sum_{k=1}^9 N_k(\xi, \eta) (V_{1,i}^k \vartheta_1^k - V_{2,i}^k \vartheta_2^k) \quad i = 1, 3 \quad (6)$$

or in matrix form

$$\mathbf{u}_i = \mathbf{N} \mathbf{u}_i^k \quad (7)$$

where

$$\mathbf{u}_i = [u_1(\xi, \eta, \zeta), u_2(\xi, \eta, \zeta), u_3(\xi, \eta, \zeta)]^T \quad (8)$$

and vector of nodal degrees of freedom is defined as

$$\mathbf{u}_i^k = [\dots, u_1^k, u_2^k, u_3^k, \vartheta_2^k, \vartheta_1^k, \dots]^T \quad k = 1, 9 \quad (9)$$

Matrix $\mathbf{N}$ is easily obtained using shape functions N_k and expressions (6).

With the assumption of zero normal stresses, local Cartesian coordinate system (Fig. 2) used for defining of these is defined as

$$\mathbf{e}'_1 = \mathbf{e}_\xi, \quad \mathbf{e}'_2 = \mathbf{e}'_3 \times \mathbf{e}'_1, \quad \mathbf{e}'_3 = \frac{\frac{\partial \mathbf{x}_i}{\partial \xi} \times \frac{\partial \mathbf{x}_i}{\partial \eta}}{\left\| \frac{\partial \mathbf{x}_i}{\partial \xi} \times \frac{\partial \mathbf{x}_i}{\partial \eta} \right\|} \quad (10)$$

and the corresponding strain field through

$$\boldsymbol{\varepsilon}' = [\varepsilon'_1 \quad \varepsilon'_2 \quad 0 \quad 2\varepsilon'_{12} \quad 2\varepsilon'_{23} \quad 2\varepsilon'_{31}]^T \quad (11)$$

Strain field in the global Cartesian system is related to nodal degrees of freedom, over strain matrix $\mathbf{B}$

$$\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{u}_i^k \quad (12)$$

Transformation of strain field tensor components $\boldsymbol{\varepsilon}'$ to $\boldsymbol{\varepsilon}$ is by means of matrix $\mathbf{T}$ [4]

$$\mathbf{T} = [\mathbf{e}'_1 \quad \mathbf{e}'_2 \quad \mathbf{e}'_3]^T \quad (13)$$

over expression

$$\boldsymbol{\varepsilon}' = \mathbf{T} \boldsymbol{\varepsilon} \mathbf{T}^T \quad (14)$$

Assuming linear elastic material response, constitutive equation relating stress tensor $\boldsymbol{\tau}'$ and strain tensor $\boldsymbol{\varepsilon}'$ in local coordinate system, is written through

$$\boldsymbol{\tau}' = \mathbf{C}' \boldsymbol{\varepsilon}' \quad (15)$$

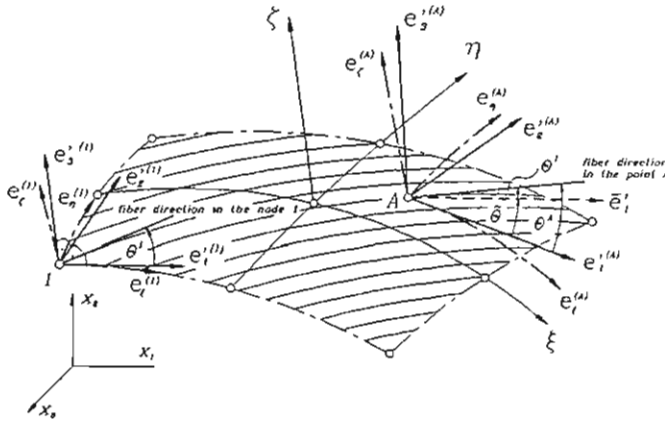


Fig. 2 Definition of fiber direction with respect to the local coordinate system

Taking into account layered setup of composite shell, material matrix $\mathbf{C}'$ becomes dependent on the ξ, η, ζ . Each layer is assumed to have orthotropic material model with respect to fiber defined coordinate system and composite laminate is formed by perfect bonding of these layers. Layer coordinate system is created with axis 1 corresponding to fiber direction and axis 2 perpendicular to it. These axes are in the plane of layer while axis 3 is perpendicular to 1, 2 forming right hand coordinate system. Material matrix expressed in layer coordinate system ${}^j\mathbf{C}''$ for j -th layer has the following form

$${}^J\mathbf{C}'' = \begin{bmatrix} C_{11}'' & C_{12}'' & 0 & 0 & 0 & 0 \\ & C_{22}'' & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & C_{44}'' & 0 & 0 \\ & \text{sim.} & & & C_{55}'' & 0 \\ & & & & & C_{66}'' \end{bmatrix} \quad (16)$$

Terms of the ${}^J\mathbf{C}''$ matrix are easily expressed by means of more convenient engineering elastic constants where E_1, E_2 stand for Young's modulus in fiber direction and perpendicular to it, ν_{21} is Poisson's ratio, and G_{12}, G_{23}, G_{31} are corresponding shear moduli.

$$\begin{aligned} C_{11}'' &= \frac{E_1}{1 - \nu_{21}^2 \frac{E_2}{E_1}}, & C_{12}'' &= \nu_{21} C_{22}'', & C_{22}'' &= \frac{E_2}{1 - \nu_{21}^2 \frac{E_2}{E_1}} \\ C_{44}'' &= G_{12}, & C_{55}'' &= k G_{23}, & C_{66}'' &= k G_{31} \end{aligned} \quad (17)$$

These values could be calculated using some of the micromechanical approaches [5,6] taking into account mechanical properties of fiber and matrix, their geometric configuration and volume fraction of the constituents. k is the transverse shear correction factor, usually taken as 5/6 [4], considering error involved by the assumption that transverse shear strain is constant over the thickness, differing from the real parabolic one [4]. With these assumptions, it is necessary to transform the material matrix from the fiber coordinate system to the local coordinate system at each integration point (Fig. 2). Fiber angle to the local $\mathbf{e}_j'$ axis has to be defined in consistent manner at each point of the shell. By using the assumption that fibers are parallel within one layer, it is possible to define the angle θ^A at each point A of the shell volume [6]. $\bar{\mathbf{e}}_j'$ vector is parallel to $\mathbf{e}_j'^{(1)}$, and defined by the cross product

$$\bar{\mathbf{e}}_j' = \mathbf{e}_2'^{(1)} \times \mathbf{e}_3'^{(A)} \quad (18)$$

Angle $\tilde{\theta}$ is between local coordinate axis $\mathbf{e}_j'^{(A)}$ and axis parallel to $\mathbf{e}_j'^{(1)}$, thus measuring angle variation in ξ, η coordinate system. Signum function is employed to assure that different shell geometries could be described correctly.

$$\tilde{\theta} = \text{sign}(\mathbf{e}_2'^{(A)} \cdot \bar{\mathbf{e}}_1') \arccos \left(\frac{\mathbf{e}_1'^{(A)} \cdot \bar{\mathbf{e}}_1'}{\|\mathbf{e}_1'^{(A)}\| \|\bar{\mathbf{e}}_1'\|} \right) \quad (19)$$

Finally, the angle between fiber direction and local coordinate system axis $\mathbf{e}_j'^{(A)}$ in point A is given by

$$\theta^A = \theta^1 + \tilde{\theta} \quad (20)$$

The procedure explained in (18), (19), (20) is to be performed at each integration point. Therefore, the transformation from fiber to local coordinate Cartesian coordinate system is to be made at each integration point

$${}^A\mathbf{C}' = {}^A\mathbf{T}_j^T {}^j\mathbf{C}'' {}^A\mathbf{T}_j \quad (21)$$

with transformation matrix ${}^A\mathbf{T}_j$ defined using θ^A [6]. ${}^A\mathbf{C}'$ has to be transformed to the global coordinate system using transformation matrix ${}^A\mathbf{G}$ [6,7], to submit stiffness matrix assembly. This matrix yields from the second-order tensor transformation with final expression

$${}^A\mathbf{C} = {}^A\mathbf{G} {}^A\mathbf{C}' {}^A\mathbf{G}^T \quad (22)$$

Taking into account layered nature of the shell (Fig.3), ζ curvilinear coordinate is defined for each of N layers through the expression

$${}^n\zeta = -1 + \frac{2}{h} \sum_{j=1}^n {}^jh, \quad n = 1, N \quad (23)$$

with h as the total thickness of the shell, and ${}^jh$ is the thickness of j -th layer [8]. Numerical Gauss integration is to be performed over each layer due to variation of material properties in individual layers. Consequently, material matrix in the global coordinate system for each layer is to be defined as

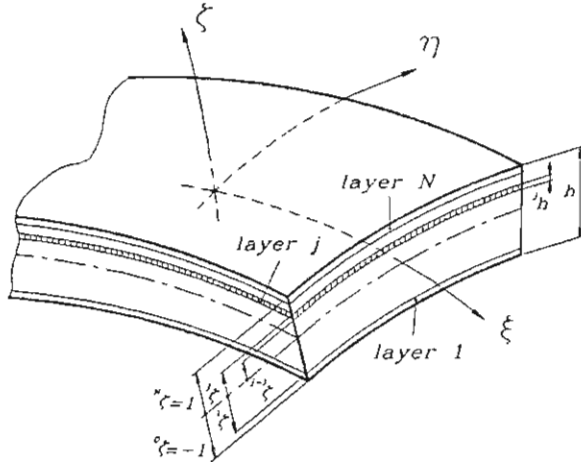


Fig. 3 Laminate geometry

$$\mathbf{C} = \begin{cases} {}^1\mathbf{C} & 0 \leq \zeta \leq {}^1\zeta \\ {}^2\mathbf{C} & {}^1\zeta < \zeta \leq {}^2\zeta \\ \dots & \dots \\ {}^N\mathbf{C} & {}^{N-1}\zeta < \zeta \leq {}^N\zeta \end{cases} \quad (24)$$

Now, stiffness matrix of the element is expressed using $\mathbf{B}$ matrix as defined before

$$\mathbf{K}^e = \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV \quad (25)$$

what in the treatment of layering effect is to be written as

$$\mathbf{K}^e = \sum_{j=1}^N \int_{V_j} {}^j\mathbf{B}^T {}^j\mathbf{C} {}^j\mathbf{B} dV_j \quad (26)$$

Full or reduced integration over ξ, η domain is to be performed, using one or two integration points over the thickness of each layer.

3. NUMERICAL EXAMPLES

All programming has been performed in MATLAB due to its superior matrix handling characteristics. Calculations have been done on PC with 80486 processor and 16 MB RAM. This amount of memory was necessary due to the nature of MATLAB which is interpreter and not a higher programming language. Therefore, all variables are in the computer memory unless otherwise ordered in the program. Solution of the finite element equation was performed using simple matrix inversion function that is intrinsic function of MATLAB.

Example 1

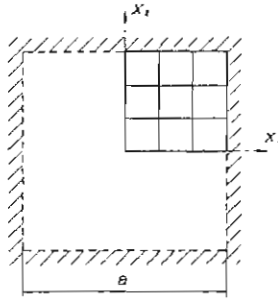


Fig. 4. Square plate (3x3 mesh)

A fully clamped plate (Fig. 4) under a constant load q with nine layered $[0/90/0/90/0/90/0/90/0]$ lay-up is examined. The geometry is defined with $a = 254 \text{ mm}$, and various thickness ratios $h/a = 0.1; 0.01; 0.001; 0.0001$. Material properties are $E_1 = 3 \times 10^7 \text{ N/mm}^2$, $E_2 = 7.5 \times 10^5 \text{ N/mm}^2$, $G_{12} = G_{13} = 4.5 \times 10^5 \text{ N/mm}^2$, $G_{23} = 3.75 \times 10^5 \text{ N/mm}^2$, $\nu_{21} = 0.25$. One quarter of the plate is discretized due to the symmetry. Deflection w at the middle point of the plate is calculated using different integration orders. Numerical results are compared in Table 1. with four-

noded, field-consistent plate/shell element [9] that discretized one quarter of the plate with 4×4 mesh. This element uses reduced $2 \times 2 \times 1$ numerical integration in optimal integration points for shear energy calculation and field redistributed shape functions to avoid membrane and shear locking. In the element presented in this paper full integration $3 \times 3 \times 1$ shows certain degree of locking in the case of very thin shells. The results are significantly improved by using reduced integration $2 \times 2 \times 1$, differing less than 2 % from the values presented in [9] using 8×8 elements.

Table 1: Non-dimensional center displacement $\bar{w} = w 10^3 E_2 h^3 / q a^4$

h/a					Ref. [9]	
	3 x 3		4 x 4		4 x 4	8 x 8
	3 x 3 x 1	2 x 2 x 1	3 x 3 x 1	2 x 2 x 1		
0.1	2.642	2.644	2.633	2.634	2.308	2.316
0.01	0.926	0.979	0.955	0.977	0.941	0.957
0.001	0.863	0.961	0.905	0.959	0.927	0.944
0.0001	0.862	0.961	0.904	0.959	0.927	0.944

Example 2

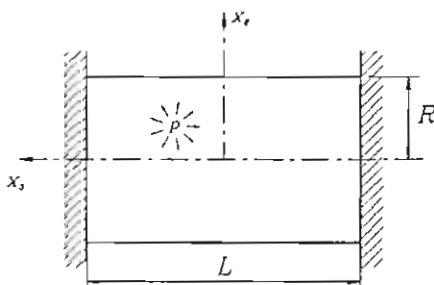


Fig. 5. Clamped cylinder shell

A fixed cylinder (Fig. 5) exposed to constant internal pressure $p = 0.0141 \text{ N/mm}^2$ is studied. Lay-ups studied are $[45/-45/45/45]$ and $[0/90/90/0]$. The geometry is defined as $L = 508 \text{ mm}$, $R = 508 \text{ mm}$, $R/h = 20; 100$. Material properties are $E_1 = 5.177 \times 10^4 \text{ N/mm}^2$, $E_2 = 1.381 \times 10^4 \text{ N/mm}^2$, $G_{12} = 8.629 \times 10^3 \text{ N/mm}^2$, $G_{13} = G_{23} = 4.349 \times 10^3 \text{ N/mm}^2$, $\nu_{21} = 0.25$. Since

the lay-ups are symmetric, only one eight of the shell is discretized. The radial displacement u_r at the middle of the shell is calculated using $2 \times 2 \times 1$ integration rule. The element selected for comparison of the results is nine-noded assumed strain element for composite plates and shells [10] based on Hellinger-Reissner principle, with strains assumed to have polynomial distribution of various order in ξ and η . This element is stated to be free of locking and with no communicable spurious zero modes. Very good agreement was established with results within 3% of the reference solution [10] what is shown in Table 2.

Table 2: Non-dimensional radial displacement $\bar{u}_r = u_r E_1 h / p R^2$

Layering setup	R/h=20		R/h = 100	
	mesh 4×4	Ref. [10] 4×4	mesh 4×4	Ref. [10] 4×4
0 / 90 / 90 / 0	1.70	1.64	1.56	1.55
45 / -45 / -45 / 45	1.96	1.97	2.18	2.20

Example 3

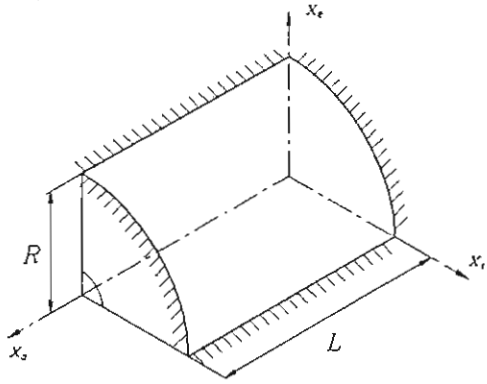


Fig. 6. 90° shell clamped at all edges

A clamped 90° section of the cylinder (Fig. 6) under constant internal pressure $p=0.0141 \text{ N/mm}^2$ is investigated. [-45/45], [45/-45/-45/45] and [0/90/90/0] lay-ups are studied. The geometry of the shell and material properties are as in Example 2. R/h ratios are given as 20 and 100. Full shell is discretized. The radial displacement u_r at the center of the shell is calculated using $2 \times 2 \times 1$ integration order. Comparison was made to the results in Ref. [10] in Table 3. In the case of [0/90/90/0] lay-up certain discrepancies appeared that increased in [45/-45/-45/45] lay-up. The reason for this effect is presumed to be parabolic interpolation of transverse shear strains [10] differing from constant one used in the element presented.

Table 3: Non -dimensional radial displacement $\bar{u}_r = u_r E_1 h / p R^2$

Layering setup	R/h=20		R/h = 100	
	mesh 4×4	Ref. [10] 4×4	mesh 4×4	Ref. [10] 4×4
-45 / 45	2.65	2.68	1.40	1.52
0 / 90 / 90 / 0	2.76	2.68	1.35	1.65
45 / -45 / -45 / 45	2.59	2.10	1.45	1.35

4. CONCLUSIONS

The results presented in the paper are obtained using nine noded element based on degeneration principle. The element proved well in the linear analysis of multilayered composite shells with certain disadvantages regarding locking, what is to be solved in the future work. Programming in MATLAB enabled very easy code writing but some unwanted side-effects have appeared, like unexpected ill-conditioned stiffness matrices in the analysis of plate problems. Further investigation and improvements in interpolation of transverse shear strains are necessary to fully utilize this code and make it suitable for wider range of applications.

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Geometrically Nonlinear Space Beams with End Releases

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Summary

A novel approach is presented for the analysis of spatial beam elements with end releases, in which for each joint in the structure an additional (slave) set of kinematic variables is introduced, which is directly related to the existing (master) set of variables at that node. This relationship is established according to the nature of the joint to be modelled and takes into account that the sliding/rotation takes place along/around the axis that is rigidly attached to the structural node and is thus not fixed in space. In this way, the most interesting releases such as revolute, spherical, prismatic and cylindrical joints can be analysed accurately and efficiently. The main concepts are applicable to any spatial beam finite element, provided that a displacement vector and a rotation matrix are defined at both end nodes. In the article, however, only the case when the rotation matrix is parametrised using the components of noncommutative rotational pseudovector is addressed in detail. The numerical example, in which a highly deformable space frame with a revolute joint is analysed, demonstrates the accuracy of the proposed procedure and its advantages compared to the "penalty" technique, in which the joint is simulated using the appropriate bending stiffness of the joint element.

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1. Introduction

There are increased industrial interests in the analysis of rotating machinery, robot arms, retractable-deployable elastic space structures and other types of structures composed of three-dimensional beam elements with particular end releases acting along/around the axes of the rotating system. Due to the improved industrial standards, which require the actual deformations of the structure to be accounted for, the finite element approach is presently being preferred to the traditional rigid body mechanics principles. Along the lines of the finite element method there have been proposed several procedures [1,2,3], mainly arising from the functional analysis in which the nonlinear constraint equations were taken into consideration through different versions of the Lagrangian multiplier technique. In the present work we propose a specific master-slave approach, in which dependent variables are written in terms of independent variables. In this approach the nonlinear constraint equations, associated with large rotations [4] are used and consistently linearised so that the full benefit of the present class of nonlinear beam elements [5,6,7,8] is obtained. The main concept should be usable in conjunction with any element, provided that an orthogonal rotation matrix and a displacement vector are defined at both end nodes of the element. The present work, however, develops this idea in a particular case when the rotation matrix is parametrised using the components of the rotational pseudovector, as in the elements of [5,6,7,8]. The procedure has been developed for prismatic joints, revolute joints (hinges), spherical joints and cylindrical joints [9].

2. Theory

If there is an element in a structure which is not connected to the other elements, it may be displaced in a manner shown in Fig 1. The following relations can be established for its left-end node:

$$\mathbf{d}_1 = \mathbf{d}_{m1} + \boldsymbol{\rho}_1 \quad (1)$$

$$\mathbf{T} = \mathbf{T}^* \mathbf{T}_m, \quad (2)$$

where $\mathbf{d}_{m1}$ and $\mathbf{T}_m$ define the displacement vector and rotation matrix of the *master* left-end node, and $\mathbf{d}_1$ and $\mathbf{T}$ define the displacement vector and rotation matrix of the *slave* left-end node. Note that in the engineering beam theory the origins of the vectors $\mathbf{d}_{m1}$ and $\mathbf{d}_1$ coincide, i.e. the gap between them in Fig. 1 has been drawn only for the sake of clarity. The columns of the rotation matrices $\mathbf{T}_m$ and $\mathbf{T}$ in (2) consist of orthonormal base vectors $\mathbf{t}_{m1}, \mathbf{t}_{m2}, \mathbf{t}_{m3}$ and $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ from Fig. 1. The rotation matrix $\mathbf{T}^*$ in (2) is the matrix that defines the rotation of the master triad, $\mathbf{t}_{m1}, \mathbf{t}_{m2}, \mathbf{t}_{m3}$, onto the slave triad, $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$. For the right-end node the equivalent equations to (1) and (2) can also be established. The rotation matrices will in the following be parametrised using the components of the rotational pseudovectors [4].

When modelling real joints, the master variables, $\mathbf{d}_{m1}$ and $\mathbf{T}_m$ (or $\mathbf{d}_{m2}$ and $\mathbf{Q}_m$), are not entirely independent of the slave variables, $\mathbf{d}_1$ and $\mathbf{T}$ ($\mathbf{d}_2$ and $\mathbf{Q}$), as they are in Fig. 1. Depending on the type of joint, some of the components of the displacement vectors, $\mathbf{d}_{m1}$ and $\mathbf{d}_1$ ($\mathbf{d}_{m2}$ and $\mathbf{d}_2$), or parameters of the rotation matrices, $\mathbf{T}_m$ and $\mathbf{T}$ ($\mathbf{Q}_m$ and $\mathbf{Q}$), are the same. Different types of joints are defined by releasing displacements or rotations along/around chosen axes which, in a geometrically nonlinear environment, move in space together with the node they are attached on. Accordingly, all the releases will be defined with respect to the master triad, $\mathbf{t}_{m1}, \mathbf{t}_{m2}, \mathbf{t}_{m3}$ ($\mathbf{q}_{m1}, \mathbf{q}_{m2}, \mathbf{q}_{m3}$).

For prismatic (sliding) joints [9], this means that the "difference displacement vector", ρ_1 (or ρ_2), between master and slave variables is, when transformed into coordinates defined by the master triad, equal to the vector of released displacements:

$$s_1 = T_m^T \rho_1, \quad s_2 = Q_m^T \rho_2, \quad (3a, b)$$

where the vector of released displacements, s_1 (or s_2), has zero components in non-released directions.

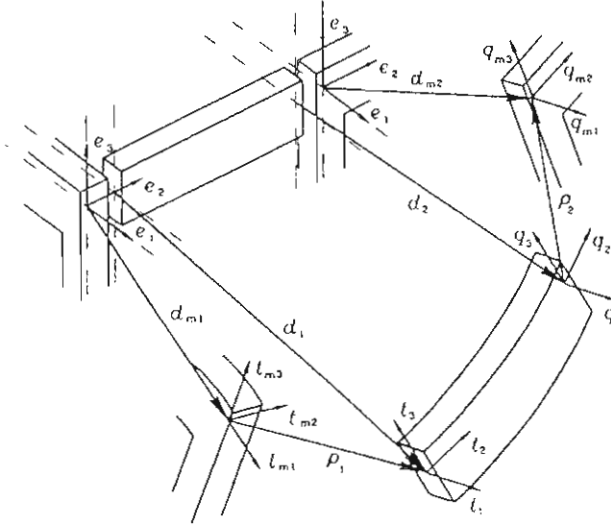


Figure 1. Master, slave and released element variables

For revolute joints [9] (hinges), the rotational pseudovector α^* (or β^*), extracted from the "difference rotation matrix", T^* (or Q^*), and transformed into coordinates defined by the master triad, must be equal to the rotational pseudovector of released rotations:

$$\psi = T_m^T \alpha^*, \quad \varphi = Q_m^T \beta^*, \quad (4a, b)$$

where the rotational pseudovector of released rotations, ψ (or φ) has zero components in non-released directions.

The first step in the finite element calculation is the application of the principle of virtual work so we need to express the variations of the slave variables in terms of the master and released variables and their variations. The variations of the variables will be referred to as the *degrees of freedom*.

The variation of the rotational pseudovector, $\delta\alpha$ (or $\delta\beta$), relates to the variation of the rotation matrix in the following way [4]

$$\delta T = S(\delta\alpha) T, \quad \delta Q = S(\delta\beta) Q,$$

where $S(\delta\alpha)$ and $S(\delta\beta)$ are the skew symmetric matrices containing the components of $\delta\alpha$ and $\delta\beta$, which are nonadditive to α and β even when $\delta\alpha$ and $\delta\beta$ tend to zero.

Inserting (3a) into (1) and bearing in mind that the inverse of an orthogonal matrix equals its transpose gives

$$d_1 = d_{m1} + T_m s_1.$$

Variation of this equation is

$$\delta d_1 = \delta d_{m1} + S(\delta \alpha_m) T_m s_1 + T_m \delta s_1,$$

where α_m is the rotational pseudovector associated with the rotation matrix T_m , and $S(\delta \alpha_m)$ is a skew symmetric matrix containing the components of $\delta \alpha_m$. The skew symmetric matrix acts as a vector product operator, therefore δd_1 may be expressed as

$$\delta d_1 = [I \quad -S(T_m s_1) \quad T_m] \begin{Bmatrix} \delta d_{m1} \\ \delta \alpha_m \\ \delta s_1 \end{Bmatrix}. \quad (5)$$

In equation (5), the slave translational degrees of freedom are expressed in terms of the master and released degrees of freedom through the linear transformation matrix which contains master and released variables. A corresponding expression also applies to the right-end node. The slave rotational degrees of freedom will be treated in a similar manner. As the rotational pseudovector associated with T^* is α^* , i.e. $T^* = \exp S(\alpha^*)$, equation (2) may be rewritten as

$$T = \exp S(\alpha^*) T_m.$$

Inserting α^* from (4a) into this equation and bearing in mind that

$$\exp S(T_m \psi) = \exp [T_m S(\psi) T_m^T] = T_m \exp S(\psi) T_m^T,$$

which is easily confirmed by matrix multiplication using a MacLaurin series expansion of the matrix exponential function, we obtain

$$T = T_m \exp S(\psi).$$

The variation of this equation [10] leads to the expression

$$S(\delta \alpha) = S(\delta \alpha_m) + S(T_m \delta \psi),$$

which can be rewritten in terms of axial vectors associated with skew symmetric matrices as

$$\delta \alpha = [I \quad T_m] \begin{Bmatrix} \delta \alpha_m \\ \delta \psi \end{Bmatrix}. \quad (6)$$

A corresponding expression also applies to the right-end node. Equations (5) and (6) define the relationships between the slave and the "master and released" degrees of freedom.

The degrees of freedom for the beam element in Fig. 1 are arranged in the following manner:

$$\begin{aligned}
 \text{Slave degrees of freedom:} \quad & \delta \mathbf{p}^T = \langle \delta d_1^T \quad \delta \alpha^T \quad \delta d_2^T \quad \delta \beta^T \rangle \\
 \text{Master degrees of freedom:} \quad & \delta \mathbf{p}_m^T = \langle \delta d_{m1}^T \quad \delta \alpha_m^T \quad \delta d_{m2}^T \quad \delta \beta_m^T \rangle \\
 \text{Released degrees of freedom:} \quad & \delta \mathbf{p}_r^T = \langle \delta s_1^T \quad \delta \psi^T \quad \delta s_2^T \quad \delta \varphi^T \rangle \\
 \text{Master \& released DOF:} \quad & \delta \mathbf{p}_{mr}^T = \langle \delta s_1^T \quad \delta \psi^T \quad \delta s_2^T \quad \delta \varphi^T \quad \delta d_{m1}^T \quad \delta \alpha_m^T \quad \delta d_{m2}^T \quad \delta \beta_m^T \rangle
 \end{aligned}$$

Using (5) and (6) and the previously introduced symbols, the slave degrees of freedom can be expressed in terms of the master and released degrees of freedom in the following way

$$\begin{Bmatrix} \delta d_1 \\ \delta \alpha \\ \delta d_2 \\ \delta \beta \end{Bmatrix} = \begin{bmatrix} T_m & 0 & 0 & 0 & I & -S(T_m s_1) & 0 & 0 \\ 0 & T_m & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & Q_m & 0 & 0 & 0 & I & -S(Q_m s_2) \\ 0 & 0 & 0 & Q_m & 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \delta s_1 \\ \delta \psi \\ \delta s_2 \\ \delta \varphi \\ \delta d_{m1} \\ \delta \alpha_m \\ \delta d_{m2} \\ \delta \beta_m \end{Bmatrix}. \quad (7)$$

The introduction of (7) into the principle of virtual work would lead to the nonlinear equilibrium equations. The linearisation of these equations, the elimination of the released degrees of freedom at the element level and the formation of the structural tangent stiffness matrix and the vector of residuals should be done according to the principles of the finite element method. All these topics are fully described in [10]. However, as they require more space than available, they are here omitted. The master, released and slave variables at the left-end node are updated using [10]:

$$d_{m1, \text{new}} = d_{m1} + \delta d_{m1}$$

$$T_{m, \text{new}} = \exp S(\delta \alpha_m) T_m$$

$$s_{1, \text{new}} = s_1 + \delta s_1$$

$$\exp S(\psi_{\text{new}}) = \exp S(\delta \psi) \exp S(\psi)$$

$$d_{1, \text{new}} = d_{m1, \text{new}} + T_{m, \text{new}} s_{1, \text{new}}$$

$$T_{\text{new}} = T_{m, \text{new}} \exp S(\psi_{\text{new}}).$$

The master, released and slave variables at the right-end node are updated using the same equations with the corresponding right-end node symbols.

3. Example

Fig. 2 shows a right-angled frame and gives the adopted dimensions and geometric and material properties. The frame is loaded by means of a conservative force F of magnitude $2\pi \cdot 10^5$ which acts at the connection of the two legs of the frame in the negative direction of the base vector e_3 . The legs are connected to each other via a revolute joint.

The proposed non-linear master-slave procedure has been mounted in the finite element system LUSAS [11]. For the analysis, each leg of structure was modelled with the aid of four first-order Timoshenko beam elements which are embedded within a co-rotational framework [7,12]. The mesh is shown in Fig. 3a. The revolute joint between the two legs of the frame allows the rotation around the axis that initially coincided with the local y axis of the element 5 at node ⑤ (see Fig. 3b).

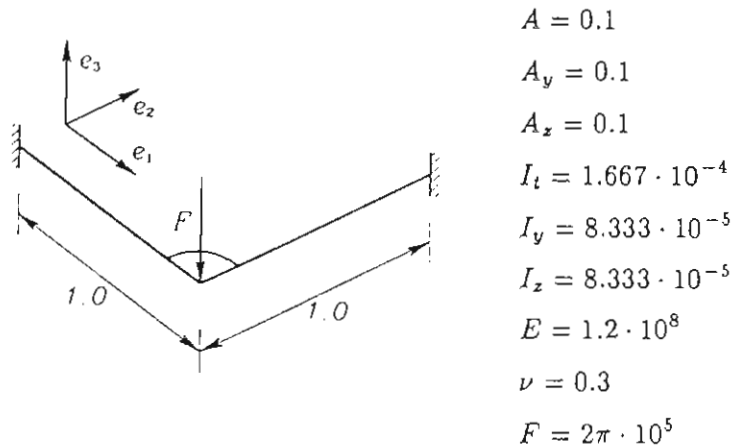


Figure 2. Right-angle planar frame with perpendicular point load

The full Newton-Raphson solution procedure was adopted and the iterations were terminated once the Euclidean norm of the iterative displacement change (considering only translational variables) was less than 10^{-5} .

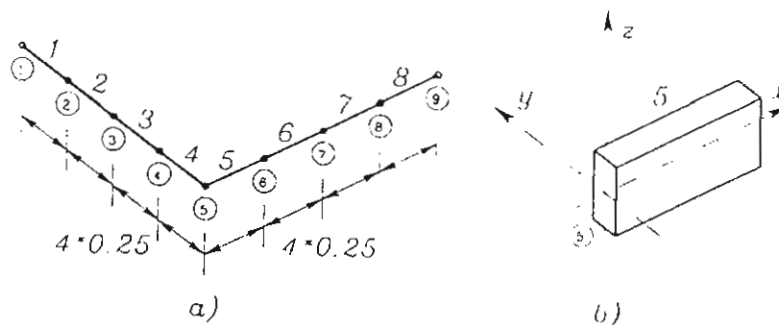


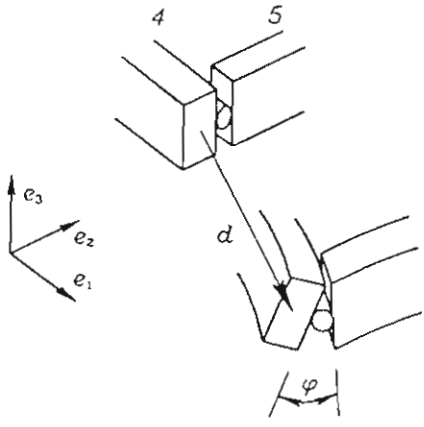
Figure 3. Right-angle planar frame
 a) Finite element mesh, b) Local coordinate axes of element 5

The final configuration was reached in five equal loading steps with the results for the displacement and the released rotation at node ⑤ being given in Table 1, which also shows the required number of iterations for each load step.

As a check on the results, the joint was simulated via a small element e of the length 0.0001 between elements 4 and 5. Rotation around the axis that initially coincided with the local y axis of the element

5 was allowed by nullifying the corresponding moment of inertia of the element e (see figure in Table 2). All the other translations and rotations within this element were reduced by enlarging the remaining geometric properties by the factor of 100 with respect to those from Fig. 2. This process can be seen as a form of "penalty" technique. The results are tabulated in Table 2.

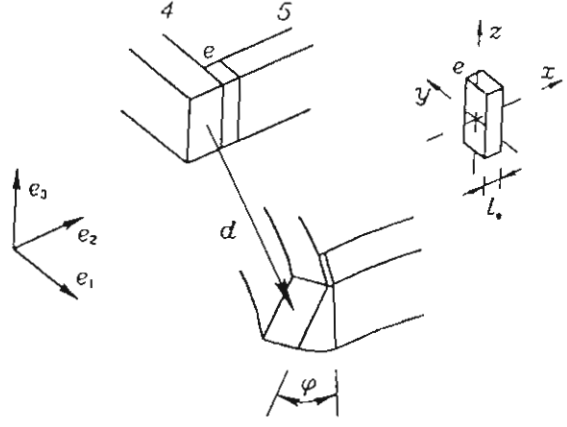
Table 1 – Revolute joint



$$d = ue_1 + ve_2 + we_3$$

Increment	Loading factor	Number of iterations	u	v	w	φ
1	0.2	5	-0.0147	0.0152	-0.1595	0.2441
2	0.4	5	-0.0492	0.0509	-0.2861	0.4495
3	0.6	5	-0.0884	0.0915	-0.3752	0.6072
4	0.8	5	-0.1249	0.1291	-0.4364	0.7256
5	1.0	5	-0.1565	0.1616	-0.4794	0.8151

Table 2 – Simulation of revolute joint



$$d = ue_1 + ve_2 + we_3$$

$$A = 10.0$$

$$A_y = 10.0$$

$$A_z = 10.0$$

$$I_t = 1.667 \cdot 10^{-2}$$

$$I_y = 0.0$$

$$I_z = 8.333 \cdot 10^{-3}$$

$$l_e = 0.0001$$

Increment	Loading factor	Number of iterations	u	v	w	φ
1	0.2	9	-0.0147	0.0153	-0.1596	0.2448
2	0.4	10	-0.0494	0.0512	-0.2867	0.4540
3	0.6	9	-0.0892	0.0923	-0.3764	0.6187
4	0.8	10	-0.1262	0.1305	-0.4381	0.7458
5	1.0	8	-0.1583	0.1635	-0.4813	0.8442

In the final configuration, the results for the two models differ by $\sim 0.4\%$ for the w component of the displacement vector, by $\sim 1.2\%$ for its u and v components and by $\sim 3.6\%$ for the released rotation, φ . The "penalty" simulation required $\sim 84\%$ more iterations to achieve equilibrium.

4. Conclusion

The paper has outlined the idea for a nonlinear master-slave procedure that can be used for joints with three-dimensional beam elements and large rotations. The idea is independent of the original element formulation, but in the present work only the case when the rotation matrix is parametrised using the nonadditive rotational pseudovector has been considered.

To test the technique, a form of penalty technique has been used, in which a small element, endowed with artificially small stiffnesses has simulated a revolute joint. Close agreement in results has been obtained, yet the penalty technique has required significantly more iterations. Besides, due to additional joint elements, the "penalty system of equations" to be solved has had more degrees of freedom. The additional advantages of the proposed method and the drawbacks of the penalty technique were examined in the reference [10], along with the detailed theoretical exposition and more numerical examples.

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A NON-LINEAR FINITE ELEMENT MODEL FOR THIN ELASTIC SHELLS

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Abstract - *A geometrically non-linear finite element formulation for thin shells is presented. It is based on the shell theory for thin elastic shells undergoing large rotations. Large rotations are parametrized with two Euler angles. Consistent linearization of the adopted shell model is performed, which leads to a numerical algorithm with a quadratic rate of asymptotic convergence. Two examples are provided and compared with the solutions reported in recent literature to illustrate the approach.*

1. INTRODUCTION

The objective in this paper is to present a simple geometrically non-linear finite element formulation for thin elastic shells. The underlying shell theory is derived from the three-dimensional continuum theory. The adopted assumptions on the displacement field cause that the displacement based finite element formulations suffer from the transverse shear locking. It is therefore essential that the solutions are based on mixed variational formulations. A current 'standard' to avoid transverse shear locking is ANS (Assumed Natural Strain) approach [1]. An approach which relies on rotations around fixed axes is chosen for the parametrization of the shell director rotation. The rotation matrix is expressed with two independent angles, as originally introduced by Ramm [2]. Local Cartesian frames are defined over numerical integration points. A consistent linearization of the shell model leading to the Newton-Raphson numerical procedure is performed in order to obtain an efficient algorithm with a quadratic rate of asymptotic convergence.

2. FUNDAMENTAL EQUATIONS

The attention is restricted to the material description. Material points of the shell body are identified by a set of curvilinear coordinates $\{\xi^1, \xi^2, \xi^3\}$. Coordinates ξ^1 and ξ^2 parametrize the mid-surface of the shell, while ξ^3 is a parameter which defines the shell thickness. Two basic assumptions are adopted: a linearly varying displacement field in the thickness direction and an inextensibility of the shell director field. The position vectors in the reference and deformed configuration then read

$$R(\xi^i) = X(\xi^\alpha) + \xi^3 T(\xi^\alpha), \quad r(\xi^i) = X(\xi^\alpha) + u(\xi^\alpha) + \xi^3 t(\xi^\alpha), \quad (1)$$

with $\xi^3 \in [-h/2, h/2]$ and $\|T\| = \|t\| = 1$. Here h is the shell thickness, X denotes the mid-surface position vector at the reference configuration, u the displacement vector of the mid-surface, while T and t denote the shell director at initial and deformed configuration respectively. The position vectors, along with the displacement field, are related to the fixed Euclidean base $\{E_1, E_2, E_3\}$. A motion of a single director at a particular material point of the shell mid-surface is regarded as a rotation of a unit vector. All its possible configurations define a unit sphere $S^2 := \{t \in R^3, \|t\| = 1\}$. Transformation between any two vectors, elements of S^2 , is given by rotation matrix Q . In the present model Q is expressed as a sequence of two rotations around fixed axes. This parametrization was introduced in 1976 by Ramm. The position of a particular shell director at initial and deformed configuration is $T = Q(\Psi, \Omega)E_2$ and $t = Q(\Psi + \alpha, \Omega + \beta)E_2$ respectively. An explicit form of t is given as

$$t = \begin{Bmatrix} \cos(\Omega + \beta) \\ \cos(\Psi + \alpha)\sin(\Omega + \beta) \\ \sin(\Psi + \alpha)\sin(\Omega + \beta) \end{Bmatrix}, \quad (2)$$

where angles Ω and Ψ define the position of the shell director at reference configuration.

The Green-Lagrange strains are defined as the difference between two metrics, i.e.

$$E_{ij} = 1/2(g_{ij} - G_{ij}), \quad g_{ij} = g_i \cdot g_j, \quad G_{ij} = G_i \cdot G_j, \quad (3)$$

where $g_i = r_{,i}$ and $G_i = R_{,i}$. Here $(\bullet)_{,i}$ denotes the partial derivative with respect to ξ^i .

The mid-surface tensor at the reference configuration is given as $A_{\alpha\beta} = X_{,\alpha} \cdot X_{,\beta}$ and its

contravariant part is obtained by an inversion $[A^{\alpha\beta}] = [A_{\alpha\beta}]^{-1}$. Using (1) in (3) the strains E_{ij} may be expressed as a quadratic function of ξ^3 , i.e. $E_{ij} = 1/2[\epsilon_{ij} + \xi^3 \kappa_{ij} + (\xi^3)^2 \rho_{ij}]$. The strain measures provide a quadratic representation for normal strains and linear representation of transverse shear strains through the thickness, ($\rho_{\alpha 3} = 0$), while $E_{33} = 0$. If the gradients of the transverse shear strains are small and the initial thickness is only slightly varying, the quadratic terms for normal strains, ρ_{ij} , and linear terms for transverse shear strains, $\kappa_{\alpha 3}$, can be neglected. The membrane, transverse shear and bending strain measures then take the form

$$\epsilon_{\alpha\beta} = \frac{1}{2}(X_{,\alpha} \cdot u_{,\beta} + X_{,\beta} \cdot u_{,\alpha} + u_{,\alpha} \cdot u_{,\beta}), \quad \gamma_\alpha = 2E_{\alpha 3} = [X_{,\alpha} \cdot (t - T) + u_{,\alpha} \cdot T] \quad (4)$$

$$\kappa_{\alpha\beta} = \frac{1}{2}[X_{,\alpha} \cdot (t_{,\beta} - T_{,\beta}) + X_{,\beta} \cdot (t_{,\alpha} - T_{,\alpha}) + u_{,\alpha} \cdot t_{,\beta} + u_{,\beta} \cdot t_{,\alpha}]. \quad (5)$$

Having defined the strain measures, the principle of virtual work can be written. This is a governing equation for the displacement based finite element formulation. The two dimensional principle of virtual work is

$$G(r, \delta r) = \int_{S_0} (n^{\alpha\beta} \delta \epsilon_{\alpha\beta} + m^{\alpha\beta} \delta \kappa_{\alpha\beta} + q^\alpha \delta \gamma_\alpha) dS_0 - G_{ext}(\delta r) = 0, \quad (6)$$

where stress resultants $n^{\alpha\beta}$, $m^{\alpha\beta}$ and q^α are expressed as

$$n^{\alpha\beta} = \int_{-h/2}^{h/2} S^{\alpha\beta} \mu d\xi, \quad m^{\alpha\beta} = \int_{-h/2}^{h/2} S^{\alpha\beta} \mu \xi d\xi, \quad q^\alpha = \int_{-h/2}^{h/2} S^{\alpha 3} \mu d\xi. \quad (7)$$

Here $\mu = \sqrt{G/A}$, with $G = \det[G_{ij}]$ and $A = \det[A_{\alpha\beta}]$ and $S^{\alpha\beta}$ are the components of the 2nd Piolla-Kirchhoff stress tensor.

A simple hyperelastic constitutive law is employed. For the defined strain measures the thickness stretch does not appear as a kinematic variable. It is therefore obtained by employing an assumption of zero stress in the direction of ξ^3 and the condensation of the material law. The constitutive relations read

$$n^{\alpha\beta} = \frac{Eh}{1-\nu^2} C^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta}, \quad m^{\alpha\beta} = \frac{Eh^3}{12(1-\nu^2)} C^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}, \quad q^\alpha = kGA^{\alpha\beta} \gamma_\beta, \quad (8)$$

$$C^{\alpha\beta\gamma\delta} = \nu A^{\alpha\beta} A^{\gamma\delta} + \frac{1}{2}(1-\nu)(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma}), \quad (9)$$

where E , G , k and ν are elastic and shear modulus, shear correction factor and Poisson's ratio respectively.

The definition of the displacement based problem is: Find a configuration $\mathbf{r}$, such that $G(\mathbf{r}, \delta\mathbf{r}) = 0$ for any $\delta\mathbf{r}$. Since the solution of the above non-linear problem is obtained by the Newton-Raphson method, a linearization of $G(\mathbf{r}, \delta\mathbf{r})$ is required. The Newton-Raphson procedure at the configuration $\mathbf{r} = \bar{\mathbf{r}}$ may be expressed in the form $DG(\bar{\mathbf{r}}^k, \delta\mathbf{r})[\Delta\mathbf{r}] = -G(\bar{\mathbf{r}}^k, \delta\mathbf{r})$, where the right side has the standard interpretation of the residual forces. The tangent operator $DG(\bar{\mathbf{r}}^k, \delta\mathbf{r})[\Delta\mathbf{r}]$ may be split into a material and a geometric part

$$D_M G(\mathbf{r}, \delta\mathbf{r})[\Delta\mathbf{r}] = \int_{S_0} (Dn^{\alpha\beta}[\Delta\mathbf{r}]\delta\epsilon_{\alpha\beta} + Dm^{\alpha\beta}[\Delta\mathbf{r}]\delta\kappa_{\alpha\beta} + Dq^\alpha[\Delta\mathbf{r}]\delta\gamma_\alpha) dS_0 \quad (10)$$

$$D_G G(\mathbf{r}, \delta\mathbf{r})[\Delta\mathbf{r}] = \int_{S_0} (n^{\alpha\beta} D(\delta\epsilon_{\alpha\beta})[\Delta\mathbf{r}] + m^{\alpha\beta} D(\delta\kappa_{\alpha\beta})[\Delta\mathbf{r}] + q^\alpha D(\delta\gamma_\alpha)[\Delta\mathbf{r}]) dS_0. \quad (11)$$

3. FINITE ELEMENT DISCRETIZATION

The finite element discretization is introduced in a standard manner. Here a four-node element is used with isoparametric interpolation

$$\begin{aligned} \mathbf{X} &= \sum_{I=1}^4 N^I(\xi^1, \xi^2) \mathbf{X}_I, & \mathbf{T} &= \sum_{I=1}^4 N^I(\xi^1, \xi^2) \mathbf{T}_I, & \mathbf{u} &= \sum_{I=1}^4 N^I(\xi^1, \xi^2) \mathbf{u}_I, \\ t &= \sum_{I=1}^4 N^I(\xi^1, \xi^2) t_I, & \delta\mathbf{u} &= \sum_{I=1}^4 N^I(\xi^1, \xi^2) \delta\mathbf{u}_I, & \delta t &= \sum_{I=1}^4 N^I(\xi^1, \xi^2) \delta t_I. \end{aligned} \quad (12)$$

where N^I denotes the shape function and nodal values are denoted by a subscript I . Each node has five degrees of freedom - three components of $\mathbf{u}_I$ and two rotational components α_I and β_I . The transverse shear strains are interpolated as proposed in [1]. Using upper interpolation functions, along with their derivatives (with respect to ξ^a) a discrete formulation of a first variation of a partially mixed functional can be constructed. The linearization of this equation is required within the iterative Newton-Raphson procedure. Local Cartesian frames are defined over numerical integration points in order to simplify the governing expressions and prepare the formulation to include inelastic material models.

4. NUMERICAL EXAMPLES

EXAMPLE 1: Stretching of a cylinder with free ends. The geometry and loading conditions are shown in Fig.1. The radius of the cylinder is $R=4.953$, length is $L=10.35$ and thickness is $t=0.094$. The elasticity properties are $E=10.5 \times 10^3$ and $\nu=0.3125$. The initial value of the force is $F_0=50.0$. This example was considered in /3/, /4/ and /5/.

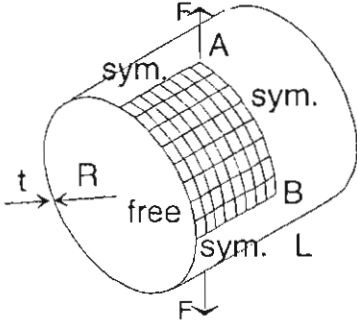


Fig.1: Cylinder with free ends: Geometry and loading.

Three different meshes 8×8 , 16×16 and 32×32 are considered in the present analysis. Computation is performed by load control and the standard check of the residual forces is performed with convergence tolerance 1.0×10^{-7} . Force F versus displacement curves of points A and B are presented in Fig.2.

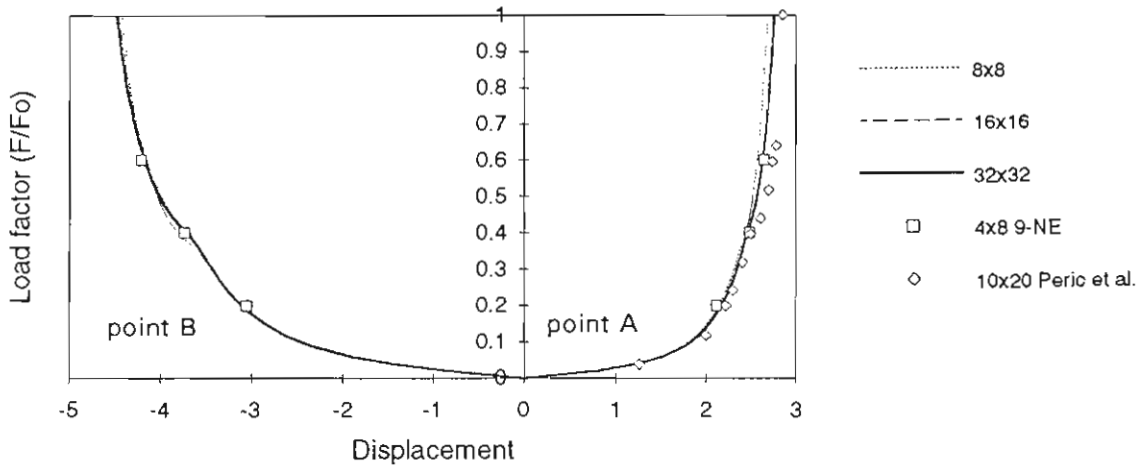


Fig.2: Cylinder with free ends: Load versus displacement curves for points A and B.

Solutions obtained with 16×16 and 32×32 meshes are in very good agreement with the solutions obtained by Sansour and Bufler /5/ who used 9 node element (denoted as 9-NE). In Fig.3 four different equilibrium configurations are depicted.

EXAMPLE 2: Clamped rubber hemisphere under the point load. This example, considered in /6/ and /7/, examines the collapse of a clamped spherical shell of a rubber-like material. The radius of the sphere is $R=26.36 \text{ mm}$ and thickness is $t=4.4 \text{ mm}$. Elasticity properties are $E=4.0 \times 10^3 \text{ kPa}$, $G=1.333 \times 10^3 \text{ kPa}$ and $\nu=0.5$.

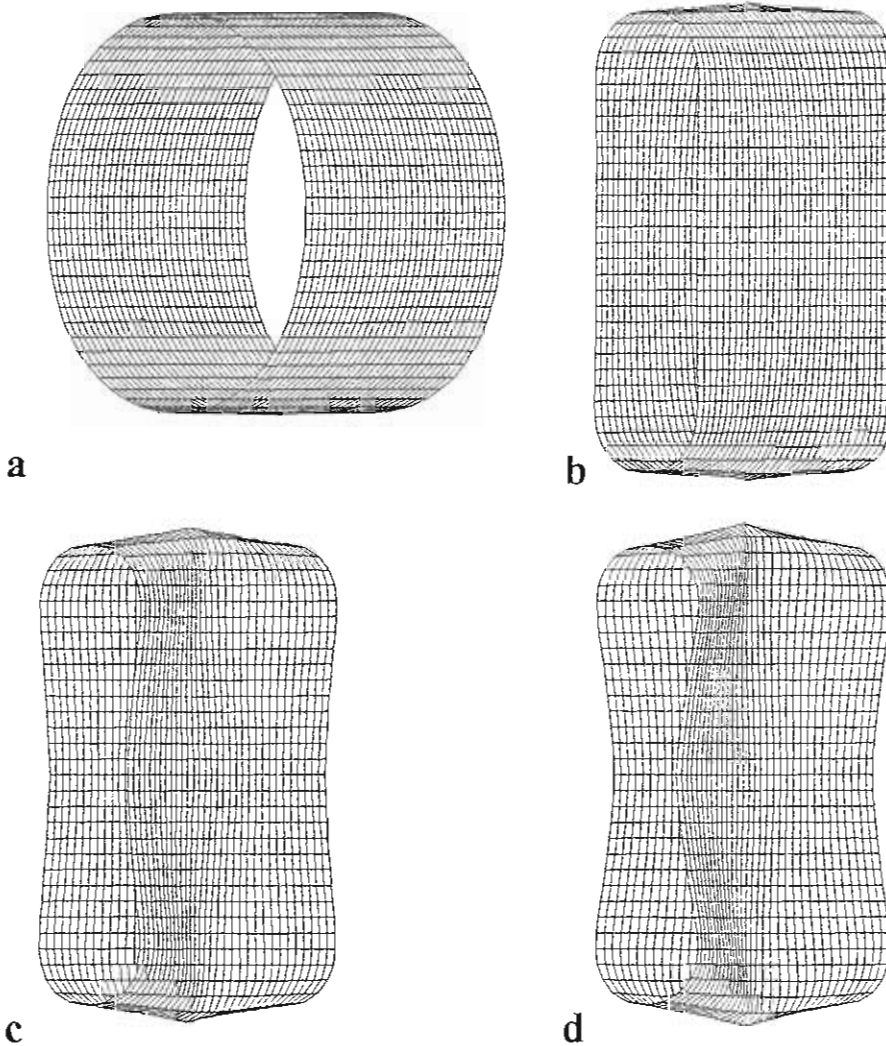


Fig.3: Cylinder with free ends: Deformed mesh (16 x 16) at a) $F/F_0=0.04$, b) $F/F_0=0.33$, c) $F/F_0=0.60$, d) $F/F_0=1.0$.

In the analysis the shear correction factor is increased 100 times in order to suppress the transverse shear deformations, see /6/. One quarter of the shell is discretised with 16 x 16 elements. The load versus deflection curve is plotted in Fig.5, along with

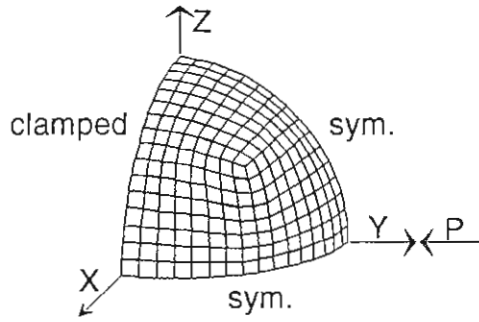


Fig.4: Rubber ball: Geometry and loading.

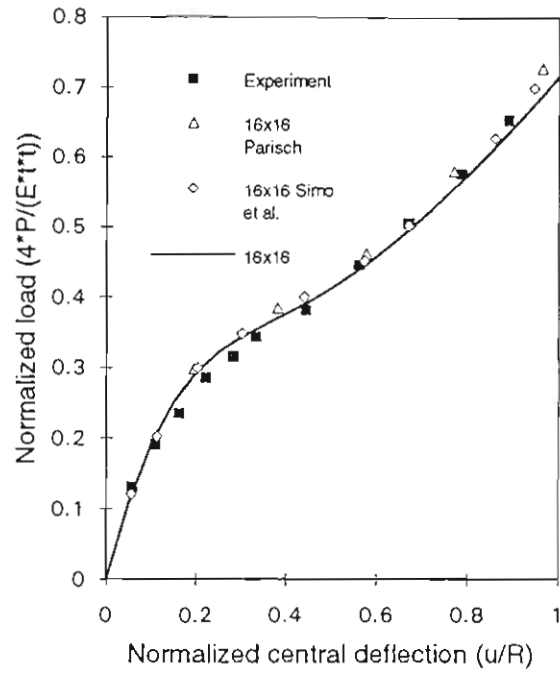
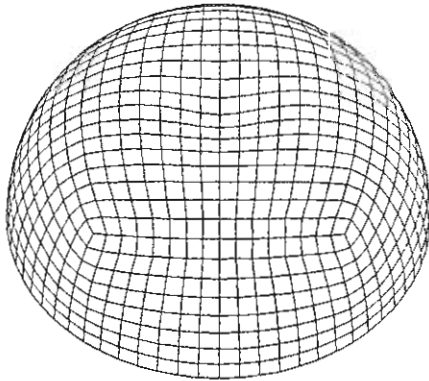
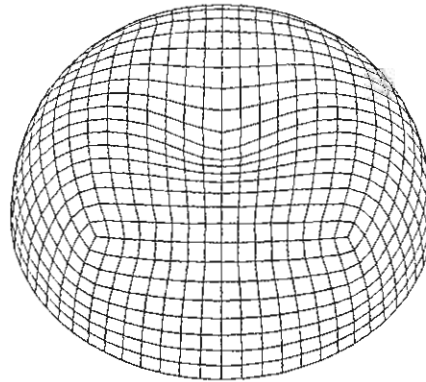


Fig.5: Rubber ball: The normalized load-deflection curve.

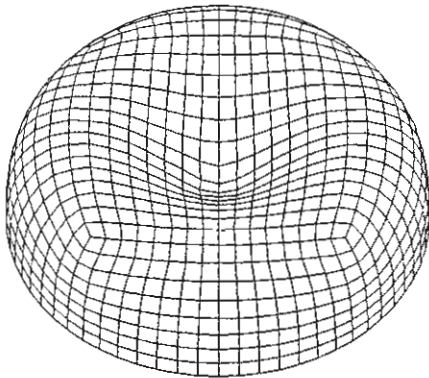
a



b



c



d

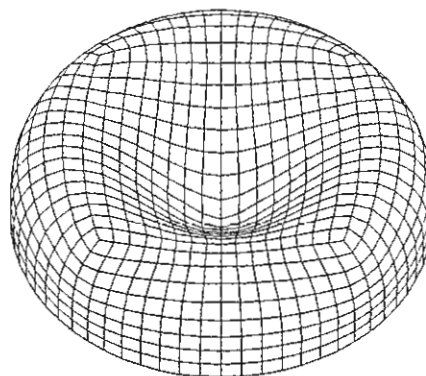


Fig.6: Rubber ball: Deformed mesh at a) $u=R/4$, b) $u=R/2$, c) $u=3/4R$ and d) $u=R$.

experimental results (taken from /6/) and numerical results from /6/ and /7/. The excellent agreement is obtained with an experiment. In Fig.6 four different equilibrium configurations of the rubber hemisphere are compared.

5. CONCLUSION

The shell theory for thin elastic shells undergoing large rotations, with the basis on the Green-Lagrange strain measures is sketched out, along with its simple finite element formulation for the four node isoparametric stress resultant shell element. Using this formulation two tests have been presented. The results are in good agreement with those reported in recent literature. An improvement of its behaviour can be obtained by employing mixed interpolation for the membrane and/or bending part of a functional.

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ON GEOMETRICALLY NONLINEAR ELASTIC-PLASTIC COMPUTATION OF COMPOUND SHELL STRUCTURES

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Summary. The contribution presents a numerical concept for the geometrically nonlinear elastic-plastic analysis of shell structures. The constitutive equation for an elastic-plastic material model was derived using von Mises yield criterion and assuming isotropic strain hardening. A doubly curved layered finite element based on a tensor-oriented displacement formulation was presented. Its efficiency is demonstrated by the computation of compound shell structures in the form of torispherical heads. The hoop stress distribution and the spreading of plastic regions through the shell thickness were computed.

1. Introduction

A compound shell consisting of different shell segments, such as cylindrical, spherical, conoidal and toroidal shapes, may be found in a variety of shell structures used in various fields, such as aerospace, food processing, chemical, nuclear, and oil industries. The present paper is concerned with the axisymmetric analysis of a thin internally pressurized shell, which consists of a toroidal and spherical geometry in conjunction with a cylindrical shell. Geometrically nonlinear elastic-plastic computation was performed. The geometrically nonlinear Donnell-Marguerre theory with total Lagrangean formulation and elastic-plastic material model based on the classical flow theory with isotropic strain hardening and von Mises yield criterion were applied. A doubly curved layered finite element which provides an accurate description of the shell geometry was presented. The element was implemented in the FEMAS computer program [1] developed at the Institute of Statics and Dynamics, Ruhr University Bochum, Germany, and all computations presented were performed using this program package.

2. Theoretical concept

The finite element equation for the solution of geometrically and physically nonlinear problems was derived from the discretized incremental principle of virtual work [2, 3] and may be expressed in the following form

$$\left(k_{phM}^N + k_{\sigma M}^N\right)^+ \bar{v}_N = f_{eM} - f_{iM} \quad (1)$$

where k_{phM}^N and $k_{\sigma M}^N$ denote physically nonlinear stiffness matrix and initial stress matrix,

respectively, while f_{eM} and f_{iM} represent the vectors of external and internal nodal forces. $\dot{v}_N^+$ is the vector of the incremental degrees of freedom.

The element matrices may be written as

$$k_{phM}^N = \iint_A \left(\bar{N}^{(\alpha\beta)N} \dot{\alpha}_{(\alpha\beta)M}^+ + \bar{M}^{(\alpha\beta)N} \dot{\omega}_{(\alpha\beta)M}^+ \right) dA \quad (2)$$

$$k_{\sigma M}^N = \iint_A \left(\bar{N}^{(\alpha\beta)} \dot{\alpha}_{(\alpha\beta)M}^{++N} + \bar{M}^{(\alpha\beta)} \dot{\omega}_{(\alpha\beta)M}^{++N} \right) dA . \quad (3)$$

The vectors of external and internal forces are

$$f_{eM} = \iint_A \left(p^\alpha \dot{u}_{\alpha M}^+ + p^3 \dot{u}_{3M}^+ \right) dA + \int_C \left(n^\alpha \dot{u}_{\alpha M}^+ + n^3 \dot{u}_{3M}^+ + m^\alpha \dot{\omega}_{\alpha M}^+ \right) ds \quad (4)$$

$$f_{iM} = \iint_A \left(\bar{N}^{(\alpha\beta)} \dot{\alpha}_{(\alpha\beta)M}^+ + \bar{M}^{(\alpha\beta)} \dot{\omega}_{(\alpha\beta)M}^+ \right) dA . \quad (5)$$

In the above expressions dA denotes the differential middle surface element of the undeformed configuration and ds is the differential element of the undeformed boundary curve C . The notation $(\bar{\dots})$ indicates the fundamental state [2, 3] representing the loaded and deformed shape of the shell in a state of equilibrium and the variables marked with $(\dot{\dots})$ and $(\dot{\dots})^+$ refer to linear and quadratic functions of incremental displacements. $\dot{u}_{\alpha M}^+$ and $\dot{u}_{3M}^+$ are the shape functions of the incremental displacement components, while $\dot{\omega}_{\alpha M}^+$ is the shape function of the tangential components of the incremental rotation vector [4]. The tangential and normal components of the surface load vector are p^α and p^3 , while n^α and n^3 stand for the tangential and normal components of the boundary forces and m^α are the tangential components of the boundary moment vector. The values $\dot{\alpha}_{(\alpha\beta)M}^+$, $\dot{\alpha}_{(\alpha\beta)M}^{++N}$, $\dot{\omega}_{(\alpha\beta)M}^+$ and $\dot{\omega}_{(\alpha\beta)M}^{++N}$ designate the shape functions of the incremental strain, which can be found for the Donnel-Marguerre theory of small rotation by means of the following kinematic equations [5]

$$\dot{\alpha}_{(\alpha\beta)}^+ = \frac{1}{2} \left(\dot{u}_{\alpha|\beta}^+ + \dot{u}_{\beta|\alpha}^+ - 2\dot{u}_3^+ b_{\alpha\beta} + \dot{u}_{3|\beta}^+ \bar{u}_{3|\alpha} + \dot{u}_{3|\alpha}^+ \bar{u}_{3|\beta} \right) \quad (6)$$

$$\dot{\alpha}_{(\alpha\beta)}^{++} = \frac{1}{2} \dot{u}_{3|\alpha}^+ \dot{u}_{3|\beta}^+$$

$$\dot{\omega}_{(\alpha\beta)}^+ = - \left(\dot{u}_{3|\alpha\beta}^+ + \dot{u}_{\lambda|\alpha}^+ b_{\beta}^\lambda + \dot{u}_{\lambda|\beta}^+ b_{\alpha}^\lambda + \dot{u}_\lambda^+ b_{\alpha|\beta}^\lambda - \dot{u}_3^+ b_{\alpha\lambda} b_{\beta}^\lambda \right) \quad (7)$$

$$\dot{\omega}_{(\alpha\beta)}^{++} = 0$$

where $\alpha_{(\alpha\beta)}$ denotes the first and $\omega_{(\alpha\beta)}$ the second strain tensor. u_α and u_3 are the tangential

and normal components of the displacement vector, while $b_{\alpha\beta}$ represents the curvature tensor and vertical slash covariant derivatives. The shape functions of the incremental stress resultants are designated by $\tilde{N}^{+(\alpha\beta)N}$ and $\tilde{M}^{+(\alpha\beta)N}$ and may be obtained by means of the following expressions [6]

$$\begin{aligned}\tilde{N}^{+(\alpha\beta)} &= \overset{1}{E}{}^{\alpha\beta\rho\lambda} \tilde{\alpha}_{(\rho\lambda)}^+ + \overset{2}{E}{}^{\alpha\beta\rho\lambda} \tilde{\omega}_{(\rho\lambda)}^+ \\ \tilde{M}^{+(\alpha\beta)} &= \overset{2}{E}{}^{\alpha\beta\rho\lambda} \tilde{\alpha}_{(\rho\lambda)}^+ + \overset{3}{E}{}^{\alpha\beta\rho\lambda} \tilde{\omega}_{(\rho\lambda)}^+\end{aligned}\quad (8)$$

with material tensors which refer to middle surface

$$\overset{n}{E}{}^{\alpha\beta\rho\lambda} = \int_{-h/2}^{h/2} C_{ep}^{\alpha\beta\rho\lambda} (\theta^3)^{n-1} d\theta^3; \quad n = 1, 2, 3 \quad (9)$$

where h is the shell thickness and θ^3 the distance from the middle surface. In equations (8) $\tilde{N}^{+(\alpha\beta)}$ is the pseudo stress resultant and $\tilde{M}^{+(\alpha\beta)}$ the moment tensor. $C_{ep}^{\alpha\beta\rho\lambda}$ in equation (9) is the material response which depends geometrically on the metric of the undeformed middle surface, while physically it may be a function of the current stress, strain states and the deformation history.

The incremental stress at each point of the shell continuum can be expressed by the constitutive law

$$\tilde{\sigma}^{+(\alpha\beta)} = C_{ep}^{\alpha\beta\rho\lambda} \tilde{\gamma}_{(\rho\lambda)}^+ \quad (10)$$

where $\tilde{\gamma}_{(\rho\lambda)}^+$ is the incremental strain which is given by

$$\tilde{\gamma}_{(\rho\lambda)}^+ = \tilde{\alpha}_{(\rho\lambda)}^+ + \theta^3 \tilde{\omega}_{(\rho\lambda)}^+ \quad (11)$$

The stresses and the stress resultants of the fundamental states are computed as the sum of all hitherto computed increments.

The global finite element equation of the computation model is derived upon assembling all the element matrices

$$\mathbf{K}_T \tilde{\mathbf{V}}^+ = \mathbf{F}_E - \mathbf{F}_I \quad (12)$$

where

$\mathbf{K}_T$ - tangential stiffness matrix

$\mathbf{F}_E$ - vector of external forces

$\mathbf{F}_I$ - vector of internal forces

$\tilde{\mathbf{V}}^+$ - vector of the incremental degrees of freedom.

The geometrically nonlinear elastic-plastic problem is solved by means of equation (12) using incremental-iterative methods, which must be coupled with the integration of the constitutive equations of the elastic-plastic material model [7].

Taking an elastic-plastic material behavior into account, the layered finite element LNACS48 (Figure 1) was derived on the basis of the NACS48 [8] doubly curved shell element with four nodes and 48 degrees of freedom. The layered model enables numerical integration over the shell thickness which postulates a linear variation of a given variable within each layer [6]. For each material point, the nonlinear incremental constitutive equation is solved by the Euler-forward integration approach [7] and the spreading of plastic regions through the shell thickness can be described.

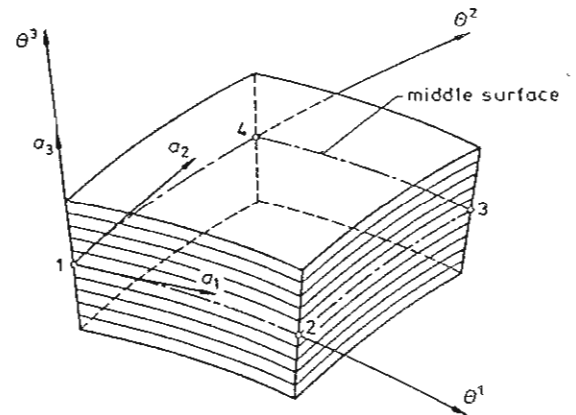


Figure 1 Layered shell element

By means of appropriate geometric functions, the curvilinear formulation [9] provides an accurate description of the shell geometry, which is an advantage over isoparametric elements. For instance, the geometric functions such as the metric and curvature tensors and Christoffel symbols of cylindrical, toroidal and spherical shells provide an accurate description of compound torispherical structures [5]. The boundary conditions for the juncture of different geometries are described by equalizing the physical nodal displacement components. Besides the shell geometry, the formulation presented enables separate implementation of the shape functions, shell theory and constitutive law in the finite element equation.

Elastic-plastic material model

An elastic-plastic material model for isotropic strain hardening may be described by the yield surface

$$f(\sigma^{ij}, \gamma_{ij}^p) = 0 \quad (13)$$

with the current state of stress σ^{ij} and the plastic strain γ_{ij}^p which is defined by the associated flow rule in the form

$$d\gamma_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma^{ij}} \quad (14)$$

where $d\lambda$ is a scalar function describing the history of the deformation process, which can be expressed as

$$d\lambda = \frac{1}{H} \frac{\partial f}{\partial \sigma^{kl}} \dot{\sigma}^{kl} \quad (15)$$

with H as the hardening modulus. After lengthy derivations [10], the general expression for the elastic-plastic material tensor is given in the following form

$$C_{ep}^{ijkl} = C^{ijkl} + \frac{C^{ijop} \frac{\partial f}{\partial \sigma^{op}} \frac{\partial f}{\partial \sigma^{st}} C^{stkl}}{\frac{\partial f}{\partial \sigma^{mn}} \frac{\partial f}{\partial \gamma_{mn}^p} - \frac{\partial f}{\partial \sigma^{mn}} \frac{\partial f}{\partial \sigma^{op}} C^{mnop}} \quad (16)$$

where C^{ijkl} is the elastic material tensor [4]

The elastic-plastic material model for the von Mises yield criterion and isotropic strain hardening applied in this paper may be written as

$$f(\sigma^{ij}, \gamma_{ij}^p) = J_2 - \frac{1}{2} \sigma_Y^2 = 0 \quad (17)$$

where J_2 denotes the second invariant of the stress deviator tensor S^{ij} , while σ_Y is the uniaxial yield stress. The hardening modulus is determined from the uniaxial elastic-plastic material behavior in the form

$$H = \frac{4J_2}{3 \left(\frac{1}{E_t} - \frac{1}{E} \right)} \quad (18)$$

Using (18), the elastic-plastic material tensor of the isotropic-hardening von Mises model may be expressed as

$$C_{ep}^{ijkl} = C^{ijkl} - \frac{12G^2 \left(\frac{1}{E_t} - \frac{1}{E} \right)}{4J_2 + 12GJ_2 \left(\frac{1}{E_t} - \frac{1}{E} \right)} S^{ij} S^{kl} \quad (19)$$

where E and E_t denote the elastic modulus and the elastic-plastic tangent modulus, respectively, while G represents the shear modulus.

3. Torispherical shell under internal pressure

Assuming linear strain hardening, which corresponds to the constant tangent modulus E_t , geometrically nonlinear elastic-plastic computations of torispherical shells (Figure 2) were performed. The computation model represents a segment of the shell defined by the angle ψ (Figure 3) with the boundary conditions corresponding to axisymmetry. The equilibrium iterations on the structural level were carried out by the Newton-Raphson approach. The computed hoop stress resultant presented in Figure 4 for the torispherical shell ($D/h = 500$, $R_t/D = 0.1$, $R_s/D = 1$, $E = 206010$ MPa, $\sigma_Y = 235$ MPa) shows good agreement of the solution obtained by the layered finite element LNACS48 with the Rensch's solution [11].

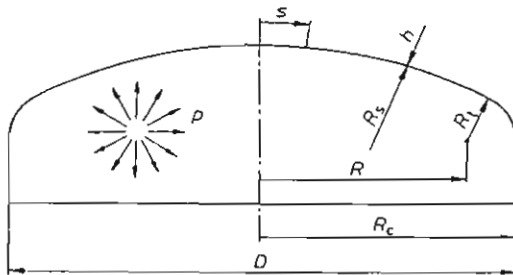


Figure 2 Torispherical shell geometry

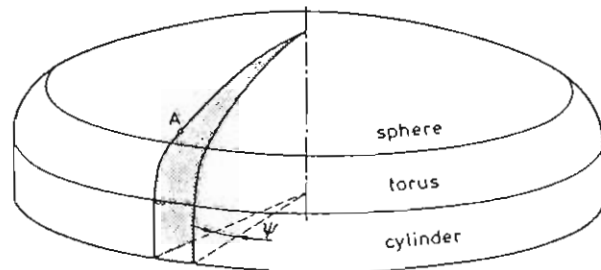


Figure 3 Computation model

Geometrically linear and geometrically nonlinear elastic-plastic computations of a shell with the geometric characteristics $D/h = 1000$, $R_i/D = 0,1$, $R_s/D = 1$ were performed for different tangent moduli E_t assuming constant yield stress $\sigma_Y = 207$ MPa. Other material data include modulus of elasticity $E = 207$ GPa and Poisson's ratio $\nu = 0,3$. The internal pressure-normal displacement curves for the nodal point A of the computation model are shown in Figure 5. Figure 6 shows the hoop stress resultant distributions obtained by geometrically nonlinear computation at internal pressure $p = 0,15$ MPa, while the corresponding plastic regions on the deformed configuration are presented in Figure 7. Figure 7 also shows the spreading of plastic regions through the shell thickness due to increase in the load levels. As may be seen, the plastic region for the strain hardening material with $E_t = 0,40 E$ at internal pressure $p = 0,15$ MPa differs very slightly from that for the elastic-ideal plastic material model at internal pressure $p = 0,14$ MPa. It is evident from Figure 6 that an increase in E_t produces an increase in the

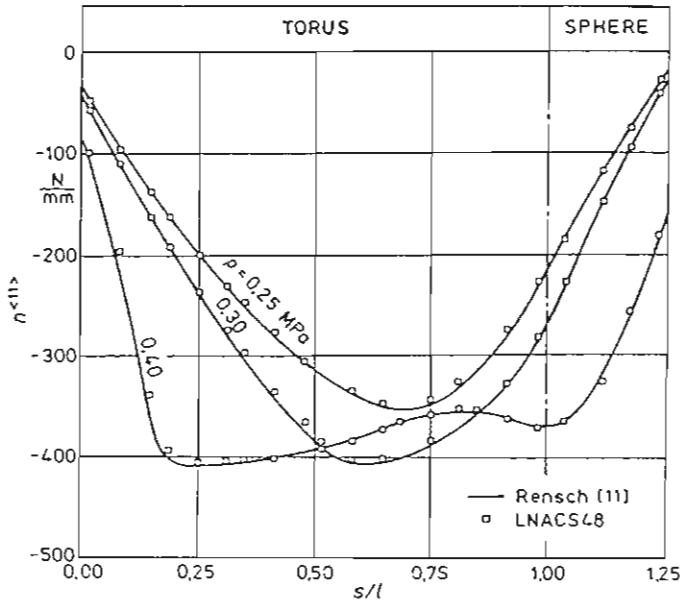


Figure 4 Hoop stress resultant distribution for elastic, perfectly plastic material model

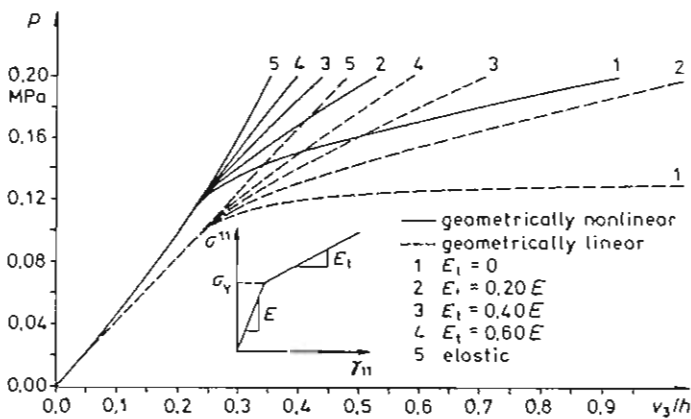


Figure 5 Internal pressure-normal displacement curves in nodal point A of the computation model

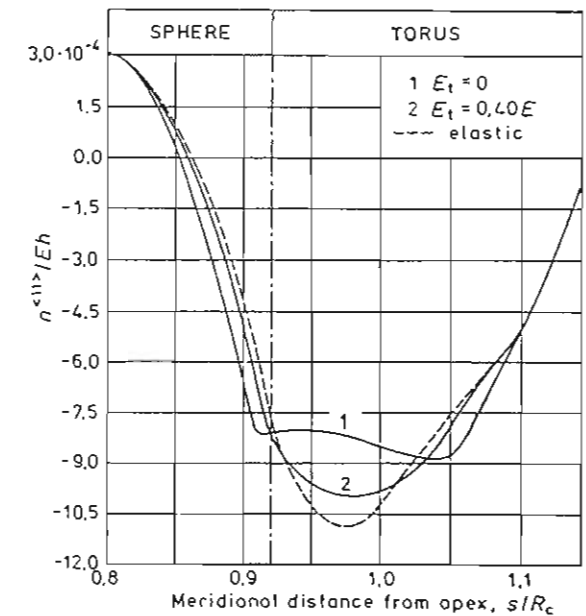


Figure 6 Hoop stress resultant distribution at pressure $p = 0,15$ MPa

internal hoop stresses, which is coupled with a redistribution of the bending moments. The effect of strain hardening and yield stresses on the distribution of internal forces and the influence of the size of plastic regions on the differences in the values obtained by geometrically linear and nonlinear computations were analyzed in [10] and [12].

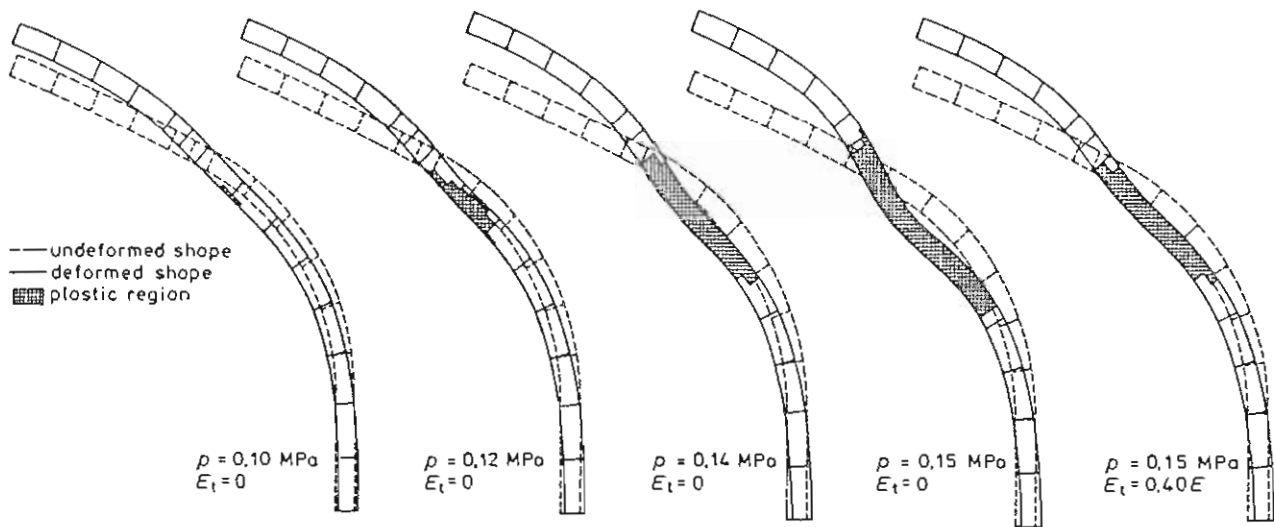


Figure 7 Spreading of plastic regions through the shell thickness

4. Conclusions

The finite element formulation which provides a solution of the nonlinear problems of shell structures with combined geometric and material nonlinearities was presented. In contrast to the isoparametric concept, the tensor-oriented way offers high precision concerning the description of geometry and loading and its advantage also lies in the ability to combine various shell theories, shape functions and constitutive models. The layered approach enables computation of the plastic regions through the shell thickness. The numerical examples show the efficiency of computation strategy presented for the geometrically nonlinear elastic plastic analysis of compound shell structures. The geometrically nonlinear Donnell-Marguerre theory and von Mises yield criterion with isotropic strain hardening were assumed.

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Non-linear concrete model with curved reinforcement

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SUMMARY

In practical examples of reinforced concrete structures the reinforcement is generally such that it cannot be simply described by the concrete finite element mesh, since it either does not correspond to the finite element boundaries or it requires a very dense mesh. This paper presents a numerical model applied to the plane reinforced concrete structures with curved reinforcing bars. The paper presents the expressions used for modelling generally curved bars embedded in a concrete eight-node Serendipity element. The reinforcement is given in a global coordinate system separately of the finite 2D element mesh. The transformation of the global coordinates of reinforcement points into the local coordinate system of the concrete finite element was performed using the Newton iterative method. Finally, this model was applied to several actual examples. In these examples the validity of the model is firstly tested by comparing the obtained results with the results from literature, subsequently, a nonlinear analysis of the reinforced concrete beam, dimensioned according to the classical theory, has been performed.

1. INTRODUCTION

One of the main advantages of isoparametric elements is their adaptability regarding the description of the model geometry. This means that a relatively small number of elements can be used to achieve a good description of the model geometry which makes it possible to use relatively large finite elements. In order to maintain this advantage, it is necessary to obtain the geometry and the reinforcement location separately of the finite 2D element mesh. This can be achieved by embedding into a concrete two-dimensional finite element the generally curved reinforcement which is given in a global coordinate system independently of the finite 2D element mesh.

This paper presents the model with the one-dimensional isoparametric three-node generally oriented reinforcing bars embedded into a concrete two-dimensional eight-node isoparametric finite element. A similar model was published by Elwi and Hrudey^[1]. Before that, Philips and Zienkiewicz^[2] and Elwi and Murray^[3] separately developed a model in which the reinforcement is aligned with one of the local isoparametric element coordinate. Embedded representations that allow for a bar placed at an angle to the local isoparametric element axes of the 2D element was presented by Chang^[4], which have been restricted to problems having straight reinforcing bars and rectilinear meshes.

2. MATERIAL MODELLING

A brief description of the material models adopted in this analysis is presented herein. Further details may be found in literature^[5].

Compressive behaviour of concrete

In the present analysis an elasto-viscoplastic approach^[6-7] was employed to model the compressive behaviour of the concrete. The yield condition was formulated in terms of the first two stress invariants^[5-6,8].

Beyond the yield surface, a viscoplastic response^[5-7] was assumed for concrete in compression. The standard associated flow rule^[8-9] was assumed valid for yielded concrete.

Concrete crushing is a strain controlled phenomenon which is monitored by a fracture surface similar to the yield surface. The crushing condition can be obtained by simply converting the yield criterion, described in terms of stresses, directly into the strains^[6].

Tensile behaviour of concrete

The response of concrete under tensile stresses was assumed to be linear elastic until the fracture surface is reached. A smeared crack approach^[6-7] was used to model the behaviour of cracked concrete with opening and closing of tensile cracks, taking into account the decrease in the tensile strength^[6-7]. Cracks were assumed to form in planes perpendicular to the direction of the maximum principal tensile stress as soon as this stress reaches the concrete tensile strength. The shear stiffness in the cracked zones was modelled by the reduction of the shear modulus^[6-7].

Reinforcing steel

Reinforcing bars exhibit an uniaxial behaviour in the longitudinal direction and they were both modelled by uniaxial elasto-viscoplastic model^[5-7].

3. REINFORCEMENT MODELLING

The parent element is determined by the node coordinates (x, y) of the global coordinate system (Figure 1).

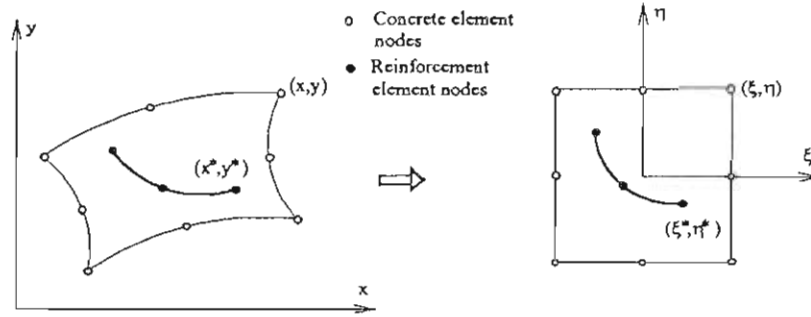


Figure 1 Reinforcement location within the concrete finite element

Local coordinates of the concrete finite element are denoted by (ξ, η) . Employing the procedure of isoparametric mapping, the global coordinates at any element point can be expressed by the shape functions $N_i(\xi, \eta)$, $(i = 1, \dots, n)$, and by the global coordinates of the element nodes (x_i, y_i) as:

$$x = \sum_{i=1}^n N_i(\xi, \eta) x_i; \quad y = \sum_{i=1}^n N_i(\xi, \eta) y_i \quad (1)$$

Differential element of coordinates dx and dy can be expressed in the form:

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} x_i \\ \sum_{i=1}^n \frac{\partial N_i}{\partial \xi} y_i & \sum_{i=1}^n \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \quad (2)$$

where $\mathbf{J}$ is the jacobian matrix.

Once the parent element mesh has been created, the reinforcing bar may be specified by locating a set of nodes. The reinforcement simulation was performed using one-dimensional three-node isoparametric elements: in order to define them uniquely, it is necessary to know the coordinates of the central node apart from those of the boundary nodes. In isoparametric elements the middle node should be at equal distance from the boundary nodes. Since their coordinates cannot be determined analytically, they are obtained numerically by interpolation at the center of the arch. The model provides the possibility for interpolation along the line for the rectilinear reinforcement and along the square parabola for curved reinforcement^[5].

The reinforcement location is determined by the global coordinates of the reinforcement nodes (x_j^*, y_j^*) , $(j = 1, 2, 3)$. The arbitrary point coordinate can be expressed by the shape function of the 1D element $h_j = h_j(\zeta)$, where ζ is the independent normalized coordinate, and by the global coordinates of the reinforcement nodes as:

$$x = \sum_{j=1}^3 h_j x_j^*; \quad y = \sum_{j=1}^3 h_j y_j^* \quad (3)$$

The computation of the reinforcement stiffness requires integration along the reinforcement element, so it is necessary to know the differential element of length ds :

$$ds = \sqrt{\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2} d\zeta \quad (4)$$

and the values of $\sin\beta$ and $\cos\beta$, where β is the angle between the global axis x and the tangent on the element axis:

$$\begin{aligned}\sin\beta &= \frac{dy}{ds} = \frac{dy}{d\zeta} \frac{d\zeta}{ds} = \left(\sum_{j=1}^3 \frac{dh_j}{d\zeta} y_j^* \right) \frac{1}{\sqrt{\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2}} \\ \cos\beta &= \frac{dx}{ds} = \frac{dx}{d\zeta} \frac{d\zeta}{ds} = \left(\sum_{j=1}^3 \frac{dh_j}{d\zeta} x_j^* \right) \frac{1}{\sqrt{\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2}}\end{aligned}\quad (5)$$

A differential element of volume dV_s can be expressed in terms of length ds and cross-section area of the bar A_s as:

$$dV_s = A_s ds \quad (6)$$

4. MAPPING FROM THE GLOBAL INTO LOCAL COORDINATE SYSTEM

The computation of the reinforcement element stiffness requires integration along the reinforcement element, which is performed numerically in the Gaussian points, where all other values, as well strains and stresses, are computed. In order to determine the influence of the reinforcement upon the concrete element, it is necessary to add the respective local coordinates of the parent element (ξ_p, η_p) to the Gaussian points of the reinforcement with the known global coordinates (x_p, y_p) . The global coordinates (x_p, y_p) of an arbitrary point of a 2D element can be expressed as:

$$x_p = \sum_{i=1}^n N_i(\xi_p, \eta_p) x_i, \quad y_p = \sum_{i=1}^n N_i(\xi_p, \eta_p) y_i \quad (7)$$

where $N_i(\xi_p, \eta_p)$ are the shape functions of the concrete element, and (x_i, y_i) are the reinforcement nodes coordinates. Since the shape functions are parabolic it is not possible to express them explicitly (ξ_p, η_p) by (x_p, y_p) . Therefore the mapping procedure from the global into local coordinate system is performed numerically.

According to expression (7) a system of two non-linear equations is obtained:

$$x_p - \sum_{i=1}^n N_i(\xi_p, \eta_p) x_i = 0 \quad ; \quad y_p - \sum_{i=1}^n N_i(\xi_p, \eta_p) y_i = 0 \quad (8)$$

The solution may be found using a Newton iterative procedure. The solution after $j+1$ iterations is given by:

$$\begin{Bmatrix} \xi_p \\ \eta_p \end{Bmatrix}^{j+1} = \begin{Bmatrix} \xi_p \\ \eta_p \end{Bmatrix}^j + \begin{Bmatrix} \Delta\xi_p \\ \Delta\eta_p \end{Bmatrix}^{j+1} \quad (9)$$

The increment of coordinates $\Delta\xi_p$ and $\Delta\eta_p$ can be determined from the equation:

$$\begin{Bmatrix} \Delta\xi_p \\ \Delta\eta_p \end{Bmatrix}^{j+1} = -\mathbf{J}^{-1} \begin{Bmatrix} x_p - \sum_{i=1}^n N_i(\xi_p^j, \eta_p^j) x_i \\ y_p - \sum_{i=1}^n N_i(\xi_p^j, \eta_p^j) y_i \end{Bmatrix} \quad (10)$$

where $\mathbf{J}^{-1} = \left[\mathbf{J}(\xi_p^j, \eta_p^j) \right]^{-1}$.

5. REINFORCEMENT STRAINS

The axial force upon the reinforcement element causes strain in the direction of the element axis. The differential element of the length ds after the strains becomes $(ds + \varepsilon_s ds)$, where ε_s represents a reinforcement strain. If the coordinate system axes are x, y and the displacement u, v are parallel to them, and if ζ is the normalized coordinate of the reinforcement element, then the following expression is valid:

$$(ds + \varepsilon_s ds)^2 = \left(dx + \frac{\partial u}{\partial \zeta} d\zeta \right)^2 + \left(dy + \frac{\partial v}{\partial \zeta} d\zeta \right)^2 \quad (11)$$

where dx and dy are the increments of the respective coordinates. Developing expression (11) and neglecting members $\left(\frac{\partial u}{\partial \zeta} \cdot \frac{d\zeta}{ds} \right)^2, \left(\frac{\partial v}{\partial \zeta} \cdot \frac{d\zeta}{ds} \right)^2$ we obtain:

$$\varepsilon_s = \frac{dx}{ds} \left(\frac{\partial u}{\partial \zeta} \cdot \frac{d\zeta}{ds} \right) + \frac{dy}{ds} \left(\frac{\partial v}{\partial \zeta} \cdot \frac{d\zeta}{ds} \right) \quad (12)$$

Since the displacements u, v can be represented by the shape functions $N_i(\xi, \eta)$ of a two-dimensional concrete element and by the concrete element node displacements $u_i, v_i, (i=1, \dots, n)$, after the necessary transformations^[5,10], the strain ε_s can be represented in the form:

$$\varepsilon_s = \mathbf{B}_s^T \cdot \begin{Bmatrix} u \\ v \end{Bmatrix} \quad (13)$$

where the strain matrix $\mathbf{B}_s^T$ is:

$$\mathbf{B}_s^T = \begin{bmatrix} \frac{\partial N_1}{\partial x} \cos^2 \beta + \frac{\partial N_1}{\partial y} \sin \beta \cos \beta, & \frac{\partial N_1}{\partial y} \sin^2 \beta + \frac{\partial N_1}{\partial x} \sin \beta \cos \beta, \\ \frac{\partial N_2}{\partial x} \cos^2 \beta + \frac{\partial N_2}{\partial y} \sin \beta \cos \beta, & \frac{\partial N_2}{\partial y} \sin^2 \beta + \frac{\partial N_2}{\partial x} \sin \beta \cos \beta, \dots \\ \frac{\partial N_n}{\partial x} \cos^2 \beta + \frac{\partial N_n}{\partial y} \sin \beta \cos \beta, & \frac{\partial N_n}{\partial y} \sin^2 \beta + \frac{\partial N_n}{\partial x} \sin \beta \cos \beta \end{bmatrix} \quad (14)$$

Since the stiffness matrix of the reinforcement element is obtained by numerical integration in the Gaussian points of 1D element, the values of the Cartesian derivations $\frac{\partial N_i}{\partial x}$ and $\frac{\partial N_i}{\partial y}, (i=1, \dots, n)$, in matrix $\mathbf{B}_s$ are computed in those Gaussian points. Their position is defined by the normalized coordinate ζ , which can be used to determine the respective global coordinates. Since 2D element shape functions are dependent upon the local coordinates (ξ, η) , it is necessary to perform the mapping of the global coordinates of the reinforcement Gaussian points into the local coordinate system $\xi-\eta$, as presented in Item 4.

6. EQUILIBRIUM EQUATIONS

The incremental internal virtual work in the reinforcement element has the following form:

$$\delta \Delta W = \int_{V_s} (\sigma_s + \Delta \sigma_s) \delta \Delta \varepsilon_s dV_s \quad (15)$$

where σ_s is the normal stress in the reinforcement and $\Delta \sigma_s$ is the stress increment which can be represented as:

$$\Delta \sigma_s = E_s \cdot \Delta \varepsilon_s \quad (16)$$

In expression (16) E_s is the tangential modulus of elasticity of the $\sigma_s - \varepsilon_s$ relation of the reinforcement, and the increment of specific strain $\Delta \varepsilon_s$ is:

$$\Delta \epsilon_s = \mathbf{B}_s^T \{\Delta \delta\} \quad (17)$$

If the values for $\Delta \sigma_s, \Delta \epsilon_s$ are introduced into (15), we obtain:

$$\delta \Delta W = \{\delta \Delta d\}^T \mathbf{K}_s \{\Delta d\} \quad (18)$$

where $\{\delta \Delta d\}$ is the vector of the displacement increment of the 2D element nodes, and $\mathbf{K}_s$ is the stiffness matrix of the reinforcing element:

$$\mathbf{K}_s = \int_{\zeta} \mathbf{B}_s^T E_s B_s \frac{ds}{d\zeta} A_s d\zeta \quad (19)$$

which is obtained by the numerical integration along the reinforcing bar.

The stiffness matrix of concrete element with embedded reinforcement can be presented as:

$$\mathbf{K} = \mathbf{K}_c + \mathbf{K}_s \quad (20)$$

where $\mathbf{K}_c$ is concrete stiffness matrix^[5-6] and $\mathbf{K}_s$ is reinforcement stiffness matrix.

Vector of residual forces due to reinforcement strain $\{Q_s\}$ can be obtained using expression for the incremental internal virtual work:

$$\delta \Delta W = \int_{V_s} \sigma_s \delta \Delta \epsilon_s dV_s = \{\delta \Delta d\}^T \{Q_s\} \quad (21)$$

where:

$$\{Q_s\} = \int_{\zeta} \mathbf{B}_s^T \sigma_s A_s \frac{ds}{d\zeta} d\zeta \quad (22)$$

Global load vector (vector of residual forces) of concrete element with embedded reinforcement $\{Q\}$ is determined according to expression:

$$\{Q\} = \{Q_c\} + \{Q_s\} \quad (23)$$

where $\{Q_c\}$ is a vector of external forces and residual forces on concrete element^[5,9].

7. EXAMPLES

Example 1:

The validity of the model was tested on the example of a thick ring^[1] (Figure 3). The ring was reinforced with one layer in which the stress should be uniform. The example was tested for $A_s/L=0.025$, $E_s/E_c=8$ and $\nu=0.25$, where A_s is the cross-sectional area of the reinforcement, E_s and E_c are the moduli of elasticity of the reinforcement and concrete and ν is the Poisson coefficient.

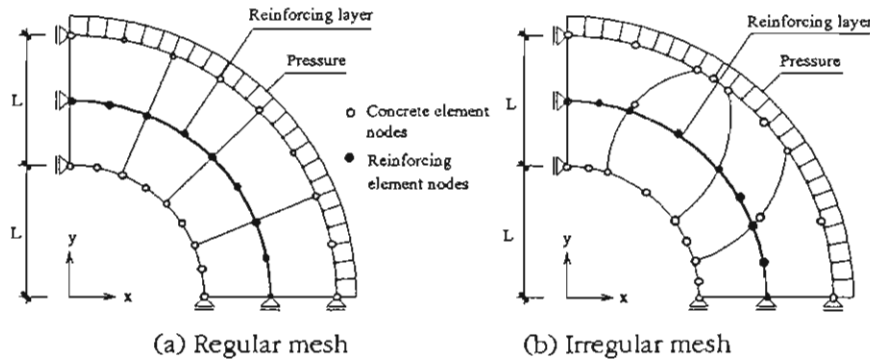


Figure 3 Quarter Ring Problem

Firstly, the concrete elements were selected so that the reinforcement was parallel with the local coordinate axes of the 2D element (Fig. 3a), while later the same problem was analyzed with a 2D element mesh placed so that the reinforcing element is actually a curved element generally oriented

on a 2D concrete element (*Fig. 3b*). In both cases the results agree with those presented in ref.[1]. For the regular mesh the stress is practically constant, whereas in the irregular mesh the maximum deviation was 2.5%.

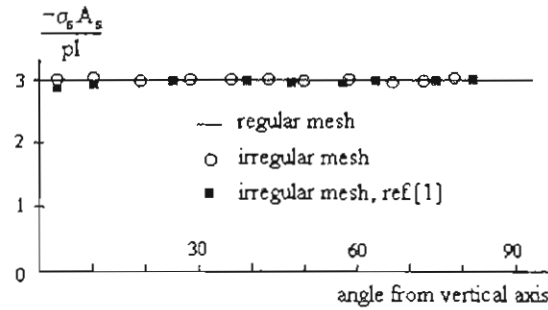


Figure 4 Steel Stress

Example 2:

The analysis included a reinforced concrete beam with a square cross-section $b/h=20/70$ cm, shown in *Figure 5*. The uniaxial compressive strength was $f'_c=40$ MPa, while reinforcement yield stress was $f_y=400$ MPa. The reinforcement was dimensioned according to preliminary computation of the classical theory for reinforced concrete^[1]. The selected tensile reinforcement was $5\phi 19$ ($14,18 \text{ cm}^2$), the compressive one was $2\phi 14$ ($3,08 \text{ cm}^2$) and the shear bars were $2\phi 14$ ($3,08 \text{ cm}^2$). The example was analysed for two types of secondary reinforcement (stirrups and longitudinal reinforcement): (1) $\phi 10$ ($0,79 \text{ cm}^2$) and (2) $\phi 14$ ($1,54 \text{ cm}^2$). The stirrups were placed at the distance of 20 cm. The geometrical characteristics and the distribution of the reinforcement are presented in *Figure 5*.

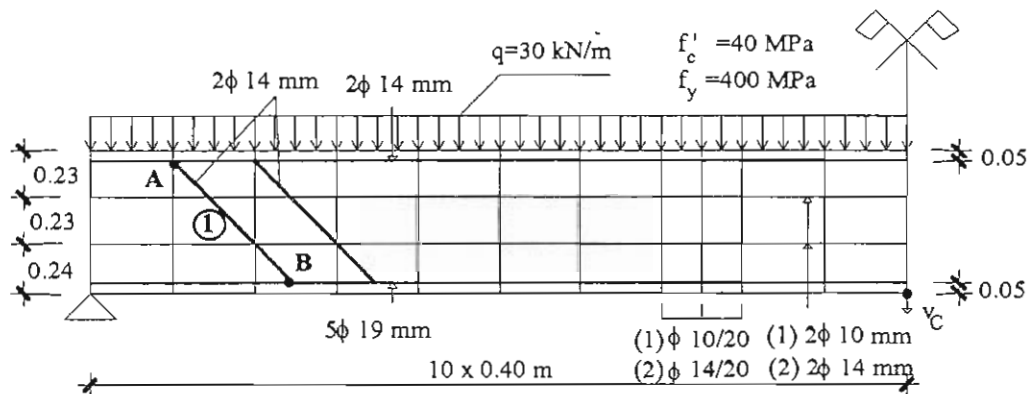


Figure 5 Reinforced Concrete Beam

Figure 6(a) presents the load-displacement response in point C for two cases of secondary reinforcement. It can be noted that with a stronger secondary reinforcement the concrete failure occurs at the load factor 1,4. The failure was expected at factor 1,6 according to the assumption that the tensile reinforcement fails in the middle of the span. The failure however occurred earlier due to the influence of the principal shear stresses in the zone immediately next to the support. Various alternatives of the secondary reinforcement were used in order to avoid early failure of the structure caused by principle shear stresses. The *Figure 6(c)* presents the state of cracks before the structure failure for case (2), whereas *Figure 6(b)* shows the stress in the reinforcement no. 1. The results show that the tensile stress does not occur immediately at the beginning, since the surrounding material

does not allow it. This problem is not caused by the length of anchoring, but by the strain field in that area.

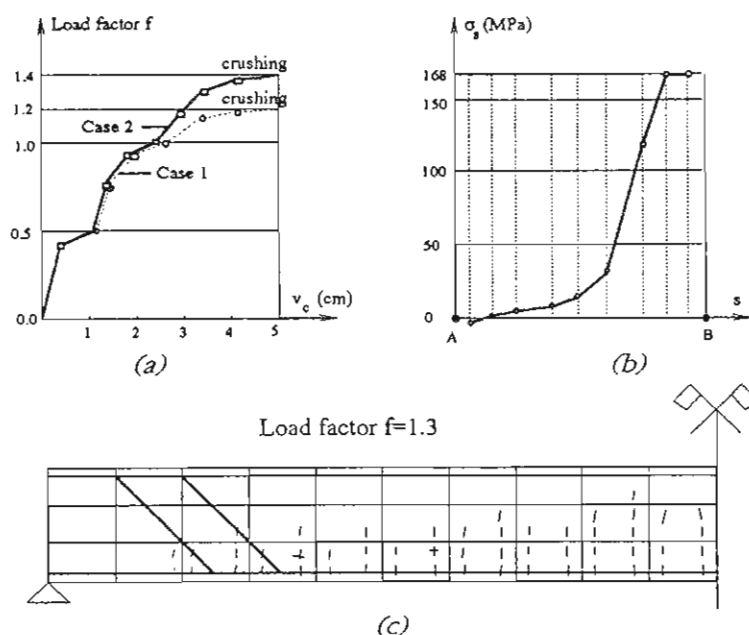


Figure 6 (a) load-displacement response, (b) stress in shear bar, (c) concrete cracking for $f=1.3$

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Finite element analysis of posttensioned concrete structures

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SUMMARY

This work presents a non-linear numerical model developed for the computations of posttensioned plane structures. Generally curved prestressing tendons and reinforcing bars are embedded into the concrete and they are modelled independently of the concrete mesh using one-dimensional curvilinear elements. Among the losses which influence the decrease in the prestress force, it is possible to compute the losses caused by friction between tendons and the concrete, the losses which result from the concrete deformation and the losses in the anchorage zone. The computation for posttensioned structures is organized in phases. In the phase which precedes the prestressing of the tendons the structure is computed taking into account the dead load and one part of the permanent load. In the prestressing phase the tendons are tensioned individually. The prestress force can be applied at once or incrementally. In the third phase, which follows the tensioning of all tendons, the structure is computed taking into account the remaining part of the dead load and the live load.

Certain possibilities of the developed model have been illustrated on one numerical example from literature.

1. INTRODUCTION

The model presented in this paper is applied to non-linear analysis of posttensioned concrete structures. Numerical modelling of non-linear prestressed structures has been studied by many authors^[1-6]. The models differ according to the method used to describe tendon geometry and whether they can or cannot compute the prestress losses.

The starting assumption in this paper is that the prestressed tendons are embedded into the concrete^[7-10]. The tendons were modelled by one-dimensional isoparametric three-node elements. The influence of the prestressed tendon upon the concrete is modelled by distributed normal and tangential load along the tendon and the two forces concentrated on the anchors.

The stress in prestressed tendons is decreased due to various influences which directly affects the bearing capacity of the structure. The model developed in this paper makes it possible to compute standard losses caused by the short-term linear and non-linear deformation of concrete, losses in the anchorage zone caused by concrete deformation and friction losses.

The computation for posttensioned structures is organized in phases: the phase preceding prestressing (Phase I), the prestressing phase (Phase II) and the phase following prestressing (Phase III).

2. NUMERICAL MODEL FOR PRESTRESSING

2.1 The Mathematical Model for the Prestress Force Transfer

The force at any cross-section in a tendon depends upon the prestress force and losses^[11-12]. Generally, it is necessary to determine the prestress influence in the internal points of the tendon if the external forces at the tendon ends are known.

The friction between the tendon and the concrete causes losses in the prestress force. *Figure 1* shows the forces acting on an infinitesimal element ds of a curved tendon. The prestress force at the distance s from the beginning of the tendon will be denoted by $S(s)$. This prestressing results in a distributed load within the tendon, i.e. normal $p_n(s)$ and tangential $p_t(s)$ on the concrete element. Normal load at any cross-section of the tendon depends upon the curvature radius $r(s)$ of the tendon and the intensity of the prestress force at that section^[4,11-12]:

$$p_n(s) = \frac{S(s)}{r(s)} \quad (1)$$

while tangential load is the function of the normal stress and friction coefficient μ between the tendon and concrete^[4,11-12]:

$$p_t(s) = \mu p_n(s) \quad (2)$$

The force in the tendon at a distance s from beginning of the tendon can be expressed as:

$$S(s) = S_A - \int_A^s p_t(s) ds \quad (3)$$

where S_A is known prestress force at one end of the of the tendon.

If equations (1) and (2) are introduced into equation (3), after the necessary transformations, the force in the tendon at any point, at distance s from the beginning, can be expressed as:

$$S(s) = S_A e^{-\mu \int_A^s \frac{ds}{r(s)}} \quad (4)$$

where S_A is known prestress force at one end of the tendon and ds is the differential element of length of a curved tendon.

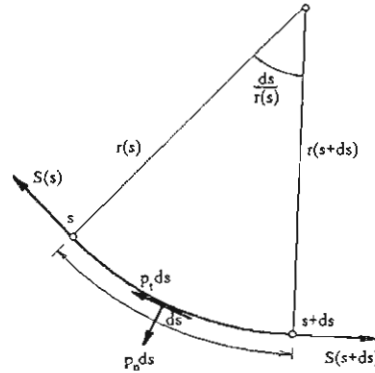


Figure 1 Forces acting on an infinitesimal element of curved tendon

2.2 Determination of Tendon Geometry

Prestressed tendons can occupy a general position within the concrete element; they can be either straight or parabolic. They can also consist of straight and parabolic parts (Figure 2). All tendons can be simulated in this way, whether straight or parabolic, if the incontinuity of the first derivation is obtained during the transfer from the region of one curvature into another.

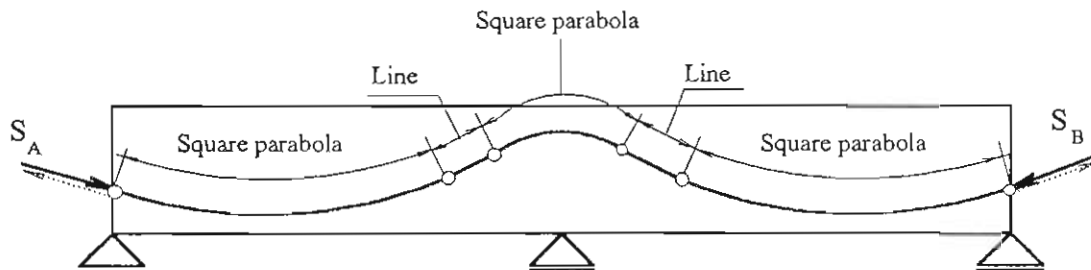


Figure 2 Possibility of tendon determination

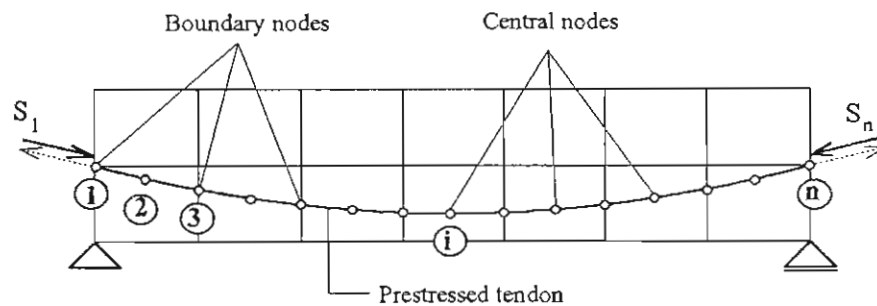


Figure 3 Tendon position determined by nodes

The tendons are discretized with one-dimensional isoparametric three-node finite elements which can occupy a general position on a two-dimensional concrete element^[8-9]. The tendon position is determined by nodes whose coordinates are defined in a global coordinate system. In order to ensure continuity between the tendon elements, the boundary nodes have to be placed at the

intersection point of the tendon and the boundaries^[7] of a two-dimensional concrete element (*Figure 3*). The paper uses an iterative procedure for interpolation coordinates of the central node^[8-10].

2.3 Losses in the prestress force

The model developed in this paper make it possible to compute standard losses caused by short-term linear and non-linear concrete deformation, losses in the anchorage zone caused by concrete deformation and the losses caused by friction.

Standard losses caused by concrete deformation are directly related to the applied model for concrete, reinforcement and prestressed tendons^[13-15]. Concrete is modelled by a non-linear elasto-viscoplastic two-parameter model^[13-15], whereas the reinforcement and the prestressed tendons are modelled by an uniaxial elasto-viscoplastic model.

Losses in the anchorage zone caused by concrete deformation are obtained directly. In the zone where the prestress force is introduced into the structure, local concentration of compressive stresses occurs in the direction of the tendon and significant tensile stresses occur in the direction perpendicular to the tendon. At those points farthest from the place where the force is applied, the state becomes more stable. Due to the described concentration of stress, the prestress force is significantly decreased at the beginning of the tendon. The described effects occur along the so-called anchorage length^[12]. The accuracy of the obtained results depends upon the density of the finite elements mesh in this zone.

The computation of losses caused by friction is described in detail in Item 2.1.

3. NUMERICAL PROCEDURE

The computation of posttensioned structures is performed in phases. The load can be applied incrementally in each phase.

In this paper the initial stiffness method^[16] was applied for solving the system of non-linear equations within each load increment. Consequently, the tangential stiffness matrix was formed at the beginning of each increment and it was kept in the subsequent iterations.

Phase I

In the first phase the structure was computed taking into account the load it carries before the prestressing of tendons. A concrete or reinforced concrete structure was analyzed herein. According to the known geometry and load we form the global stiffness matrix K_I for phase I according to expression:

$$K_I = K_C + K_R \quad (5)$$

where K_C is concrete stiffness matrix^[8] and K_R is reinforcement stiffness matrix^[7-10].

Global load vector (vector of residual forces) F_I is determined according to expression:

$$F_I = F_C + F_R \quad (6)$$

where F_C is a vector of external forces and residual forces on concrete element^[8,13-15], while F_R is vector of residual forces due to reinforcement strain^[7-8].

Phase II

In the second phase the tendons were tensioned individually. The prestress force can be applied at once or incrementally, and, thus, gradual prestressing procedures can be simulated. During the prestressing phase the respective tendon was not treated as a structural element. Its geometry was used actually to compute the initial influence of prestressing which is modelled as a fictitious distributed load. In subsequent iteration, the tendon functions as a classical reinforcement with a given initial stress. During the successive prestressing of tendons, the tendon which is currently being prestressed does not influence the stiffness of the structure, while the previously prestressed tendons take over the stresses as a classical reinforcement.

The global stiffness matrix in this phase can be presented in the following form:

$$K_{II}^i = K_I + \sum_{j=1}^{i-1} K_P^j \quad (7)$$

where: i - tendon index, i.e. index of a group of tendons which are being prestressed,

K_{II}^i - global stiffness matrix at the moment of prestressing the i -th group of tendons,

K_I - global stiffness matrix of the phase I,

K_P^j - stiffness matrix of one tendon or group of tendons which started functioning as a classical reinforcement^[7-10].

The loading vector can be presented in the following form:

$$F_{II}^i = F_I + \sum_{j=1}^i \Delta F_{II}^j \quad (8)$$

where: F_{II}^i - global vector of loading at the moment of prestressing the i -th group of tendons,

F_I - load vector after the phase I,

ΔF_{II}^j - vector of equivalent load which represents the influence of the prestressing force of a given tendon upon the concrete structure^[8].

When the prestressing force is introduced into the structure gradually, vector ΔF_{II}^j is applied incrementally and not at once.

Phase III

The prestressing of all tendons is followed by the third phase in which the structure is computed taking into account the remaining part of the dead load and the live load. Concrete, reinforcement and all prestressed tendons which function as a classical reinforcement, contribute to the stiffness of the structure. The load is applied incrementally until failure. The stiffness matrix K_{III} in this phase is:

$$K_{III} = K_C + K_R + \sum_{j=1}^n K_P^j \quad (9)$$

where n is the number of prestressed tendons, i.e. the number of prestressed tendon groups.

The loading vector can be presented as:

$$F_{III} = F_I + \sum_{j=1}^n \Delta F_{II}^j + \Delta F_{III} \quad (10)$$

where ΔF_{III} is the part of the loading vector which resulted from the load taken over by the structure after the completed prestressing. When the load is applied incrementally vector ΔF_{III} is applied in increments.

4. ILLUSTRATIVE NUMERICAL EXAMPLE

The validity of the model was tested on a prestressed beam which had been analyzed in references^[8-10] and, due to the shortage of space, that example is not presented herein.

In this paper certain possibilities of the developed model have been illustrated on a posttensioned nonprismatic continuous beam^[1]. *Figure 4* shows the beam dimensions, tendon layout and load.

The load was considered in the three phases:

Phase I (the phase preceding prestressing): own weight of the beam (25 kN/m^3) and uniformly distributed dead load $p_d = 20 \text{ kN/m}$.

Phase II (prestressing phase): load considered in Phase I and prestressing force $S = 2000 \text{ kN}$ applied at one end of the tendon while the other end is anchored to concrete.

Phase III (under service-load condition): load considered in Phase II, uniformly distributed live load $p_l = 20 \text{ kN/m}$ and concentrated load $P = 200 \text{ kN}$.

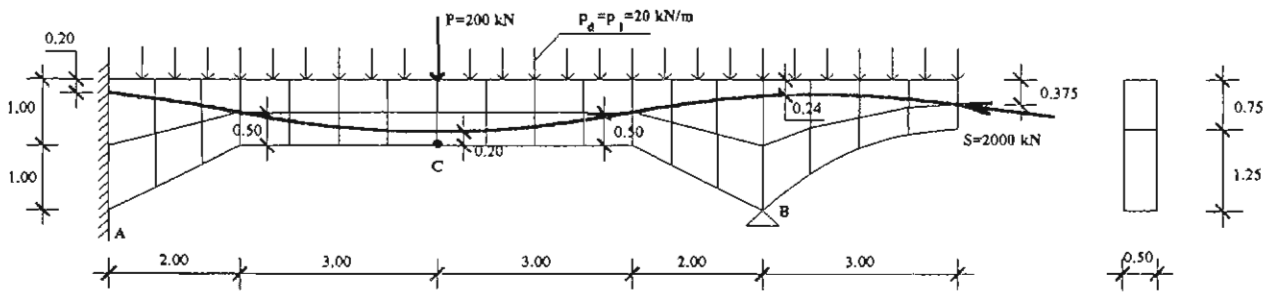


Figure 4 Nonprismatic prestressed continuous beam

Material parameters for concrete and prestressing steel are given in *Table 1*.

Table 1 Material parameters for *Example 1*

Concrete		Prestressing steel	
Young's modulus, E_c	28000 N/mm ²	Young's modulus, E_s	224000 N/mm ²
Uniaxial compressive strength, f_c	30 N/mm ²	Yield stress, f_y	1500 N/mm ²
Ultimate compressive strain, ϵ_u	0.0035	Area thickness, A_s	2000 mm ²
Poisson's ratio, ν	0.25		
Tensile strength, f_t	3.0 N/mm ²		

Table 2 presents the results obtained for reactions for support A for linear and non-linear analyses which are compared with the results for the linear analyses. The results for the linear analysis agree well with those from ref.^[1]. However, the non-linear analysis shows that cracks occur already in the phase of tendon prestressing due to the great prestressing force, which causes deviations in the rate of reactions on the bearing. In addition, the model presented in this paper takes into account the losses caused by the prestress force due to non-linear deformation of the concrete and the losses in the anchorage zone which affected the reaction rate.

Table 2 Reactions for support A (kN) for beam shown in Figure 4

Load case	This paper						Ref. [1]		
	Nonlinear			Linear			Linear		
	F_x	F_y	M	F_x	F_y	M	F_x	F_y	M
Phase I	107	180	296	107	180	296	105	180	298
Phase II	-308	250	575	-469	273	655	-479	273	647
Phase III	-142	476	1098	-251	485	1093	-260	487	1093

Figure 5 presents the force along the prestressed tendon for several cases of load. After prestressing, the force in the tendon decreases due to non-linear deformation of concrete and great concentration of compressive stresses in the anchorage zone. In the third phase the prestressed tendon starts functioning as a reinforcement which influences the changes of the force in the tendon. Load-displacement response of the beam in point C is shown in Figure 6.

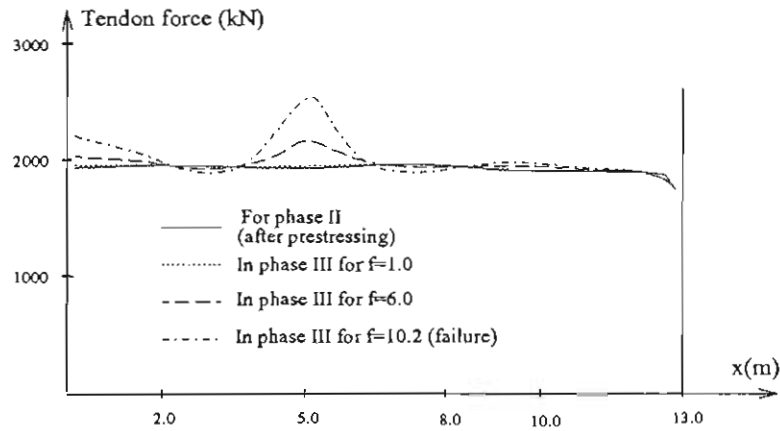


Figure 5 The force in prestressed tendon for different cases of loading

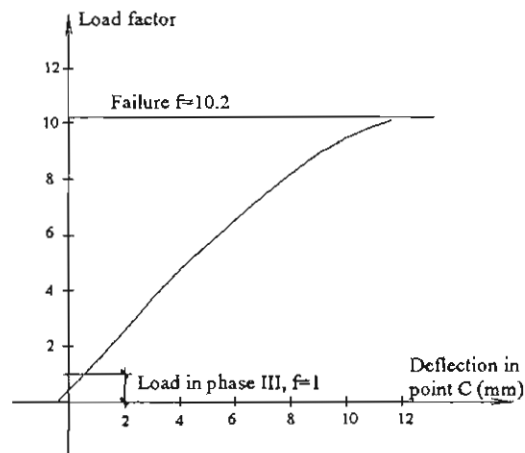


Figure 6 Load-displacement response of the beam

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NUMERICAL METHOD FOR SOLVING LINEAR AND NON-LINEAR SOLID BODY PROBLEMS

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Summary

In this paper a relatively new method for solid body stress analysis is presented. The method is very simple and easy to use: first, the time is subdivided into a number of time steps and solid into a number of control volumes of arbitrary shape. Then, the equations governing momentum and energy balance, written in an integral form, are solved over each control volume at every time interval, resulting in a set of (in general) non-linear algebraic equations, whose solution is obtained by employing an efficient segregated algorithm.

The method was developed at Mechanical Engineering Faculty of Sarajevo University in 1987 and since then has been applied to a number of both linear and non-linear solid body deformation problems, including dynamic fracture in pressurised pipes, prediction of residual stresses in a welded work-piece etc. In this paper a review of applications is presented and calculations are compared with analytical solutions and experimental data. Results show that the method can be considered to be a serious alternative to the well established finite element methods in many areas of solid body stress analysis.

1. Introduction

In the last several decades the finite element methods have become a synonym for the solid body stress analysis. Encouraged by the success of the finite volume methods in fluid flow predictions, a group from Mechanical Engineering Faculty of Sarajevo University has developed a new, finite-volume based numerical method for stress analysis equally applicable to linear and non-linear materials. In what follows an outline of the method is given and some results illustrating the method's abilities are presented. More details about the method can be found in [1-6].

2. Mathematical Model

The behaviour of an arbitrary part of a solid body of volume V bounded by the surface S (with unite vector $\mathbf{n}$ pointing outwards) at any instant of time t can be described by equations of momentum and thermal energy balance

$$\frac{\partial}{\partial t} \int_V \rho \frac{\partial \mathbf{u}}{\partial t} dV = \int_S \mathbf{T} \cdot \mathbf{n} dS + \int_V \mathbf{f} dV; \quad \frac{\partial}{\partial t} \int_V \rho c T dV = \int_S \mathbf{q} \cdot \mathbf{n} dS + \int_V h dV, \quad (1)$$

where ρ is the density, $\mathbf{u}$ is the displacement velocity vector, $\mathbf{T}$ is the stress tensor, $\mathbf{f}$ is the resultant body force, T is the temperature, c is the specific heat, $\mathbf{q}$ is the heat flux vector and h is the internal energy heat source. In order to close the above system of equations the following constitutive relations for the thermo-elasto-plastic solid are used:

$$d\mathbf{T} = 2\mu d\mathbf{D} + \lambda \text{tr}(d\mathbf{D}) \mathbf{I} - (3\lambda + 2\mu) \alpha dT \mathbf{I} - \frac{9\mu^2}{3\mu + H'} \frac{\mathbf{T}^d : d\mathbf{D}}{\bar{\sigma}} \mathbf{T}^d; \quad \mathbf{q} = k \text{grad } T, \quad (2)$$

where

$$\mathbf{D} = \frac{1}{2} [\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^T]; \quad \mathbf{T}^d = \mathbf{T} - \frac{1}{3} \text{tr}(\mathbf{T}) \mathbf{I}; \quad \bar{\sigma} = \left(\frac{3}{2} \mathbf{T}^d : \mathbf{T}^d \right)^{1/2} \quad (3)$$

are the deformation (strain) tensor, the stress deviator and the effective stress (for von Mises yield criterion), respectively, μ and λ are the Lamé's coefficients, α is the linear thermal expansion coefficient, H' is the plastic modulus and k is the thermal conductivity.

By introducing constitutive relations (2) into equations governing momentum and energy balance (1) a closed system of 2 (generally non-linear and coupled) equations with two unknown functions of spatial coordinates and time ($\mathbf{u}$ and T) is obtained.

To complete the mathematical model, initial and boundary conditions have to be specified. As initial conditions, temperature and displacements in the whole solution domain at the initial instant of time have to be given. The boundary conditions ought to be specified at all boundaries in terms of temperature and/or heat flux and displacements and/or forces (stresses).

3. Numerical Solution Procedure

In order to solve system of equations (1), the time domain is subdivided into a number of time intervals δt and the space domain into N contiguous control volumes (CV), which can be of an arbitrary topology, as illustrated in Fig. 1. By assuming a suitable variation of dependent variables in time and space (e.g. piece wise linear spatial variation and fully implicit time differencing, as in the present study), equations (1) are integrated over time interval δt and over each control volume resulting in a system of $4N$ (generally non-linear and coupled) algebraic equations of the form

$$a_{\phi 0} \phi_{P_0} - \sum_{j=1}^n a_{\phi j} \phi_{P_j} = b_{\phi}, \quad (4)$$

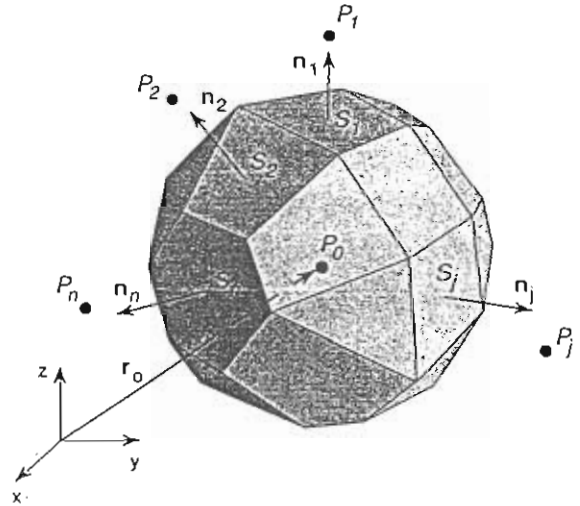


Fig.1 Control volume of arbitrary topology.

where ϕ stands for Cartesian displacement components u_i ($i=1,2,3$) or temperature T , a_{ϕ} 's are always positive, and n is the number of cell faces of a control volume (Fig. 1).

Direct solution of the above system is impossible if eq. (4) are non-linear and requires a large computer memory if they are linear. For that reason the present method uses the following segregated approach. First, all dependent variables are given the corresponding initial values. Then the boundary conditions which correspond to the first time step are applied and the sets of equations for each individual dependent variable are linearised and temporarily decoupled by using the

dependent variables initial values to calculate coefficients a_ϕ and b_ϕ . In such a manner four independent sets of N linear algebraic equations with N unknowns with a sparse diagonally dominant symmetric coefficient matrix are obtained. These sets are then solved sequentially in turn until a converged solution is reached, i.e. a desired accuracy is obtained. In the next time step the whole procedure is repeated, except that the initial values are replaced by the values from the previous time level.

Systems of linear equations are solved by efficient conjugate gradient solvers. Since coefficients are only approximated (based on the values of dependent variables from the previous iteration/time step), there is no need to solve system of linear equations to a tight tolerance. Normally, reduction of residuals for one order of magnitude is sufficient. However, in order to promote stability of this solution method, an under-relaxation is often necessary.

Finally, it is important to mention that the method is fully conservative, because all forces and heat fluxes acting on the internal CV faces cancel out and the overall momentum and heat balance reduces to the balance of forces and heat fluxes acting on the solution domain boundaries. This enables physically meaningful results to be obtained even when very coarse meshes are employed. In addition, the fully implicit time differencing used avoids a stability-related time step restrictions. In principle, it allows any magnitude of the time step to be used, and in practice it is limited only by the required temporal accuracy. If only steady state is sought, the temporal accuracy is not important and the solution can be obtained by using a single, infinite time step.

4. Application of the Method

The method described in the previous sections has been applied to a number of both linear and non-linear solid body deformation problems [1-6], few of which will be presented in this paper.

4.1 Flat plate with circular hole subjected to uniform tension

An infinite plate with a circular hole of the radius $R = 0.5$ m in its centre is loaded by uniform tension of $t_x = 10^4$ Pa in x direction. Taking into account the symmetry of the problem, a quadratic portion of the plate with dimensions $a \times b = 2 \times 2$ m is used for the numerical analysis. In order to eliminate the influence of the finite solution domain dimensions the tractions calculated from the analytical solution [7] are imposed at the boundaries. The plate material is assumed to be elastic with modulus of elasticity $E = 10^7$ Pa and Poisson coefficient $\nu = 0.3$.

Stress distribution calculated on a uniform 2944 CV numerical mesh is compared with the analytical solution and an excellent agreement is obtained, as illustrated in Fig. 2. In order to illustrate the methods geometrical flexibility, the results obtained on non-uniform mesh comprising of only 76 CV of different topology (with 3, 4 and 5 cell faces) (Fig. 3) are compared with results from uniform 736 CV mesh (Fig. 4). The slightly better results of the locally refined mesh can be attributed to the mesh spacing near the hole being a bit finer than in the case of uniform mesh. At the same time this is an example of the local grid refinement, where fine mesh is used only in regions of high solution variable gradients.

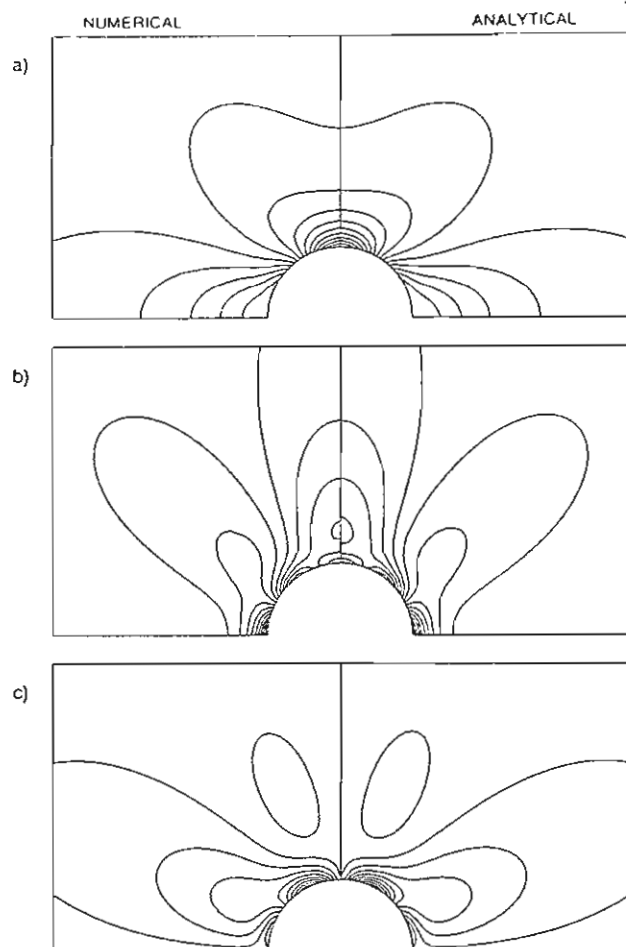


Fig. 2. Comparison of numerical and analytical stress fields. (a) σ_{xx} contour levels: from 1000 to 29000, step 2000 Pa; (b) σ_{yy} contour levels: from -9500 to 4500, step 1000 Pa; (c) σ_{xy} contour levels: from -8500 to 1500, step 1000 Pa.

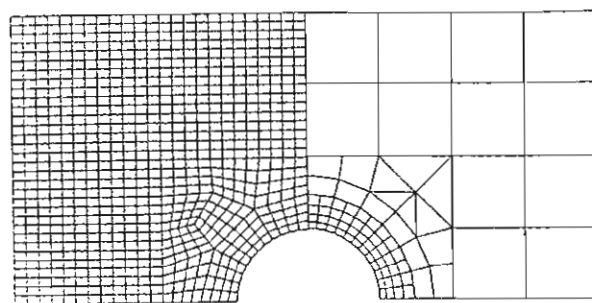


Fig. 3. Uniform 736 CV and locally refined 76 CV mesh with CVs of different topology.

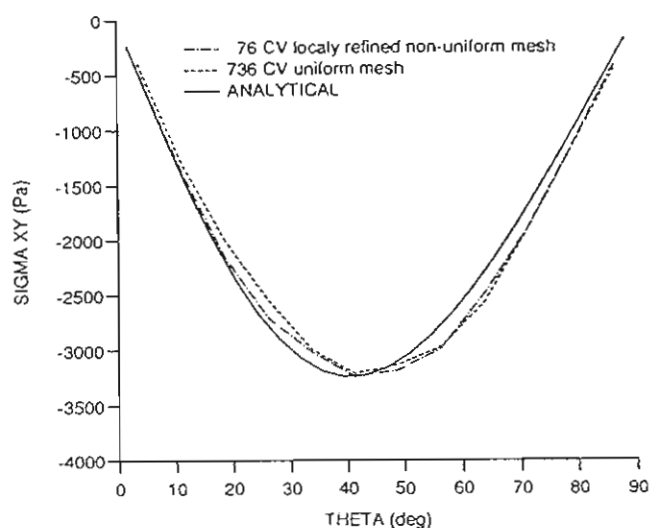


Fig. 4. Comparison of shear stress variation along the circular section $r=1.25 R$ obtained by using uniform and locally refined non-uniform mesh.

4.2 Rapid crack propagation along pressurised pipe

Under certain conditions, pressurised gas and water distribution pipes can fail catastrophically by rapid crack propagation (RCP) and this problem is recognised as one of the most important tasks of dynamic fracture mechanics. Since an appropriate analytical solution is not available, this problem has to be solved numerically. The only method yet adopted resorts to a dynamic shell finite-element simulation [8]. The application of this approach is restricted by the pipe wall thickness. The only alternative attempt so far has been conducted by the present authors [4, 5].

In this section the present method is used for full 3D, both quasi-steady and transient analysis of an axial crack propagating rapidly along an air pressurised pipe of linearly elastic material ($E = 2.5 \cdot 10^9$ Pa, $\nu = 0.4$ and $\rho = 940$ kg/m³). Calculations are carried out on the range of crack speeds from 0 to 285 m/s.

A fixed mesh 'node release' technique is employed for the simulation of a transient crack, initiating from the stationary crack of zero length. The crack is propagated along the straight path at a constant speed and the crack driving force is generated. Due to the symmetry, only half of the pipe is considered and the crack path is represented as a boundary. The running crack is modelled by

sequential release of the boundary cell faces along the crack path, progressively changing boundary conditions. A 'holding back force' concept [5] is used to simulate the crack propagation along newly released cell-faces. A uniform mesh is used throughout, as recommended for the simulation of transient RCP [9]. Each time crack tip cell faces are released, dissipated energy is calculated by both global energy balance and the local work done by the holding back forces. Both methods agree if the solution is stable [9]. The pipe is initially pressurised to p_0 , but as the crack appears the decompression wave is generated behind which pressure falls to p_I [10]. This analysis relies on prescribed interaction between the pipe and its gas content: uniform internal pressure p_I is specified ahead of the moving crack front, whilst behind it the pressure is assumed to decay linearly to zero over an outflow length l (Fig. 5). These assumptions are based on 1D fluid flow analysis [10, 11] backed up by experimental observations [12].

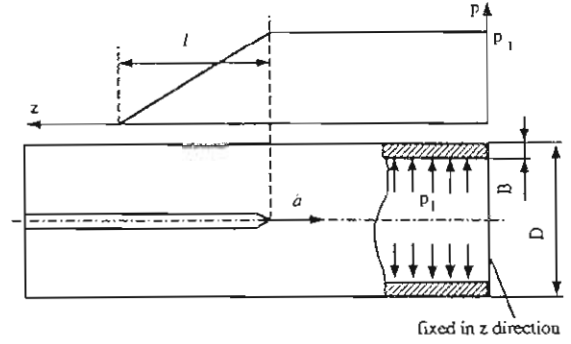


Fig. 5. Pipe geometry and loading conditions ($D = 0.25$ m, $D/B = 11$, $l = 1.5 D$).

In practice, one is generally concerned with computation of the crack driving force G , derived from the global energy balance:

$$G = \frac{dW_e}{dA} - \frac{dE_s}{dA} - \frac{dE_k}{dA}, \quad W_e = \int_V p \mathbf{n} \cdot \dot{\mathbf{u}} dS dt, \quad E_s = \frac{1}{2} \int_V \mathbf{T} : \mathbf{D} dV, \quad E_k = \frac{1}{2} \int_V \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dV \quad (5)$$

where $dA = B \dot{a} dt$ is a unit area of crack advance, B is the pipe wall thickness, $\dot{a}$ is the crack speed, W_e is external work and E_s and E_k are strain and kinetic energy, respectively. By equating G and the material's dynamic crack resistance, a critical pressure can be determined, below which RCP is not sustainable.

Typical transient results are presented in Fig. 6. It can be seen that the solution approaches the steady state after the crack has propagated for a certain distance. Figure 7 shows the crack opening profile at various time during RCP at 50 m/s. Clearly, inside the 'steady state window' the crack profile does not change and it translates along the pipe (12.5 and 15 ms).

A very good agreement between the transient and quasi-steady results is demonstrated in Fig. 8.

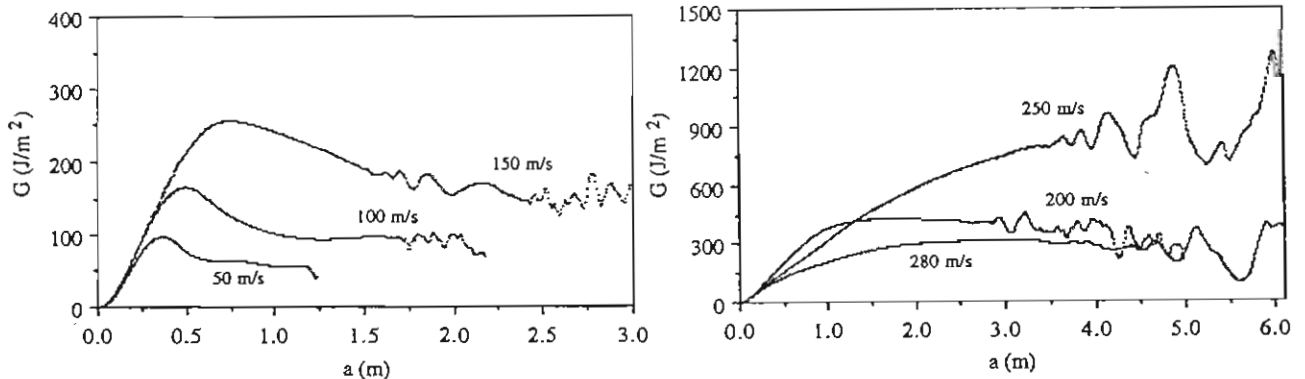


Fig. 6. Crack driving force vs. crack length; transient results.

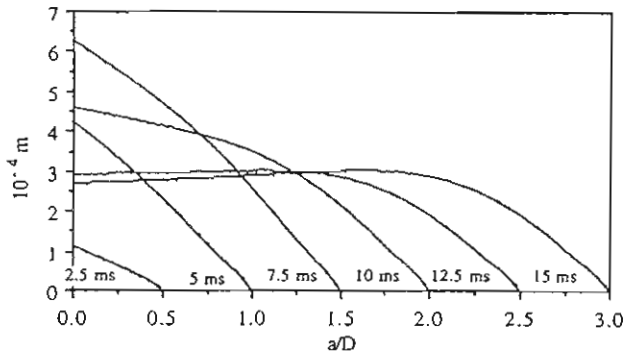


Fig. 7. Crack opening displacement vs. crack length.

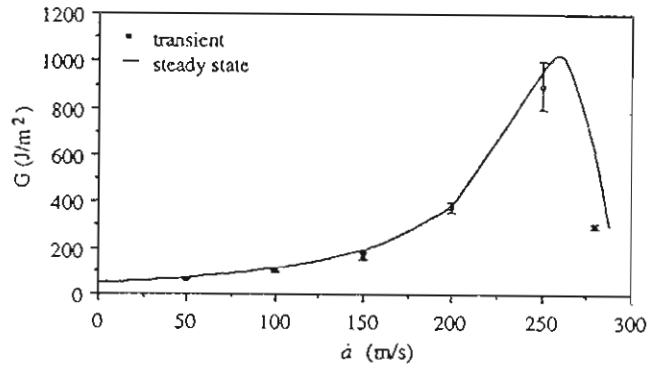


Fig. 8. Comparison between transient and quasi-steady results.

However, some discrepancies develop around the region where G reaches its peak. Here, the validity of steady state assumptions is somewhat questionable. The rising part of G is a decompression dominated region, whilst the falling part is limited by inertia. In the peak region these two strongly counteract.

In order to enhance the method's efficiency a multigrid acceleration technique [6] is employed, resulting in a drastic reduction of CPU time, e.g. the quasi-steady simulation at $\dot{a} = 50$ m/s was 34 times faster when using 4 levels of mesh refinement, compared to a single fine mesh (16500–24000 CVs, depending on l) solution.

4.3 Welding of a thin plate

To illustrate the method's applicability to thermal stress analysis, predictions of unsteady temperature and stress fields generated during one-pass arc-welding of a rectangular notched steel plate are presented in this section. Since the plate is thin, it is assumed that the heat generated by electric arc and temperature are uniformly distributed across the thickness of the plate and that the plane stress conditions prevail. Heat gains due to the electric arc and heat losses at the bottom and top surfaces are taken into account through volume source term h . In the region where temperature reaches the melting point the latent heat of melting is included into the internal energy balance.

Solution domain ($a \times b = 0.5 \times 0.165$ m) is subdivided by a 40×15 CV mesh uniform in x and non-uniform in y direction (finer near x axis where higher gradients of dependent variables are expected). The welding period is subdivided into 1200 equal time intervals of the size $\delta t = 0.125$ s, while during the cooling period 35 time intervals increasing from $\delta t = 0.2$ s to $\delta t = 200$ s are used. As the initial condition temperature equal to the ambient temperature and zero displacements in the whole solution domain are used and zero tractions at all boundaries and the boundary conditions of the third kind (prescribed heat transfer coefficient) are employed. The plate material is assumed to be thermo-elasto-plastic. More details about this case can be found in [2].

A large amount of data relevant to the analysis of the welding process can be extracted from the calculated results. However, only contours of temperature and stresses at time $t = 123$ s are presented in Fig. 9. Temperature field is typical for a low speed welding process, showing a well established quasi-steady temperature distribution around the melted pool. The normal stress contours show an expected behaviour, with clearly indicated regions of compressive ($\sigma < 0$) stresses in the vicinity of

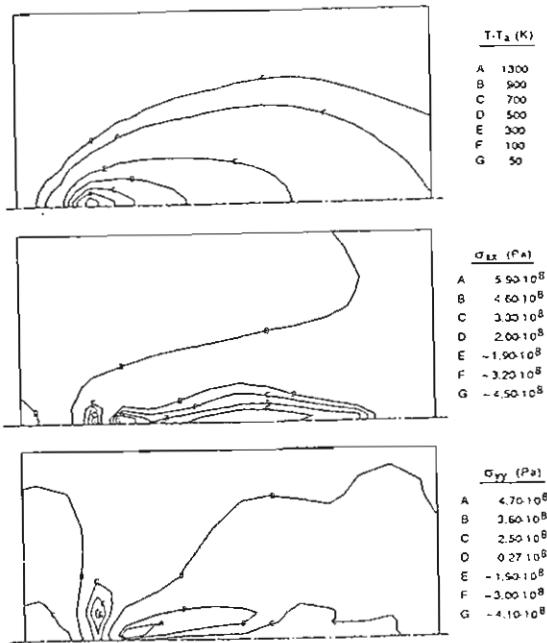


Fig. 9. Contours of temperature and normal stresses at $t = 123$ s.

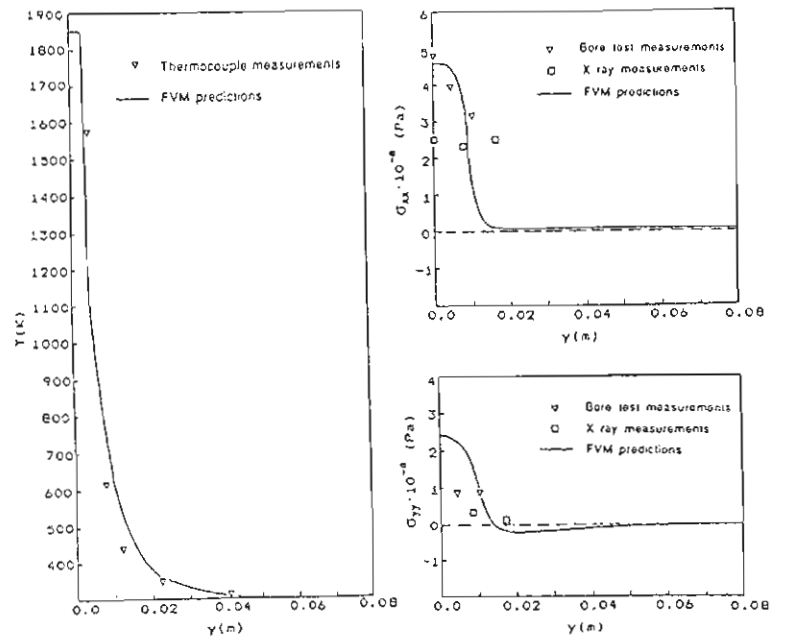


Fig. 10. Temperature profile at $x = 0.25$ m and $t = 75$ s and residual stresses at $x = 0.25$ m; comparison with experiments [8].

the melted pool, where the temperature is the highest, and extensive stresses ($\sigma > 0$) in the rest of the plate.

In Fig. 10 results of calculations are compared with measurements reported in ref. [13]. First, the temperature distribution at the $x = 0.25$ m at the moment when the electric arc is at that cross-section are compared with thermocouple measurements. After that, the predicted residual stress distribution at the same cross section are compared with measurements obtained with the aid of abrasive bore test and x-ray diffraction technique. In all cases agreement between predictions and measurements can be considered satisfactory, taking into account a considerable dispersion of experimental results and uncertainty about the heat transfer coefficient.

5. Conclusions

A recently developed numerical method for stress analysis has been outlined and its applicability to the solution of a range of both steady and transient problems involving various elastic and elasto-plastic materials has been demonstrated.

The main features of the numerical method presented in this paper is its simplicity, stemming out of its straight forward physical interpretation (balance of forces and heat fluxes acting on a CV). This enables the method's easy application to different materials and initial and boundary conditions. Equally important are method's stability and conservativeness, which result in a stable solution and a reasonable accuracy even when calculations with large Courant number and coarse mesh are conducted.

The overall efficiency of the method, especially in case of linear materials, is significantly improved by employing a multigrid acceleration technique.

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NUMERICAL MODELLING OF TWO-DIMENSIONAL REINFORCED CONCRETE STRUCTURES

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Summary

A two-dimensional nonlinear finite element analysis of reinforced concrete structures is presented. The concrete material is modelled including the biaxial nonlinear stress-strain behaviour, tensile cracking, compression crushing, strain-softening, post-tensile cracking and post-compression crushing. A numerical example illustrates the applicability of the model presented.

1.0 The concrete material model

The material model is hypoelastic and orthotropic, based on a uniaxial stress-strain relation generalized to take into account biaxial stress conditions^{[6][14][1]}. Biaxial stress-strain relations are derived from uniaxial stress-strain relation (Fig. 1).

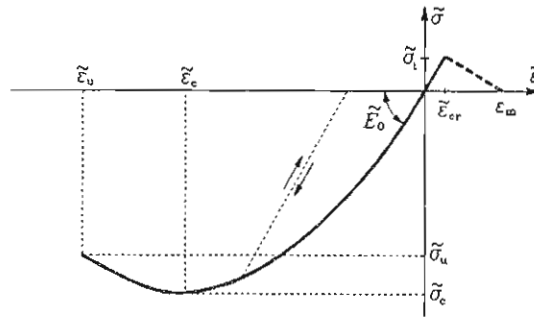


FIG. 1 Uniaxial stress-strain law.

Uniaxial stress-strain law is defined by the following parameters: $\tilde{\epsilon}_{cr}$ is the strain corresponding to the uniaxial tensile strength $\tilde{\sigma}_t$, $\tilde{\epsilon}_c$ is the strain corresponding to the compressive strength $\tilde{\sigma}_c$, and $\tilde{\epsilon}_u$ is the ultimate compressive strain. ϵ_m is the strain when the tensile stress becomes zero.

In **compression** the stress-strain curve is assumed to be as follows^[1]

$$\tilde{\sigma} = \frac{(\tilde{E}_0/\tilde{E}_s)(\tilde{\epsilon}/\tilde{\epsilon}_c)\tilde{\sigma}_c}{1 + A(\tilde{\epsilon}/\tilde{\epsilon}_c) + B(\tilde{\epsilon}/\tilde{\epsilon}_c)^2 + C(\tilde{\epsilon}/\tilde{\epsilon}_c)^3}, \quad \tilde{E}_s = \frac{\tilde{\sigma}_c}{\tilde{\epsilon}_c} \quad (1)$$

A, B and C are constant parameters which are defined from the uniaxial experimental tests for the concrete under consideration.

$$A = \frac{(\tilde{E}_0/\tilde{E}_u) + (p^3 - 2p^2)(\tilde{E}_0/\tilde{E}_s) - (2p^3 - 3p^2 + 1)}{(p^2 - 2p + 1)p} \quad (2)$$

$$B = 2 \frac{\tilde{E}_0}{\tilde{E}_s} - 3 - 2A, \quad C = 2 - \frac{\tilde{E}_0}{\tilde{E}_s} + A, \quad \tilde{E}_u = \frac{\tilde{\sigma}_u}{\tilde{\epsilon}_u}, \quad p = \frac{\tilde{\epsilon}_u}{\tilde{\epsilon}_c}$$

For unloading conditions and repeated loading, the initial Young's modulus $\tilde{E}_0$ is assumed.

The orthotropic stress-strain law for the two-dimensional case of plane stress can be written in an incremental form as^[16]:

$$\{d\sigma\} = [c]\{d\varepsilon\}, \quad \{d\sigma\}^T = \langle d\sigma_1, d\sigma_2, d\tau_{12} \rangle, \quad \{d\varepsilon\}^T = \langle d\varepsilon_1, d\varepsilon_2, d\gamma_{12} \rangle \quad (3)$$

$$[c] = \frac{1}{1 - \nu_1\nu_2} \begin{bmatrix} E_1 & \nu_2 E_1 & 0 \\ \nu_1 E_2 & E_2 & 0 \\ 0 & 0 & (1 - \nu_1\nu_2) G_{12} \end{bmatrix}, \quad [c]^{-1} = \begin{bmatrix} 1/E_1 & -\nu_2/E_2 & 0 \\ -\nu_1/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \quad (4)$$

If we approximate

$$E_1\nu_2 \approx E_2\nu_1 \equiv E_{12}\nu_{12} \quad (5)$$

where E_{12} and ν_{12} are the weighted Young's modulus and the weighted Poisson's ratio^[8]

$$E_{12} = \frac{|\sigma_1|E_1 + |\sigma_2|E_2}{|\sigma_1| + |\sigma_2|} = 2(1 + \nu_{12})G_{12}, \quad \nu_{12} = \frac{|\sigma_1|\nu_1 + |\sigma_2|\nu_2}{|\sigma_1| + |\sigma_2|} \quad (6)$$

then, symmetric matrix $[c]$ is obtained

$$[c] = \frac{1}{1 - \nu_1\nu_2} \begin{bmatrix} E_1 & \nu_{12}E_{12} & 0 \\ \nu_{12}E_{12} & E_2 & 0 \\ 0 & 0 & (1 - \nu_1\nu_2)G_{12} \end{bmatrix} \quad (7)$$

Usually it is assumed that $\nu_1 \approx \nu_2$ ^[1]. Parameters E_1 , E_2 , E_{12} , ν_{12} and G_{12} in matrix $[c]$ are stress-dependent material properties. The orthotropic directions 1 and 2 coincide with the principal stresses for uncracked concrete. When a crack occurs, the material directions coincide with the crack direction and direction perpendicular to the crack plane.

The strain increments $d\varepsilon_1$ and $d\varepsilon_2$ are expressed by the stress increments $d\sigma_1$ and $d\sigma_2$ according to Eqs. (3) and (4).

$$d\varepsilon_1 = \frac{d\sigma_1}{E_1} - \frac{\nu_2 d\sigma_2}{E_2}, \quad d\varepsilon_2 = -\frac{\nu_1 d\sigma_1}{E_1} + \frac{d\sigma_2}{E_2} \quad (8)$$

The tangential moduli E_1 and E_2 are defined by Eq. 8. Assuming that $d\sigma_2 = 0$ in the first Eq. (8), an equivalent strain^{[6][14]} increment $d\varepsilon_1^e$ is obtained

$$d\sigma_2 = 0: \quad d\varepsilon_1 \rightarrow d\varepsilon_1^e = \frac{d\sigma_1}{E_1} \quad (9)$$

The relationship (9) between the principal stress σ_1 and the equivalent strain ε_1^e is called equivalent uniaxial stress-strain curve^[14]. The analogous relationship is obtained for the second direction, i.e.

$$E_i = \frac{d\sigma_i}{d\varepsilon_i^e}; \quad (i = 1, 2) \quad (10)$$

From Eq. (10) it can be seen that Poisson's ratio is neglected. Two independent equivalent uniaxial stress-strain relations (Fig. 3) for a biaxial state may be obtained for current principal

stresses (σ_1, σ_2) using a biaxial failure envelope. The failure envelope, shown in Fig. 2, is taken from the experimental results reported by Kupfer^[9].

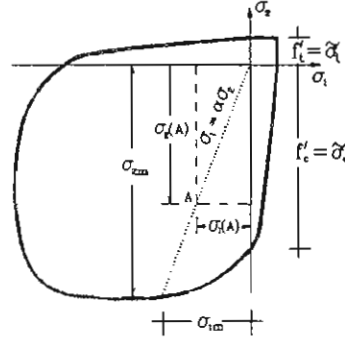


FIG. 2 Kupfer's biaxial failure envelope.

The compressive strength σ_{1m} and σ_{2m} for a given proportional loading $\alpha = \sigma_1/\sigma_2$ of principal stresses can be obtained from Fig. 2^[3]:

$$\sigma_{1m} = \alpha \sigma_{2m}, \quad \sigma_{2m} = \frac{1 + 3.65\alpha}{(1 + \alpha)^2} f'_c; \quad (0 \leq \alpha \leq 1) \quad (11)$$

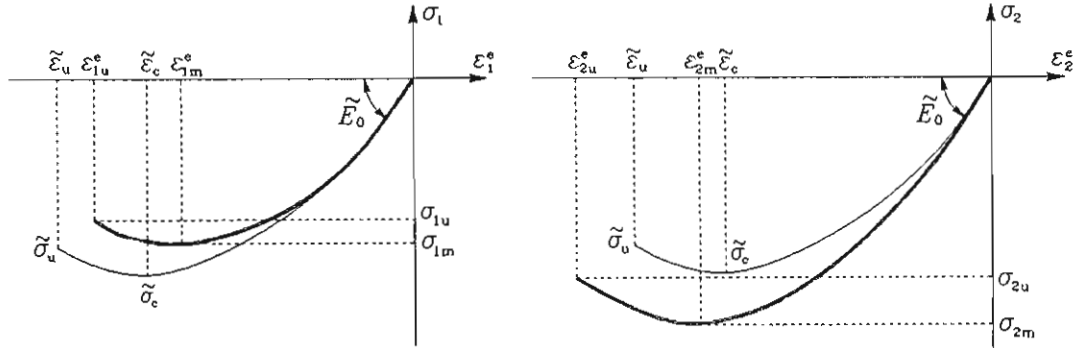


FIG. 3 Equivalent uniaxial stress-strain curves for the proportional loading.

In order to construct two equivalent uniaxial stress-strain curves (Fig. 3), it is necessary to define equivalent strains $(\epsilon_{1m}^e, \epsilon_{2m}^e)$, as well as the ultimate stresses $(\sigma_{1u}, \sigma_{2u})$ and corresponding equivalent strains $(\epsilon_{1u}^e, \epsilon_{2u}^e)$. Ultimate stresses $(\sigma_{1u}, \sigma_{2u})$ can be expressed in the same manner as Eq. (11):

$$\sigma_{1u} = \alpha \sigma_{2u}; \quad \sigma_{2u} = \frac{1 + 3.65\alpha}{(1 + \alpha)^2} \sigma_{2m}; \quad (0 \leq \alpha \leq 1) \quad (12)$$

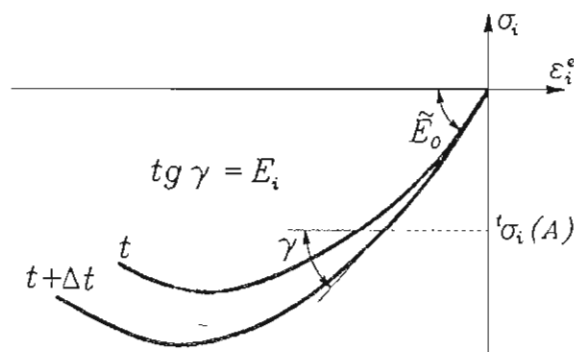
The equivalent strains $(\epsilon_{1m}^e, \epsilon_{2m}^e)$ are assumed to be^[3]

$$\epsilon_{1m}^e = \tilde{\epsilon}_c \left[-1.6 \left(\frac{\sigma_{1m}}{f'_c} \right)^3 + 2.25 \left(\frac{\sigma_{1m}}{f'_c} \right)^2 + 0.35 \frac{\sigma_{1m}}{f'_c} \right] = \tilde{\epsilon}_c k_1, \quad \epsilon_{2m}^e = \tilde{\epsilon}_c \left(3 \frac{\sigma_{2m}}{f'_c} - 2 \right) = \tilde{\epsilon}_c k_2 \quad (13)$$

and for the equivalent ultimate strains $(\epsilon_{1u}^e, \epsilon_{2u}^e)$ coefficients k_1 and k_2 are used according to (13)

$$\epsilon_{1u}^e = \tilde{\epsilon}_u k_1, \quad \epsilon_{2u}^e = \tilde{\epsilon}_u k_2 \quad (14)$$

The uniaxial curves in Fig. 3 are not changed in the case of the proportional loading. However, non-proportional loadings cause the change of uniaxial stress-strain curves. In this case there is one curve for state "t" and another curve for the new state "t + Δt" after the stress increment has been applied. The moduli E_i ($i = 1, 2$) are in this case defined as shown in Fig. 4^[14].

FIG. 4 Determination of the moduli E_i for a non-proportional loading.

In case of negative E_i , it is assumed that $E_i = 0$ for calculation of the stiffness matrix^{[14][1]}.

Compression – tension: The reduction of the compressive strength of a cracked concrete is less than about 20%^[10]. This reduction is here approximated by a linear function^{[4][10]} of the strain component perpendicular to the crack direction (Fig. 5).

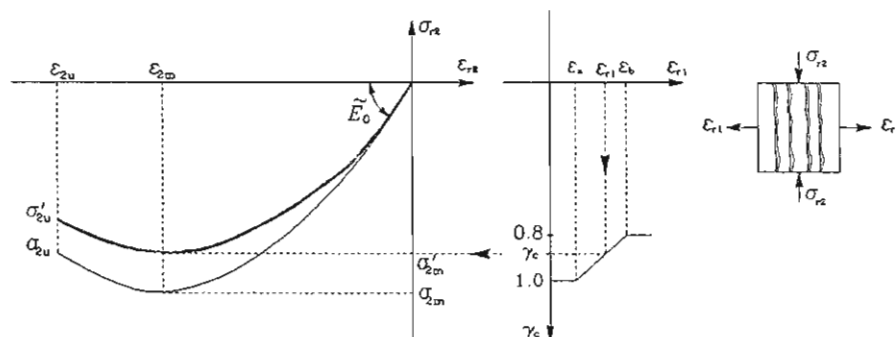


FIG. 5 Reduction of the compressive strength of cracked concrete.

Constants σ'_{2m} and σ'_{2u} are defined by a factor γ_c (Fig. 5):

$$\sigma'_{2m} = \gamma_c \sigma_{2m}, \quad \sigma'_{2u} = \gamma_c \sigma_{2u} \quad (15)$$

Tensile stress: From the failure envelope (Fig. 2), it can be seen that increasing the compressive stress, σ_2 , tensile strength σ'_t is reduced. This can be expressed by the following equation^[4]:

$$\sigma'_t = \frac{2 - \sigma_2/\tilde{\sigma}_c}{3 + \sigma_2/\tilde{\sigma}_c} \tilde{\sigma}_t \quad (16)$$

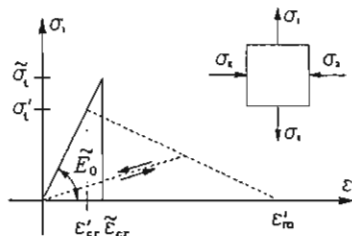


FIG. 6 Stress-strain law in tension.

The strain ϵ'_m (See Fig. 6) is defined as^[2]:

$$\epsilon'_m = \frac{2G_f}{\sigma'_t h_c} \quad (17)$$

where G_f is the fracture energy and h_c is crack band width. The fracture energy is determined from the experimental evidence. Crack band width depends on the element size, element

shape, integration scheme, and on the particular problem under the consideration. Sometimes a rough estimation that a crack band width is a square root of an element area A_e ^{[4][15]} is used.

$$h_c = \sqrt{A_e} \quad (18)$$

Due to the rough crack surfaces and the Dowel action of the reinforcement bars intersecting the cracks, internal forces can be transferred parallelly and perpendicularly to the crack planes.

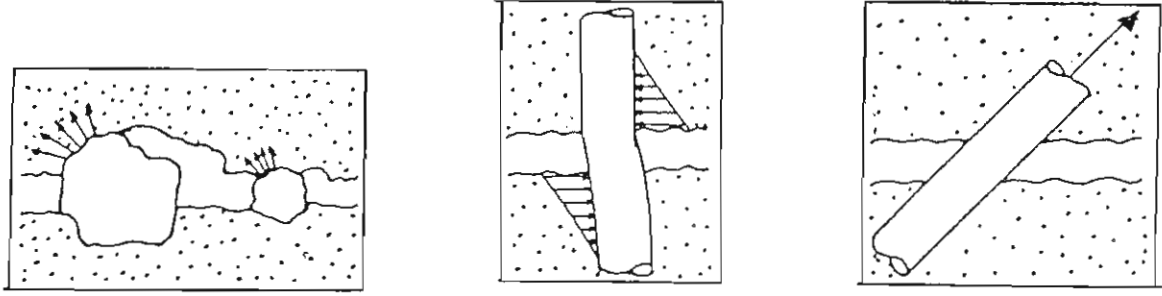


FIG. 7 Possible shear stiffness of cracked concrete.

Shear stiffness of cracked concrete in the case of a nonrotational crack model^[15] is reduced using the Kolmar's formula $G_c = \alpha G$ ^[10], where

$$\alpha = -\frac{\ln\left(\frac{\varepsilon_{mr}}{c_1}\right)}{c_2}; \quad \varepsilon_{mr} (\%) \quad (19)$$

$$c_1 = 7 + 5 \frac{\rho - 0.005}{0.015}; \quad c_2 = 10 - 2.5 \frac{\rho - 0.005}{0.015}; \quad \rho \leq 0.02 \quad (20)$$

G is the shear modulus of uncracked concrete, ε_{mr} is the mean tensile strain of reinforcement, and ρ is the reinforcing ratio of all the reinforcements crossing the crack.

2.0 Reinforcement modelling

The formulation for a two-dimensional, generally curved reinforcing bars embedded in a finite element mesh is used (Fig. 8)^[7].

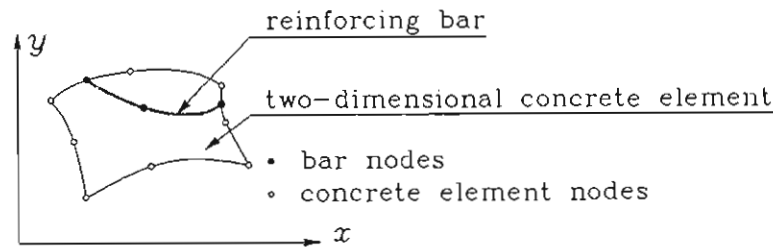


FIG. 8 8-node two dimensional isoparametric plane stress concrete element and 3-node one-dimensional reinforcing element.

Reinforcing bars are described in global coordinates. The incremental form of the constitutive relation for a reinforcing bar can be written as

$$\Delta\sigma_s = E_s \Delta\varepsilon_s \quad (21)$$

where, E_s is the tangential modulus for the unidirectional stress-strain relation. When the effect of a bond slip is accounted for, the bond stress-slip relation is also needed:

$$\Delta\sigma_b = E_b \Delta w_b \quad (22)$$

Δw_b is the slip increment along a bar, E_b is the tangential modulus of the bond stress-slip relation, which can be obtained experimentally. Fig. 9 shows some mean bond stress-slip relations^[12]:

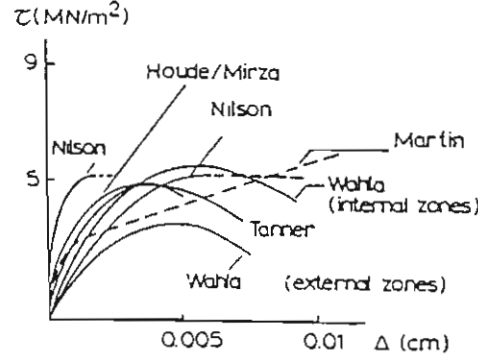


FIG. 9 Mean bond stress-slip relations.

A drawback of this approach is that the bandwidth of the stiffness matrix increases^[7].

3.0 Finite element solution procedure

Discretized equilibrium equations are

$$[{}^tK]\{\Delta U^{(k)}\} = \{{}^{t+\Delta t}R\} - \{{}^{t+\Delta t}F^{(k-1)}\} \quad (23)$$

$[{}^tK]$ is the nonlinear stiffness matrix at time t . To simplify calculation of $[{}^tK]$, a loading function ${}^t f^{[1]}$ is used

$${}^t f = \sqrt{\frac{1}{2} {}^t s_{ij} {}^t s_{ij}}; \quad {}^t s_{ij} = {}^t \sigma_{ij} - \frac{1}{3} \delta_{ij} {}^t \sigma_{kk} \quad (24)$$

Unloading condition is

$${}^t f < {}^t f_{max} \quad (25)$$

Stresses $\{{}^{t+\Delta t}\sigma\}$ at time $t + \Delta t$ are calculated using the material moduli corresponding to the current strain. (Any new conditions of the tensile failure and crushing have to be taken into account.):

$$\{{}^{t+\Delta t}\sigma\} = \{{}^t\sigma\} + [c] \{\Delta\varepsilon\} \quad (26)$$

4.0 Numerical example

A simply supported reinforced concrete deep beam^{[11][5][17]} is analysed by the proposed material model. The beam is loaded by the two forces P (Fig. 10).

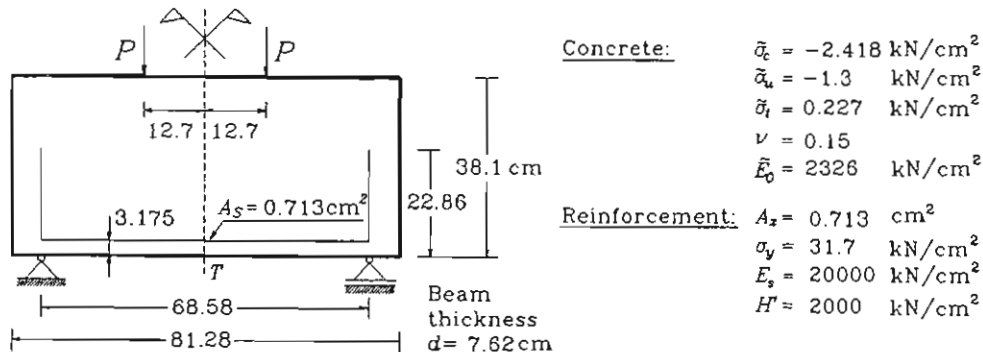


FIG. 10 Problem details of the deep beam.

Fig. 11 shows a finite element idealization for a half of the deep beam with its reinforcement.

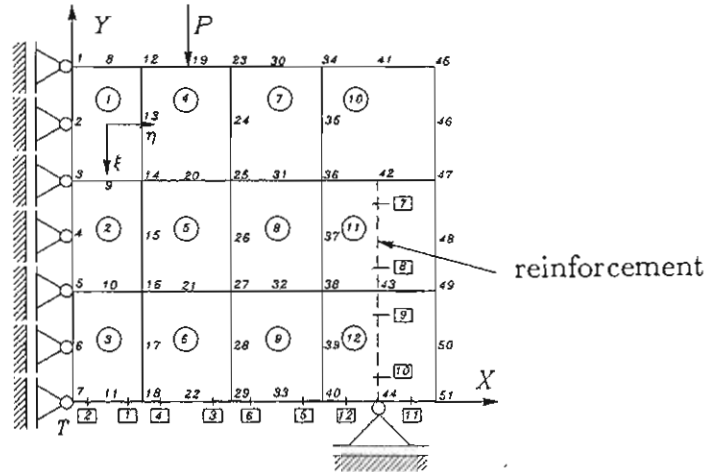


FIG. 11 Finite element for mesh for a half of the deep beam.

Fig. 12 shows the load-displacement curve for point T in the middle of the beam. The beam was considered experimentally by Ramakrishnan and Ananthunarayana^[13]. Our results for various ways of modelling tension softening are compared to the numerical^[11] and experimental result^[13].

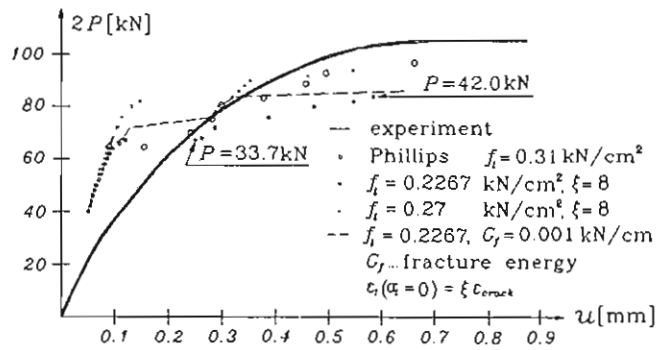


FIG. 12 Load-displacement curve for point T .

One possible cause of the discrepancy between numerical and experimental results at the beginning of loading is due to an unreliable measurement of very small displacements.

Fig. 13 shows cracks distribution for two different values of loading force P . For the force of 33.7 kN there are only cracks in the middle of the beam. A numerical collapse of the deep beam is observed under the loadings of $P = 42.0$ kN, when a large cracking zone is seen. In this analysis 22 load steps were used and on average 5 equilibrium iterations were needed.

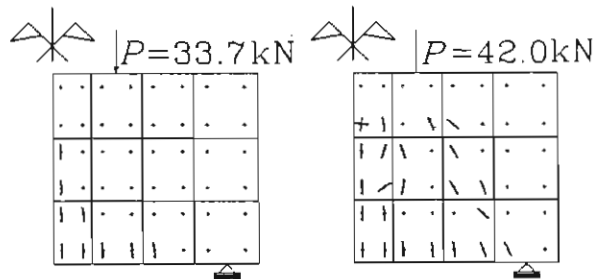


FIG. 13 Cracks distribution for two different loadings.

5.0 Conclusions

The presented model is relatively simple and models all dominant effects of the nonlinear concrete behaviour. Only the uniaxial experimental data are needed to define the concrete

material model which also includes post-cracking and crushing behaviour. The stress-strain law has been set-up with reference to the fixed axes of orthotropy in the directions defined by the crack. The biaxial stress-strain law is based on the failure envelope, which is constructed for loading in principal stress directions.

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Elastic Analysis of Flexibly Jointed Laminated Timber Plates

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ABSTRACT

A mechanical model is developed which permits the analysis of arbitrarily layered plate-like timber-composites taking into account the flexibility of the joint interfaces. This is an extension of the classical laminate theory allowing for relative displacements of the bounding surfaces of the individual layers. The effectiveness of the adopted approach is demonstrated on two simple examples.

1 INTRODUCTION

Plate-like timber structures consist of an arbitrary but usually small number of board layers across the thickness and characteristically have a variable orientation of the board angle of neighbouring layers. These board layers may be rigidly glued together or flexibly connected by mechanical fasteners such as nails or screws (Fig. 1). The application of connection techniques permits the construction of large-span roof structures, walls, ceilings and even bridge decks. The use of the mechanical connection technique in particular is comparatively simple and economic and is therefore of interest also for small timber building companies.

The flexible joint of timber beams has been treated by many researchers during the last three decades^{5,7}. The mechanical behaviour of the one-dimensional joint interface of beams has been studied by Kneidl (1991)¹ and Norlin (1993)⁶, taking into account the nonlinear constitutive behaviour of the fasteners.

For two-dimensional multilaminate timber structures the specific nature of the joint interface is important. Experimental as well as theoretical and numerical work using the FE method has been carried out mainly on laminated veneer timber plates where the layers were rigidly glued together. The elastic behaviour of such an anisotropic plate is very similar to a plate made of fibre-reinforced-plastics and therefore essentially requires an analogous theoretical treatment^{e.g. 2,4}.

The flexibility of the joint interface may play a dominant role when the timber laminate is made of board layers which are connected by mechanical fasteners. This effect seems to have not been investigated systematically up to now. In consequence, it is the aim of the present work to build a mechanical model which takes into account this flexibility of the joint interface of arbitrarily laminated timber structures⁸. The constitutive models of the board layers and of the joint interfaces are restricted to anisotropic elasticity. Furthermore, the present investigation is restricted to rectangular plates for simplicity but may be extended in a straightforward manner also to shell-like structures.

2 SETUP OF LAMINATED TIMBER STRUCTURES

2.1 Lamina Orientations and Lamina Behaviour

We consider a laminated timber plate element made of several board layers of varying orientation (Fig. 1). The boards of each layer are arranged in parallel, in contact with each other, but are not connected otherwise. Consequently, the effective in-plane shear stiffness is very small. The lateral membrane stiffness of the boards is an order of magnitude smaller than in longitudinal direction due to the natural growth conditions of trees. These conditions result in an orthogonally anisotropic material behaviour of each board layer which is treated accordingly³. Thereby the in-plane contact of the neighbouring boards is not modelled directly but is taken into account in an average sense by applying reduced elastic moduli.

2.2 Flexible Connection of Layers

The exclusive use of mechanical fasteners opposite to rigid glued connections is assumed for the through-the-thickness connection of the individual board layers. The characteristic feature of this connection technique is its flexibility mainly concerning relative shear between the pairs of contact surfaces of the board layers (Fig. 1). The joint flexibility is usually different for shear action parallel and orthogonal to the board orientation because of the anisotropic behaviour of the timber itself. Since the board orientations are generally different at a joint interface and also the board thicknesses may vary the effective total joint stiffness results from an interplay of the partial joint stiffnesses of the contributing layers.

The flexible joint of the plate laminas results in a discontinuous distribution of in-plane deformations and in a stepped piecewise distribution of transversal deformations (Fig. 2c). In contrast, the classical laminate theory assumes uniform transverse shear deformation across the thickness (Fig. 2a) or allows in an extended version stepped variation of the shear angle of the layers, but no relative displacements at the joint interfaces (Fig. 2b).

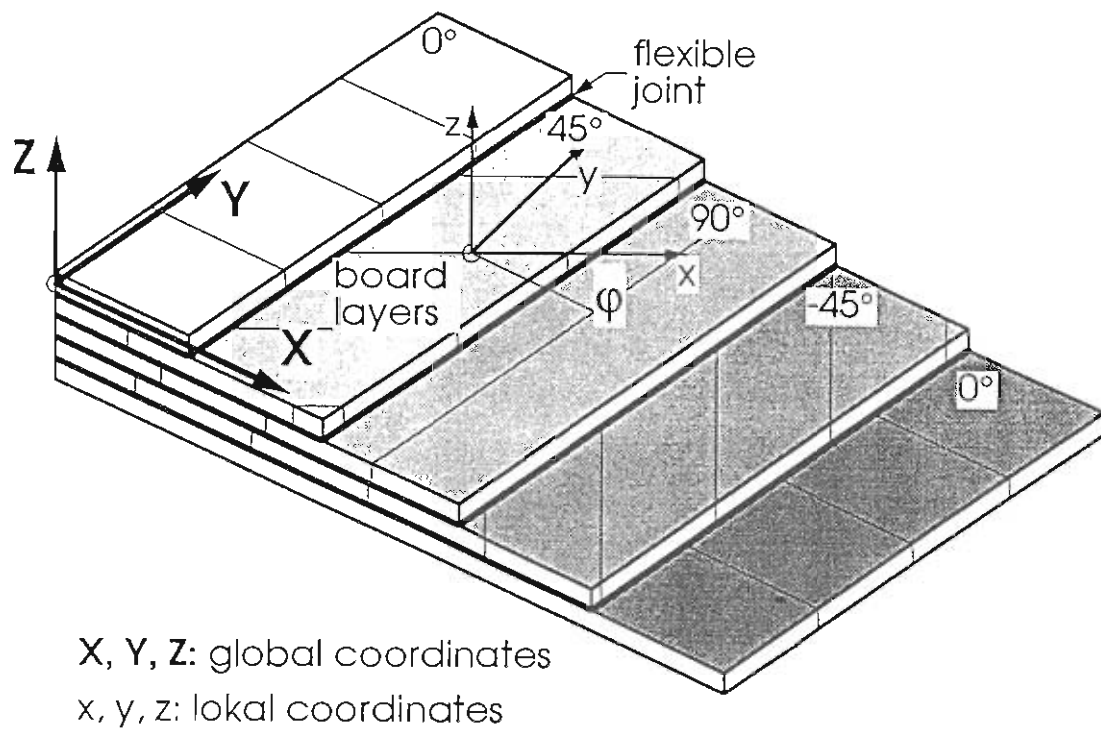


Fig. 1. Setup of a laminated timber plate with star-shaped board (=fibre) directions

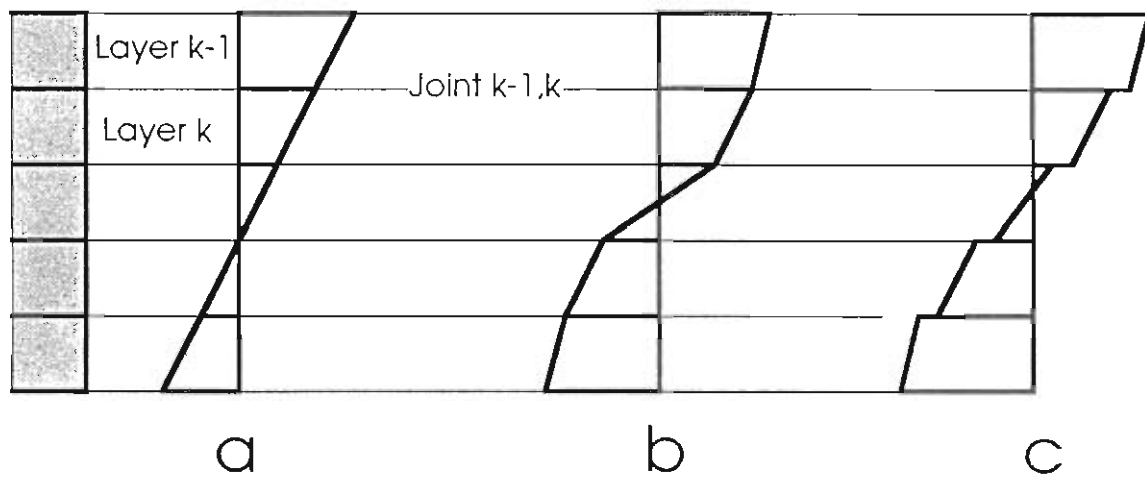


Fig. 2. Displacement assumptions across the thickness of a board lamina
 a. shear-flexible composite, b. shear-flexible layers, c. flexibly jointed layers

3 MODELLING OF PLATE LAMINAS

In the following, the individual board layers are modelled independently as a stack of disconnected homogeneous anisotropic shear-elastic (Reissner-Mindlin type) plates. The membrane contribution is simultaneously modelled because unavoidable coupling with bending arises after the layers are joined together. The according differential equations are presented because of reasons of completeness, although they can be found in many places^{e.g. 2,3}.

3.1 Kinematics of the Plate Laminas

The in-plane displacement components vary linearly across the thickness. Thickness stretch is not taken into account but is neglected. Substituting this displacement assumptions into the cartesian 3D strain-displacement relations we arrive at the generalized strain vector of the plate lamina (Eq. 1).

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & f & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & f \end{bmatrix} \cdot \begin{bmatrix} U_0 \\ V_0 \\ W_0 \\ \beta_y \\ \beta_x \end{bmatrix} = \mathbf{B} \cdot \mathbf{U} \quad \text{Eq. 1}$$

3.2 Equilibrium of the Plate Laminas

$$\hat{\mathbf{B}}^T \cdot \mathbf{S} + \mathbf{p} = \mathbf{0} \quad \text{with} \quad \mathbf{p}^T = [p_x p_y p_z m_y m_x] \quad \text{Eq. 2}$$

In an analogous manner starting from the cartesian 3D equilibrium equations (stress-load relations), after integrating over the thickness, we arrive at the equilibrium equations of the plate expressed in membrane, bending and shear section forces (Eq. 2). The according operator matrix $\hat{\mathbf{B}}$ arises from the kinematic operator $\mathbf{B}$ (Eq. 1) by substituting there $f=-1$ instead of $f=1$.

3.3 Anisotropic–Elastic Constitutive Behaviour of the Plate Laminas

$$\sigma_L = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}_L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}_L \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}_L = \mathbf{Q}_L \cdot \varepsilon_L \quad \text{Eq. 3}$$

Each lamina is orthogonally anisotropic–elastic and requires 6 independent effective material constants, referred to the local (L) material principal axes of the board layer, when the condition of zero thickness strain is enforced (Eq. 3).

$$\mathbf{S} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_y \\ M_x \\ M_{xy} \\ Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{26} & 0 & 0 & 0 & 0 & 0 \\ A_{61} & A_{62} & A_{66} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} & 0 & 0 \\ 0 & 0 & 0 & D_{21} & D_{22} & D_{26} & 0 & 0 \\ 0 & 0 & 0 & D_{61} & D_{62} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_{55} & E_{54} \\ 0 & 0 & 0 & 0 & 0 & 0 & E_{45} & E_{44} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \mathbf{Q}_G^k \cdot \varepsilon \quad \text{Eq. 4}$$

After integrating over the thickness and appropriate coordinate transformation we arrive at the global generalized force–generalized strain matrix of a plate lamina (Eq. 4). We notice that membrane, bending and transverse shear effects are uncoupled due to the assumed material homogeneity of the board layers themselves.

3.4 Resulting Differential Equation of the Individual Plate Laminas

$$\underbrace{(\hat{\mathbf{B}}^T \cdot \mathbf{Q}_G^k \cdot \mathbf{B})}_{\mathbf{K}^k} \cdot \mathbf{U}^k + \mathbf{p}^k = 0$$

Eq. 5

Combining the results of the previous subsections yields the finally needed equi-

librium equations of the plate lamina k expressed in midsurface displacement components, with the 5 by 5 symmetric stiffness operator matrix $\mathbf{K}$ (Eq. 5).

4 MODELLING OF THE FLEXIBLE JOINT

To our knowledge experimental information is available only concerning the flexible joint of parallel board layers but none concerning relative shear between board layers crossing under oblique angles. In consequence, it is necessary to build a consistent theoretical model which takes these effects into account at least approximately. Thereby the flexibility constants known from experiments related to parallel board layers should be incorporated, including the differences of relative shear in longitudinal and orthogonal direction of the boards. The main idea is to split the total flexibility of the joint interface into contributions resulting from the upper board layer and into contributions resulting from the lower of each two neighbouring board layers and taking the arithmetic mean value. These flexibility contributions may be determined on their own, as if the layers were parallel, dependent on the individual thickness, angle of orientation, material constants of the board layer and the properties of the chosen mechanical fasteners.

4.1 Kinematics of the Flexible Joint – Interface

$$\Delta \mathbf{u} = \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix}^{k-1, k} = \begin{bmatrix} 1 & 0 & 0 & -\frac{h^{k-1}}{2} & 0 & -1 & 0 & 0 & -\frac{h^k}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{h^{k-1}}{2} & 0 & -1 & 0 & 0 & -\frac{h^k}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u^{k-1} \\ v^{k-1} \\ w^{k-1} \\ \beta_y^{k-1} \\ \beta_x^{k-1} \\ u^k \\ v^k \\ w^k \\ \beta_y^k \\ \beta_x^k \end{bmatrix} = \mathbf{T} \cdot \mathbf{U}^{k-1, k} \quad \text{Eq. 6}$$

The relative displacements $\Delta \mathbf{u}$ at the joint interface are related to the primary plate midsurface deformation parameters of the neighbouring board layers (Eq. 6). The excentric position of the joint interface relative to the plate midsurfaces is taken into account. The joint interface in Fig. 3 is symbolically drawn as a dark shaded solid brick for reasons of clearness only. In reality it has of course no volume because the outer board surfaces are initially in direct contact.

4.2 Equilibrium of the Flexible Joint – Interface

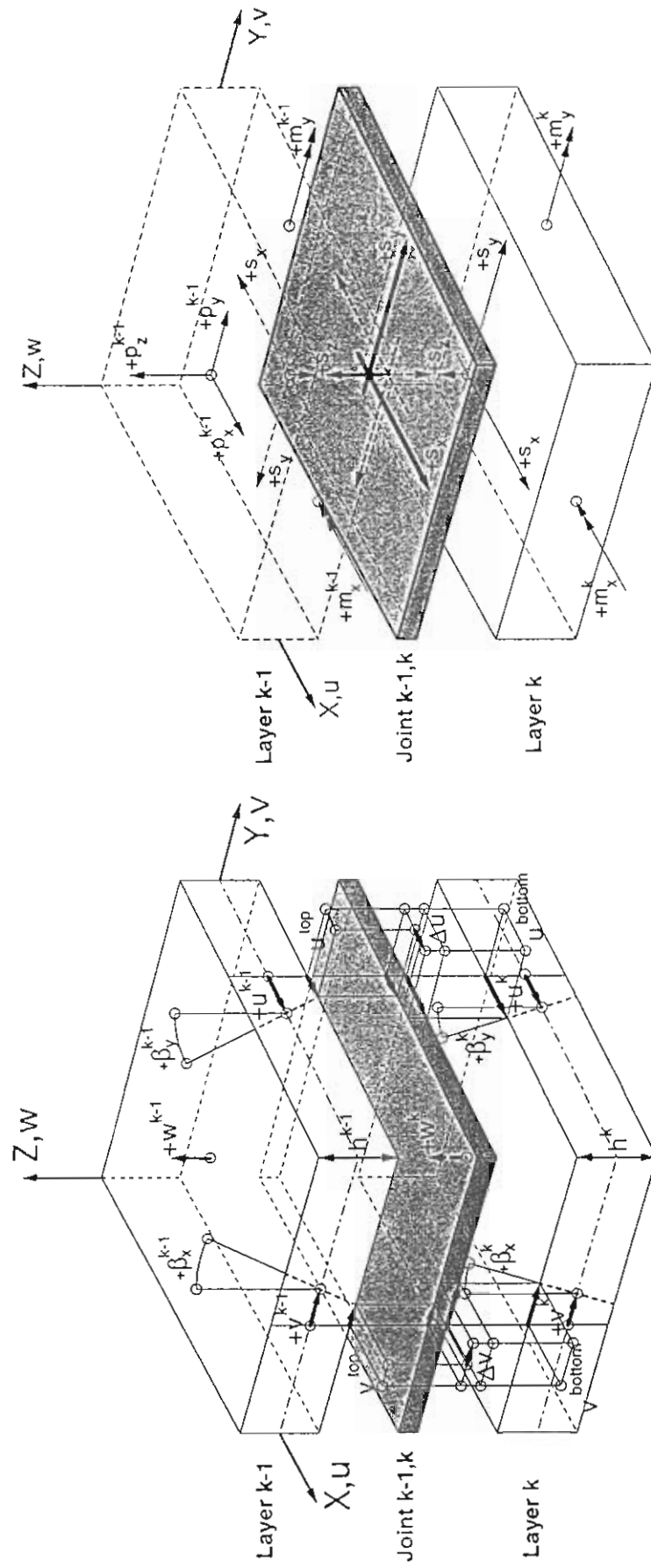


Fig. 3. Relative displacements at a flexible joint connecting two board layers k-1 and k

Fig. 4. Interface forces at a flexible joint connecting two board layers k-1 and k

The shear forces at the joint interface are transformed into plate midsurface loadings, again taking the excentricities with respect to the plate midsurfaces accurately into account. This results in area-related elastic tangential and normal force loadings and moment loadings of the neighbouring plate midsurfaces (Eq. 7 and Fig. 4).

$$\mathbf{T}^T \cdot \mathbf{s} = \mathbf{p}^{k-1, k} \quad \text{Eq. 7}$$

4.3 Anisotropic-Elastic Constitutive Behaviour of the Flexible Joint - Interface

$$\mathbf{s}_L = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}_L = \begin{bmatrix} c_{\parallel} & 0 & 0 \\ 0 & c_{\perp} & 0 \\ 0 & 0 & c_w \end{bmatrix}_L \cdot \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix}_L = \mathbf{c}_L \cdot \Delta \mathbf{u}_L \quad \text{Eq. 8}$$

The total flexibility of the joint interface is defined as the arithmetic mean value of the partial flexibilities relating to each of the two connected board layers. The uncoupled local stiffness behaviour corresponds to Eq. 8 above, relating shear and normal contact stresses to relative displacements at the joint interface. Each of these local stiffness matrices is transformed into global coordinates using the individual board angles (Eq. 9) and result in partial stiffness contributions (Eqs. 10).

$$\mathbf{t} = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Eq. 9}$$

$$\left. \begin{aligned} \Delta \mathbf{u}^{k-1} &= 0.5 \mathbf{c}^{-(k-1)} \cdot \mathbf{s}^{k-1} = 0.5 (\mathbf{t}^T \cdot \mathbf{c}_L \cdot \mathbf{t})^{-(k-1)} \cdot \mathbf{s}^{k-1} \\ \Delta \mathbf{u}^k &= 0.5 \mathbf{c}^{-k} \cdot \mathbf{s}^k = 0.5 (\mathbf{t}^T \cdot \mathbf{c}_L \cdot \mathbf{t})^{-k} \cdot \mathbf{s}^k \end{aligned} \right\} \quad \text{Eq. 10}$$

Equilibrium at the joint interface and the assumption of adding-up the partial flexibilities (serial spring model, Eq. 11) finally yields the total elastic constitutive matrix $\mathbf{C}$ of the flexible joint (Eq. 12).

$$\mathbf{s}^{k-1} = \mathbf{s}^k = \mathbf{s} \quad \text{and} \quad \Delta \mathbf{u} = \Delta \mathbf{u}^{k-1} + \Delta \mathbf{u}^k \quad \text{Eq. 11}$$

$$\mathbf{s} = \underbrace{2 (\mathbf{c}^{-(k-1)} + \mathbf{c}^{-k})^{-1}}_{\mathbf{C}} \cdot \Delta \mathbf{u} \quad \text{Eq. 12}$$

4.4 Stiffness of the Flexible Joint – Interface

Combining the joint stiffness matrix $\mathbf{C}$ derived above with the kinematic and equilibrium relations Eq. 6 and 7 finally yields the 10 by 10 symmetric joint stiffness matrix $\mathbf{k}$. Equivalent internal elastic plate midsurface loadings are a function of the primary plate midsurface displacement parameters and the geometry and material parameters of the joint interface hidden in the matrices $\mathbf{T}$ and $\mathbf{C}$ (Eq. 13).

$$\underbrace{(\mathbf{T}^T \cdot \mathbf{C} \cdot \mathbf{T})}_{\mathbf{k}} \cdot \mathbf{U}^{k-1, k} = \mathbf{p}^{k-1, k} \quad \text{Eq. 13}$$

5 COUPLED SYSTEM OF FLEXIBLY JOINTED LAMINATED PLATES

Combining the stiffness contributions of the board layers (Eq. 5) and those of the joint interfaces (Eq. 13) it becomes possible to describe the behaviour of the flexibly jointed layered board-composite. Fig. 5 shows symbolically the assemblage of stiffness contributions for a flexibly jointed 5-layer board-composite. The load vector of each plate lamina consists of two parts, firstly, the usual external load components of the board layers and secondly, the internal load components resulting from the elastic shear resistance of the joint interfaces. The overall stiffness matrix of the board-composite is obtained by successively adding the 10 by 10 joint stiffness matrices $\mathbf{k}$ to the 5 by 5 plate stiffness matrices $\mathbf{K}$ of those neighbouring board layers which are connected by that joint. The coupling effect of the joint matrices ultimately results in the two-dimensional flexibly jointed board-composite. The same assemblage process of the contributing stiffnesses is further denoted in Eq. 14 below.

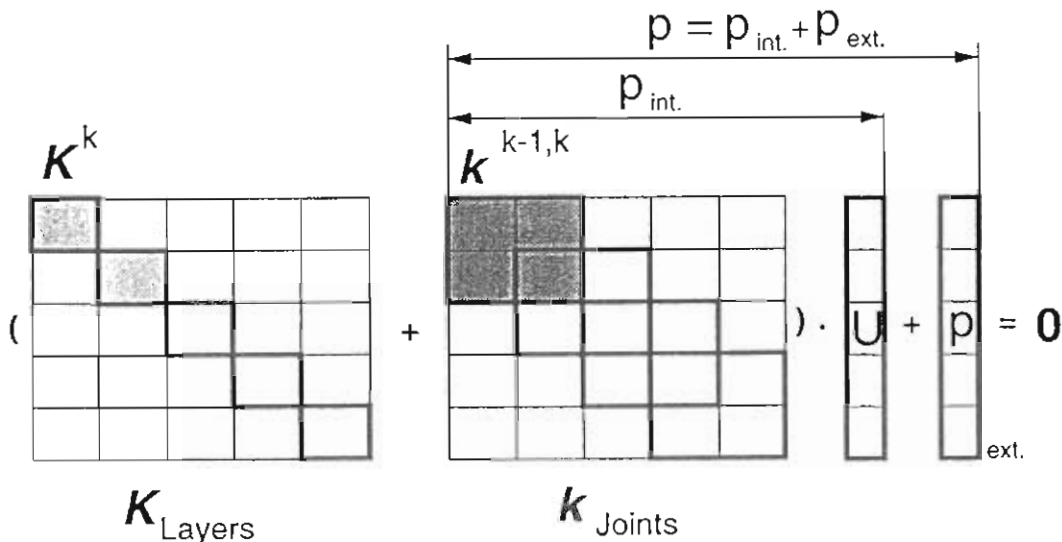


Fig. 5. Superposition of stiffness matrices of the coupled system

$$K_{Composite} = \underbrace{\sum_1^k K_{Layer}}_{K_{Layers}} + \underbrace{\sum_{1,2}^{k-1,k} k_{Joint}}_{k_{Joints}} \quad \text{Eq. 14}$$

6 NUMERICAL SOLUTION AND EXAMPLE PROBLEM

The resulting system of differential equations of the layered board-composite is directly discretized using trigonometric double series expansions of all external and internal system variables. A small computer program was built which permits easy analysis of arbitrarily layered rectangular board-composite plates.

The capability of the adopted approach is demonstrated by two simple examples. A simply supported square plate under uniform vertical loading consists of two board layers which are flexibly connected. In the first case the orientation of the board layers is diagonal ($45^\circ/-45^\circ$), in the second case the board layers are parallel to each other ($0^\circ/0^\circ$) and essentially correspond to the one-dimensional case of flexibly connected beams (Fig. 6).

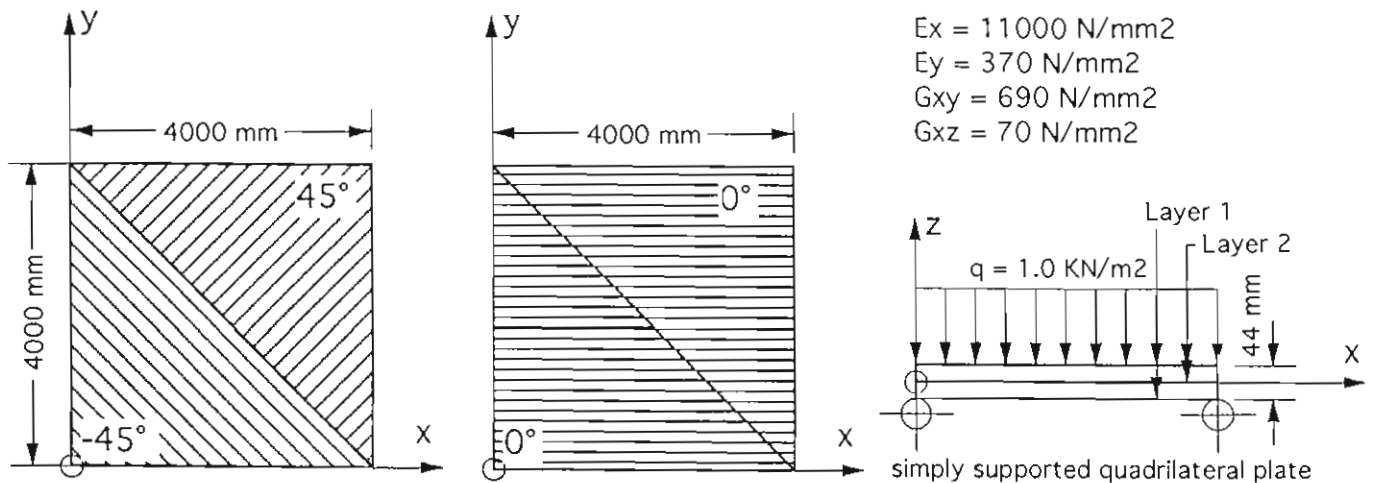


Fig. 6. Quadrilateral plate consisting of two flexibly connected board layers
a. diagonal layers, b. parallel layers

The joint stiffness is varied and the maximum displacement at the plate center is recorded as an indicator of the effectivity of the joint on the load bearing behaviour of the plate (Fig. 7). For both cases the results show characteristic common features. The joint behaves rigid for a wide range of stiffness values (left part of Fig. 7). However, as soon as the joint stiffness falls below a certain value the joint behaves practically completely weak (right part of Fig. 7). In the second case of

parallel layers this means, that the center displacement increases by a factor of 4, making the joining of layers highly effective. In contrast, the first case of diagonal layers shows only a small difference of about 40% in the maximum displacements for rigid versus weak joints. This is mainly due to the mutual supporting effect of each layer in favour of the other in the corners of the square plate. This special arrangement makes the board-composite very stiff in advance with only little activation of elastic shear forces at the joint interface. This example shows that support conditions in interaction with orientations of the board angles have a striking effect on the behaviour of the joint interface. The limiting cases of completely rigid and completely weak interfaces were checked by performing computations with ABAQUS⁹ and show practically full agreement of results.

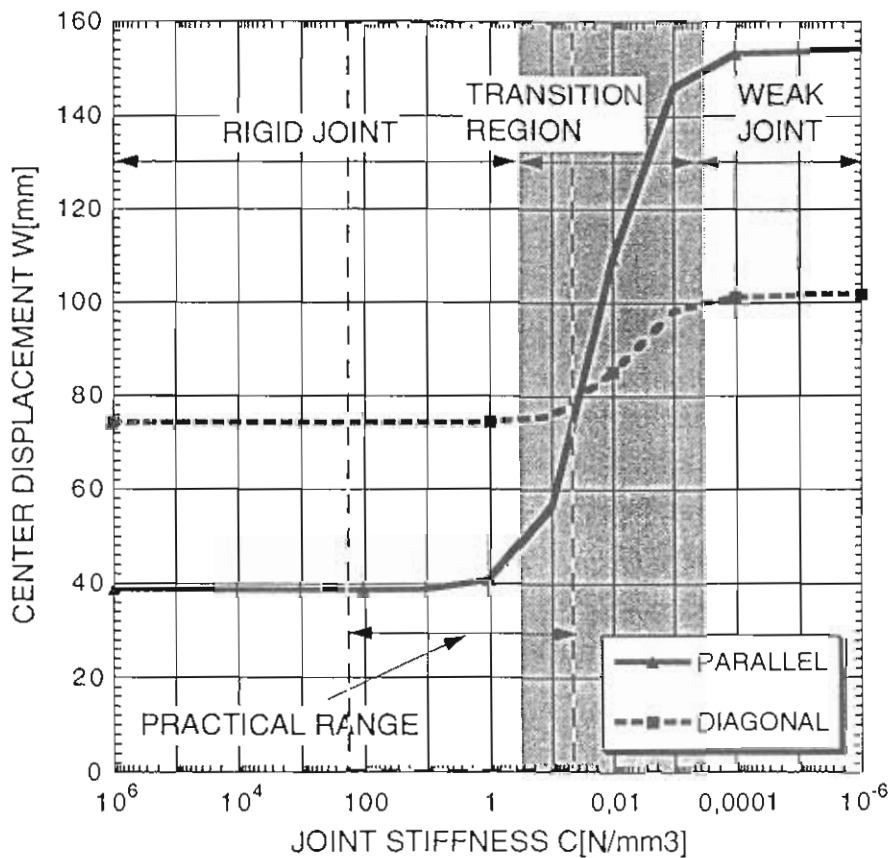


Fig. 7. Center displacements as a function of the joint stiffness

7 CONCLUSIONS

The presented model of a two-dimensional timber structure with flexible joint interfaces is a contribution to support increased future use of two-dimensional timber structural elements from the theoretical side. To be able to carry out more realistic limit load analyses it would be necessary to take into account the non-linear material behaviour of the board layers and of the flexible joints in special.

8 APPENDIX — NOTATION

L, G	Subscripts for local and global coordinates
$k-1, k$	Superscripts of plate laminas (board layers)
h	thickness of an individual plate lamina
φ	angle of orientation of boards in a plate lamina
U^k	generalized midsurface-displacement vector of a plate lamina
$B, \hat{B}$	differential operator matrices of plate laminas
S, ε	generalized internal force and strain vectors of a plate lamina
p	total load vector of a plate lamina
σ_L, ε_L	local 3D stress and strain vectors
Q_L	local 3D orthogonally anisotropic elastic constitutive matrix
Q_G^k	global generalized force-generalized strain matrix of a plate lamina
A, D, E	generalized elastic constants for membrane, bending and transverse shear
K^k	(differential operator) stiffness matrix of a plate lamina
Δu	relative displacement vector at joint interface of two plate laminas
$U^{k-1, k}$	collection of generalized displacement vectors of two plate laminas
s	vector of interface forces between two plate laminas
$p^{k-1, k}$	generalized load vector due to elastic forces at a joint interface
c	partial elastic stiffness matrix of a flexible joint with parallel board layers
C	total elastic stiffness matrix of a flexible joint
T	transformation matrix taking into account eccentricities of joint interface
t	local-global coordinate transformation matrix of joint constitutive matrix
k	global joint stiffness matrix

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Boris Vidak, Pavao Marović

Some parametric analyses of contact elements

SUMMARY

This paper presents some results of performed parametric analyses of the previously developed contact elements which are used to discretize the space between two mechanically interacting bodies or adjoining and/or disjoining non miscible media.

The following characteristics and properties have been varied in the performed parametric analyses: the type and the rule of numerical integration, the influence of Poisson's ratio, the thickness of the contact zone, the influence of the size of non-linearity of the contact line and the number of elements in contact.

The presented parametric analyses have been performed on simple and small sample problems, in which a large number of parameters can be varied economically.

1. INTRODUCTION

Contact elements discretize the space between two mechanically interacting bodies or adjoining and/or disjoining non miscible media.

The joint concept has been cast into several different types of elements, under different names: contact element, interface element, joint element, gap element, point element, boundary slip element, thin-layer element, thin element, etc. - thus giving an idea of some confusion in the engineering community. A good attempt to systematize all contact elements, so far known, is shown in Ref. [1, 2].

The idea of performing parametric analyses on our previously developed contact - interface element [3, 4] arises from the main differences between different contact elements and difficulties in their use.

In the performed parametric analyses we varied the next characteristics and properties: (a) the type of numerical integration - Gauss or Lobatto; (b) the rule of used numerical integration - 2×2 or 3×3 ; (c) the influence of Poisson's ratio - 0.0 and 0.15; (d) the thickness of the contact element; (e) the influence of the size of non-linearity of the contact line - straight or non-straight with some deviations; (f) the number of elements in contact.

The presented parametric analyses have been performed on simple and small sample problems in which a large number of parameters can be varied economically and some of the obtained results and comparisons will be presented in the paper.

2. ANALYZED CONTACT ELEMENTS

In order to make the paper clearer, the main characteristics of the used contact element will be briefly given in this Section.

The formulation of the contact elements is evaluated from the relative node displacements of the finite elements among which it is incorporated [5].

According to the isoparametric formulation, the geometry of the contact element can be written only as a coordinate ξ function (ξ is a coordinate in the direction of the contact element) as follows:

$$x(\xi) = N_i(\xi)x_i, \quad y(\xi) = N_i(\xi)y_i, \quad i = 1, 2, 3 \quad (1)$$

where $N_i(\xi)$ are interpolation functions and the pairs (x_i, y_i) are coordinates of the nodes.

Following the standard procedure, deformations in the local coordinate system are expressed as a product of the deformation matrix **B** and displacement vector **a**:

$$\varepsilon' = \mathbf{B}\mathbf{a} \quad (2)$$

where the matrix **B** is evaluated as a quotient of the matrix of interpolation functions **N** and the width of the contact element w .

The stress-deformation relation for linear elastic behaviour has a form:

$$\sigma = \begin{Bmatrix} \tau_s' \\ \sigma_n' \end{Bmatrix} = \begin{bmatrix} G_s & 0 \\ 0 & E_n \end{bmatrix} \begin{Bmatrix} \gamma_s' \\ \varepsilon_n' \end{Bmatrix} = \mathbf{D}\varepsilon \quad (3)$$

where **D** is a matrix of material characteristics of the contact element, with E_n modulus of elasticity and G_s shear modulus.

The stiffness matrix of the contact element is evaluated by a standard procedure from one of the equilibrium conditions, and generally has a form:

$$\mathbf{K} = \int_{\epsilon} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \quad (4)$$

Introducing into Eq. (4), matrix **B** and matrix **D**, which is defined in local coordinate system according to Eq. (3), matrix **K** at last yields:

$$\mathbf{K} = w \int_{-1}^1 \mathbf{B}^T \mathbf{D} \mathbf{B} \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \right]^{1/2} d\xi \quad (5)$$

The implementation of the adopted contact element in the static two-dimensional analysis is described in Refs. [3, 4] and will not be repeated here.

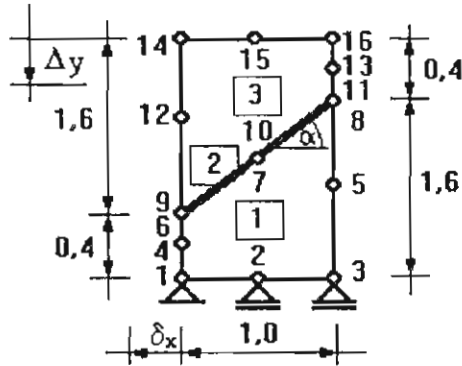
3. PARAMETRIC ANALYSES

The parametric analyses were performed varying the next characteristics and properties: (a) the type of numerical integration - Gauss or Lobatto integration rule; (b) the rule of the used numerical integration - 2 x 2 or 3 x 3 integration rule; (c) the influence of Poisson's ratio - 0.0 or 0.15; (d) the thickness of the contact element, w ; (e) the influence of the size of non-linearity of the contact line - straight or non-straight with some different deviations; (f) the number of elements in the contact zone.

These analyses have been performed on a few very simple and small sample problems, because a large number of parameters can be varied economically on them.

3.1 EXAMPLE No. 1

This problem has been taken from Ref. [1], and the geometry of the problem together with the element characteristics is shown in Figure 1. It represents two finite elements connected at an angle. The upper element is subjected to prescribed vertical displacement.



Basic element

$E = 10^5 \text{ MPa}$
 $\nu = 0.0 (0.15)$
 $t = 1.0$
 $\rho = 0.0$
 $k_r = 0.0$
 $H = 2.0 \text{ m}$
 $\Delta_Y = -0.2 \text{ m}$

Contact element

$\alpha = 50^\circ$
 $w = 0.06 \text{ m}$
 $K_s = 0.5 \cdot 10^4 \text{ MN/m}^3$
 $K_n = 10^4 (10^8) \text{ MN/m}^3$
 $G = K_s w = 300 \text{ MPa}$
 $E_1 = K_n w = 600 \text{ MPa}$
 $E_2 = K_n w = 6 \cdot 10^6 \text{ MPa}$
 $\sigma = 1000 \text{ MPa}$
 $\tau = 1300 \text{ MPa}$

Fig. 1 Geometry and the material characteristics for Example No. 1

The analytical results are obtained due to the formulae taken from [1]:

Compression force:

$$F_y = \frac{\Delta_y}{\frac{H}{E_c} + \left(\frac{\sin^2 \alpha}{K_s} + \frac{\cos^2 \alpha}{K_n} \right) \cos \alpha} \quad (6)$$

Normal and tangential forces:

$$\begin{pmatrix} \tau_s \\ \sigma_n \end{pmatrix} = F_y \cos \alpha \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \quad (7)$$

Displacement:

$$\delta_x = \begin{pmatrix} \tau_s / K_s \\ \sigma_n / K_n \end{pmatrix}^T \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \quad (8)$$

This problem was subjected to the previously mentioned analyses, varying the type of numerical integration (Gauss or Lobatto) and the Poisson's ratio (ν , ranging from 0.0 to 0.15). The results, horizontal displacement of node 9, δ_x , and normal and tangential stresses in the contact element, obtained using 3×3 integration rules are shown in Tables 1 and 2.

Table 1 Numerical analysis using Gauss integration rule (3×3)

ν	w_y	$K_n = 10^4$			$K_n = 10^8$		
		δ_x	τ_s	σ_n	δ_x	τ_s	σ_n
0,15	10^{-1}	-0,05347	-806,50	-677,50	-0,13121	-1030,10	-903,90
	10^{-2}	-0,05187	-807,70	-673,40	-0,13173	-1029,40	-860,90
	10^{-3}	-0,05171	-807,60	-673,00	-0,13178	-1029,40	-857,50
	10^{-4}	-0,05170	-807,60	-673,00	-0,13183	-1029,40	-857,30
	10^{-5}	-0,05169	-807,60	-673,00	-0,13179	-1029,40	-857,00
0,00	10^{-1}	-0,05346	-806,50	-677,30	-0,13117	-1030,10	-906,30
	10^{-2}	-0,05187	-807,70	-673,40	-0,13171	-1029,30	-863,50
	10^{-3}	-0,05171	-807,60	-673,00	-0,13173	-1029,30	-858,70
	10^{-4}	-0,05170	-807,60	-673,00	-0,13182	-1029,30	-858,00
	10^{-5}	-0,05169	-807,60	-673,00	-0,13177	-1029,30	-857,90
exact [1]		-0,05181	-807,23	-677,34	-0,13260	-1031,55	-865,57

Table 2 Numerical analysis using Lobatto integration rule (3 x 3)

ν	w_y	$K_n = 10^4$			$K_n = 10^8$		
		δ_x	τ_s	σ_n	δ_x	τ_s	σ_n
0,15	10^{-1}	-0,05334	-808,60	-676,90	-0,13116	-1030,40	-883,70
	10^{-2}	-0,05186	-807,70	-673,40	-0,13173	-1029,50	-859,00
	10^{-3}	-0,05171	-807,60	-673,00	-0,13178	-1029,40	-857,00
	10^{-4}	-0,05169	-807,60	-673,00	-0,13176	-1029,10	-858,10
	10^{-5}	-0,05170	-807,60	-672,90	-0,13179	-1029,40	-856,00
0,00	10^{-1}	-0,05332	-808,60	-676,80	-0,13116	-1030,40	-884,80
	10^{-2}	-0,05186	-807,70	-673,40	-0,13173	-1029,50	-859,00
	10^{-3}	-0,05171	-807,60	-673,00	-0,13178	-1029,40	-856,50
	10^{-4}	-0,05169	-807,60	-673,00	-0,13176	-1029,40	-859,10
	10^{-5}	-0,05169	-807,60	-673,00	-0,13179	-1029,40	-855,70
exact [1]		-0,05181	-807,23	-677,34	-0,13260	-1031,55	-865,57

3.2 EXAMPLE No. 2

This problem has also been taken from Ref. [1]. The geometry of the problem with two different meshes (the former with two basic elements without the contact element, and the latter with two basic and one contact element) together with the element characteristics are shown in Figure 2. The applied load is a uniformly distributed load over the upper element.

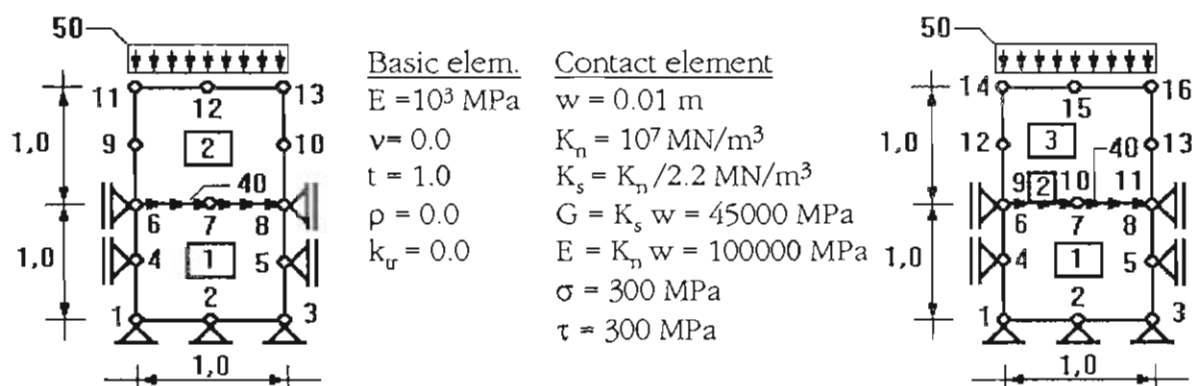


Fig. 2 Geometry and the material characteristics for Example No. 2

The obtained results, horizontal and vertical stresses in basic elements, for 2 x 2 Gauss and Lobatto integration rules for both cases, i.e. with and without the contact element, are shown in Table 3, and for 3 x 3 rules in Table 4.

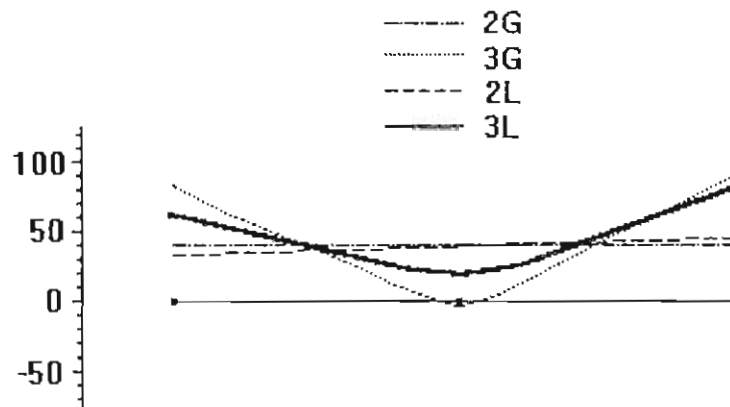
Table 3 Numerical analysis using 2 x 2 Gauss and Lobatto integration rule

Gauss (2x2)	only FE		with CE		Lobatto (2x2)	only FE		with CE	
	σ_x	σ_y	σ_x	σ_y		σ_x	σ_y	σ_x	σ_y
1	-12,00	-53,10	0,10	-44,70	1	-22,50	-150,00	-22,50	-150,00
2	-22,60	-50,70	18,20	-48,80	2	-22,50	-150,00	-22,50	-150,00
3	-3,00	-46,90	-15,10	-55,30	3	-22,50	-150,00	-22,50	-150,10
4	7,60	-49,30	-33,20	-51,20	4	-22,50	-150,00	-22,50	-150,10
5	-15,10	-50,00	0,00	-50,00	5	-22,50	-150,00	-22,50	-150,00
6	3,60	-50,00	0,00	-50,00	6	0,00	-150,00	0,00	-150,00
7	10,90	-50,00	0,00	-50,00	7	-22,50	-150,00	-22,50	-150,00
8	-3,30	-50,00	0,00	-50,00	8	0,00	-150,00	0,00	-150,00

Table 4 Numerical analysis using 3 x 3 Gauss and Lobatto integration rule

Gauss (3 x 3)	only FE		with CE		Lobatto (3 x 3)	only FE		with CE	
	σ_x	σ_y	σ_x	σ_y		σ_x	σ_y	σ_x	σ_y
1	-11,00	-54,60	-4,10	-45,90	1	-8,40	-56,10	-6,70	-44,90
2	-20,20	-52,80	5,20	-47,80	2	-23,10	-53,70	8,00	-47,40
3	-29,40	-50,90	14,50	-49,60	3	-37,70	-51,20	22,70	-49,90
4	-7,40	-49,60	-7,40	-49,60	4	-7,50	-49,70	-7,50	-49,70
5	-7,40	-49,60	-7,40	-49,60	5	-7,50	-49,70	-7,50	-49,70
6	-7,40	-49,60	-7,40	-49,60	6	-7,50	-49,70	-7,50	-49,70
7	-4,10	-45,90	-11,00	-54,60	7	-6,70	-45,00	-8,40	-56,10
8	5,10	-47,80	-20,30	-52,80	8	7,90	-47,40	-23,10	-53,70
9	14,30	-49,60	-29,50	-50,90	9	22,60	-49,90	-37,80	-51,20
10	-25,60	-50,00	15,50	-49,70	10	-37,60	-50,70	22,70	-49,50
11	-6,50	-49,50	6,60	-49,50	11	-9,40	-49,70	8,10	-49,70
12	7,70	-49,70	-7,20	-50,00	12	13,50	-49,50	-11,90	-50,70
13	-5,20	-50,60	-5,20	-50,50	13	-7,60	-50,40	-7,50	-50,30
14	-0,10	-50,20	-0,10	-50,20	14	-0,70	-49,90	-0,70	-49,90
15	0,10	-50,60	0,20	-50,50	15	0,80	-50,40	0,80	-50,30
16	15,40	-49,70	-25,70	-50,00	16	22,60	-49,50	-37,70	-50,70
17	6,60	-49,50	-6,50	-49,50	17	8,00	-49,70	-9,40	-49,70
18	-7,20	-50,00	7,70	-49,70	18	-11,90	-50,70	13,50	-49,50

The shear stress distribution in the contact element for both cases, with and without the contact element, for both Gauss and Lobatto and 2 x 2 and 3 x 3 integration rules, is shown in Figure 3.

**Fig. 3 Shear stress distributions for different types and rules of integration**

3.3 EXAMPLE No. 3

Example No. 3 is the same as Example No. 2, but the domain is divided into eight basic elements and four contact elements with three additional vertical lines. This problem has also been taken from Ref. [1]. The outer geometry of the problem, the element characteristics and the loading conditions are the same as in Figure 2. This example is the expansion of Example No. 2 which investigates the influence of the number of elements in contact. The shear stress distribution in the contact elements for both Gauss and Lobatto and 2 x 2 and 3 x 3 integration rules, is shown in Figure 4. This influence can be seen by comparing Figures 3 and 4 where shear stress distributions in contact for different types and rules of integration are shown.

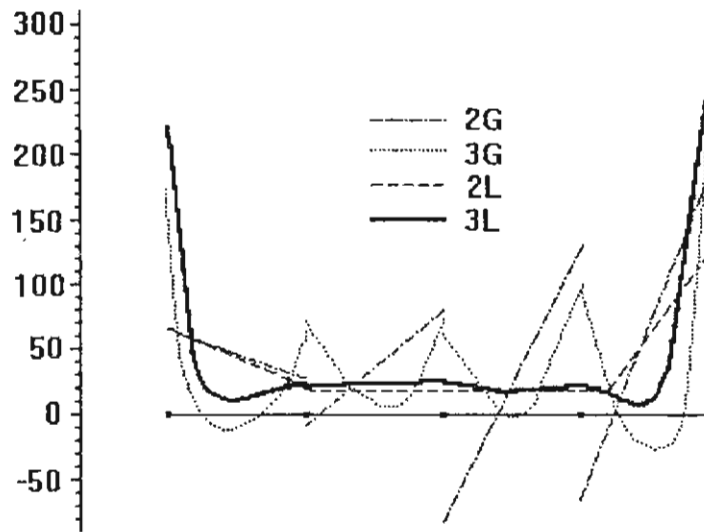


Fig. 4 Shear stress distributions for different types and rules of integration

3.4 EXAMPLE No. 4

Example No. 4 is basically taken from Refs. [3, 4], but here the influence of non-linearity of the contact line is investigated. The principal discretization and problem characteristics are shown in Figure 5.

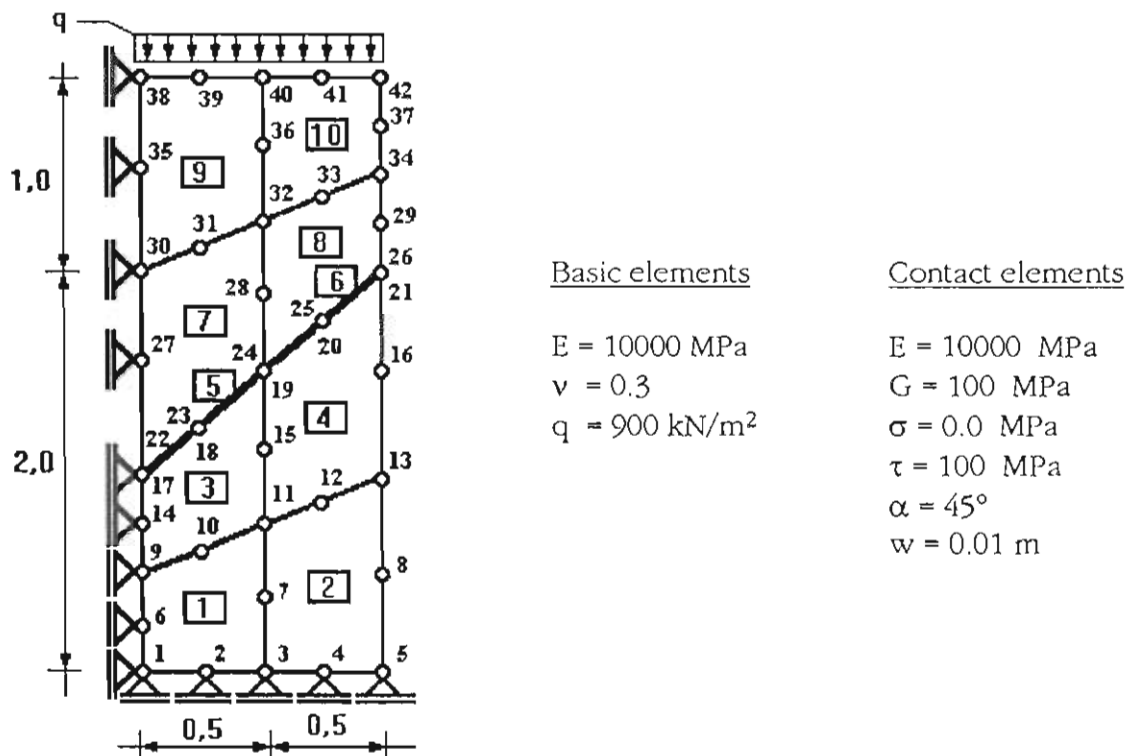


Fig. 5 Geometry and the material characteristics for Example No. 4

In the first case the contact line was ideally straight. In the second case, nodes 21 and 26 have been raised for 10 cm from the straight line while the pair of nodes 17, 22 and 19, 24 remained at the same place as in the first case. In the third case a pair of nodes 19, 24 has been lowered for 1 cm, and a pair of nodes 21, 26 has been raised for 1 cm; both from the straight line as in the first case.

The obtained normal and tangential stresses in the contact elements are shown in Table 5. The significant influence of non-linearity of the contact line can be seen from there.

Table 5 Normal and tangential stresses due to the influence of non-linearity of contact line

Gauss points	case (a)		case (b)		case (c)	
	τ (MPa)	σ (MPa)	τ (MPa)	σ (MPa)	τ (MPa)	σ (MPa)
1	-22,70	-2269,70	-16,50	-2064,40	-22,90	-2338,70
2	-100,00	-796,80	-100,00	-837,20	-100,00	-816,40
3	-100,00	-553,80	-100,00	-2249,30	-100,00	-1300,40
4	-100,00	-553,80	0,00	0,00	0,00	0,00
5	-100,00	-427,40	-100,00	-135,00	-100,00	-374,30
6	-100,00	-58,90	-100,00	-231,90	-100,00	-74,50

4. CONCLUSIONS

This paper presents the performed parametric analyses where: (a) the type of numerical integration - Gauss or Lobatto; (b) the rule of used numerical integration - 2×2 or 3×3 ; (c) the influence of Poisson's ratio - 0.0 and 0.15; (d) the thickness of the contact element; (e) the influence of the size of non-linearity of the contact line - straight or non-straight with some deviations; (f) the number of elements in contact, were varied to obtain some preliminary information how, where and when to apply our previously developed contact element [3, 4].

Only a part of the obtained results is presented here and some more details and results about these parametric analyses can be found in Ref. [6].

The following conclusions can be drawn from all previously performed parametric analyses:

- as thickness approaches a very small value ($w_y < 10^{-2}$) similar results are obtained in spite of varying Poisson's ratio (0.0 or 0.15) and the type of numerical integration (Gauss or Lobatto);
- the stress distribution in the basic elements surrounding the contact zone is slightly better for 3×3 Lobatto integration rule than for both Gauss integration rules (2×2 and 3×3); noting that sometimes there is no difference whether there exists a contact element or not, and sometimes the differences are drastical;
- for 2×2 Lobatto integration rule the obtained results for stresses and displacements in the basic elements are too stiff;
- in the case of a smaller number of contact elements in the contact zone, the shear stress distributions in the contact zone are similar for 2×2 Lobatto and 2×2 Gauss integration rules (the same applies to 3×3 Lobatto and 3×3 Gauss rules), but in both cases 2×2 integration rules give a more uniform distribution than 3×3 rules;
- when there are more contact elements in the contact zone, the shear stress distributions are similar for both Lobatto integration rules (2×2 and 3×3 rule) and give more uniform distributions in the middle of the zone with some concentrations at the ends; both Gauss integration rules give non uniform distribution (almost oscillatory, but the mean values are almost equal to the values obtained by Lobatto rules);
- the non-linearity of the contact line gives a great influence at the stress values, as expected.

In the end, it could be suggested that for each real situation it should be better to perform several different analyses in order to obtain more reliable and more accurate results.

The analyzed contact element is still being studied for an efficient future implementation in some more sophisticated finite element programmes.

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NUMERIČKO MODELIRANJE SPOJA LJUSKE I STUPA

SAŽETAK

Numerička analiza konstrukcija sastavljenih od ljuski i greda podrazumijeva specifične probleme povezivanja različitih konačnih elemenata u kompatibilnu cjelinu. U ovom radu je predstavljen algoritam zasnovan na novim izoparametrijskim konačnim elementima ljuske sa šest stupnjeva slobode u svakom čvoru i standardnim grednim konačnim elementima. Osnovni problem, koji se odnosi na osiguravanje potpune kompatibilnosti pomaka i zaokreta u zajedničkom čvoru, riješen je formulacijom rotacijskog stupnja slobode oko normale na srednju plohu ljuske. Posebno je analiziran spoj ljuske i stupa koji prenosi moment torzije u "ravninu" ljuske. Usporedba s analitičkim rješenjem potvrđuje točnost i pouzdanost korištenog numeričkog postupka.

NUMERICAL MODELLING OF JOINT OF SHELL AND COLUMN

SUMMARY

Numerical analysis of structures consisting of shells and beams implies specific problems of connecting various finite elements into a compatible entire. This work presents the algorithm based upon new isoparametric finite shell elements with six degrees of freedom per node and standard finite beam elements. The main problem related to the satisfying the fully compatibility of displacements and rotations in the common nodes, has been solved by formulating of the rotational degree of freedom around the normal to the mid-surface of the shell. Specially, the joint of shell and column which transfers torsional moment to the shell mid-surface is considered. The compare with analitical solution confirms the accuracy and reliability of employed numerical process.

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1. UVOD

Suvremeni pristup u analizi složenih inženjerskih konstrukcija predstavlja simuliranje ponašanja istovremeno cijele konstrukcije. Numeričko modeliranje konstrukcija sastavljenih od ljuski i stupova podrazumijeva realnu interpretaciju fizikalnih i geometrijskih svojstava cijelog nosivog sustava. Korištenjem metode konačnih elemenata u rješavanju takvih problema, svi traženi zahtjevi se mogu zadovoljiti. Važno je pri tome osigurati potpunu kompatibilnost pomaka i zaokreta na spoju različitih tipova konačnih elemenata.

U tu svrhu je razvijen programski paket SHELL6 [8] koji koristi nove izoparametrijske konačne elemente ljuske i standardne gredne elemente, tako da su zadovoljeni uvjeti kompatibilnosti na spojevima različitih elemenata konstrukcije. Ovim programom je moguće vršiti linearnu analizu svih tipova konstrukcija sastavljenih od ljuski (ploča, zidova), greda i stupova.

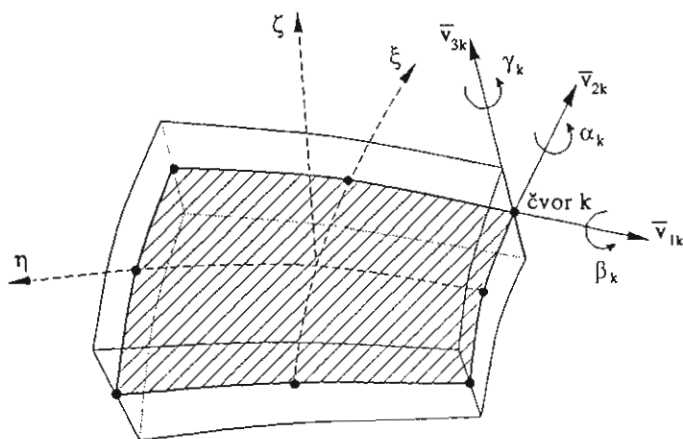
Novi konačni element ljuske uključuje i zaokret oko normale na njezinu srednju plohu. U slučaju značajnijeg utjecaja torzije stupa na ljusku, javlja se problem u inženjerskoj praksi poznat kao "problem bušenja ploče". Za ilustraciju tog fenomena analiziran je kružni disk koji je oslonjen na vertikalni stup kružnog poprečnog presjeka izložen djelovanju momenta torzije. U slučaju malog promjera stupa, u njegovoj neposrednoj okolini se javlja koncentracija posmičnih naprezanja koju potvrđuje analitičko rješenje i numerička analiza.

Pažljivim numeričkim modeliranjem spoja ljuske, pod djelovanjem momenta torzije, i stupa, te usporedbom s analitičkim rješenjem, može se pokazati da predloženi numerički model odlično opisuje stvarno ponašanje konstrukcije.

2. NUMERIČKI MODEL SPOJA LJUSKE I STUPA

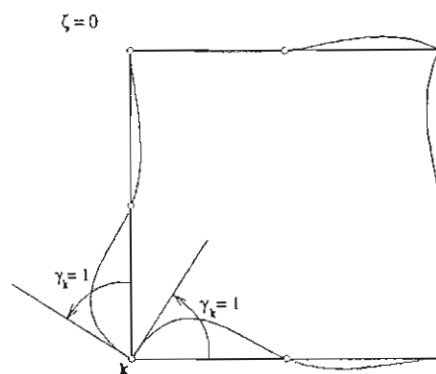
2.1 Konačni element ljuske

Ovdje je ukratko opisan 8-čvorni konačni element ljuske s tri stupnja slobode za pomake i tri stupnja slobode za zaokrete (crtež 2.1) koji se temelji na izoparametrijskim C^0 elementima ljuske sa pet stupnjeva slobode u svakom čvoru a koji su izvedeni iz prostornih konačnih elemenata [2].



Crtež 2.1 Rotacijski stupnjevi slobode u čvoru k

(Fig. 2.1 Rotational degrees of freedom in node k)



Crtež 2.2 Deformacija elementa uslijed zaokreta γ_k

(Fig. 2.2 Deformation of element owing to the rotation γ_k)

Novi rotacijski stupanj slobode elementa ljuske proizlazi iz jednadžbi ravnoteže Cosserat-ovog kontinuuma [1] napuštajući pri tome uobičajenu pretpostavku da je tenzor naprezanja σ_{ij} simetričan. Antisimetrični dio tenzora naprezanja (skew σ_{ij}) proizvodi zaokret γ_k u ravnini ljuske oko normale na njezinu srednju plohu (crtež 2.2) [6]. Ovaj zaokret sudjeluje kod modeliranja spoja ljuske i stupa. Njegov utjecaj se integralno uzima u obzir na elementu kao doprinos lokalne rotacije krutog tijela, a opisuje se istim baznim funkcijama kao i ostali stupnjevi slobode elementa ljuske.

Prema tome, polje pomaka u, v i w konačnog elementa ljuske izražava se linearnom kombinacijom pomaka i zaokreta čvorova elementa pomoću baznih funkcija. Doprinos čvora $\mathbf{k}$ vektoru pomaka točke je:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} N_k & 0 & 0 & N_k z \frac{h_k}{2} \bar{v}_{1k}^x & -N_k z \frac{h_k}{2} \bar{v}_{2k}^x & 0 \\ 0 & N_k & 0 & N_k z \frac{h_k}{2} \bar{v}_{1k}^y & -N_k z \frac{h_k}{2} \bar{v}_{2k}^y & 0 \\ 0 & 0 & N_k & N_k z \frac{h_k}{2} \bar{v}_{1k}^z & -N_k z \frac{h_k}{2} \bar{v}_{2k}^z & 0 \end{bmatrix} \begin{bmatrix} U_k \\ V_k \\ W_k \\ \alpha_k \\ \beta_k \\ \gamma_k \end{bmatrix} \quad (2.1)$$

gdje su N_k bazne funkcije koje za čvorove na vrhovima imaju oblik:

$$N_k = \frac{1}{4} (1 + \xi \xi_k) (1 + \eta \eta_k) (\xi \xi_k + \eta \eta_k - 1) \quad ; \quad k = 1, 3, 5, 7 \quad (2.2)$$

a za čvorove na sredinama stranica:

$$N_k = \frac{\xi_k^2}{2} (1 + \xi \xi_k) (1 - \eta^2) + \frac{\eta_k^2}{2} (1 + \eta \eta_k) (1 - \xi^2) \quad ; \quad k = 2, 4, 6, 8 \quad (2.3)$$

$$\xi_k, \eta_k = \pm 1$$

Uslijed zaokreta γ_k oko normale u čvoru $\mathbf{k}$ dobiva se nova komponenta u vektoru deformacija [2]. Ta komponenta se javlja u membranskom stanju naprezanja i predstavlja lokalnu rotaciju krutog tijela [6], a u jednadžbi (2.4) je označena s φ_z . U predstavljenom numeričkom modelu se koriste elementi ljuske s analitičkom integracijom po debljini ljuske, pa vektor deformacija ima oblik:

$$\mathbf{\epsilon}^* = [\epsilon_x, \epsilon_y, \gamma_{xy}, \varphi_z, K_x, K_y, K_{xy}, \gamma_{xz}, \gamma_{yz}]^T \quad (2.4)$$

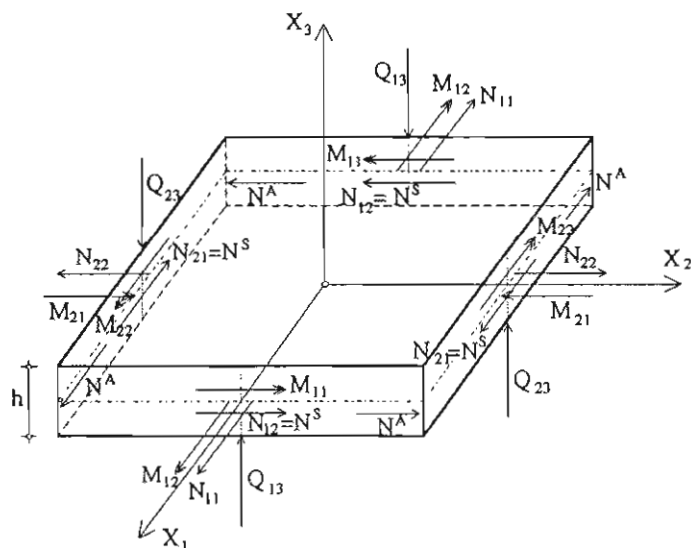
Konstitutivni zakon napisan u lokalnom koordinatnom sustavu elementa x', y', z' ima oblik:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy}^S \\ \tau_{xy}^A \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \mathbf{d} \cdot \mathbf{\epsilon} = \begin{bmatrix} E_1 / (1 - \nu_{12} \nu_{21}) & E_2 \nu_{12} / (1 - \nu_{12} \nu_{21}) & 0 & 0 & 0 & 0 \\ E_2 \nu_{12} / (1 - \nu_{12} \nu_{21}) & E_2 / (1 - \nu_{12} \nu_{21}) & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_1^* \cdot G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_2^* \cdot G_{23} \end{bmatrix} \begin{bmatrix} \epsilon_x + z' K_x \\ \epsilon_y + z' K_y \\ \gamma_{xy} + z' K_{xy} \\ \varphi_z \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \quad (2.5)$$

Analitičkom integracijom jednadžbe (2.5) po debljini elementa dobiva se devet rezultanti naprezanja:

$$\mathbf{N} = [N_x, N_y, N_{xy}^S, N_{xy}^A, M_x, M_y, M_{xy}, Q_{xz}, Q_{yz}]^T = \mathbf{D} \cdot \mathbf{\epsilon}^* \quad (2.6)$$

gdje su, u matrici svojstava materijala $\mathbf{D}$, odvojeni membranski članovi od progibnih i posmičnih sadržani u matrici $\mathbf{d}$. U jednadžbi (2.6) N_{xy}^S označava membransku posmičnu silu koja nastaje od simetričnog dijela tenzora naprezanja, a N_{xy}^A je membranska posmična sila koja nastaje uslijed antisimetričnog dijela tenzora naprezanja. Crtež 2.3 prikazuje svih devet rezultanti naprezanja.



Crtež 2.3 Rezultante naprezanja u točki konačnog elementa ljuske

(Fig. 2.3 Stress resultants in finite shell element point)

2.2 Formulacija problema

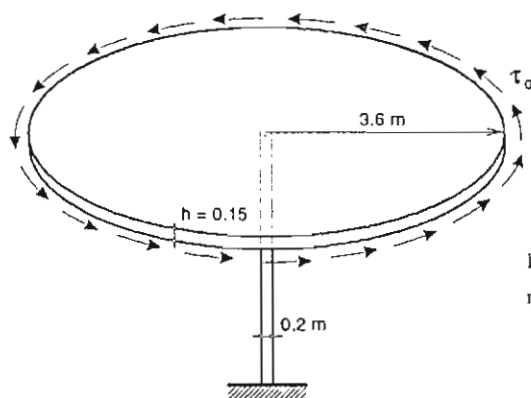
Novi konačni element ljuske sa šest stupnjeva slobode u svakom čvoru daje mogućnost numeričke analize ljuske u koju se unosi koncentrirani moment koji djeluje oko normale na srednju plohu ljuske (drilling degree of freedom). Takvi slučajevi nastupaju kod konstrukcija sastavljenih od ljuski i greda (stupova) kod kojih se unose koncentrirani momenti u srednju plohu ljuske na mjestima njihovog međusobnog spoja.

Kod toga se javlja efekt "momenta bušenja" koji predstavlja koncentraciju posmičnih naprezanja u ljusci po opsegu priključka grednog elementa. Moment oko normale na srednju plohu ljuske uvodi se u volumen elementa ljuske preko samo jednog čvora, zajedničkog s grednim elementom. Kada je element ljuske manjih dimenzija (slučaj gušće mreže), moment se distribuira po manjem volumenu pa je lokalni efekt koncentriranih naprezanja izraženiji, što je u skladu s formulacijom rotacijskog stupnja slobode bušenja. Ovaj efekt analogan je djelovanju koncentrirane sile na veoma malu površinu.

Postavlja se pitanje točnosti i prihvatljivosti numeričkih rješenja dobivenih za ovakve slučajeve opterećenja ljuske. U tom cilju je analiziran kružni disk pod djelovanjem momenta torzije oslonjen na stup (crtež 2.4).

2.3 Zaokret kružnog diska oslonjenog na kruti stup

Zadatak koji se rješava prikazan je na crtežu 2.4. Kružni disk polumjera 3.6 m, debljine 0.15 m, opterećen jednolikim posmičnim naprezanjem po vanjskom opsegu, oslonjen je u sredini na stup kružnog poprečnog presjeka polumjera 0.1 m. Model na crtežu 2.4 simulira vezu stupa i ploče u čiju ravninu se unosi koncentrirani moment, s tim da se razmatra dio ploče između stupa i vanjskog ruba ploče.



Svojsiva materijala:

$$E = 30000 \text{ kN/m}^2$$

$$G = 15000 \text{ kN/m}^2$$

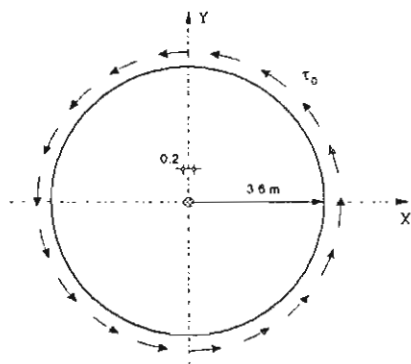
Ekvivalentno opterećenje momentom torzije:

$$M_t = 100.0 \text{ kNm}$$

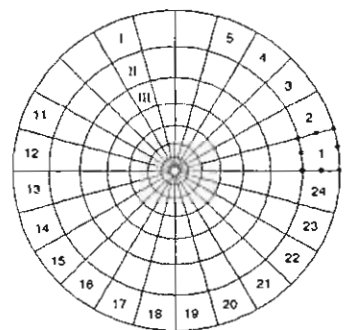
Crtež 2.4 Kružni disk na vertikalnom upetom stupu

(Fig. 2.4 Circular disk on the vertical column)

Na crtežu 2.5 prikazan je kružni disk opterećen posmičnim naprežanjem τ_0 po vanjskom opsegu, sve točke diska po vanjskom rubu su slobodne.



Crtež 2.5 Kružni disk oslonjen na stup
(Fig. 2.5 Circular disk supported by column)



Crtež 2.6 Diskretizacija diska konačnim elementima
(Fig. 2.6 Discretisation of circular disk by finite elements)

Diskretizacija ploče konačnim elementima prikazana je na crtežu 2.6. Konstrukcija je u transversalnom smjeru podijeljena na 24 8-čvorna konačna elementa ljuske, a u radialnom smjeru na 7 prstenova elemenata.

Da bi se ispitao efekt "momenta bušenja", izvršena je numerička analiza kružne ploče s različitim promjerima kružnog stupa u sredini koji s ljuske preuzima koncentrirani moment torzije. Tako su analizirani slučajevi kada je radijus stupa jednak unutrašnjem radijusu prvog vanjskog prstena, zatim unutrašnjem radijusu drugog prstena, i tako redom do radijusa stupa 0.1 m koji je jednak unutrašnjem radijusu sedmog prstena. Na taj način je analiziran utjecaj gustoće mreže konačnih elemenata ljuske u istom promatranom području oko stupa na pomake konstrukcije. Dobiveni pomaci i zaokreti čvora 1 za sve analizirane slučajeve prikazani su u tablici 1:

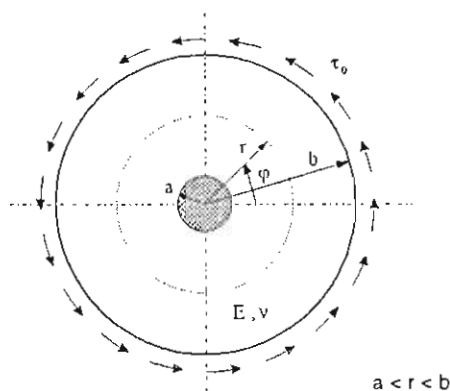
Tablica 1.

ČVOR 1	1 prsten	2 prstena	3 prstena	4 prstena	5 prstenova	6 prstenova	7 prstenova
v_1 [mm]	0.66	2.02	5.06	12.89	38.11	156.90	1404.0
φ_1 [rad]	0.39	0.74	1.59	3.79	10.84	43.91	390.5

3. ANALITIČKO RJEŠENJE KRUŽNOG DISKA KONSTANTNE I PROMJENJIVE KRUTOSTI OPTEREĆENOG MOMENTOM TORZIJE

Za kružni disk debljine h prikazan na crtežu 3.1 potrebno je odrediti polje pomaka i naprežanja ako je po vanjskom rubu radijusa b opterećen posmičnim naprežanjem τ_0 . Ekvivalent djelovanju posmičnog naprežanja τ_0 je moment torzije:

$$M_t = \tau_0 (2 \cdot b \cdot \pi) \cdot h \cdot b \quad (3.1)$$



Crtež 3.1
(Fig. 3.1)

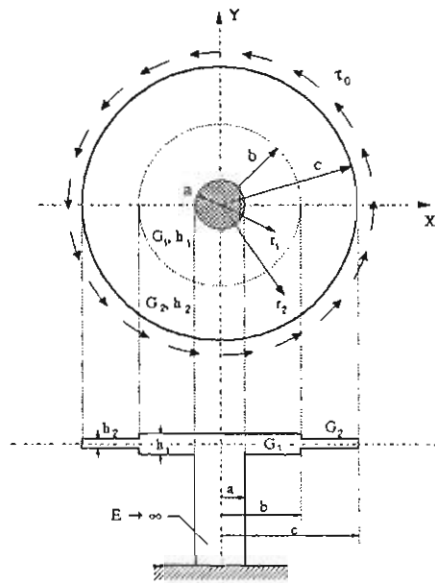
Zbog dužine izvoda, ovdje se daje samo konačno rješenje za pomak, i posredno za zaokret:

$$v(r) = \frac{M_t}{4\pi G \cdot h} \cdot \frac{r^2 - a^2}{a^2 \cdot r^2} \cdot r, \quad \varphi(r) = \frac{dv(r)}{dr} \quad (3.2)$$

dok se za komponente naprezanja dobivaju izrazi:

$$\sigma_r = 0; \quad \sigma_\varphi = 0; \quad \tau_{r\varphi} = \frac{M_t}{2\pi \cdot r^2 \cdot h} \quad (3.3)$$

Sada se može dobiti analitičko rješenje za kružni disk promjenjive krutosti opterećen posmičnim naprežanjem po obodu, kao što prikazuje crtež 3.2.



a - radijus stupa

b - radijus unutrašnjeg dijela ploče

c - radijus vanjskog dijela ploče

G_1, h_1 - modul posmika i debljina unutrašnjeg dijela ploče

G_2, h_2 - modul posmika i debljina vanjskog dijela ploče

$$a < r_1 \leq b$$

$$b < r_2 \leq c$$

Crtež 3.2

(Fig. 3.2)

Za unutrašnji prsten $a < r \leq b$ vrijedi rubni uvjet za $r=a$: $v(a) = 0$, pa pripadajuće rješenje ima oblik:

$$v(r_1) = \frac{M_t}{4\pi G_1 \cdot h_1} \cdot \frac{r_1^2 - a^2}{a^2 \cdot r_1^2} \cdot r_1, \quad \varphi(r_1) = \frac{dv(r_1)}{dr_1} \quad (3.4)$$

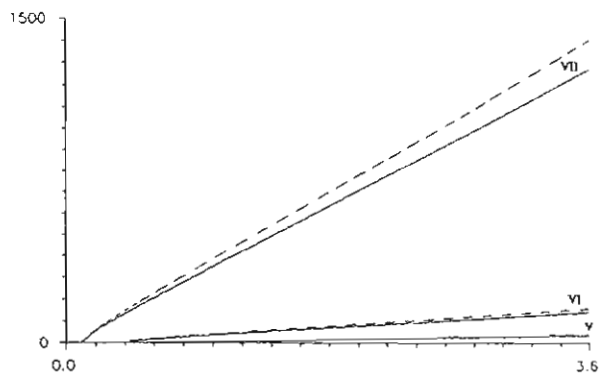
Rješenje za područje kružnog diska $b < r \leq c$ u tom slučaju glasi:

$$v(r_2) = \frac{M_t}{4\pi G_1 \cdot h_1} \cdot \frac{b^2 - a^2}{a^2 \cdot b^2} \cdot r_2 + \frac{M_t}{4\pi G_2 \cdot h_2} \cdot \frac{r_2^2 - b^2}{b^2 \cdot r_2^2} \cdot r_2, \quad \varphi(r_2) = \frac{dv(r_2)}{dr_2} \quad (3.5)$$

4. USPOREDBA ANALITIČKOG I NUMERIČKOG RJEŠENJA

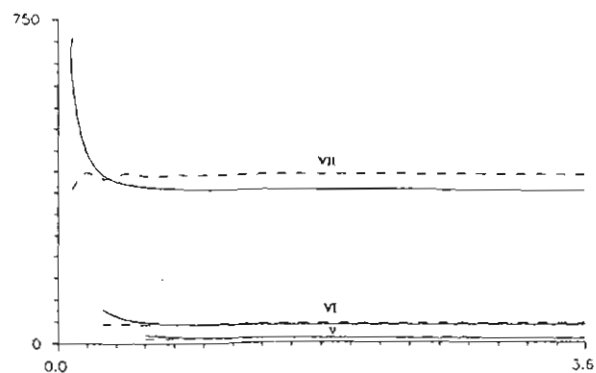
Vrši se usporedba numeričkog rješenja dobivenog programom SHELL6 [8] s analitičkim rješenjem kružnog diska opterećenog momentom torzije. Na crtežima 4.1 i 4.2 prikazani su dijagrami pomaka i dijagrami zaokreta, respektivno, u ovisnosti o promjeru stupa i to za promjer stupa veličine sedmog, šestog i petog prstena. Isprekidanom crtom označeno je numeričko rješenje a punom crtom analitičko rješenje dobiveno izrazima (3.2).

Iako numeričko rješenje dosta dobro prati analitičko za različite promjere stupa, iz gornjih dijagrama vidljivo je izvjesno odstupanje i to naročito za slučaj kad je radijus stupa jednak unutrašnjem radijusu sedmog prstena konačnih elemenata.



Crtež 4.1 Dijagram pomaka u ovisnosti o promjeru stupa

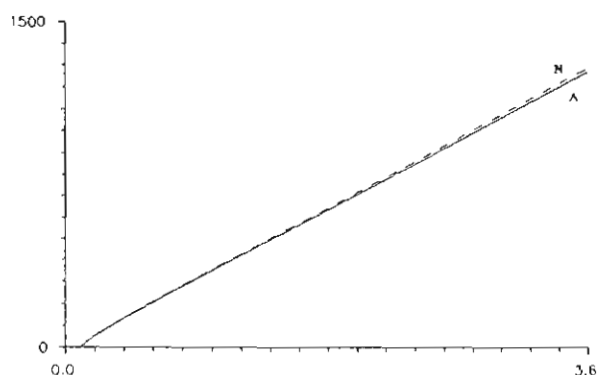
(Fig. 4.1 Diagram of displacement related to column radius)



Crtež 4.2 Dijagram zaokreta u ovisnosti o promjeru stupa

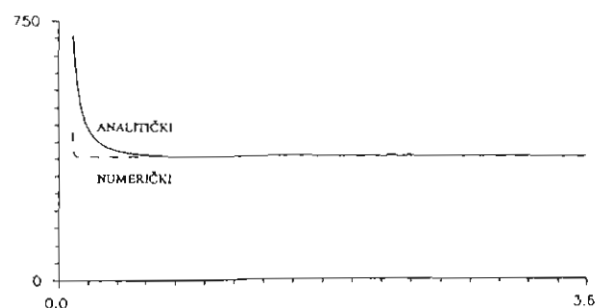
(Fig. 4.2 Diagram of rotation related to column radius)

Iz tog razloga je numerički proračun ponovljen za taj promjer stupa ali s gušćom mrežom konačnih elemenata ljuske. Uz korištenje osne simetrije, dovoljno je promatrati kružni isječak što omogućava izbor vrlo guste mreže. Na crtežima 4.3 i 4.4 prikazani su dijagrami pomaka odnosno dijagrami zaokreta na kružnom isječku s centralnim kutom od 2° dobiveni numerički programom SHELL6 i analitički pomoću izraza (3.2).



Crtež 4.3 Dijagrami pomaka na kružnom isječku

(Fig. 4.3 Diagrams of displacement on the circular cut)



Crtež 4.4 Dijagrami zaokreta na kružnom isječku

(Fig. 4.4 Diagrams of rotation on the circular cut)

Dijagrami 4.3 i 4.4 ukazuju na gotovo identičnu podudarnost numeričkog rješenja dobivenog gustom mrežom konačnih elemenata i analitičkog rješenja kružnog diska. Postoji samo izvjesno odstupanje u dijagramu zaokreta i to u području neposredne blizine stupa gdje postoji koncentracija posmičnih naprezanja. Bitno je naglasiti da samo pomaci predstavljaju rješenje diferencijalne jednadžbe kružnog diska dok je funkcija zaokreta zapravo prva derivacija rješenja. U numeričkoj interpretaciji problema to znači da zaokret treba pratiti preko drugih derivacija baznih funkcija konačnog elementa. Važno je istaknuti da su u prikazanoj formulaciji dodatnog rotacijskog stupnja slobode u ravnini ljuske korištene iste bazne funkcije kao i za ostale stupnjeve slobode (tj. Lagrangeovi polinomi II stupnja), što predstavlja osnovnu prednost i kvalitetu ove formulacije. Prikazana analiza potvrđuje točnost i pouzdanost predloženog numeričkog postupka.

5. NUMERIČKI MODEL SPOJA KRUŽNOG DISKA I STUPA

Kada se moment torzije prenosi s ploče na stup malog poprečnog presjeka u odnosu na radijus ploče, dolazi do efekta "bušenja" ploče. Kao što proizlazi iz rezultata izvršene analize, taj efekt se može spriječiti jednostavno povećanjem promjera stupa.

U građevinskoj praksi se često vrše ojačanja ploče u neposrednoj blizini stupa. Stoga je zgodno analizirati utjecaj veće krutosti ljuske oko stupa na pomake kružnog diska.

Ponovno su izvršeni numerički proračuni za najmanji poprečni presjek stupa uz povećavanje krutosti elemenata sedmog prstena za 10, 100, 1000 i 10000 puta. Dobiveni rezultati su dati u tablici 2. a uspoređeni su s analitičkim rješenjem dobivenim pomoću izraza (3.5). Vidljivo je da se postiže isti

efekt kao za povećanje promjera stupa (vidi tablicu 1.) ako se poveća krutost elemenata oko stupa za 100 - 1000 puta. Za dovoljno veći modul elastičnosti i modul posmika elemenata oko stupa, eliminira se pojava "klizanja" spoja ljuske i stupa, te se dobivaju rezultati prihvatljive točnosti.

Tablica 2.

$(E, G) \times 10^n$		$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
v_1	NUMERIČKI	1404.0	274.30	159.40	147.90	146.70
	ANALITIČKI	1272.3	253.67	151.81	141.62	140.60
φ_1	NUMERIČKI	390.5	76.52	44.60	41.40	41.08
	ANALITIČKI	353.95	71.01	42.71	39.88	39.60

6. ZAKLJUČAK

U ovom radu je predstavljen numerički model na osnovu kojeg je nastao programski paket SHELL6 [8] kojim je moguće vršiti linearnu analizu svih tipova konstrukcija sastavljenih od ljuski, ploča, zidova, greda i stupova. Da bi bila osigurana potpuna kompatibilnost na spoju grednih i ljuskastih elemenata, uobičajeni izoparametrijski konačni elementi ljuske s pet stupnjeva slobode upotunjeni su s rotacijskim stupnjem slobode kao zaokretom oko normale na srednju plohu ljuske [6]. Formulacija šestog stupnja slobode temelji se na neovisnosti polja pomaka i zaokreta. Korištenjem jednadžbi ravnoteže Cosseratovog kontinuuma, ovaj stupanj slobode je implementiran u konačni element ljuske koji se temelji na jednadžbama klasičnog kontinuuma, uz analitičku integraciju po debljini ljuske, te je opisan Lagrangeovim polinomima drugog stupnja [2]. Takve bazne funkcije predstavljaju izvjesno ograničenje za primjenu potpune formulacije rotacijskog stupnja slobode u "ravnini" ljuske.

Uspriko tome, na temelju izvršene analize i usporedbe s analitičkim rješenjem, može se zaključiti da se navedenom formulacijom mogu veoma uspješno analizirati problemi u kojima se kao opterećenje javlja, uz ostalo, i koncentrirani moment koji djeluje direktno u smjeru rotacijskog stupnja slobode "bušenja" [5]. Takvi se problemi javljaju na spoju ljuske i stupa koji prenosi moment torzije u "ravninu" ljuske. Kod većih vrijednosti koncentriranog momenta, ili kod veoma male površine poprečnog presjeka stupa u odnosu na površinu ljuske, mora se zadovoljiti kriterij da konačni element ljuske na kojeg se priključuje stup mora imati dimenzije reda veličine poprečnog presjeka stupa i povećanu krutost 100 - 1000 puta. Ukoliko i dalje dolazi do pojave efekta "bušenja" ljuske, radi se o konceptijskom problemu izbora promjera stupa te se mora odabrati stup većeg poprečnog presjeka.

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A SIMPLIFIED MODEL FOR NUMERICAL ELASTOPLASTIC PLANE STRESS ANALYSIS

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Summary

A simplified numerical model is presented based on the backward Euler iteration in which the complete iterative process is restricted to the plane $\sigma_3=0$ in the principal stress space. Initial stiffness method is used, implying purely elastic prediction in every iteration step, in conjunction with the modified radial return. The yield limit update in the plastic range is performed as isotropic hardening based on the equivalent total strain. The procedure is illustrated in the common example of the tension strip with circular notches.

1. Introduction

Numerical modelling of solids and structures, primarily using the finite element method, has strongly developed in recent years in the field of elastoplastic analysis. Following the pioneering work by Wilkins [1], the iterative procedures are developed in which the prediction and plastic correction phases repeatedly follow each other, until the prescribed convergence criterion is satisfied. The predictor phase is characterised by the stiffness operator (initial, tangent stiffness, or combination thereof [2]), and its considerable improvement is achieved by the introduction of the recent consistent tangent operators which increase the efficiency of the iteration [3], [4], [5]. The plastic correction phase takes place in those parts of the structure (finite elements, Gauss points) in which prediction violates the yield criterion, and it consists in returning the state of stress to the yield limit. Such a reduction of the calculated stresses of course violates the equilibrium, and the unbalanced residual forces appear, which give rise to the new iteration step i. e. evaluation of the incremental displacement field. Along with the original "radial return" [1], other similar procedures are also developed and assessed [6], [7].

The plane stress is often, within the context of elastic analysis, regarded as a special case of a general plane situation (plane stress - plane strain - axial symmetry), with the appropriate modifications of the elasticity matrix. In the plastic range however it is characterised by some special features [8], and deserves particular attention of its own [9], [10]. It also holds having in mind that the state of stress in the layers of some important engineering structures such as thin plates and shells, essentially corresponds to the plane stress conditions.

Although the methodology of analysis in this field is well established [9], [4], a motivation is always present to achieve acceptable engineering results with reduced programming and computational effort. In this paper an attempt is presented to perform numerical elastoplastic plane stress analysis in the simplest possible fashion, neglecting any drift from the plane $\sigma_3=0$ in the principal stress space in both prediction and correction phase, which would otherwise be connected to the deviatoric plane. Furthermore, the equivalent plastic strain, to be determined by numerical integration as the hardening criterion, is replaced by the equivalent total strain, obtained using the "overall algorithmic Poisson's ratio", the value of which is monotonically increasing towards 0.5 in the plastic range.

2. Iterative procedure

In the following the proposed iterative procedure will be presented as split into its principal steps: prediction, yield limit update (hardening), plastic correction (stress return), and finally evaluation of the residual forces and convergence criterion.

2.1. Prediction phase

As mentioned above, in this step it is possible to use initial stiffness or some of the formulations of the elastoplastic tangent stiffness. The latter have been present for some time in the computational literature [11], and have been considerably improved by the introduction of the so called consistent tangent operators which preserve the quadratic rate of convergence [3].

However, in this paper the initial stiffness method is applied, due to the simplicity of its programming implementation, which implies purely elastic prediction in every iteration step. Accordingly, it is reasonable to preview relatively coarse tolerance in the convergence criterion, or alternatively very high number of iteration steps. It should be noted that, within the proposed procedure, it is essential to keep the initial prediction after application of every new load step purely elastic (and not elastoplastic corresponding to the last converged state), in view of the possible elastic unloading.

The displacement field is computed by solving the global system of equations

$$[K]\{d\}=\{R\} \quad (1)$$

where $\{d\}$ denotes the vector of the unknown nodal displacements, $\{R\}$ the vector of given nodal forces, and $[K]$ the reduced global stiffness matrix. In the first step the nodal forces correspond to the initial load or the specified load increment, while in further steps they represent the residual nodal forces obtained after the return of stress in the previous step. Except for the initial elastic prediction, $\{d\}$ in (1) in fact represents the incremental displacement field to be added to the previously computed displacements to obtain the accumulated total displacements. On the element level, the displacement increment yields the corresponding deformation increment

$$\{\Delta\epsilon\}=[B]\{\Delta\delta\} \quad (2)$$

from which on one hand the new total deformation follows as

$$\{\epsilon\}^{i+1}=\{\epsilon\}^i+\{\Delta\epsilon\} \quad (3)$$

and on the other hand the stress increment and the accumulated total stress state

$$\{\sigma\}^{i+1}=\{\sigma\}^i+[D]\{\Delta\epsilon\} \quad (4)$$

Here $[B]$ denotes the strain-displacement matrix, and $[D]$ the elasticity matrix for plane stress.

2.2. Yield limit update

After evaluating the assumed deformation $\{\epsilon\}^i$ and the principal deformations in the plane of the element $\epsilon_{1,2}$, it is necessary to evaluate ϵ_3 and the equivalent uniaxial strain. To this end we define the elastoplastic (overall algorithmic) Poisson's ratio

$$\nu_{ep}=0.5-\epsilon_0(0.5-\nu)/\epsilon_{eq} \quad (\epsilon_{eq}>\epsilon_0) \quad (5)$$

the value of which is monotonically increasing in the elastoplastic range from ν to 0.5 (Fig. 1).

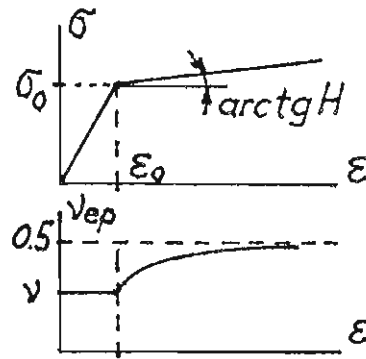


Figure 1. The elastoplastic Poisson's ratio

From the elastic prediction it follows

$$\epsilon_3 = -\nu(\epsilon_1 + \epsilon_2) \quad (6)$$

and the equivalent total strain is calculated as

$$\epsilon_{eq} = [K(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)]^{1/2} \quad (7)$$

with the factor K following from the comparison with the uniaxial case

$$K = (1 + 2\nu_{ep}^2)^{-1} \quad (8)$$

($K=2/3$ for purely plastic deformation). For the given $\epsilon_{1,2}$ equations (5), (6) taking ν_{ep} in place of ν and (7) give ϵ_{eq} by the local iteration on the element (Gauss point) level, which in turn serves as the base for the updating of the yield limit. Isotropic hardening scheme is applied (expansion of the von Mises ellipse in the plane $\sigma_1\sigma_2$), and for the bilinear σ - ϵ diagram the yield limit is calculated as

$$\sigma_0^* = \sigma_0 + (\epsilon_{eq} - \epsilon_0)H \quad (9)$$

with H denoting the inclination of the straight portion in the plastic range of the uniaxial σ - ϵ diagram (Fig. 1). The expression (9) of course holds for the case $\epsilon_{eq} > \epsilon_0$ and $\Delta\epsilon_{eq} > 0$, and in the case of unloading the previously computed σ_0^* is kept unchanged.

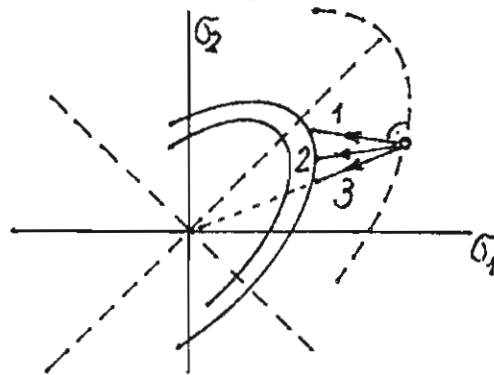
2.3. Stress return

In this step, referred to in the computational literature also as "plastic relaxation" or "return mapping", the previously computed excess of the stress state over the yield limit is reduced by means of a suitable projection of the point representing the stress state, to the updated yield limit. In present numerical model the return of stress is defined by

$$\{\Delta\sigma\} = \Delta\lambda[D]\{\partial Q/\partial\sigma\} \quad (10)$$

where Q denotes the plastic potential and $\Delta\lambda$ follows from the plastic consistency condition, i. e. the condition that after return the stress state lays on the yield limit. It shows [12] that, thanks to the particular structure of the matrix $[D]$, the return (10) appears equivalent in the principal stress space and in the general stress component space, and that the principal directions of the predicted and returned stress states are equal. Furthermore, in the context of the associated plasticity (plastic potential coinciding with the yield function) and von Mises

yield criterion it can easily be shown that the return (10) is purely radial ($\Delta\sigma_1/\Delta\sigma_2 = \sigma_1/\sigma_2$) for $\nu=0.5$, and normal to the plastic potential in the prediction point for $\nu=0$ (Fig. 2).



1	$\nu = 0$
2	$0 < \nu < 0,5$
3	$\nu = 0,5$

Figure 2. Return of stress to the yield limit

It is clearly noticed that for $\nu < 0.5$ the regions exist from which the return (10) would "miss" the ellipse, the case corresponding to the complex solution of quadratic equation for $\Delta\lambda$, or it would give physically unacceptable projection to the opposite quadrant of the ellipse. For this reason it is convenient to modify the return (10) for the case of extremely large overshoots, as to subdivide it in several steps with gradual corrections of the return direction, which would in the limit for $\nu=0$ correspond to the return trajectory normal to the plastic potential (steepest descent path). Alternatively it is possible to formulate the return to the closest point of the ellipse using a simple iterative algorithm. More detailed analysis of this problem and the mentioned possibilities for its overcoming is beyond the scope of this paper.

2.4. Residual forces

If the stress state $\{\sigma\}$ in a finite element is given, corresponding element nodal forces are determined [2] by the expression

$$\{f\}^e = \int_{\Omega} [B]^T \{\sigma\} d\Omega \quad (11)$$

Force on a node of the finite element mesh is obtained by the summation over all elements connected in the node under consideration (and in the sense of the degree of freedom under consideration) from the equilibrium condition

$$F^e = -\sum f_i^e \quad (12)$$

Finally, the residual force vector represents the difference between the forces $\{F\}^e$ and the specified nodal forces corresponding to external loading

$$\{R\} = \{F\}^e - \{F\} \quad (13)$$

Non-zero components of the residual force vector $\{R\}$ appear in the nodes where the elements are connected in which the stress reduction preceded. The vector $\{R\}$ gives rise to the new iteration step, i.e. it becomes the right-hand side of the global equation system for determining the new incremental displacements as mentioned above, unless the convergence is indicated in which case the new load level can be applied.

2.5. Convergence criterion

It is usual to terminate the iteration when some convenient norm of the error is reduced to within the prescribed tolerance limits. In the present model use is made of the Euclidean norm of residual forces. Let us note that it is alternatively possible to use energy norm, or maximum or average overprediction over the yield limit in the plastified elements. In the experimental phase of programme testing it proves convenient to have complete insight to the computational results in every iteration step, to be able to react in case of unforeseen errors.

3. Numerical example

The above described iterative elastoplastic algorithm is programmed as the extension of an existing finite element program for elastic plane stress analysis. In the programme triangular finite elements (CST) are used and the profile equation solver with Gauss elimination. The experimental version programmed on a desk computer is modest in capacity and speed of computation, but it provides graphic support in both problem definition and visualisation of results.

As an illustrative example the common problem of tension strip with circular notches is solved (Fig. 3a). For symmetry reasons one quarter of the strip is modelled. The details concerning the finite element mesh and the initial loading which produced the first plastic deformation are displayed in fig. 3b. The bilinear elastoplastic diagram is assumed (Fig. 3c).

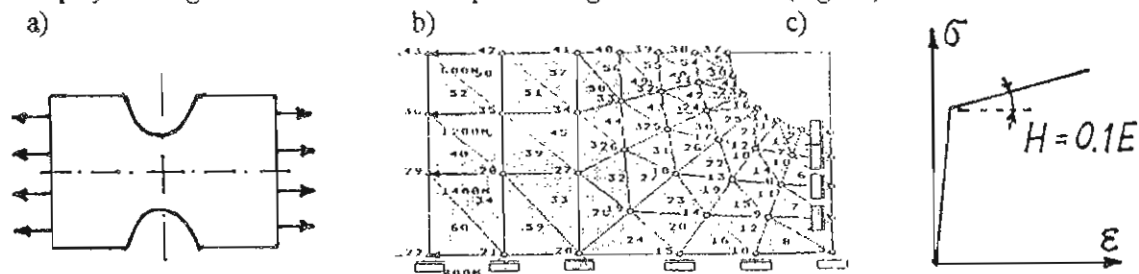


Figure 3. Numerical example - a) Notched tension strip, b) Finite element mesh, c) Material model

The loading is increased in three steps with thirty per cent of the previous load added in every step. Reduction of the residual force norm was typically 30 - 50 per cent in every iteration step. For every load step five iteration steps were performed, which reduced the initial residual force norm to just under 10 per cent. The results are partly presented in figures 4 and 5. In fig. 4 the non-linear part of the load - characteristic displacement diagram is shown (Fig. 4a), and the plastic zone after fifth iteration step for third load level (Fig. 4b).

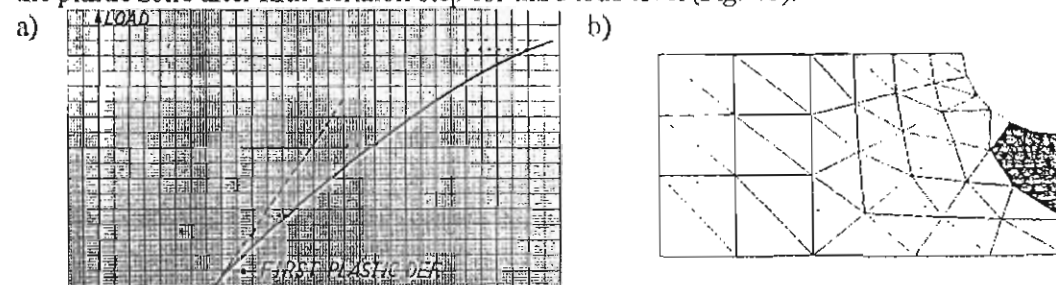


Figure 4. a) Non-linear part of the load - displacement diagram, b) Plastic zone

In figure 5 the stress states in finite elements after fifth iteration step for third load level are presented. The discrete points correspond to the states of stress in every finite element represented in the $\sigma_{eq} = c_{eq}$ diagram (Fig. 5a), and in the plane σ_1, σ_2 (Fig. 5b). Black spots

correspond to the predicted state, whereas white spots represent the state after the return of stress to the yield limit.

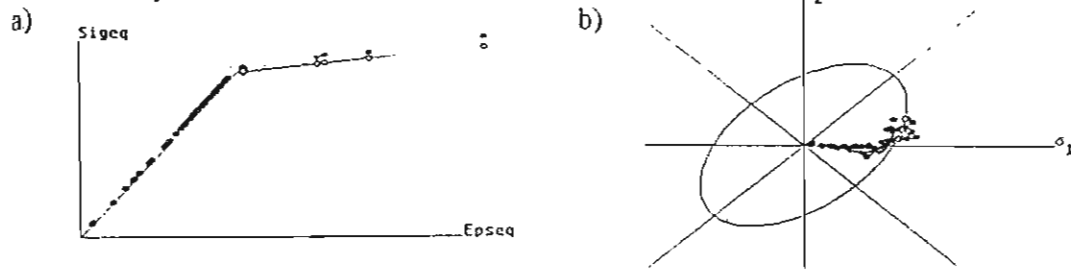


Figure 5. State of stress in finite elements, a) Diagram $\sigma_{eq} - \epsilon_{eq}$, b) Representation in the plane σ_1, σ_2

4. Concluding remarks

The method described in this paper can be regarded as an alternative to the existing procedures of the elastoplastic plane stress computation. Numerical simulation gives the physically meaningful results comparable to those obtained by other methods. A considerable improvement in terms of the efficiency and accuracy of iteration would probably be achieved by introducing the tangent stiffness in place of initial stiffness. For the realistic engineering computations it is also essential to improve the capacity and speed of the existing software solution, because with the simple CST finite elements only with very fine mesh the acceptable accuracy can be expected. In future work it would be interesting to analyse the influence of stress return schemes upon the computational results. Finally, further evaluation and improvement of the present model is planned to take place in connection with its application in the analysis of elastoplastic plate bending.

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AN ANALYSIS OF PLASTIC SHAKEDOWN BY THE NONRATCHETTING CONDITION

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ABSTRACT. The nonratchetting condition for reversed plasticity is established for an elastic-perfectly plastic solid body subjected to cyclic/steady loads such as to produce plastic shakedown (i. e. alternating plastic strains take place within one part of the body and elastic shakedown within other part). In one example this condition is used to determinate the plastic shakedown boundary.

1. INTRODUCTION

The more recent investigations (Ponter and Karadeniz, 1985; Polizzotto, 1993 a, b) concerned with the behavior of elastic-plastic solid bodies under cyclic loading have understood the term "shakedown" or "plastic shakedown" as the steady state characterized by absence of plastic strain accumulation in a complete cycle. It has a more general meaning than the classical elastic shakedown which is defined as the long term structural behavior when a plastic strain rate vanishes.

The Bleich-Melan and Koiter theorems and their extensions to hardening behavior provide necessary and sufficient conditions for elastic shakedown. Similar theorems for shakedown (plastic shakedown) were established by Polizzotto (1993 b).

The plastic shakedown limit loads have been determinated by the upper bound shakedown theorem (Ponter and Karadeniz, 1985), by the fictitious holonomic solution (Vranković, 1991) and by the optimisation methods (Polizzotto, 1993 a, b).

The aim of the present paper is to assesse the plastic shakedown limit load by the nonratchetting condition.

In the sequel, boldfaced letters will be used to denote vectors and matrices.

2. THE STATIONARY RESPONSE TO CYCLIC/STEADY LOADS

Let us consider an elastic-perfectly plastic solid body of volume V referred to three orthogonal Cartesian axes $\mathbf{x}=(x_1, x_2, x_3)$, let it is bounded by the surface S which may be divided

into a part S_u with constrained displacements and a part S_t on which a surface tractions are prescribed ($S_u \cup S_t = S$). The body is loaded by body forces and imposed strains (e. g., thermal) in V , by tractions on S_t , all of which vary cyclically with time t and with the same time period Δt .

All these actions can be collectively denoted as

$$\mathbf{F}(t) = \mathbf{F}^c(t) + \mathbf{F}^0 \quad (1)$$

where $\mathbf{F}^c(t)$ denotes cyclically variable load (mechanical or kinematical) and $\mathbf{F}^0$ a constant load (mechanical).

According to the cyclic plasticity theory (see, e.g., Gokhfeld and Cherniavsky, 1980) an elastic-plastic body subjected to cyclic loads responds in a non-periodic way during the transient period, while after, the stress response σ and the plastic strain rate response $\dot{\epsilon}^p$ become cyclic, with the same period Δt of the loads, i. e.

$$\begin{aligned} \sigma(\mathbf{x}, t) &= \sigma(\mathbf{x}, t + \Delta t), \\ \dot{\epsilon}^p(\mathbf{x}, t) &= \dot{\epsilon}^p(\mathbf{x}, t + \Delta t) \end{aligned}$$

(steady-state response).

The actual stress response $\sigma(\mathbf{x}, t)$ is the sum of two parts, the one elastic stress response of body to $\mathbf{F}(t)$, $\sigma^E(\mathbf{x}, t)$ (when the existence of a yield surface is ignored), the other one the actual residual stress $\sigma^R(\mathbf{x}, t)$. These stresses can be written as

$$\sigma(\mathbf{x}, t) = \sigma^E(\mathbf{x}, t) + \sigma^R(\mathbf{x}, t) \quad (2)$$

$$\sigma^E(\mathbf{x}, t) = \sigma^c(\mathbf{x}, t) + \sigma^0(\mathbf{x}) \quad (3)$$

$$\sigma^R(\mathbf{x}, t) = \varrho(\mathbf{x}) + \tau(\mathbf{x}, t) \quad (4)$$

where $\sigma^c(\mathbf{x}, t)$ and $\sigma^0(\mathbf{x})$ are the elastic stress responses to $\mathbf{F}^c(t)$ and $\mathbf{F}^0$ respectively, $\varrho(\mathbf{x})$ denotes the residual stresses at the instant at which transient period terminates and the steady state one begins, $\tau(\mathbf{x}, t)$ denotes the residual stresses corresponding to the net plastic strains occurring in the steady cycle.

The equations governing the steady-state response can be written in the form

$$\Phi(\sigma) \leq 0, \quad \dot{\epsilon}^p = \frac{\partial \Phi}{\partial \sigma} \dot{\lambda}, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} \Phi(\sigma) = 0 \quad (5)$$

$$\sigma = \sigma^E + \sigma^R, \quad \mathbf{C}^T \sigma^R = 0, \quad \dot{\epsilon}^R = \mathbf{A} : \dot{\sigma}^R + \dot{\epsilon}^p \quad (6)$$

$$\dot{\epsilon}^R = \mathbf{C} \dot{\mathbf{u}}^R, \quad \Delta \epsilon^p = \int_0^{\Delta t} \dot{\epsilon}^p dt, \quad \Delta \mathbf{u}^R = \int_0^{\Delta t} \dot{\mathbf{u}}^R dt \quad (7)$$

$$\int_0^{\Delta t} \dot{\sigma}^R dt = 0, \quad \mathbf{C} \cdot \Delta \mathbf{u}^R = \Delta \epsilon^p, \quad \text{all in } V \quad (8)$$

$$\sigma^R \cdot n = 0 \quad \text{on} \quad S_t \quad (9)$$

$$\dot{u}^R = 0, \quad \Delta u^R = 0 \quad \text{on} \quad S_u \quad (10)$$

Here above $\Delta \epsilon^p$ denotes the increment of plastic strains compatible with the cycle net residual displacement Δu^R , $\dot{\lambda}$ is the plastic multiplier rates vector, C denotes the compatibility differential operator, A is the elastic compliance fourth-order tensor, n denotes the unit external normal to the boundary surface S and $\Phi(\sigma)$ is the yield function by hypothesis convex.

The solution of Eqs. (5) - (10) has the following properties:

- the plastic strain rates $\dot{\epsilon}^p$ are uniquely determined at all in V
- residual stresses σ^R are uniquely determined only within the region V_p , where $\dot{\epsilon}^p$ is not identically vanishing
- the steady-state solution is independent of any initial plastic strains as well as of the load cycle origin.

The uniquely determined plastic strain rates have three possible alternatives:

- elastic shakedown, when $\dot{\epsilon}^p = 0$
- alternating plasticity, when $\dot{\epsilon}^p \neq 0$ but $\Delta \epsilon^p = 0$
- incremental collapse (ratchetting), when $\Delta \epsilon^p \neq 0$

3. A NECESSARY AND SUFFICIENT CONDITION OF PLASTIC SHAKEDOWN AND NONRATCHETTING CONDITION

Combining Eqs. (2) - (4), the actual stress response can be written as

$$\sigma(x, t) = \sigma^E(x, t) + \sigma^R(x, t) = \hat{\sigma}(x, t) + \tau(x, t) \quad (11)$$

where $\hat{\sigma}$ denotes the post-transient stress which is defined as

$$\hat{\sigma}(x, t) = \sigma^c(x, t) + \sigma^0(x) + \rho(x) \quad (12)$$

The post-transient stress response depends on the value of load parameters.

In case of elastic shakedown, $\hat{\sigma}$ lies inside the yield surface everywhere in V , otherwise, $\hat{\sigma}$ exceeds the yield surface at least somewhere in V .

A suitable description of the yield violation is the fictitious yield surface $\Phi(\sigma) = y$ (Fig. 1), homothetic to yield surface $\Phi(\sigma) = 0$ (similar as a isotropic hardening has transformed the actual yield surface into the expanded one) where y (or plastic overpotential) is defined as

$$y = \max_{t \in (0, \Delta t)} \Phi[\hat{\sigma}(x, t)] \quad (13)$$

The fictitious yield surface $\Phi(\sigma) = y$ has to satisfy a similar yield equations, as the real one $\Phi(\sigma) = 0$ makes (5). The condition when the post-transient stress $\hat{\sigma}$ find itself in a balanced

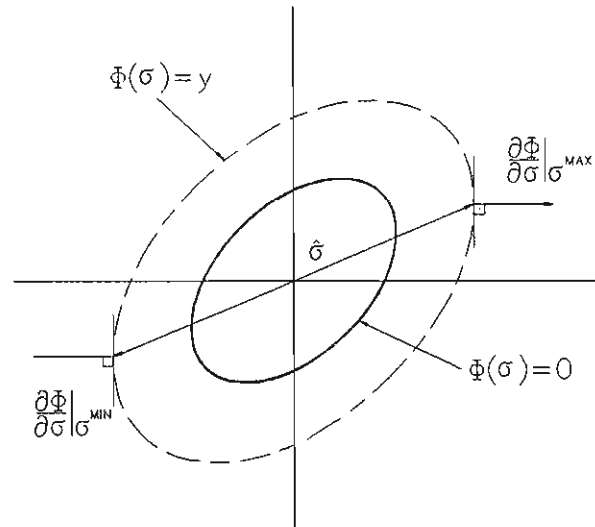


Fig. 1 Yield surface, fictitious yield surface and neutral configuration

position with the respect to the fictitious yield surface will be referred as the neutral configuration (Polizzotto 1993 b).

In the case when $\hat{\sigma}$ exceeds the yield surface somewhere in V , the body contains a nonempty region V_e where it responds elastically and region V_F where alternating plastic strains take place ($V_e \cup V_F = V$). It can be proven that such an elastic region can exist only if alternating plasticity occurs in a region V_F .

A necessary and sufficient condition for such behavior of body (plastic shakedown) is that there exists some time-independent self-stress field $\mathbf{q}$, such as on superposing it to the elastic stress field σ^E , the post-transient stress field $\hat{\sigma} = \sigma^E + \mathbf{q}$ finds itself in a neutral configuration everywhere within region V_F and inside yield surface in the remaining part (Polizzotto 1993 b), i. e.

$$\begin{aligned} \Phi(\hat{\sigma}) &= y & \text{in } V_F \\ \Phi(\hat{\sigma}) &\leq 0 & \text{in } V_e = V - V_F \end{aligned} \quad (14)$$

For the assessment of the plastic shakedown limit load were used methods in which the region V_F had been removed and an elastic shakedown limit load problem was solved only for $V_e = V_s$ loaded by the given load (Ponter-Karadeniz 1985, Polizzotto 1993 a, b).

Another approach to the same problem is the implementation some properties of the alternating plasticity which takes place within the region V_F .

An alternating plasticity is the plastic deformation process such that, in one cycle, the increment of plastic strains becomes zero and its maximum amplitude remains below a certain limit, and is characterized by the condition that

$$\Delta \boldsymbol{\varepsilon}^p = \int_0^{\Delta t} \dot{\boldsymbol{\varepsilon}}^p dt = \mathbf{0} \quad \text{in} \quad V_F \quad (15)$$

Thus the integral of each component of plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ is zero, indicating that any plastic deformation that occurs in a cycle is reversed within the same cycle, i.e.

$$\boldsymbol{\varepsilon}^p(t) = \boldsymbol{\varepsilon}^p(t + \Delta t) \quad \text{in} \quad V_F \quad (16)$$

It is easy to recognize that due to the nonratchetting condition (16), the extremes of post-transient stress history are related by

$$\hat{\boldsymbol{\sigma}}^{MAX} = -\hat{\boldsymbol{\sigma}}^{MIN} \quad \text{in} \quad V_F \quad (17)$$

(see also Ponter-Karadeniz, 1985)

Combining (14) and (17), a necessary and sufficient condition of plastic shakedown can be reformulated as

$$\begin{aligned} \Phi(\hat{\boldsymbol{\sigma}}^{MAX}) &= \Phi(\hat{\boldsymbol{\sigma}}^{MIN}) \quad \text{in} \quad V_F \\ \Phi(\hat{\boldsymbol{\sigma}}) &\leq 0 \quad \text{in} \quad V_S \end{aligned} \quad (18)$$

By the application of reformulated plastic shakedown condition (18), the plastic shakedown limit load is possible computed on the mathematically simpler way.

4. NUMERICAL EXAMPLE

The two bar problem is solved with the aim to confirm of this new approach to the computation of the plastic shakedown limit load.

The geometry and loading are shown in Fig. 2 a,b; the bar 1, of cross section A, has uniform temperature $T_1 = T_1(t)$, cyclically variable with period Δt and temperature amplitude ΔT ; the bar 2, of cross section $4A$, has constant uniform temperature T_2 . A constant mechanical load F is applied to the rigid block.

In the context of discrete systems and perfect plasticity, the yield function $\Phi(\hat{\boldsymbol{\sigma}})$ is defined by

$$\Phi(\hat{\boldsymbol{\sigma}}) = \mathbf{N}^T \boldsymbol{\sigma}^c + \mathbf{N}^T \boldsymbol{\sigma}^0 + \mathbf{N}^T \mathbf{Z} \mathbf{N} \boldsymbol{\lambda} - \sigma_y; \quad \boldsymbol{\lambda} \geq \mathbf{0} \quad (19)$$

Here above $\mathbf{N}^T = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}$ is the outward normal to the surface $\Phi(\boldsymbol{\sigma})=0$, $\mathbf{Z}$ is the matrix from relation $\boldsymbol{\varrho} = \mathbf{Z} \boldsymbol{\varepsilon}^p$ and σ_y is yielding stress.

The cyclic stress $\mathbf{N}^T \boldsymbol{\sigma}^c$ may be vary within a given elastic domain $\mathbf{M}$

$$\mathbf{M} = [\dots M_i \dots] ; \quad M_i = \max_t N_i \sigma^c(t) \quad (20)$$

It follows that the yield function of post-transient stresses is

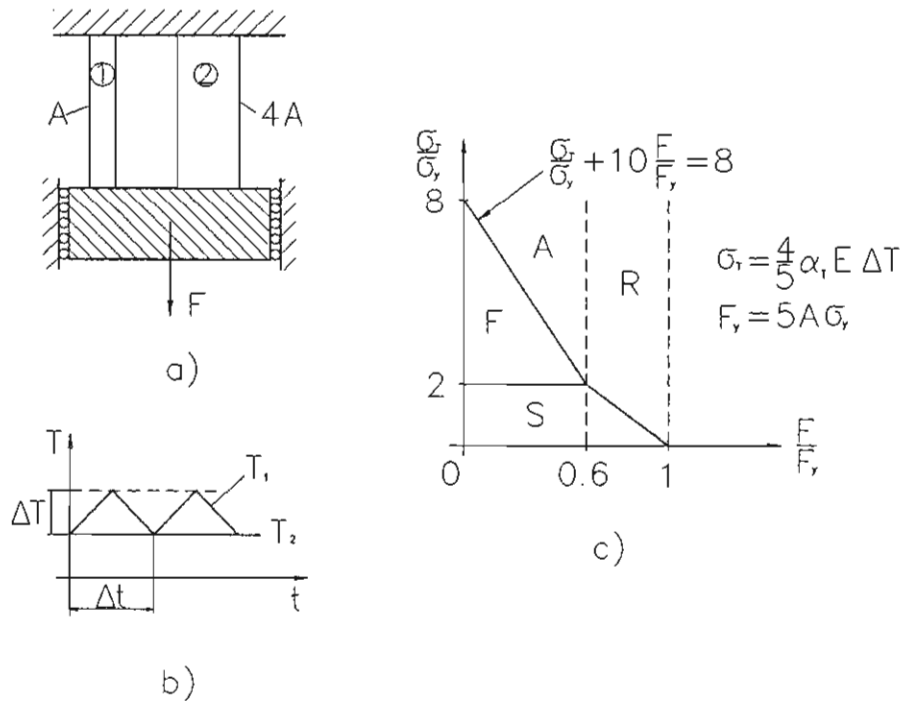


Fig. 2 Two bar structures

- a) geometry and loading
- b) thermal loading history
- c) limit load domains

$$\Phi(\hat{\sigma}) = \mathbf{M} + \mathbf{N}^T \sigma^0 + \mathbf{N}^T \mathbf{Z} \mathbf{N} \lambda - \sigma_y ; \quad \lambda \geq 0 \quad (21)$$

The necessary and sufficient condition of plastic shakedown (18) for two bar problem can be written as

$$\frac{F}{5A} + \mathbf{Z}_1 \lambda - \sigma_y = \sigma_T - \frac{F}{5A} + \mathbf{Z}_2 \lambda - \sigma_y \quad \text{in bar 1} \quad (V_F)$$

$$\frac{\sigma_T}{4} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \frac{F}{5A} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \begin{Bmatrix} \mathbf{Z}_3 \\ \mathbf{Z}_4 \end{Bmatrix} \lambda - \sigma_y \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \leq 0 \quad \text{in bar 2} \quad (V_S)$$

$$\lambda \geq 0$$

where $\mathbf{Z}_1, \mathbf{Z}_4$ are the rows of matrix $\mathbf{N}^T \mathbf{Z} \mathbf{N}$, $\sigma_T = \frac{4}{5} E \alpha_T \Delta T$ is a max thermoelastic stress and $F_y = 5 A \sigma_y$ is a plastic collapse limit load. By eliminating the common non-negative vector λ from above equations, the following expression for the plastic shakedown boundary is obtained (Fig. 2c)

$$\frac{\sigma_T}{\sigma_y} + 10 \frac{F}{F_y} \leq 8$$

what is in perfectly agreement with the previous results (Ponter and Karadeniz, 1985; Polizzotto 1993 b).

5. CONCLUSIONS

In an elastic-perfectly plastic solid body subjected to cyclic/steady loads above the elastic shakedown limit may be exist the plastic shakedown.

Thus, the body is divided into two regions, $V_F \cup V_S = V$, where V_F is the region where alternating plasticity takes place and V_S is the region of elastic shakedown. The nonratchetting requirement of reversed plasticity for the region V_F is used as the additional condition to determinate the plastic shakedown boundary.

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Numerical analysis of crack propagation in gear tooth root

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Abstract

The paper is concerned with the two-dimensional numerical analysis of crack propagation in a gear tooth root, which is based on the theory of linear elastic fracture mechanics. The stress intensity factor K is calculated for different crack lengths by means of the finite element method (BERSAFE) and the boundary element method (BEASY). The analyses of results show a good correlation between two numerical methods. The service-life of a gear is determined on the basis of numerically determined stress intensity factors, i.e. calculated is the number of loading cycles required for a crack to extend from some initial length to the critical length where the complete failure of the tooth is expected.

1.0 Introduction

The strength verification process of gear teeth that is based on the theory of fracture mechanics marks a new approach in the design phase of gear assemblies. Classical strength verification procedures according to standards DIN, AGMA, ISO, etc. are based on the comparison of the actual stress in tooth root with the permissible stress. Both stresses are determined without any regard to a possible crack occurring in the tooth root. The methods based on the theory of fracture mechanics allow for numerical simulation of crack growth from some initial crack, which may be caused due to various reasons (mechanical or thermal treatment, material defects, overloads, etc.). These methods provide the means for a relatively reliable determination of the service-life of a gear with a crack in the tooth root [4, 5].

1.1 Basis of the linear elastic fracture mechanics (LEFM)

The theory of linear elastic fracture mechanics assumes that the area of inelastic straining is relatively small comparing to the crack length, e.g. $\omega/a < 0.4$ where ω is the length of the plastic region in the crack plane and a is the crack

length l . Therefore the effect of inelastic straining is neglected and only elastic strain distribution in the area of the crack tip is considered. The components of the stress tensor near the crack tip (see Fig. 1) can be calculated with $1, 2$:

$$\sigma_{ij} = \frac{1}{\sqrt{2\pi \cdot r}} \left[K_I \cdot f_{ij}^I(\theta) + K_{II} \cdot f_{ij}^{II}(\theta) + K_{III} \cdot f_{ij}^{III}(\theta) \right] \quad (1)$$

where f_{ij} ($i, j = x, y, z$) are dimensionless functions that depend on angle θ $1, 2$ and K_i ($i = I, II, III$) are the stress intensity factors that correspond to different crack opening modes, as it is shown on Fig. 1.

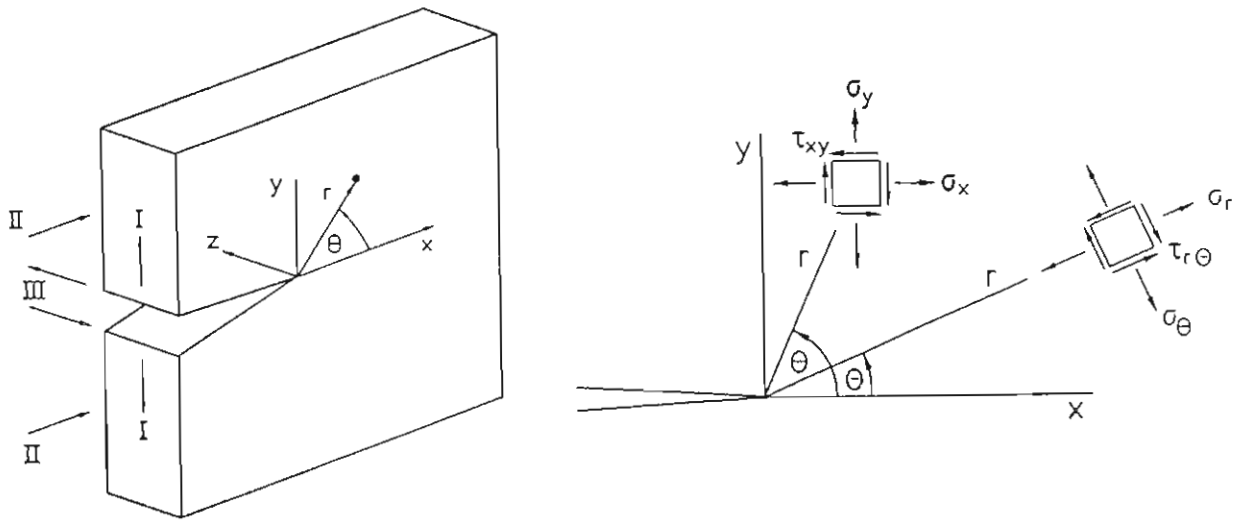


Figure 1: Crack opening modes and components of stress tensor near the crack tip

The equivalent stress intensity factor K can be defined as $1/$:

$$K = \sqrt{(K_I^2 + K_{II}^2) + (1 + \nu) \cdot K_{III}^2} \quad \text{plane stress} \quad (2)$$

$$K = \sqrt{(K_I^2 + K_{II}^2) \cdot (1 - \nu^2) + (1 + \nu) \cdot K_{III}^2} \quad \text{plane strain} \quad (3)$$

where ν is the Poisson's ratio. By introducing the fracture energy release rate G the equivalent stress intensity factor which corresponds to further crack opening can be written also as $1/$:

$$K = \sqrt{E \cdot G} \quad \text{plane stress} \quad (4)$$

$$K = \sqrt{\frac{E \cdot G}{(1 - \nu^2)}} \quad \text{plane strain} \quad (5)$$

The dependence of the stress intensity factor K on the crack length can be for simple cases determined using the methodology given in /1, 2, 3/. For more complicated cases it is necessary to use alternative methods (FEM, BEM) /4, 5/.

1.2 The crack growth by the fatigue process

The fatigue crack growth in a tooth root is based on the fatigue crack mechanics, which is used for modelling of the progressive crack growth due to dynamic loading. It is important to note that the equivalent loading might not cause any measurable crack propagation under quasi-static conditions. The results of fatigue crack investigations are frequently given in diagrams, which relate the crack propagation speed da/dN to the fatigue stress intensity factor ΔK (Fig. 2).

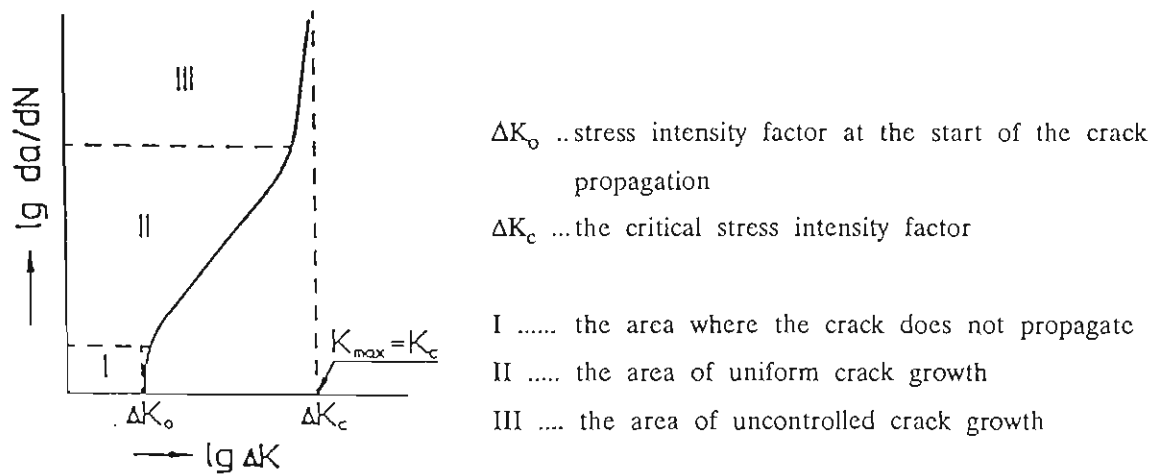


Figure 2: Diagram of the fatigue crack propagation speed

Most fracture mechanic investigations are conducted in the area II, in which it is assumed that the dependence between $\log(da/dN)$ and $\log \Delta K$ is linear and can be described with the Paris equation /1/, /2/:

$$\frac{da}{dN} = C \cdot [\Delta K(a)]^m \quad (6)$$

where C and m are coefficients of the Paris equation. They depend on the material and can be determined experimentally, usually by means of the three-point bending test according to the standard ASTM E 399-80 /3/. By integrating equation (6) one can obtain equation:

$$\int_0^N dN = \frac{1}{C} \cdot \int_{a_0}^{a_c} \frac{da}{[\Delta K(a)]^m} \quad (7)$$

which indicates that if C , m and $\Delta K(a)$ are known, the number of loading cycles N , that is required for a crack to propagate from the initial crack length a_0 to the critical crack length a_c , where the tooth fracture is expected, can be explicitly determined. The critical stress intensity factor K_c , which generally depends on the thickness of the component with a crack, corresponds to the critical crack length a_c . In the plane strain case the K_c becomes independent of the thickness and depends only on the material characteristics. Usually the critical stress intensity factor is referred to as the fracture toughness and is designated as K_{Ic} . The value of K_{Ic} can be determined experimentally according to the standard ASTM E 399-80 /3/ or it can be obtained from the available technical literature for well known materials /1/, /2/.

2.0. Determination of stress intensity factor by FEM method

The virtual crack extension method (VCE), as originally developed by Hellen /6/, has been used for the determination of stress intensity factor in the tooth root. The VCE method is a part of the finite element system BERSAFE and is based on the equation for the elastic fracture energy release rate /1/, /2/, /6/:

$$G = -\frac{dP}{da}, \quad (8)$$

where dP is the potential energy and da is the crack length. The lower bound value for the stress intensity factor K that is required for further crack extension can be determined with equation (4) or (5). Using the VCE method the stress intensity factor is determined in several different possible crack extension directions (Figure 3) and the crack is extended in the direction of the maximum stress intensity factor in the prescribed crack extension increment. The procedure is repeated until the stress intensity factor reaches the critical stress intensity factor K_{Ic} , when the complete tooth fracture is expected. The VCE method is based on the assumption of a linear elastic fracture problem and can be successfully applied for problems where the size of the plastic zone near the crack tip can be neglected.

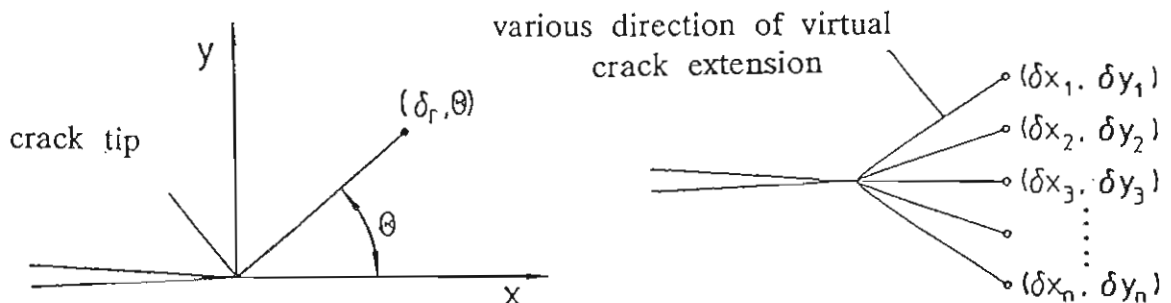


Figure 3: Virtual crack propagation in Cartesian and polar coordinates

3.0 Determination of stress intensity factor by BEM method

The program package BEASY [7] has been used for the determination of stress intensity factor in the tooth root. BEASY uses the equations provided by Irwin [8], which relate the stress intensity factors to the displacements and stress components at a point near the crack tip. The equations are valid for the local crack coordinate system, as it is shown in Fig. 1, where r is the distance of a point from the crack tip, and θ is the angle measured from the x -axis.

BEASY uses the results at the boundary and at internal points to calculate the stress intensity factors, according to the user's requirements. The value of the stress intensity factors may be determined from BEASY's displacement results or from stress results near the crack tip. In general it has been found more reliable to use the displacements. The equation for determining K_I from displacements is

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos(\frac{\theta}{2}) \cdot \left[\frac{1}{2} \cdot (\kappa - 1) + \sin^2(\frac{\theta}{2}) \right] \\ \sin(\frac{\theta}{2}) \cdot \left[\frac{1}{2} \cdot (\kappa - 1) - \sin^2(\frac{\theta}{2}) \right] \end{Bmatrix} \quad (9)$$

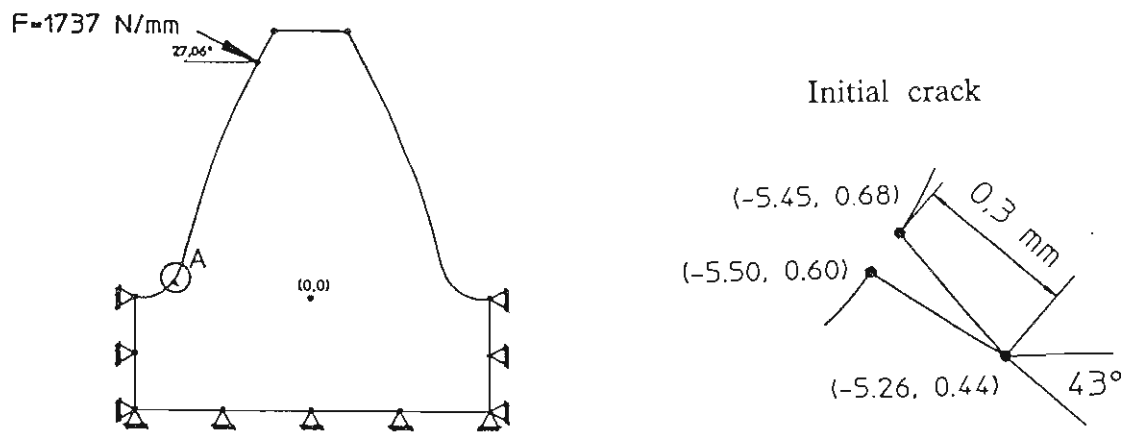
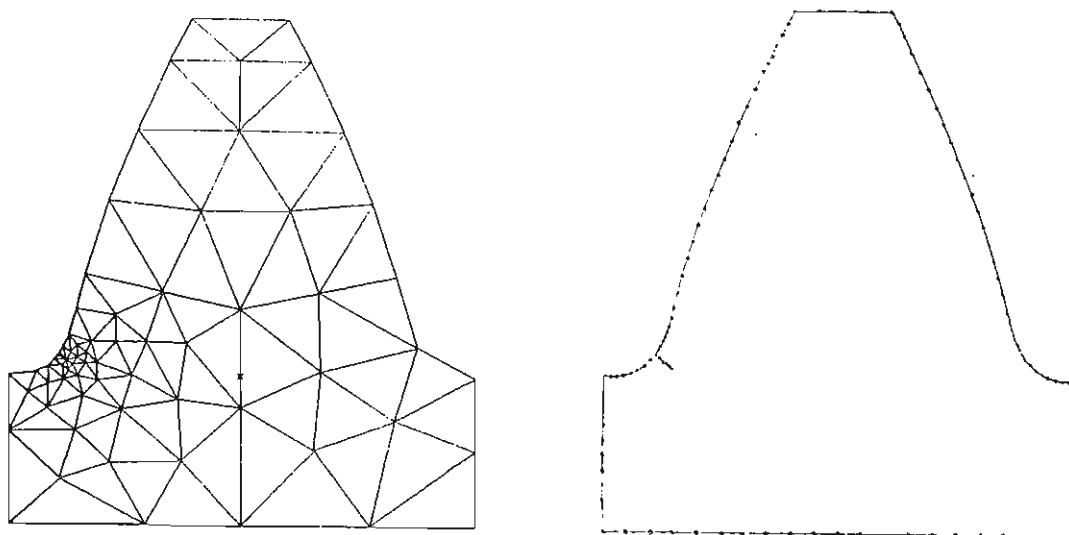
where $\kappa = (3 - 4\nu)$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for the plane stress case, with G now denoting the shear modulus.

4.0 Practical example

A spur gear pair is being analysed, with the loading torque $T = 3640$ Nm; the material is hardened steel and the complete gear data set is given in Table 1. Fig. 4 shows the gear tooth model used for the numerical analysis of the stress intensity factor in the tooth root by the finite element method and the boundary element method, with the corresponding boundary conditions (constraints, loading) and the initial crack. The finite element and boundary element discretisations of the tooth are shown on Fig. 5.

Table 1: Basic data of the spur gear pair

Magnitude	Designation	Unit	Pinion	Gear
number of teeth	z_1, z_2		11	39
normal module	m_n	mm	4,5	
tooth width	b	mm	32,5	28
centre distance	e	mm	115	
pressure angle on pitch circle	α_n	degree	24	
coeff. of profile displacement	x_1, x_2		0,526	0,0593
diameter of pitch circle	d_1, d_2	mm	49,500	175,500
contact ratio	ϵ_α		1,197	

*Figure 4: Model of the tooth and definition of the initial crack**Figure 5: The finite element mesh and the boundary element mesh*

The relation between the stress intensity factor K and the crack length a ($K=K(a)$) has been determined from a number of numerical solutions for the

stress intensity factors corresponding to different crack lengths by using the finite element method and the boundary element method. The coefficients of the Paris equation have been determined by the three-point bending experiment /4/:

$$C=0.101 \cdot 10^{-12} \quad \text{and} \quad m=2.939$$

Since the experiment required for the determination of the fracture toughness K_{Ic} was not possible, K_{Ic} was chosen from the technical literature /1/ to be:

$$K_{Ic} \approx 2429 \text{ to } 2516 \text{ Nmm}^{-3/2}$$

5.0 Analysis of the results

Fig. 6 shows the obtained relationship $K=K(a)$. It can be observed that both numerical methods, the finite element method and the boundary element method give almost identical results.

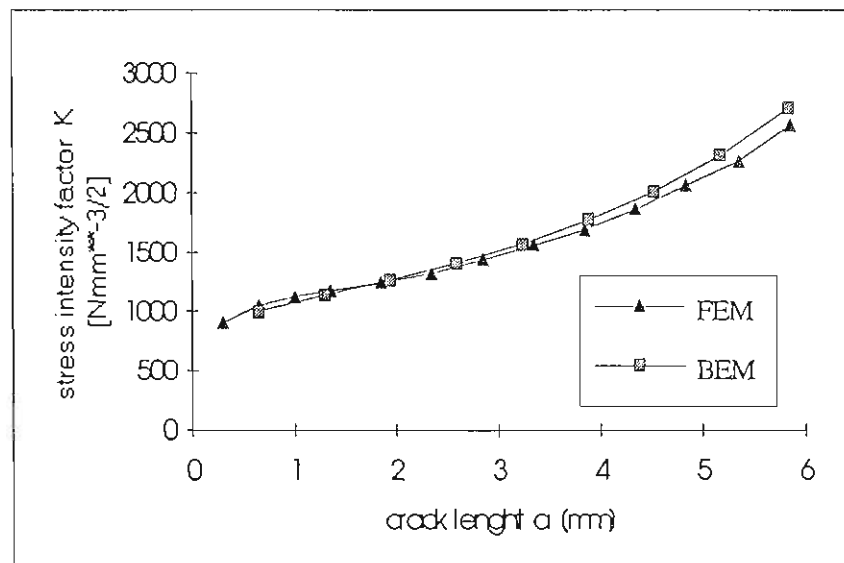


Figure 6: Comparison of stress intensity factors in the tooth root by FEM and BEM

To determine the service-life of a gear, *i.e.* the number of the loading cycles that is required for a crack to propagate from the initial length to the critical length (see equation 7), the results for the stress intensity factor obtained with the finite element method have been used. Because the gear loading is impulsive (it varies from zero load to the maximum load - see Fig. 4), it can be assumed that in equation (7) $\Delta K(a)=K(a)$. The relationship $K(a)$ follows from Fig. 6. Equation (7) can be numerically integrated and the results of such an integration are shown in Fig. 7.

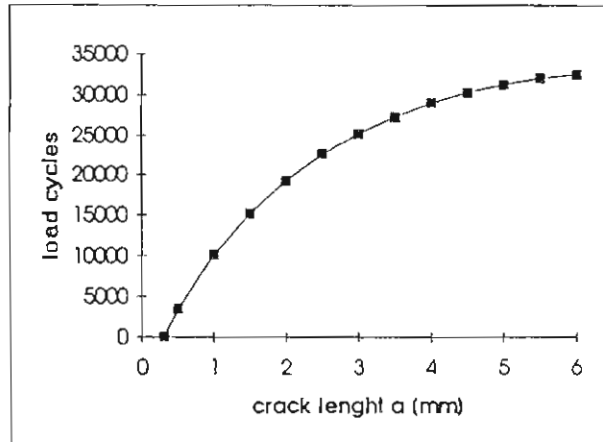


Figure 7: Crack length increase in the tooth root at different number of loading cycles

6.0 Conclusions

The presented investigation has been focused on the numerical determination of the stress intensity factor in the tooth root of a gear with the crack. Two numerical methods have been used, the finite element method and the boundary element method, and almost identical results of the two-dimensional analysis of a gear tooth with a crack in the tooth root have been obtained. Numerically determined stress intensity factors have been further used for the determination of the relation between the crack length and the number of loading cycles, which in turn defines still serviceable time period for a gear with the crack in the tooth root. Outlined procedure illustrates the crucial advantage of the numerical simulation with respect to the classical procedures as described in standards DIN, AGMA, ISO, etc., which do not provide for any consideration of the existing crack in the tooth root.

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Rapid Crack Propagation by FEM

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SUMMARY

The finite element method computer program is used to determine the material behaviour during rapid crack propagation (RCP). The generation mode, fixed mesh and step by step implicit procedure is used. The full dynamic analysis is necessary due to dynamic effects (inertia, stress wave, etc.) to determine a crack speed, crack length and loading relationship in terms of energy release rate. The modelling of the local non-linearity with a low Young's modulus local strip model is used.

1. INTRODUCTION

Presently, there is considerable interest in analysing dynamic fractures, since the hazards involved when they occur can be substantial. Although fast brittle fracture is of considerable importance in several practical applications, numerous experimental and analytical problems are not yet resolved. One of them is the dependence of crack velocity $\dot{a}$ on dynamic stress intensity factor has been subject of experimental and theoretical work. The theory of elasticity applied to the plane problem of a crack in a body yields as a solution infinite normal stresses at the crack tip. It is unreasonable to expect such an idealised stress state, even in very brittle materials, and some kind of "process" zone [15], a zone in which permanent inelastic deformation take place prior to crack extension, must exist ahead of a crack tip to lower the stress magnitude and alter the stress distribution suggested by LEFM. The phenomenon of a process zone and the parameters influencing it have not been fully understood or explained.

In the present study, the investigation of the material behaviour during rapid crack propagation was done on single edge notch tensile specimens. Under high speeds of the crack propagation the apparent toughness is largely due the dynamic effects, inertia and stress wave propagation, which are even more pronounced in the brittle materials. The numerical analyses by finite element method were implemented and results show that RCP is not covered by the LEFM approach. Due to the high strain rates and the assumption that the Young's modulus at the crack tip is lower, the values of dynamic toughness are lower and almost constant during the fracture event. The data for numerical analyses were obtained from the experiment that will be briefly described.

2. EXPERIMENT

A study of dynamic fracture requires measuring quantities such as K_{ID} -dynamic stress intensity factor or G_{ID} -dynamic energy release rate, that characterise the magnitude of the driving force in the fracture process for materials failing in a predominantly cleavage mode. For components subjected to static loading until initiation, the fracture behaviour is governed by the relationship between the crack speed $\dot{a}$ and $K_{ID}(t)$ or $G_{ID}(t)$. Experiments are not easy since they involve a very short time event. Experimental investigations of dynamic fracture have involved optical methods of measurement including reflection photo elasticity, caustics and more recently strain gauges [6]. The analysis of RCP behaviour with the indirect FEM requires accurate crack length versus time history data. For the SENT specimens with opening mode loading the grid line technique (timing lines method) was used for the crack speed measurement. The material properties must be known, with the experimental data, in order to analyse material behaviour. The material used was epoxy, with following properties: density-1120.4kg/m³; Poisson's ratio-0.38; dynamic Young's modulus-3.5GPa. All tests were made on a conventional tensile machine at the room temperature. Each specimen was clamped rigidly with two heavy grips in order to satisfy fixed-displacement conditions during RCP

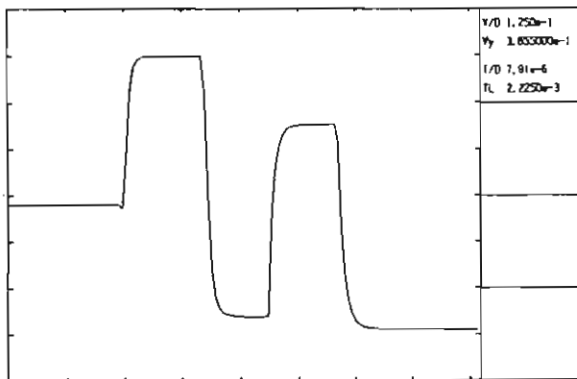


Fig.1. Typical response record.

(global energy balance is based upon the initial loading conditions during propagation). The SENT specimens were clamped with heavy clamps in order to simplify the model and to avoid cross dynamic effects (oscillations of the sample, dynamics contact stiffness, friction of a wedge, etc.). The clamps were left unlocked and rotation of grip was not suppressed, so torsion of the sample was avoided.

The chart recorder was used to record the initial load and the oscilloscope was connected to timing lines through the amplifier. The oscilloscope was set to the external the triggering and the records (figure 1) were stored on the floppy disc.

From experiments the data for analysis were obtained (breaking load, crack length history) and some observations were made: constant crack speed velocities, the crack propagates with a straight front, producing a uniform planar surface straight across the thickness and the crack speed is dependent on the breaking load.

3. FEM ANALYSIS

The solution of dynamic fracture problems becomes very difficult if the surface internal energy density per unit mass and mass per unit surface [3] are to be incorporated in the problem. An indirect method of calculation is used accompanied by some general assumptions:

- the equation of motion (1) is valid within the given region ,

$$\text{div}[\sigma] + \rho\{b\} - \rho \frac{d\{v\}}{dt} = 0 \quad (1)$$

where: $[\sigma]$ -stress tensor, ρ -density, $\{b\}$ -body forces per unit mass, $\{v\}$ -velocity field vector

- the stress tensor is symmetrical at all points of the body.

The computer program numerically solves the equation of motion written in matrix form:

$$[K]\{\delta\} + [M]\{\ddot{\delta}\} = \{F\} \quad (2)$$

where:

$[K]$ -stiffness matrix, $[M]$ -lumped mass matrix, $\{F\}$ -external force vector, $\{\delta\}$ -nodal displacement vector, $\{\ddot{\delta}\}$ -nodal acceleration vector.

From the test specimens it could be seen that the crack velocity is constant on a measured ligament (without initiation) and that the crack propagates with a straight front, producing a uniform planar surface, the crack front is straight across the specimen thickness and it is reasonable therefore to analyse the crack with two-dimensional numerical analysis. 2D isotropic constant strain triangular elements were used. Due to the symmetry only a half of specimen has been modelled by the FE mesh, and so the crack path becomes the boundary. The mesh is gradually refined on the perspective crack path. At first, the system of equations is solved for static solution of the stress-strain distribution by the iterative Gauss-Seidel scheme with relevant boundary conditions. Obtained an initial static solution a crack is propagated by the sequential release boundary nodes at times specified by the experimental data.

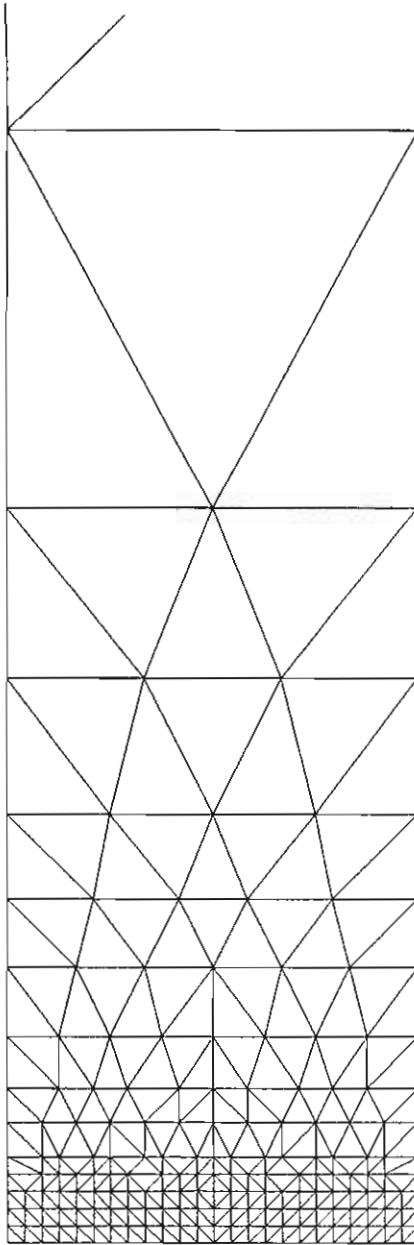


Fig.2. FE mesh sample.

The generating mode and fixed mesh (node release) procedure is used. Generation mode, from loading and boundary conditions, geometry, mechanical properties and known crack length history "generate" the corresponding dynamic crack resistance.

A propagating crack is simulated by a sequential release of nodes along the crack path, changing the boundary conditions from constrained to free. The dynamic solution is computed by the step-by-step implicit procedure [4], suitable for dynamic non-linear problems. This procedure allows changes of boundary conditions (stiffness and inertia) in time simply by accommodating updated value of [M] and [K] in the equation of motion.

From the solution for the displacements at the present and previous time step is possible to extrapolate future accelerations:

$$\ddot{\delta}(\tau) = -\frac{1}{2}(1-\tau)\tau\ddot{\delta}_- + (1-\tau)(1+\tau)\ddot{\delta}_0 + \frac{1}{2}(1+\tau)\tau\ddot{\delta}_+ \quad (3)$$

('0' present, '-' previous, '+' future)

where: $\tau = (t - t_0) / \Delta t$; Δt - time step; t - time;

and with Taylor series expansions predicts:

$$\begin{aligned} \{\dot{\delta}_+\} &= \{\dot{\delta}_0\} + \Delta t \{\ddot{\delta}_0\} - \frac{1}{24} \Delta^2 t \{\ddot{\delta}_-\} + \frac{5}{12} \Delta^2 t \{\ddot{\delta}_0\} + \frac{1}{8} \Delta^2 t \{\ddot{\delta}_+\} \\ \{\ddot{\delta}_+\} &= \{\ddot{\delta}_0\} - \frac{1}{12} \Delta t \{\ddot{\delta}_-\} + \frac{2}{3} \Delta t \{\ddot{\delta}_0\} + \frac{5}{12} \Delta t \{\ddot{\delta}_+\} \end{aligned} \quad (4)$$

These equations may be solved interactively to estimate the future displacements and velocities. Iteration continues until adequate convergence is obtained. From displacements and velocities throughout the specimen, at each time step it is possible to evaluate both, the total strain and kinetic energy. The problem becomes progressively non-linear due to changing boundary conditions.

Spurious oscillations and no energy loss is produced during the release of nodes caused by the crack tip jumping from one node to another. To overcome this problem, a "holding back force" [14] is applied on the last released node. The "holding back force" decreases gradually according to the exact position of the crack tip in every time step by conventional separation law:

$$F_h = -F_0 \left(1 - \frac{d}{DD} \right) \quad (5)$$

where:

F_h - holding back force; F_0 - force at crack tip before releasing; d - increment distance of the crack travel between nodes; DD - distance between two nodes.

The time step is limited with the following stability criterion [7] to avoid an unstable solution:

$$\Delta t = \frac{e}{10C} \quad (6)$$

where e is the smallest size of an element in the structure.

Every time a node is released, the energy absorbed by the crack propagation is computed from the global energy.

The effects of forces which are applied for only a very short period of time or change rapidly, must be considered in terms of the propagation of stress waves. According to the laws of motion particles accelerate from a stationary position and their speed is a function of the magnitude of the stress at that point [9,10]:

$$\sigma = \rho \cdot C \frac{du}{dt} \quad (7)$$

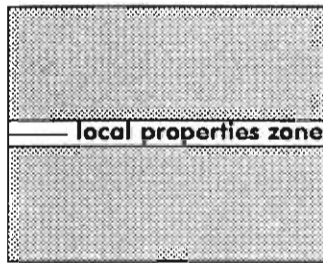
For the cases in which material non-linearity exists, the dilatation waves can be propagated at different speeds depending on the stress-strain relation:

$$C = \sqrt{\frac{1}{\rho} \frac{d\sigma}{d\epsilon}} \quad (8)$$

and for the linear case $d\sigma/d\epsilon = E$.

From other investigations [2] it can be concluded that the effects of the initiation accelerations of the crack propagation are not significant. Even from a sharp notch the crack accelerates so rapidly, to a constant speed, in such a very short period of time and crack growth that the effects on the calculated G_{ID} are negligible.

The global response of the loaded and then suddenly broken sample can be interpreted as slow rate loading and high rate unloading, which are dominated by 'static' and higher 'dynamic' Young's modulus respectively. The dynamic modulus determines the amount of retained energy available for the fracture process, since the material cannot react fast enough to reach full relaxation during the event. Since during unstable crack propagation, viscous energy dissipation is quite small [5] in the computation of G_{ID} in visco-elastic materials, only the available energy should be considered (not the total amount of energy in the sample). If the single, dynamic, modulus is used throughout computation for both parts of process, loading and unloading, the total elastic energy is similar to the true available energy [16] and the displacement is only a fraction of the true or measured displacement. The usage of a static modulus for loading and a dynamic modulus for unloading gives a false stored strain energy. The material is strain rate dependent, the visco-elastic solutions are generally idealised because the parameters of such models are unknown real material high rates. The rate dependence of the plastic flow is simulated by assuming that the cohesive stress in the yield zone depended linearly on the local rate and predicted a much lower wave speed than expected.



The fracture event is dominated by the stress-strain-strain rate relation that governs the local process at the crack tip and the capability of the remote stress field to transmit its energy into the crack tip region. Further, the parameters of the material behaviour are uncertain under dynamic conditions. In the "local strip model" local non-linear effects are

Fig.3. Local strip model modelled by considering a local strip along the perspective crack path. The large strain around crack tip would lead to a local lower modulus than of the base surrounding material. Both the strip layer and the surrounding material are taken as isotropic and linearly elastic. Any energy dissipation due to eventual plastic deformation is not considered, because no evidence of macroscopic yielding was observed on tested samples. Neglect of any plastic deformation energy dissipation is not essential in the analysis, since during loading the mechanical processes are equal to those in a non-linear elastic medium and unloading takes place mainly behind the crack tip [8].

4. CONCLUSIONS

The results of analyses are presented in figure 4 and shows the change of the dynamic crack resistance versus crack length. The analyses shows more constant G_{ID} with lower modulus in local strip.

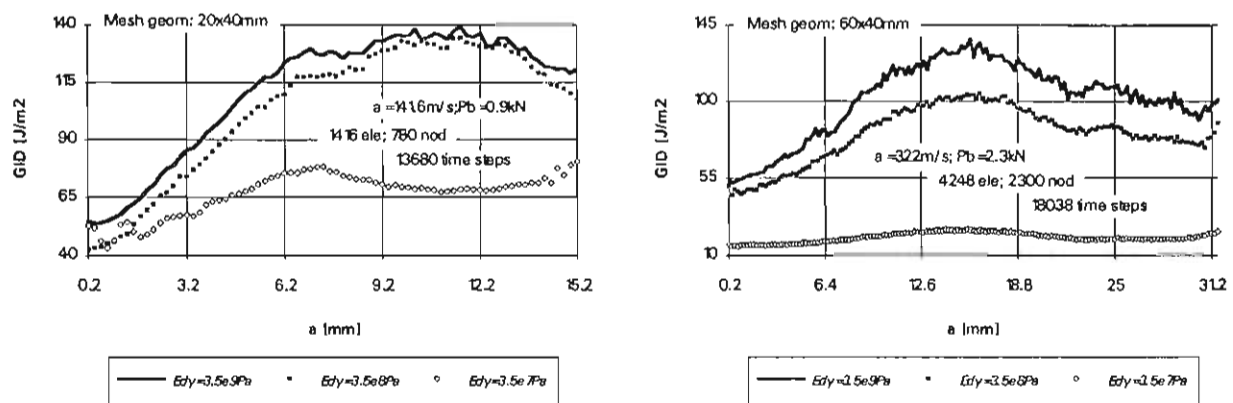


Fig.4. Dynamic crack resistance vs crack length for two samples (various modulus).

Figure 5 represents the system energy response, where are the kinetic energy data multiplied by ten to fits into the graph. The crack velocities depends on the subsequent stress distribution along the crack path. In order to have the crack moving with constant speed the steady stress distribution along the crack path must be satisfied and the constant speed requires constant G_{ID} .

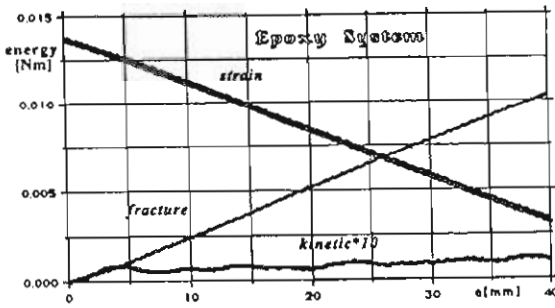


Fig. 5. Energies behaviour during event.

The crack resistance is expected to be rather constant as concluded from the fracture surface appearances. The acceleration and deceleration of the crack is induced by the non uniform surface roughness [11]. The large increase in generated G_{ID} with the crack length, for the constant crack propagation, cannot be accepted as realistic and cannot be physically explained.

Similar analysis was performed by [12] and concluded that the adequate G_{ID} results were obtained assuming 'fixed grip' boundary conditions during the fracture. However, the calculated G_{ID} cannot be affected by the way the clamped boundary was simulated until the first stress wave returns from the boundary to the crack tip. The crack resistance has enough time to rise, especially when the crack speed is high. Obviously, the problem of the increasing G_{ID} with the crack length cannot be solved simply adjusting the boundary conditions. The most important observation is that the experimental data can only be modelled by using a much lower value of the tensile wave speed (Young's modulus) than could be expected. This has been observed before, though not qualified in this way, and in [8] is suggested that it may be due to local-non-linear effects at the crack tip.

The experimental data is highly scattered even when extreme care is taken [13]. Even more problematical is the observation that measured crack speeds seem to remain remarkably constant, even though the static energy release rate is changing [12,13]. However well one performs the analysis, this always leads to multivalued G versus $\dot{a}$ curves with rapid increases of G with $\dot{a}$. Part of the difficulty in resolving the matter is that there are very few analytical results which are useful for analysing the tests, so one relies entirely on the numerical solutions. The local strip model is believed to be a good approximation of the material behaviour close to the moving crack tip, since the local modulus determine G_{ID} . The generated G_{ID} is thought to be close to the 'true' crack resistance or at least, much closer than G_{ID} obtained by using single constant modulus approximation.

The next logical step in investigation should be the implementation of incremental technique which will introduce local low modulus just near the crack tip, in order to get better simulation of RCP.

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COMPUTATIONAL AND MODELLING ASPECTS OF THE SOFTENING FIVE PARAMETER MODEL FOR CONCRETE

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Abstract

Computational and modelling aspects of the fracture energy based softening plasticity model for plain concrete, using the Willam/Warnke five parameter surface description for both initial and residual failure surfaces are discussed. A consistently linearised stress return algorithm includes separate treatments of the regular region, apex region and the transition region. Model performance is illustrated on a plain concrete notched prism example.

1 - Introduction

Fracture energy based softening plasticity models represent the simplest category of regularisation techniques to avoid mesh sensitive results when modelling localisation problems in concrete. In addition to a whole range of other regularisation techniques (gradient theories, Cosserat continuum, viscoplasticity, [1]), the fracture energy based softening plasticity models are still a subject of considerable research efforts [2-5].

Recent publications include extensions of softening Rankine plasticity models by adding a separate compressive envelope [2] and the extended Leon's model [3] based on parabolic meridians and non circular deviatoric sections. A common feature of these attempts is not only the tendency to employ more sophisticated descriptions of the initial and the residual surface, but rather the search is directed towards finding appropriate ways of controlling the evolution (shrinking) of the failure surface via the evolution of mode I or mode II fracture energy densities or some combination of both fracture energy densities.

The need to secure *quadratic convergence* of the nonlinear solution scheme on the global level has prompted research efforts into the consistent linearisation at the local Gauss point level, i.e. into the accuracy and robustness of local stress return algorithms for softening plasticity models involving complex smooth surface descriptions, as well as surfaces with singular regions.

2 - Softening Five Parameter Model

The five parameter model, formulated by Willam and Warnke [5] represents one of the most successful smooth surface descriptions for triaxial failure envelope of concrete [6], covering in a unified way the *tension-tension*, *tension-compression* and *compression-compression* regimes and requiring five parameters - uniaxial tensile f_t and uniaxial compressive strength f_c , biaxial compressive strength f_{bc} and two stress states at higher confinement level. Within the *fracture energy based regularisation* of the post peak behaviour, the simplest softening assumption is adopted here, such that the failure surface evolution law from the initial to a residual surface is controlled by the reduction of only one of the five parameters, the tensile strength f_t . An exponential softening law for the reduction of tensile strength $f_t(\kappa)$ (starting from its original value f_t') as a function of the effective plastic strain κ is adopted. For simplicity, no distinction is made between tensile (mode I) and shear fracture (mode II) energies in terms of their respective contributions to the way the failure surface changes.

$$f_t(\kappa) = f_t' e^{-\frac{h f_t'}{G_f} \kappa}$$

The total fracture energy release rate G_f is hence utilised and the characteristic length h is assumed to be related to the element size.

The softening five parameter Willam Warnke surface can be written as

$$\Phi(\sigma, \kappa) = \tau_m - f_c r(\sigma_m, \theta, \kappa) = 0$$

where $r(\sigma_m, \theta, \kappa)$ represents an elliptic interpolation between the compressive and tensile meridians for a given mean stress level. The shapes of both compressive and tensile meridians depend on the current reduced ("softened") tensile strength $f_t(\kappa)$,

$$r_t(\sigma_m, \kappa) = a_0(\kappa) + a_1(\kappa) \frac{\sigma_m}{f_c} + a_2(\kappa) \left(\frac{\sigma_m}{f_c} \right)^2 \quad \theta = 0^\circ$$

$$r_c(\sigma_m, \kappa) = b_0(\kappa) + b_1(\kappa) \frac{\sigma_m}{f_c} + b_2(\kappa) \left(\frac{\sigma_m}{f_c} \right)^2 \quad \theta = 60^\circ$$

Finally, these continuing changes of tensile and compressive meridians define in turn the evolution from the initial to the residual surface.

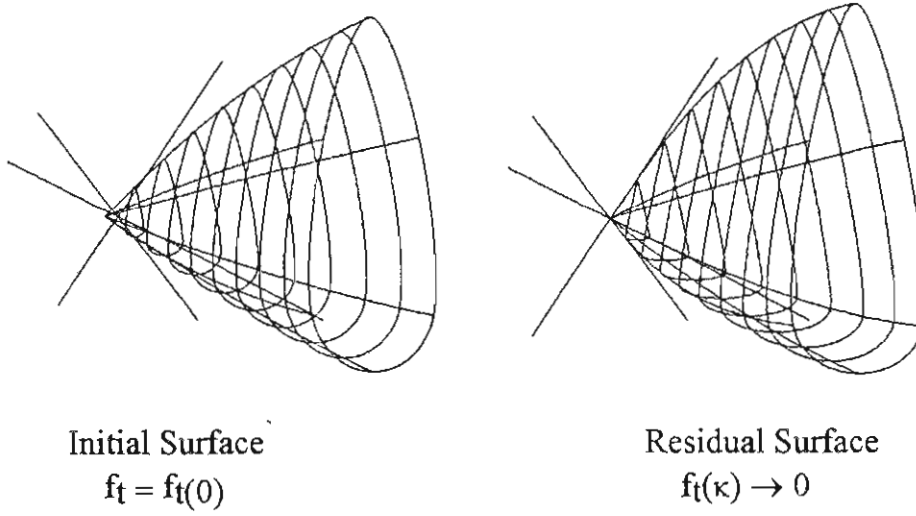


Fig 1 - Initial and residual surfaces of the softening five parameter model

3 Computational aspects of the local stress return

Although the five parameter model represents a smooth surface description with continuous gradients for all regimes except for the apex on the hydrostatic axis (corresponding to the triaxial tensile strength, f_{tt}), *three different zones* for the stress return are identified [8]

(I) *regular zone (a)* $\sigma_m \leq f_{tt}$

where the *closest point projection* algorithm starting from the elastic predictor stress state is adopted following Crisfield [7], and where the convergence is simultaneously sought for the consistency condition $\Phi = 0$, the stress residual norm and the norm on the effective plastic strain residual

$$\Phi_{n+1}(\sigma_{n+1}, \kappa_{n+1}) \leq \epsilon_\Phi \quad \left| \mathbf{r}_{n+1}^\sigma \right| \leq \epsilon_\sigma \quad \left| \mathbf{r}_{n+1}^\kappa \right| \leq \epsilon_\kappa$$

(II) *apex zone (c)* for stress states where

$$\sigma_m \geq f_{tt}(\kappa)$$

and

$$\tau_m \leq \left(\frac{K \partial \Phi / \partial \sigma_m}{\frac{3}{5} G \partial \Phi / \partial \tau_m} \right)_{f_{tt}(\kappa)} (\sigma_m - f_{tt}(\kappa)) = \text{slope}(\kappa) (\sigma_m - f_{tt}(\kappa))$$

where the stress return is forced to the current apex point $f_{tt}(\kappa)$. As the apex point shrinks during softening, a secant method is adopted such that the amount of plastic straining corresponds to the shift of the apex along the hydrostatic axis.

(III) *transition zone (b)* for stress states between zones (a) and (c)

$$\sigma_m \geq f_{tt}(\kappa) \quad \text{and} \quad \tau_m \geq \left(\frac{K \partial \Phi / \partial \sigma_m}{\frac{3}{5} G \partial \Phi / \partial \tau_m} \right)_{f_{tt}(\kappa)} (\sigma_m - f_{tt}(\kappa))$$

Separate treatment of this zone is necessary as the failure surface gradients are not defined for $\sigma_m \geq f_{tt}$ (both compressive meridian r_c and tensile meridian r_t become negative and the surface becomes concave). Therefore, an improved predictor is needed to bring the stress state into a regular regime, from where the closest point projection algorithm is continued. The improved predictor is achieved by scaling the stress tensor components onto the current failure surface.

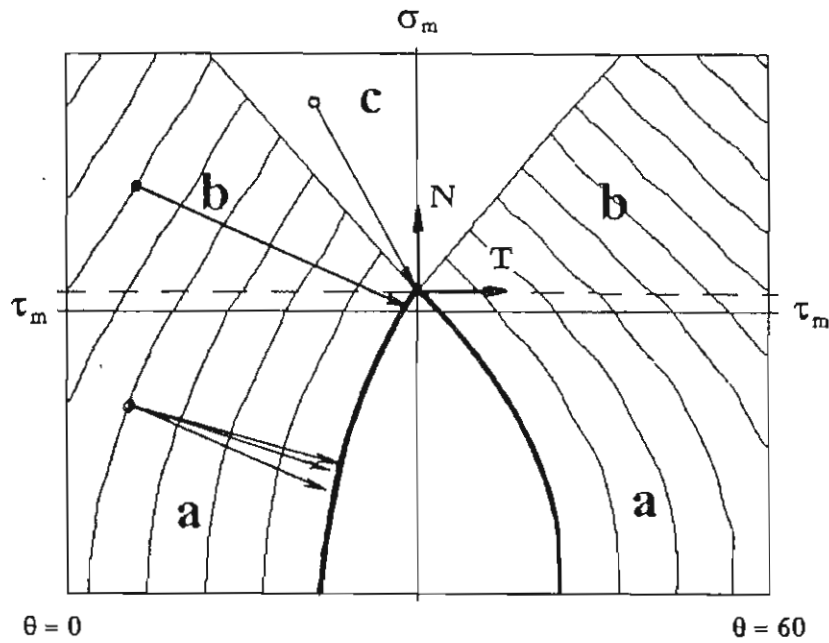


Figure 2 - Potential surfaces and stress return strategies for
(a) regular, (b) transition and (c) apex stress return zones
(potential lines in apex zone omitted)

Several features of the above stress return for a hypothetical material ($G_f = 0.5 \text{ N/mm}$, $f_c = 3 \text{ N/mm}^2$, $f_t = 1.25 \text{ N/mm}^2$, $f_{bc} = 3.48 \text{ N/mm}^2$, $E = 10 \text{ N/mm}^2$, $\nu = 0.2$) are illustrated on Figs 3 and 4. Starting from the initial state of stress at the apex, a series of normalised elastic trial states of stress are made in both normal (N) and tangential (T) direction. The number of iterations required for convergence are given on Fig 3, the relative percentage error e_σ comparing a single step return to a 1000 subincrement return on Fig 4.

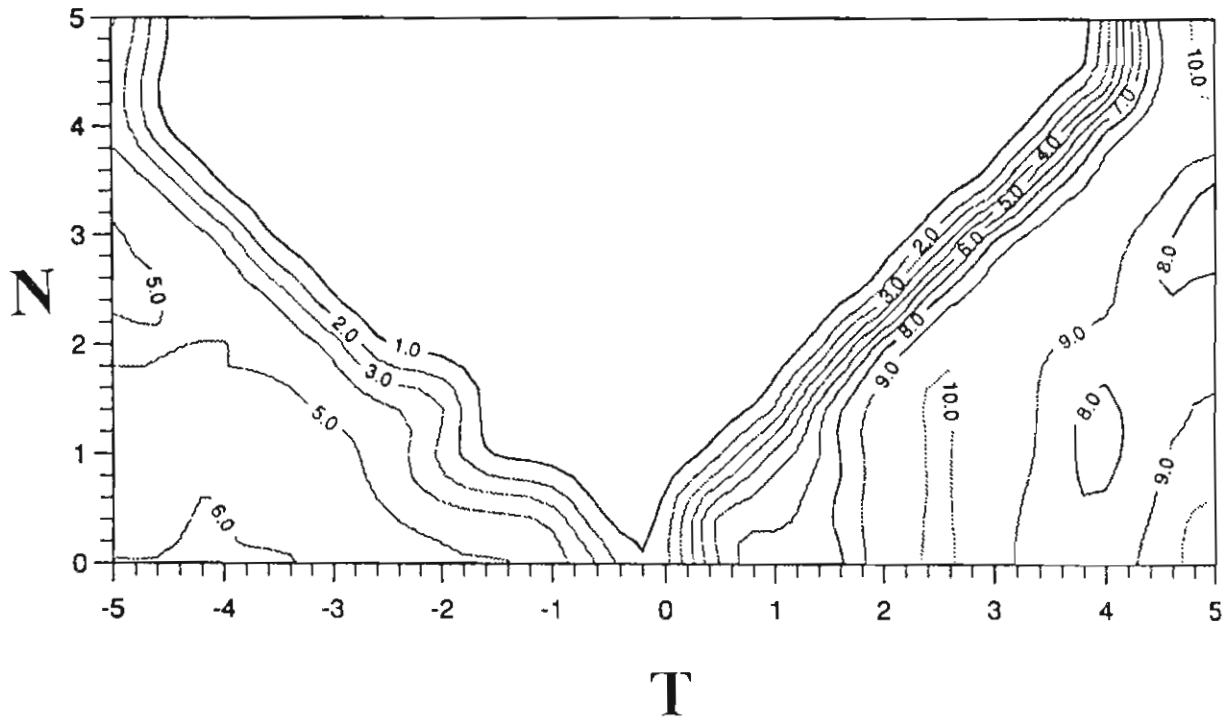


Figure 3 - Local stress return algorithm
 Number of iterations required for convergence
 (starting stress state at the apex, normalised stress increments
 in the normal (N) and tangential (T) directions)

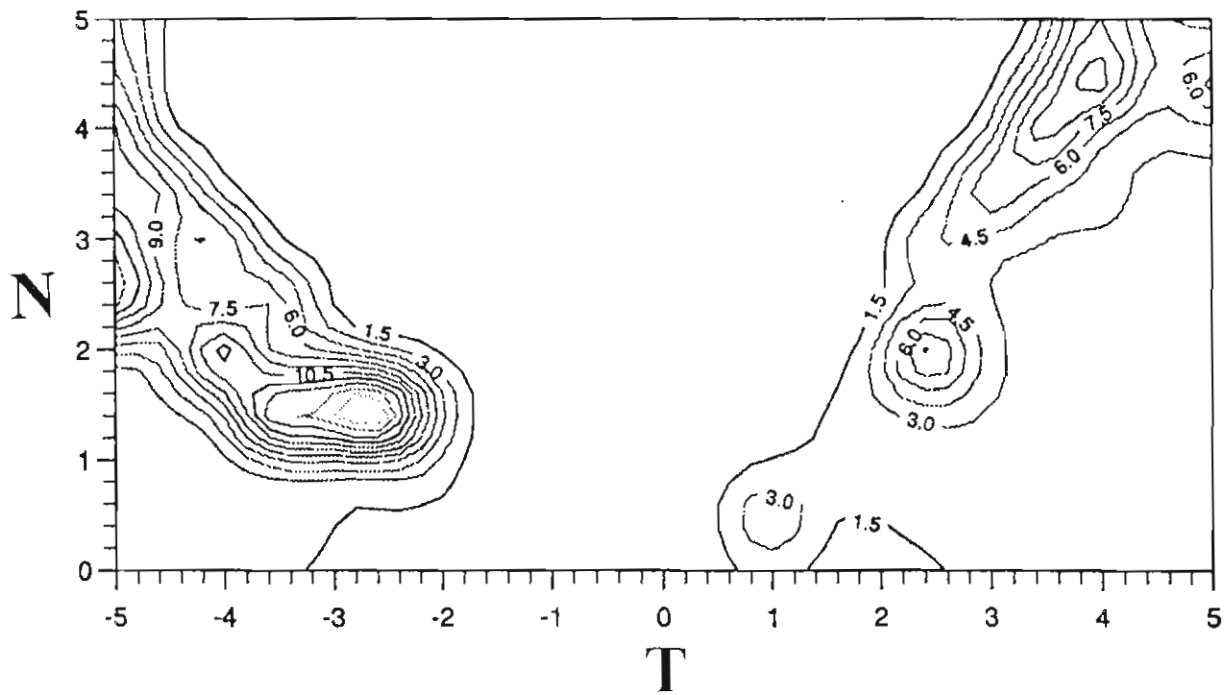


Figure 4 - Local stress return algorithm
 Relative percentage error e_σ
 (starting stress state at the apex, normalised stress increments
 in the normal (N) and tangential (T) directions)

4 Illustrative Example - Notched Prism

An example recently studied by Schlangen [10] is included for illustration purposes. Figure 5 shows a notched prism (150mm x 60mm x 50 mm, notch 5mm x 5mm) subjected to a tensile displacement control test, which was analysed using plane stress formulation and utilising double symmetry conditions

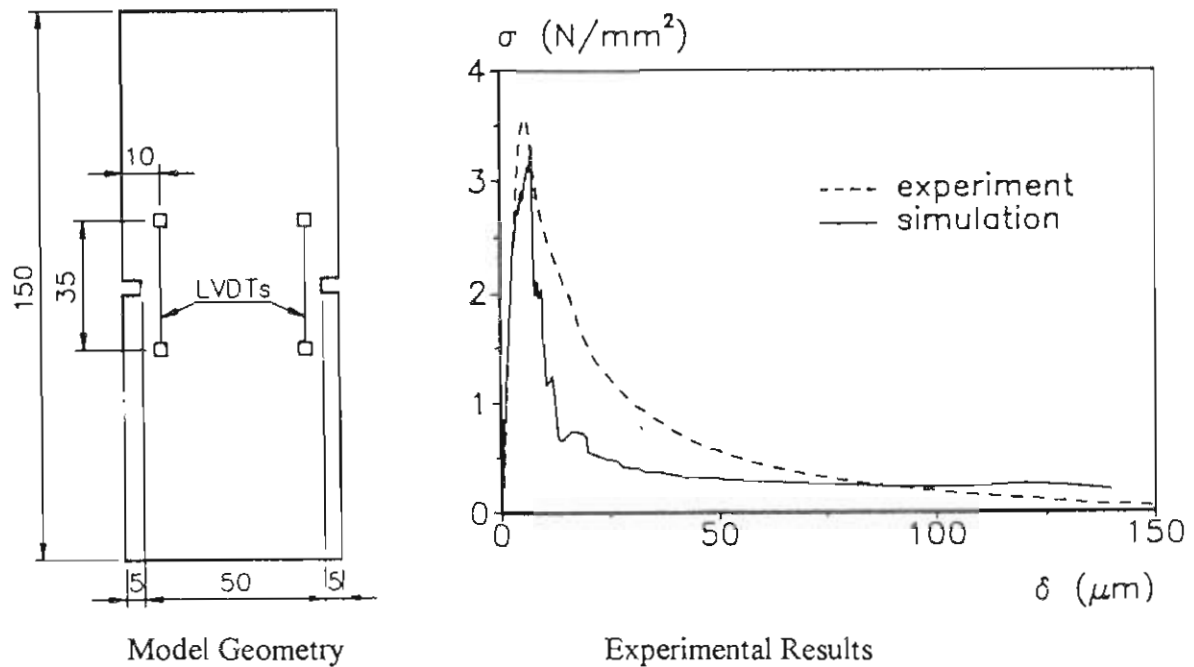


Figure 5 - Schlangen's Notched Prism Model Problem

$$\begin{aligned} G_f &= 0.1 \text{ N/mm} \\ f_t &= 3.75 \text{ N/mm}^2 \\ E &= 37700 \text{ N/mm}^2 \end{aligned}$$

$$\begin{aligned} f_c &= 30 \text{ N/mm}^2 \\ f_{bc} &= 34.8 \text{ N/mm}^2 \\ \nu &= 0.15 \end{aligned}$$

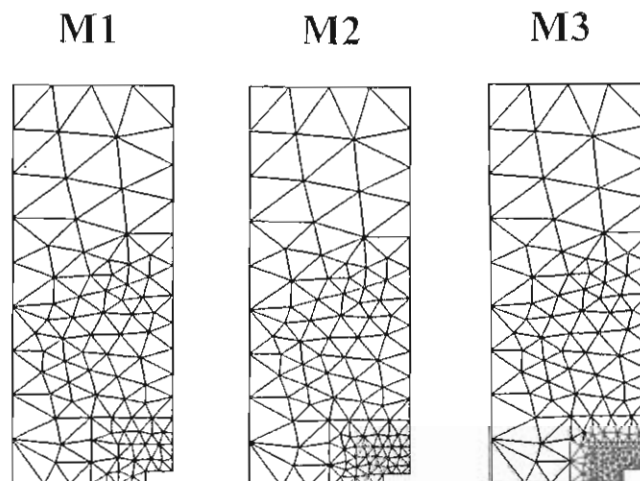


Figure 6 - Schlangen's Notched Prism Model Problem
Three levels of discretisation (meshes M1, M2 and M3)

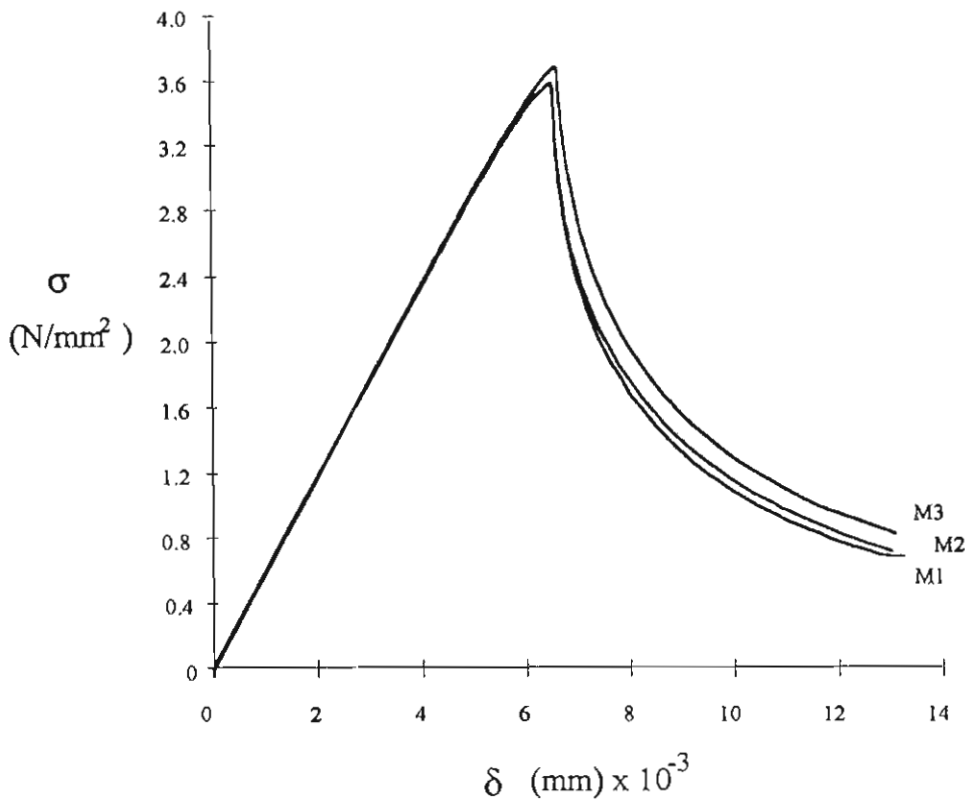


Figure 7 - *Schlagen's Notched Prism Model Problem*
Force-displacement curves for various meshes M1, M2 and M3

The force displacement response (Fig 7) for the three meshes M1, M2 and M3, used to model the double notched tensile prismatic specimen with varying mesh densities near the notch region (Fig 6), compare very well with the experimental findings and micro mechanical lattice model predictions by Schlagen. In addition, they clearly indicate the required mesh objectivity, secured through the fracture energy based regularisation of softening.

In most situations both local and global quadratic convergence was experienced. Some displacement increment cutting due to lack of convergence (both locally or globally) after a specified number of iterations, was only required near the peak response and at the tail end of the analysis.

Further model improvements are anticipated by an enhanced control of compressive strength parameters in addition to the tensile strength control only, by introducing the separate treatment of mode I and mode II energies.

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NUMERICAL MODELLING OF THE MICROSTRUCTURAL CHANGES DURING HOT WORKING

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Summary

New developments in contactless ultrasonics have enabled 'in-situ' measurements of the kinetics of phase change and recrystallization during metal forming operations at elevated temperatures. Scientists from AEA Technology have been successful in monitoring grain structure in aluminium alloy, copper, low alloy steel and nickel base alloy. It is expected that new experimental technique will be combined with inverse numerical techniques in order to provide an efficient means for estimation of material parameters involved in constitutive equations. In this paper Sellars model is presented and possibility for inverse estimation of material parameters is discussed.

1. Introduction

The microstructure that is developed during hot working processes has an important influence on the mechanical properties of the finished products. The evolution of the microstructure depends on the thermal and mechanical processing conditions. These conditions can be successfully predicted by coupled thermo-elasto-viscoplastic finite element models. The question that follows is; can the microstructural changes be predicted from the thermo-mechanical parameters provided by numerical analysis. It is well known that the higher the temperature, the larger the deformation will be and the slower the deformation rate, the more recrystallization will occur. However, these relationships have to be expressed in a quantitative way. Some material models, which provide a basic skeletal framework for such expressions, are reported in the literature [1-4] and a good general overview can be found in [2]. Extensive experimental work and complex validation procedures have to be undertaken in order to obtain data for the material models. Therefore it is not surprising that data for only a limited number of materials are available. However, new developments in contactless ultrasonics [5] and inverse numerical techniques provide a good opportunity for developing more efficient tools for investigations of microstructural changes.

In this paper the Sellars model [1], which has been implemented in our computer code, is briefly described and discussed in the light of new developments. In order to predict microstructural changes, where the most important variable is the grain size, the following have to be determined:

- the onset of recrystallization
- the recrystallization rate
- the recrystallized grain size
- grain growth after recrystallization.

In the next four sections the Sellars model is presented.

2. Dynamic structural changes

The structural changes taking place within the material during deformation result in an increase in dislocation density with strain until, in austenitic steels, a critical equivalent strain ϵ_c is reached, when the stored energy is sufficiently high to cause dynamic recrystallization. It has been experimentally observed that recrystallization starts just before the peak strain ϵ_p occurs. The relationship between the critical and peak strain is complex but it has been suggested that a linear expression

$$\epsilon_c = a \epsilon_p \quad (1)$$

is a reasonable approximation for the conditions of interest in hot working. Various values of a have been proposed in the literature but in the present computation 0.8 is used. It has also been observed that ϵ_p increases systematically with the Zener-Hollomon parameter Z and the grain size d_o . This relationship is assumed to be

$$\epsilon_p = A d_o^{1/2} Z^n \quad (2)$$

where A and n are constants. For C-Mn steel values of $4.9 \cdot 10^{-4}$ for A and 0,15 for n are proposed.

3. Static recrystallization kinetics

After deformation, softening by static recovery and recrystallization takes place. The kinetics of these two processes depends on the prior deformation conditions and the holding temperature. The static recrystallization under isothermal conditions can be described by the Avrami equation

$$X_v = 1 - \exp \left[-C \left(\frac{t}{t_f} \right)^K \right] \quad (3)$$

where X_v is the fraction recrystallized in time t , the term t_f is the time for some specified fraction F of recrystallization, $C = -\ln(1 - F)$ and K is strain dependent, i.e.

$$K = \begin{cases} 2 & ; \epsilon \leq \epsilon_c \\ 1 & ; \epsilon > \epsilon_c \end{cases} \quad (4)$$

In the above equation the time t_f has to be specified. For strains $\epsilon \leq \epsilon_c$ the time $t_{0.5}$ for 50% of recrystallization of recrystallization is given by

$$t_{0.5} = B\epsilon^{-m}d_o^pZ^q\exp\left(\frac{Q_{rex}}{R \cdot T}\right) \quad (5)$$

and for $\epsilon > \epsilon_c$ by

$$t_{0.5} = B'Z^r\exp\left(\frac{Q_{rex}}{R \cdot T}\right) \quad (6)$$

The constants for C-Mn steel in the above equations are [1]:

$$\begin{aligned} m &= 4 \\ p &= 2 \\ q &= 0 \\ Q_{rcx} &= 300 \text{ kJ/mol} \\ B &= 2.5 \cdot 10^{-19} \\ B' &= 1.06 \cdot 10^{-15} \\ r &= -0.6 \end{aligned} \quad (7)$$

4. Recrystallized grain size

Analyses of experimental results for a number of steels have shown that the statically recrystallized grain size is a function of the strain produced by the deformation process. It has been observed that the recrystallized grain size depends on the original grain size d_o if the strains are below the critical value ϵ^* and that at these levels of strain the temperature during deformation (i.e. Z) does not have a significant influence on the recrystallized grain size. The following expression has been suggested

$$d_{rex} = D d_o^{0.67} \epsilon^{-1}; \quad (\epsilon < \epsilon^*) \quad (8)$$

where the value of 0,5 [$\mu\text{m}^{0.33}$] is adopted for the constant D . The effect of original grain size vanishes at strains greater than $2 \cdot \epsilon_p$ where a steady state grain size is attained. At this stage the Zener-Hollomon parameter appears to be significant and the dependence is represented as

$$d_{rex} = D^* Z^{-u}; \quad (\epsilon > \epsilon^*) \quad (9)$$

where the value 0,15 is selected for the constant u . Experimental observations have also shown a gradual transition from strain dependence to steady state grain sizes. However, for simplicity the two stages are approximated by two straight lines where the intersection point between the two lines at ϵ^* can be obtained by using equations (8) and (9) which require

$$\epsilon^* = 0.57 d_o^{0.17} \epsilon_p \quad (10)$$

and

$$D^* = 3.6 \cdot 10^{-3} D \quad (11)$$

where ϵ_p is defined by (1). The value of ϵ^* in (10) is greater than $\epsilon_c = 0.8\epsilon_p$ for original grain sizes d_o expected in the majority of hot working operations.

5. Grain growth

Once recrystallization is complete, further grain growth takes place as a function of time and temperature. The grain growth is estimated from the following equation

$$d^{10} = d_{rex}^{10} + A' t \exp\left(\frac{-Q_{qq}}{R \cdot T}\right) \quad (12)$$

where A' and Q_{qq} depend on temperature. For a C-Mn steel the following values are assumed [3]

$$A' = \begin{cases} 5.02 \cdot 10^{53} \mu m^{10}/s; & (T < 1000^\circ C) \\ 3.87 \cdot 10^{32} \mu m^{10}/s; & (T \geq 1000^\circ C) \end{cases} \quad (13)$$

and

$$Q_{qq} = \begin{cases} 914 \text{ kJ/mole}; & (T < 1000^\circ C) \\ 400 \text{ kJ/mole}; & (T \geq 1000^\circ C) \end{cases} \quad (14)$$

6. Illustrative example

In this section it is suggested how the finite element model developed by the author in collaboration with Rockfield Software Ltd. can be applied in a combined numerical-experimental study of microstructural changes. As an example consider the investigation of the Sellars assumption that in austenitic steels dynamic recrystallization starts at a critical strain ϵ_c which depends on both the initial grain size d_o and the thermomechanical processing conditions expressed in terms of Zener-Hollomon parameter Z as follows

$$\epsilon_c = 0.8 A d_o^{1/2} Z^n \quad (14)$$

where for constants A and n values $4.9 \cdot 10^{-4}$ and 0,15 are assumed.

For the *numerical* part of the study the compression of a cylindrical specimen ($H_o = 8 \text{ mm}$; $D_o = 6 \text{ mm}$) between flat dies (relative velocity 8 mm/s) at $1000^\circ C$ was analyzed. Figure 1 shows numerically predicted distribution of the recrystallization potentials which are proportional to the ratio, $\bar{\epsilon}_{vp}\epsilon_c$, between the equivalent viscoplastic strain and the critical strain.

In a *conventional experimental* study [3] cylindrical specimens were compressed under laboratory conditions and air quenched after the test in order to freeze the microstructure developed during the deformation process. The microstructure was later metallographically examined. The experimental conditions are summarized in Fig.2 while the microstructure developed is shown in Fig.3. By comparing recrystallization potentials (Fig.1) and

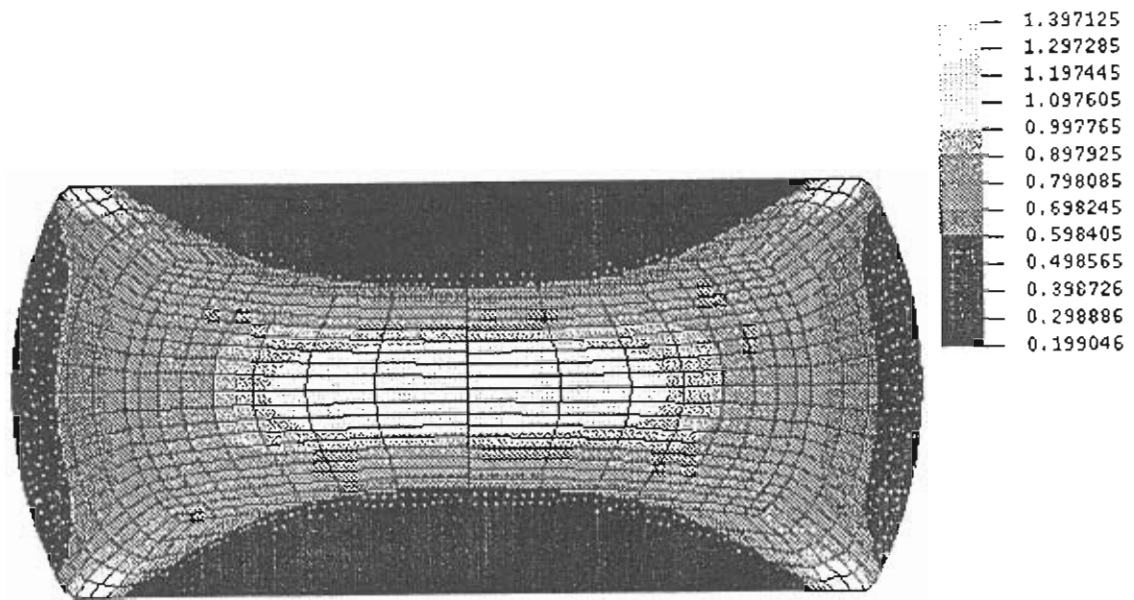


Fig.1 Distribution of recrystallization potentials

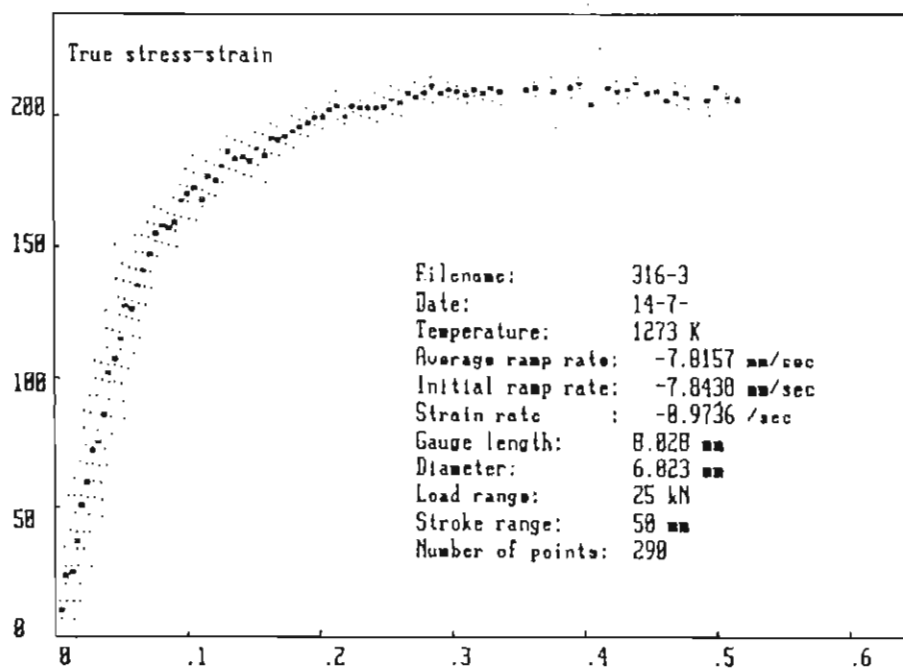


Fig.2 Loading conditions during the test

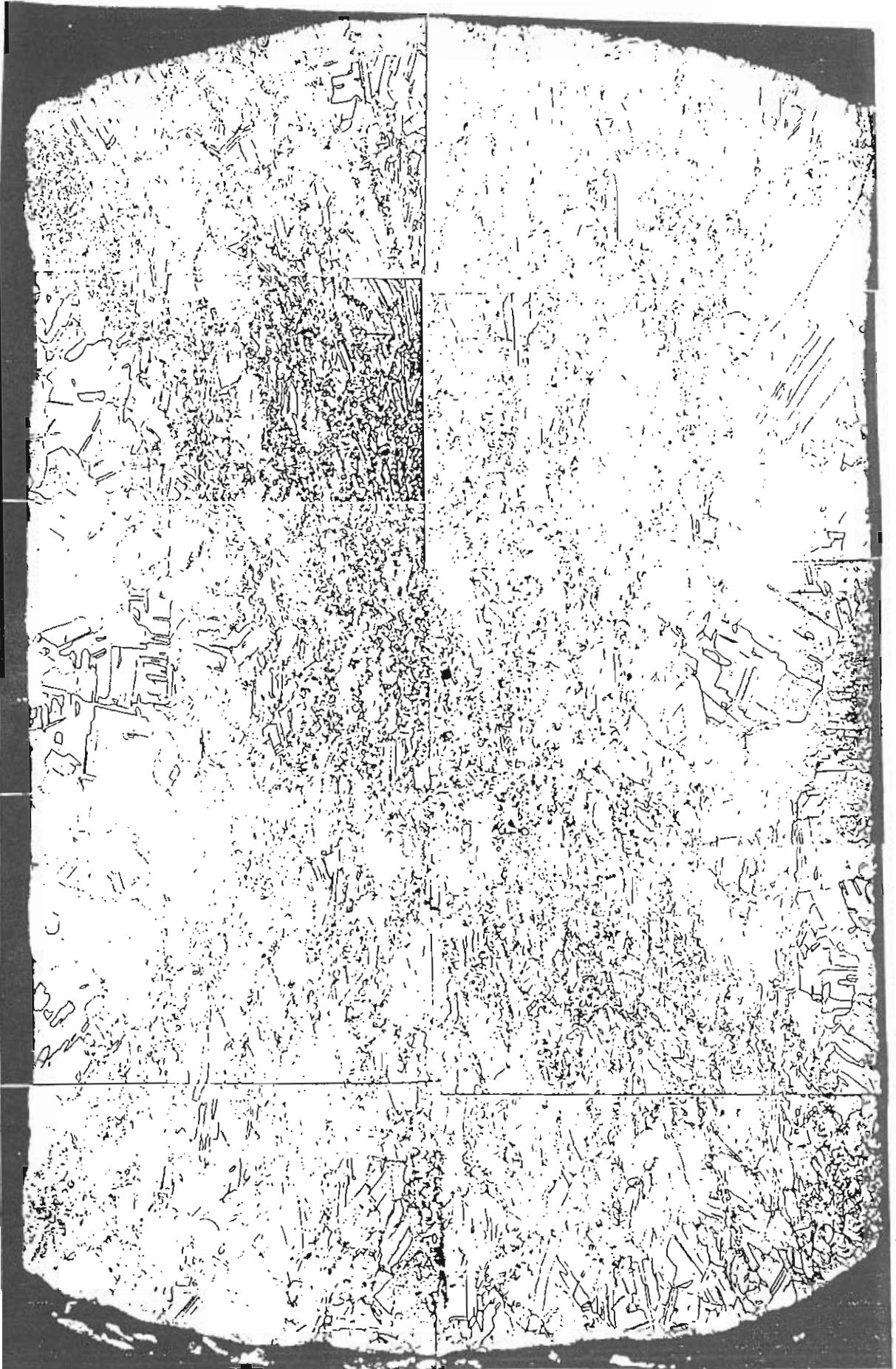


Fig.3 Microstructure developed during the cylinder compression test

metallographically examined specimen (Fig.2), the critical potential at which dynamic recrystallization occurs can be determined.

However, by employing contactless ultrasonics a combined numerical-experimental approach can be undertaken where material constants are evaluated by inverse approach from 'in-situ' measurements. In this approach thermomechanical processing conditions are evaluated by the finite element code and then input to an inverse analysis module for estimation of material constants. In the inverse procedure material constants are automatically adjusted in each iteration until calculated grain size distribution match experimental observations in a least square sense. For the automatic adjustment of material parameters modified Levenberg-Marquand scheme is employed.

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ENGINEERING APPLICATIONS OF A GEOMETRICALLY NONLINEAR ENHANCED STRAIN METHOD BASED ON A LOGARITHMIC FINITE STRAIN MODEL AND ELASTIC-INELASTIC MATERIAL LAWS

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Abstract: A robust numerical model cast into a finite element formulation is presented, which can successfully simulate various engineering problems involving solid mechanics with high degree of geometrical, material and boundary (contact) nonlinearities. The model is numerically unified for rate independent (elasto-plastic) and rate dependent (elasto-viscoplastic) material constitutive laws. Introduction of a logarithmic strain based hyperelastic finite strain model in the context of a geometrically nonlinear assumed strain method characterizes the numerical treatment of incompressibility at large deformations. Consistent linearization in all aspects of algorithmic development provides robust and quadratically convergent solutions. This is shown in general 2-D numerical examples, where simulations from a variety of engineering fields, including metal forming, structural engineering and geomechanics, are performed.

1. INTRODUCTION

The use of numerical simulations in general engineering problems involving solid mechanics has increased which has been made possible by the developing maturity of the necessary theoretical formulations and associated numerical algorithms together with the rapid improvement in computer hardware and software capabilities. The most efficient and reliable simulations rely on finite element based techniques, also utilized in the present work, which are well established as a powerful numerical framework for solving the mechanical constitutive equations. The aim of this work is to present a powerful and robust numerical model for simulation of a wide range of engineering problems described by the generally nonlinear mechanics of solids. Due to the limited space, attention is focused on the numerical examples presented in Section 3. However, the basics of the theoretical and numerical models are provided in the following section, where standard notation is used and references to more detailed descriptions are made.

2. NUMERICAL MODEL

2.1. Mechanical constitutive model

Generally, the material response to loading consist of elastic (reversible) and plastic (irreversible) deformations. In the present model, the former are assumed to be linear, which closely corresponds to the material models considered in the examples, but there are no principle obstacles to extend the model to account for nonlinear elastic response. The plastic deformations, which occur after a certain level of generalized stress has been reached, are always rate-dependent to a certain extent.

However, for the majority of materials at room temperature and especially at low strain rates, this has no practical consequences, so that mathematically well defined theory of *plasticity* can be utilized, where the yield stress is expressed as $\bar{\sigma} = f(\bar{\epsilon}^p)$. On the other hand, for materials where rate effects can not be neglected, *viscoplastic* models, $\bar{\sigma} = f(\dot{\bar{\epsilon}}^{vp}, \bar{\epsilon}^{vp})$, are employed. In the both cases the properties can be also temperature dependent.

These two models, commonly assumed in numerical applications, are also utilized in the present model. Employing the standard tensorial notation, they are summarized in Boxes 1. and 2. The so called *Perzyna* form of viscoplasticity, suitable for computational treatment, is used in the former model. Apart from the usual forms of the fluidity parameter, Υ , and the flow function, Φ , a rather specialized Sellars-Tegart law is presented, which is used for the description of metals behaviour at high temperatures, i.e. hot forming processes.^[1,4,9]

From the numerical standpoint, these material laws have to be integrated with respect to the finite time or load increments. A *fully implicit* or *Euler backward* integration method, is used to obtain the relevant incremental values. It is summarized in Boxes 3. and 4., respectively for the present models, using the standard vector/matrix notation and incremental updating of relevant values as $(\bullet)_{n+1} = (\bullet)_n + \Delta(\bullet)_{n+1}$. Overall solution is achieved by dividing an interval of interest into increments, within each a iterative solution based on the Newton method is used.

Box 1. Elasto-plastic material model

- (1) Additive strain decomposition:

$$\epsilon = \epsilon^e + \epsilon^p$$

- (2) Elastic response:

$$\sigma = C : \epsilon^e$$

$$C = \frac{\partial^2 W(x, \epsilon^e)}{\partial \epsilon^e \partial \epsilon^e} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$$

- (3) Yield function:

$$F(\sigma, \bar{\epsilon}^p) = f(\sigma) - \sigma_Y(\bar{\epsilon}^p)$$

- (4) Flow rule:

$$\dot{\epsilon}^p = \dot{\gamma} \frac{\partial Q}{\partial \sigma} = \dot{\gamma} b$$

- (5) Hardening law:

$$\dot{\sigma}_Y = \frac{\partial \sigma_Y}{\partial \bar{\epsilon}^p} \dot{\bar{\epsilon}}^p = H' \sqrt{2/3} \dot{\epsilon}^p : \dot{\epsilon}^p$$

- (6) Loading-unloading conditions:

$$\dot{\gamma} \geq 0, F \leq 0, \dot{\gamma} F = 0$$

- (7) Consistency condition:

$$\dot{\gamma} \dot{F} = 0$$

Box 2. Elasto-viscoplastic material model

- (1) Additive strain rate decomposition:

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^{vp}$$

- (2) Elastic response:

$$\dot{\epsilon}^e = C^{-1} : \dot{\sigma}$$

- (3) Flow rule:

$$\dot{\epsilon}^{vp} = \Upsilon \langle \Phi(\bar{\sigma}) \rangle \frac{\partial Q}{\partial \sigma}$$

- (4) Some forms of comonly used Υ and Φ :

(i) Exponential law

$$\Upsilon = \text{const.}; \Phi = \exp \left[m \left(\frac{f - \sigma_Y}{\sigma_Y} \right) \right] - 1$$

(ii) Power law

$$\Upsilon = \text{const.}; \Phi = \left(\frac{f}{\sigma_Y} \right)^n - 1$$

(iii) Sellars-Tegart

$$\Upsilon = C \exp \left[\frac{-Q}{R(T + 273)} \right]$$

$$\Phi = [\sinh(\alpha \bar{\sigma})]^n$$

Box 3. Integration algorithm for the elasto-plastic constitutive law

$$\Delta \sigma_{n+1} = C (\Delta \epsilon_{n+1} - \Delta \epsilon_{n+1}^p)$$

$$\Delta \epsilon_{n+1}^p = \Delta \gamma b_{n+1}$$

$$\Delta \bar{\epsilon}_{n+1}^p = \Delta \gamma \sqrt{2/3} b_{n+1}^T b_{n+1}$$

$$F_{n+1} = F(\sigma_{n+1}, \bar{\epsilon}_{n+1}^p) = 0$$

Box 4. Integration algorithm for the elasto-viscoplastic constitutive law

$$\Delta \sigma_{n+1} = C (\Delta \epsilon_{n+1} - \Delta \epsilon_{n+1}^{vp})$$

$$\Delta \epsilon_{n+1}^{vp} = \Delta \gamma b_{n+1}$$

$$\Delta \bar{\epsilon}_{n+1}^{vp} = \Delta \gamma \sqrt{2/3} b_{n+1}^T b_{n+1}$$

$$\Delta \gamma = \Upsilon_{n+1} \Phi_{n+1} \Delta t_{n+1}$$

$$\Upsilon_{n+1} = \Upsilon(T_{n+1}, \bar{\epsilon}_{n+1}^{vp})$$

$$\Phi_{n+1} = \Phi(T_{n+1}, \bar{\epsilon}_{n+1}^{vp}, \sigma_{n+1})$$

2.2. Finite strain model

The models in Boxes 1. and 2. are valid for the small strain case, but their inclusion in the finite strain model adopted in the present work is directly obtained by replacing the pair $\{\sigma, \epsilon\}$ with the conjugate pair $\{T/J, \ln U\}$, where $J = \det[F]$, whilst logarithmic stretches, $\ln U$ and the conjugate rotated Kirchhoff stress, $T = R^T \tau R$; $\tau = J\sigma$ are employed as strain and stress measures. This is justified, since the effects of finite strains defined in this way appear only at the kinematic level. The consequence of this property is that so called "consistent tangent operators"^[2-4], $\hat{c}_{EP}$ and $\hat{c}_{EVP}$ which emanate from the consistent linearization of the algorithms in Boxes 3. and 4., retain basically the same form as in the small strain case. As shown in these references, these operators are crucial if a quadratic convergence of the overall solution is to be achieved. For a detailed discussion of the adopted finite strain model, summarized in Box 5., one is referred to the pragmatic approach of Perić and co-workers^[5,6], also pursued in the present work.

Box 5. Finite strain model

(1) Polar decomposition of F :	(3) Strain energy function:
$F = \frac{\partial x}{\partial X} = RU$	$W = \mu \sum_{I=1}^3 (\ln \lambda_{(I)}^e)^2 + \frac{1}{2} \lambda (\ln J^e)^2$
(2) Computation of logarithmic stretches:	(4) Multiplicative decomposition of F :
$C = F^t F = \sum_{I=1}^3 C_{(I)} N^{(I)} \otimes N^{(I)}$	$F = F^e F^p$
$\ln U = \sum_{I=1}^3 \ln \lambda_{(I)} N^{(I)} \otimes N^{(I)} ; \lambda_{(I)} = C_{(I)}^{1/2}$	(5) Elasto-plasticity, elasto-viscoplasticity: in Boxes 1. and 2. with $\epsilon = \ln U$ $\sigma = T/\det[F^e]$

2.3. Geometrically nonlinear enhanced strain method

The models defined so far must be embedded in a variational principle suitable for a finite element formulation. The crucial task is to develop an element which provides high accuracy with relatively coarse meshes, and which treats properly the incompressible nature of the plastic flow, i.e. does not exhibit "spurious locking". In this respect, the geometrically nonlinear *assumed strain* method, proposed by Simo and Armero^[7], is introduced in the context of the present finite strain model.

The essential steps in the formulation and discretization of the problem are presented in Box 6., whilst more details can be found in the original references on the enhanced strain method^[7,8] and the logarithmic finite strain model^[5,6], as well as in the author's other references^[4,9,10].

The crucial idea is to assume the displacement gradient H to be a sum of the compatible part $\text{Grad } u$ and the enhanced part $\tilde{H}$, which also implies multiplicative decomposition $F = \tilde{F} F_x$ where $\tilde{F} = 1 + \tilde{H} F_x^{-1}$ and $F_x = \partial x / \partial X$ represent respectively the enhanced and compatible contributions. It is employed in a suitable form of the 3-field Hu-Washizu variational principle, which reduces to a 2-field (u and $\tilde{h}$) form after the stress field τ is effectively eliminated by enforcing L_2 orthogonality between the approximative spaces of stress and enhanced displacement gradient. The two remaining equations are numerically discretized in the usual finite element manner.

Standard material and geometrical parts of the element stiffness matrix are recovered consistent with the finite strain model employed and a stress return algorithm used in the integration of the constitutive equations. In the development of the material stiffness, it is assumed

Box 6. Enhanced strain model

(1) Assumed deformation gradient:

$$\mathbf{F} = \mathbf{1} + \text{Grad } \mathbf{u} + \tilde{\mathbf{H}}$$

(2) Enhanced spatial variational equations:

$$\int_{\Omega} \tilde{\nabla}(\delta \mathbf{u}) : \tilde{\boldsymbol{\tau}}(\mathbf{u}, \tilde{\mathbf{h}}) d\Omega - G_{ext}(\delta \mathbf{u}) = 0$$

$$\int_{\Omega} \delta \tilde{\mathbf{h}} : [\tilde{\boldsymbol{\tau}}(\mathbf{u}, \tilde{\mathbf{h}}) - \boldsymbol{\tau}] d\Omega = 0$$

$$\int_{\Omega} \delta \boldsymbol{\tau} : \tilde{\mathbf{h}} d\Omega = 0$$

$$\tilde{\mathbf{h}} = \tilde{\mathbf{H}} \mathbf{F}^{-1}$$

$$\tilde{\boldsymbol{\tau}}(\mathbf{u}, \tilde{\mathbf{h}}) = \mathbf{R} \frac{\partial \widehat{W}(\ln \mathbf{U})}{\partial (\ln \mathbf{U})} \mathbf{R}^T$$

(3) Finite element interpolation of $\tilde{\mathbf{H}}$:

$$\tilde{\mathbf{H}}_e = \sum_{I=1}^{\text{NENH}} \boldsymbol{\alpha}_I^e \otimes \mathbf{Q}_I^e \quad ; \quad \langle \mathbf{P}_e, \tilde{\mathbf{H}}_e \rangle_{L_2} = 0$$

(4) Element linearized operators:

$$\tilde{\nabla}^s \mathbf{u}_e^h = b_d(\widehat{\text{grad}} N_e^I) d_e \quad ; \quad \tilde{\mathbf{h}}^s = b_a(q_e^I) \boldsymbol{\alpha}_e$$

$$\tilde{\nabla} \mathbf{u}_e^h = g_d(\widehat{\text{grad}} N_e^I) d_e \quad ; \quad \tilde{\mathbf{h}} = g_a(q_e^I) \boldsymbol{\alpha}_e$$

$$\widehat{\text{grad}} N_e^I = \mathbf{F}_e^{-T} \text{Grad } N_e^I \quad ; \quad q_e^I = \mathbf{F}_e^{-T} \mathbf{Q}_I^e$$

(5) Element residual equations:

$$\mathbf{r}_d = \int_{\Omega_e} b_d^T : \tilde{\boldsymbol{\tau}} d\Omega_e - f^{ext}$$

$$\mathbf{r}_a = \int_{\Omega_e} b_a^T : \tilde{\boldsymbol{\tau}} d\Omega_e$$

(6) Newton's method of linearization:

$$\mathbf{k}_{dd} \Delta d_e + \mathbf{k}_{ad}^T \Delta \boldsymbol{\alpha}_e = -\mathbf{r}_d$$

$$\mathbf{k}_{ad} \Delta d_e + \mathbf{k}_{aa} \Delta \boldsymbol{\alpha}_e = -\mathbf{r}_a$$

$$\mathbf{k}_{dd} = \int_{\Omega_e} b_d^T (\hat{\mathbf{c}}_{E(V)P} - \mathbf{m}) b_d d\Omega_e + \int_{\Omega_e} g_d^T \hat{\sigma} g_d d\Omega_e$$

$$\mathbf{k}_{ad} = \int_{\Omega_e} b_a^T (\hat{\mathbf{c}}_{E(V)P} - \mathbf{m}) b_d d\Omega_e + \int_{\Omega_e} g_a^T \hat{\sigma} g_d d\Omega_e$$

$$\mathbf{k}_{aa} = \int_{\Omega_e} b_a^T (\hat{\mathbf{c}}_{E(V)P} - \mathbf{m}) b_a d\Omega_e + \int_{\Omega_e} g_a^T \hat{\sigma} g_a d\Omega_e$$

(7) Static condensation on the element level:

$$\tilde{\mathbf{k}}^e \Delta d_e = -\tilde{\mathbf{r}}^e$$

$$\Delta \boldsymbol{\alpha}_e = -\mathbf{k}_{aa}^{-1} (\mathbf{r}_a + \mathbf{k}_{ad} \Delta d_e)$$

$$\tilde{\mathbf{k}}^e = \mathbf{k}_{dd} - \mathbf{k}_{ad}^T \mathbf{k}_{aa}^{-1} \mathbf{k}_{ad}$$

$$\tilde{\mathbf{r}}^e = \mathbf{r}_d - \mathbf{k}_{ad}^T \mathbf{k}_{aa}^{-1} \mathbf{r}_a$$

(8) Global displacement equations:

$$\tilde{\mathbf{K}} \Delta \mathbf{d} = -\tilde{\mathbf{G}}$$

$$\tilde{\mathbf{K}} = \sum_{e=1}^{\text{NEL}} \tilde{\mathbf{k}}^e \quad ; \quad \tilde{\mathbf{G}} = \sum_{e=1}^{\text{NEL}} \tilde{\mathbf{r}}^e$$

that the elastic strains are 'small enough' compared to the (visco)plastic counterparts, which results in a particularly simple tangent operator.^[5,6]

Interelement discontinuity of the enhanced part of the displacement gradient enables static condensation on the element level and hence global displacement-type equations are obtained. These are solved in the usual incremental-iterative manner where quadratic convergence of the algorithm is ensured by consistent linearization.

It should be noted that although locking is avoided in simulations using the resulting formulation, the opposite effect of hourglassing may be encountered, depending on a particular choice of the reference configuration. To circumvent this problem, the author has originally proposed *ab initio* non-objective 'incremental enhancements'^[4,9], but following the remarks in Simo *et al.*^[8], a 5-point integration rule has now been adopted^[10], which not only enables objectivity to be retained, but also considerably larger increments to be used, particularly at high degrees of achieved deformation.

3. NUMERICAL EXAMPLES

In this section, numerical simulations from four different engineering areas are presented. The first two simulations, already presented in^[4], have been obtained using the old version of 'incremental enhancements', whilst the other two novel results emanate from the new version of the code employing the 5-point integration rule. However, preliminary re-running of the first two examples with the new code yield almost similar spatial and temporal distribution of relevant values, but the solutions are reached in less number of increments and the simulations can continue far beyond the points where the previous versions would start to diverge due to severe mesh distortions.

3.1. THIN SHEET METAL FORMING: Axisymmetric cold deep drawing.

This example, which can be considered as an intermediate stage in the forming of thin sheet metal products, has been numerically analysed by Perić and Owen^[6], where a different formulation involving a fully 3-D model and a finite strain shell element is employed. The hemispherical punch surface stretches the blank which is fully supported at the die top end, so that the axisymmetric model can be applied with appropriate boundary conditions. The geometry is shown in Fig.1. The identical material constants to the references are also employed here; $E = 69\,004\text{ N/mm}^2$, $\nu = 0.3$, the isotropic hardening according to $\sigma_Y = 589 \cdot (1.0 \cdot 10^{-4} + \bar{\epsilon}^p)^{0.216}\text{ N/mm}^2$ and Coulomb friction law, $\mu = 0.3$. The contact penalties $c_n = 10^4$ and $c_t = 10^3$ are assumed.

Fig.2. shows the deformed mesh for the ultimately achieved punch displacement of 38.55 mm , compared to the initial configuration. At this stage of punch displacement, the blank material has practically failed due to severe strain localization producing extreme thinning in the narrow region at approximately half-radius distance which closely corresponds to the experimental observations.

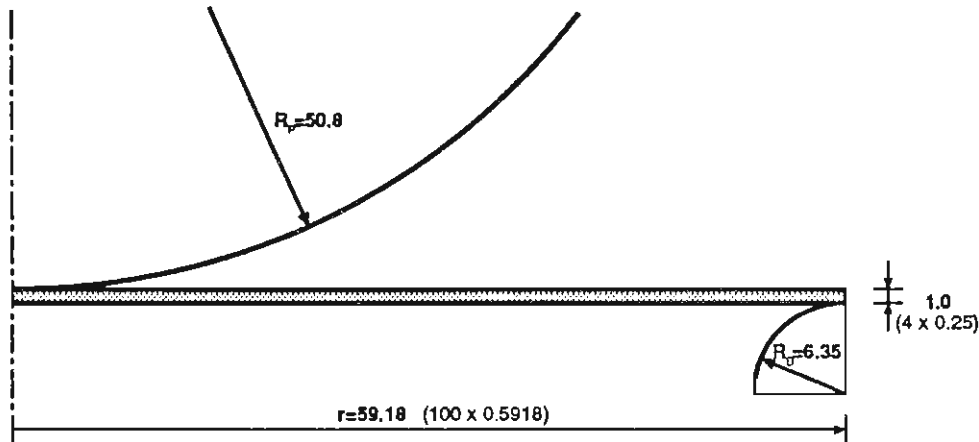


Fig.1. Axisymmetric cold deep drawing. Setup geometry (dimensions in mm)

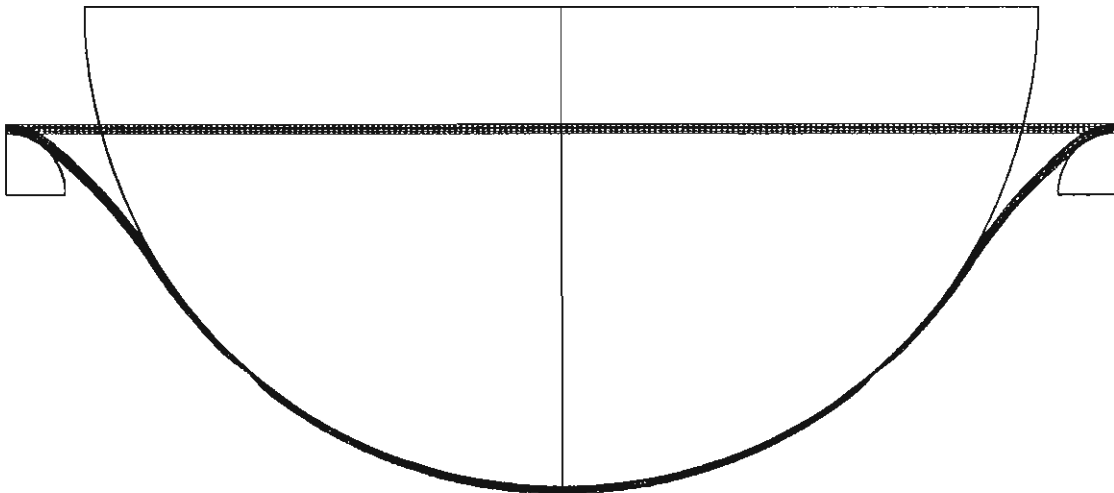


Fig.2. Axisymmetric cold deep drawing. Initial and ultimately deformed meshes

This phenomenon can be roughly seen in Fig.2., but a more precise description is provided in Figs.3.(a)-(d), where the failed region is 'zoomed' using fixed 'camera coordinates'. The sequence of deformed mesh figures for various stages of punch displacement, which also include the effective plastic strain distribution, clearly animates the transformation from a uniform strain distribution to a severe strain localization resulting in material failure.

The maximum value of the punch force is achieved at the beginning of the localization process and rapidly decreases thereafter, which is depicted in the punch force versus punch displacement diagram in Fig.4. The result is in good agreement with that of reference^[6]. The peak value is slightly larger, while the present enhanced strain model ensures a superior and more realistic load decrease curve. The thickness distribution for the failed region is shown in Fig.5. for various punch displacements, with the obvious transformation from a uniform distribution to a localized pattern.

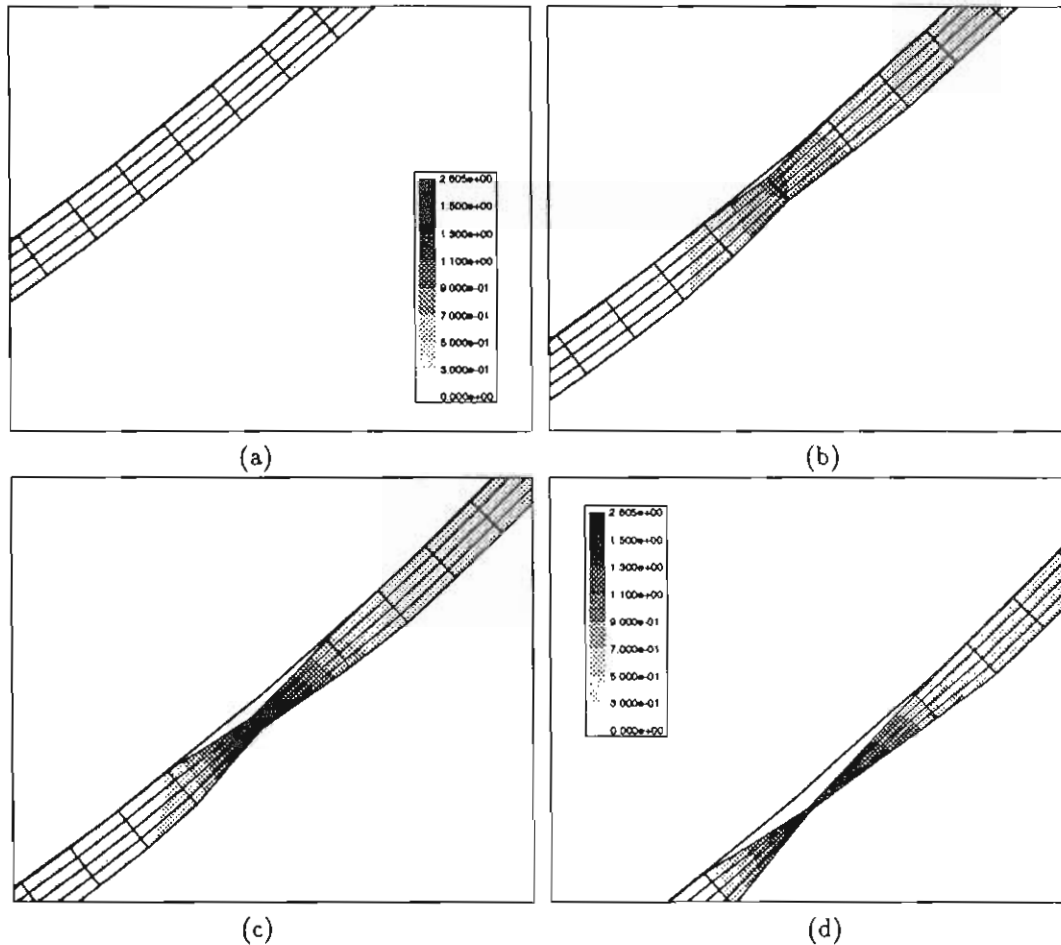


Fig.3. Axisymmetric cold deep drawing. Animation of the strain localization process. Effective plastic strain distribution for the failed region at different stages of punch displacement [mm]: (a) $\bar{u} = 34.88$; (b) $\bar{u} = 36.20$; (c) $\bar{u} = 37.22$; (d) $\bar{u} = 38.55$;

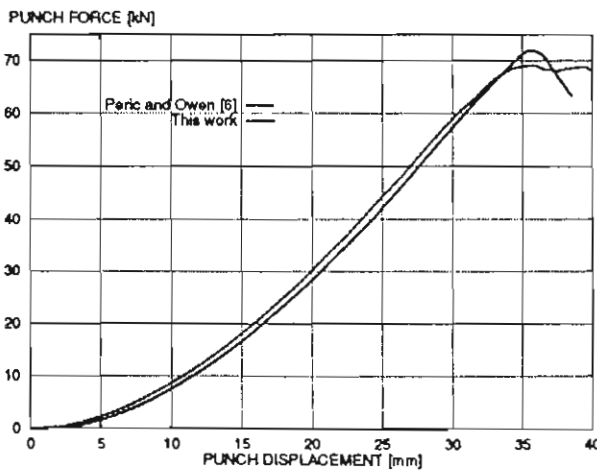


Fig.4. Axisymmetric cold deep drawing. Punch force versus punch displacement

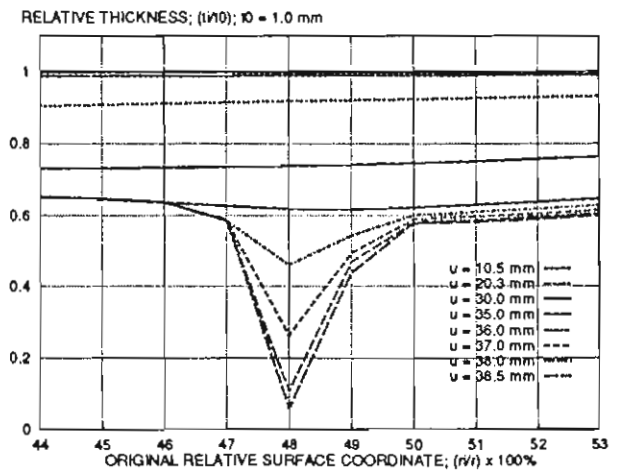


Fig.5. Axisymmetric cold deep drawing. Thickness distribution for the failed region

3.2. BULK METAL FORMING: Plane strain hot forging with flash.

Plane strain conditions are assumed in the hot forging of the industrial component of initial radius $R = 20 \text{ mm}$. The material elastic constants are $E = 70\,000 \text{ N/mm}^2$, $\nu = 0.3$, while the viscoplastic flow is assumed to follow the Sellars-Tegart interpolation function with material coefficients $C = 1.2 \cdot 10^{12} \text{ s}^{-1}$, $\alpha = 1.12 \cdot 10^{-2} \text{ mm}^2/\text{N}$, $n = 3.5$, $Q = 352\,690 \text{ J/mol}$ and Von Mises flow rule at isothermal conditions, $T = 1100^\circ\text{C}$. A Coulomb frictional law ($\mu = 0.3$) and rigid tool are assumed (for details on the frictional contact numerical treatment employed in this and previous example, please see^[4] and references therein). Fig.6. shows the initial and various deformed meshes together with effective plastic strain distribution.

The evolution of contact zones and normal and tangential tractions at the interface are presented in Fig.7. The distribution of tangential tractions remains approximately constant during the deformation process, even when the normal contact pressure increases. This justifies the usual assumptions of perfectly sticking contact conditions, usually made in bulk forming simulations in the presence of high contact pressures.

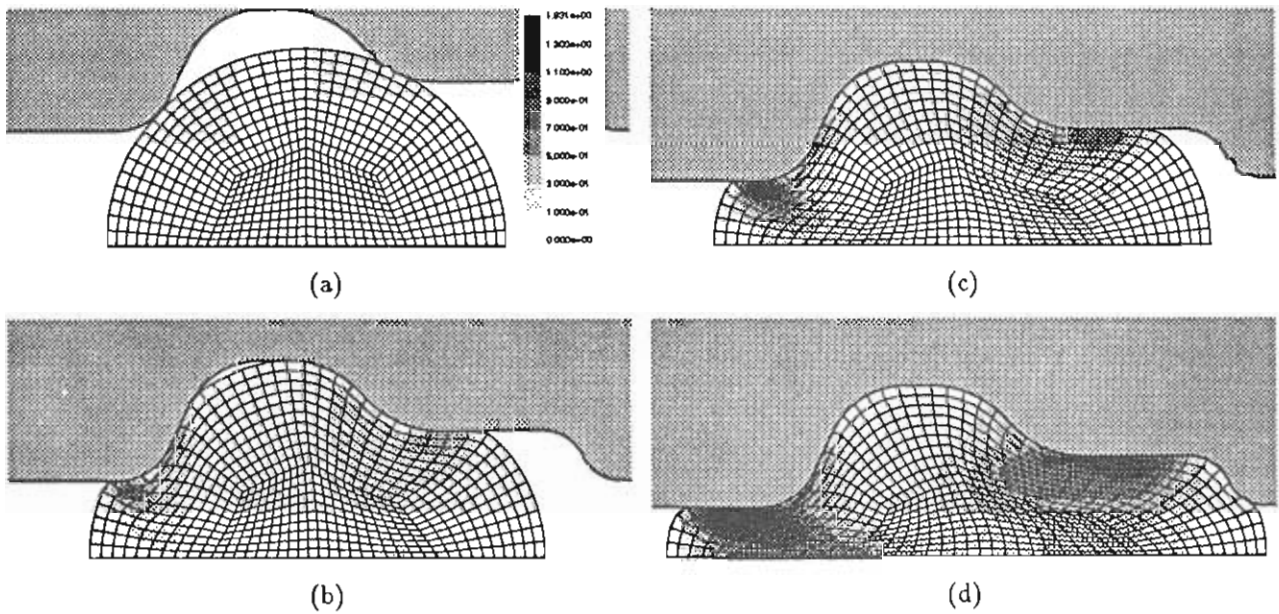


Fig.6. Plane strain hot forging. Effective plastic strain distribution for one half of the cross-section at different stages of tool displacement [mm]: (a) initial configuration; (b) $\bar{u} = 2.5$; (c) $\bar{u} = 4.1$; (d) $\bar{u} = 6.9$.

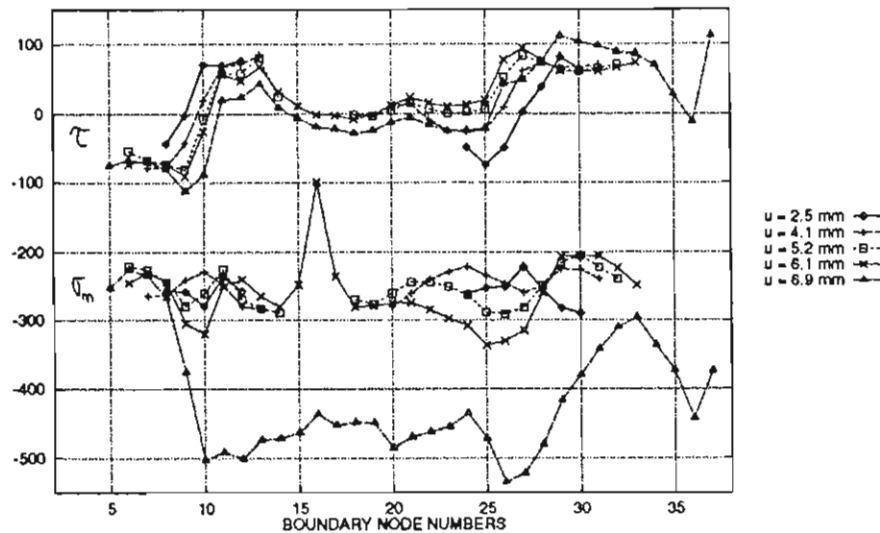


Fig.7. Plane strain hot forging. Contact traction distribution for boundary nodes numbered from the left end, along the surface to the right end, at different stages of punch displacement. Original internodal distance is $(R \pi / n_{\text{SEG}}) \approx 1.57 \text{ mm}$.

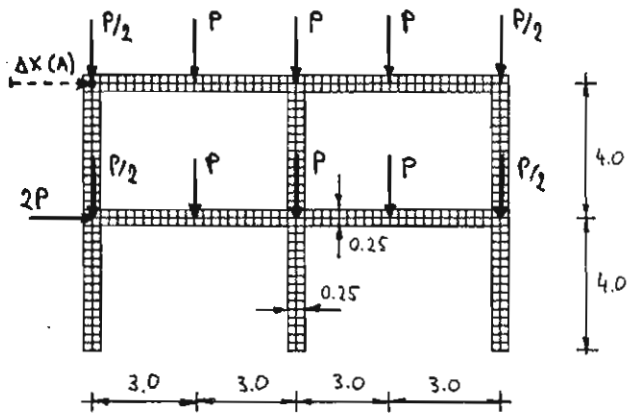


Fig.8. Steel frame. Initial configuration [m] and loading

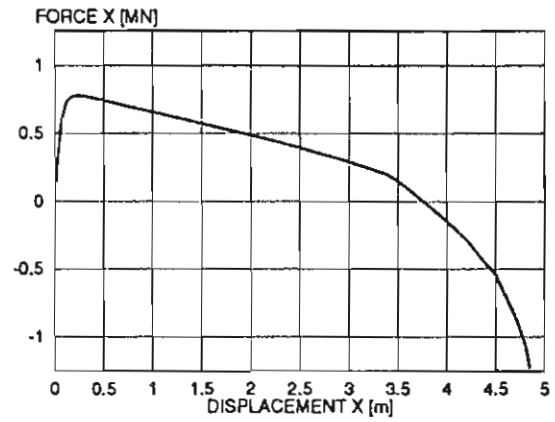


Fig.9. Steel frame. Load-displacement curve

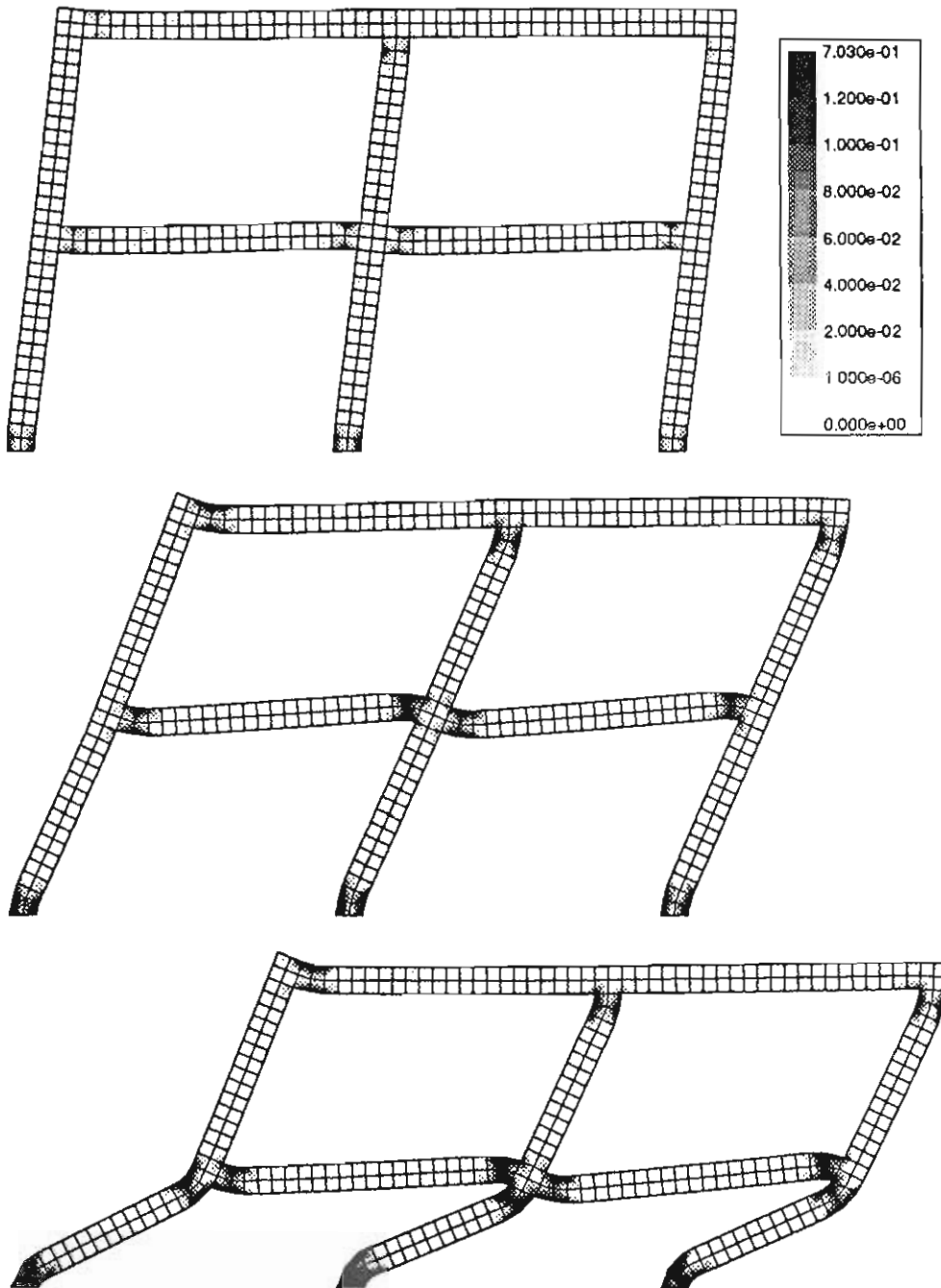


Fig.10. Steel frame. Deformed configuration and effective plastic strain distribution at different stages of imposed displacement [m]: (a) $\bar{u} = 1.0$; (b) $\bar{u} = 3.0$; (c) $\bar{u} = 4.8$.

3.3. STRUCTURAL ENGINEERING: Plane stress steel frame instability analysis.

An example similar to this has been analysed employing Cosserat continuum models in^[11] and references therein. A plane stress analysis is performed in this work, where the geometric properties have been chosen to approximately resemble A and I in the references. Thus $b/h = 0.03/0.50 [m/m]$ is assumed for both the columns and girders, where b becomes the thickness for the present model, while only two elements are employed for the through- h discretization. An elastic-perfect Von Mises plastic model ($E = 210\,000\text{ N/mm}^2$, $\sigma_Y = 300\text{ N/mm}^2$) is assumed. The full initial configuration is shown in Fig.8., where $P = 0.1 [MN]$ is taken.

Firstly, the critical load factor has been found to be 2.36, which is slightly above the values in the references. Then, in another simulation, the critical load is applied to all the points apart from the top left horizontal force (point A), where a prescribed horizontal displacement is enforced. The load-displacement curve of such an analysis is depicted in Fig.9., where a clear force decrease after the critical displacement of about $0.25 [m]$ can be seen. Figs.10.(a)-(c) show deformed configurations with plastic strain contours of the collapsing frame.

3.4. GEOMECHANICS: Soil collapse under rigid footing.

This example of a rigid footing ($w=61.71\text{ in}$) on a stratum of soil ($w=349.7\text{ in}$, $d=144\text{ in}$) is taken from^[16](see also references therein). Due to obvious symmetry, this represents half of the real problem whose discretization is shown in Fig.11.(a). Plane strain is assumed with the lower boundary fully restrained and the lateral boundaries x-restrained. An associative Drucker-Prager law is employed with the parameter values $E = 30\,000\text{ psi}$, $\nu = 0.3$, $\alpha = 0.112$ and $k = 9.22\text{ psi}$. The plasticity data correspond to a conical yield surface coinciding with the outer apices of the Mohr-Coulomb hexagon defined by $\phi = 15.382^\circ$, $c = 7.549\text{ psi}$. The soil self-weight is neglected.

The quasi-velocity field is depicted in Figs.11.(b)-(d) for the various stages of imposed settlement. The equivalent plastic strain distribution for the ultimate displacement and symmetrically-mirrored solution is shown in Fig.12. The patterns of these results closely correspond to a real soil-collapse situation, with the narrow slip-line shaped zones of plastification emanating from the corners of the rigid footing. The load-displacement curve can be seen in Fig.13., compared to the other numerical prediction, as well as to the Prandtl and Terzaghi solutions.

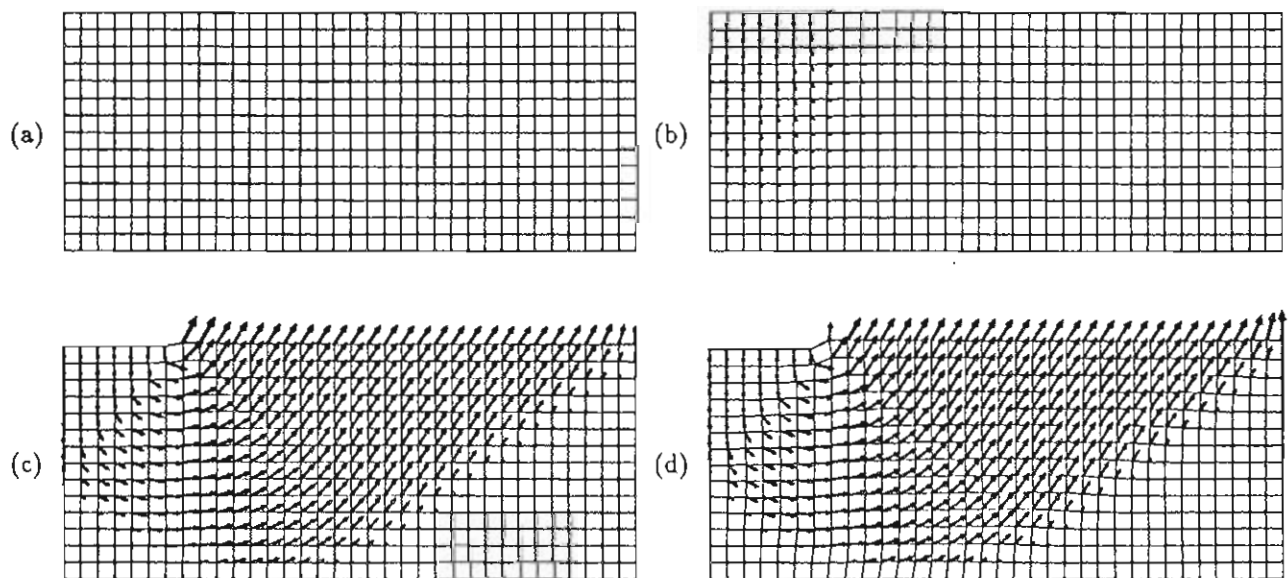


Fig.11. Rigid footing. Quasi-velocity field at different stages of stratum settlement [in]: (a) initial configuration; (b) $\bar{d} = 0.25$; (c) $\bar{d} = 1.05$; (d) $\bar{d} = 2.50$;

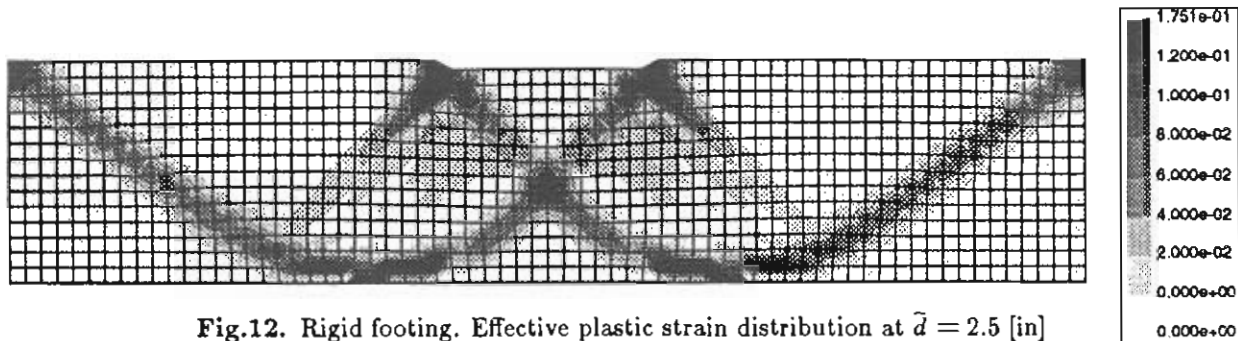


Fig.12. Rigid footing. Effective plastic strain distribution at $\bar{d} = 2.5$ [in]

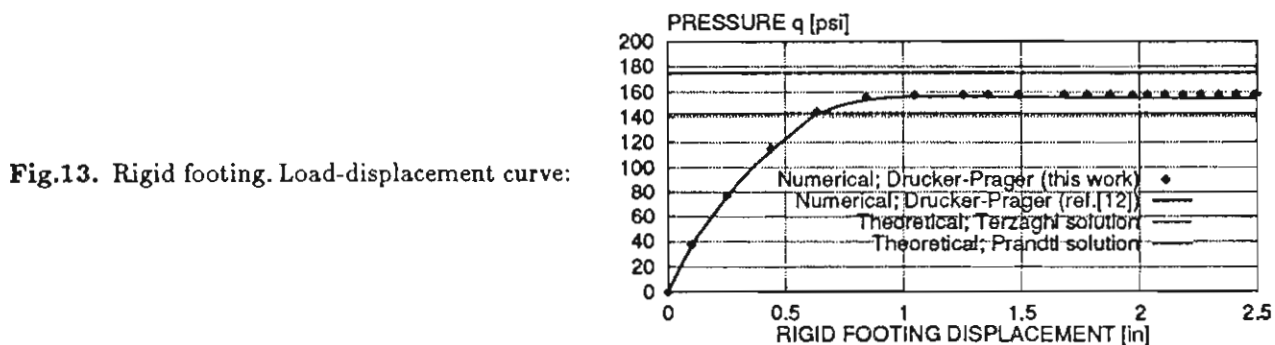


Fig.13. Rigid footing. Load-displacement curve:

4. CONCLUDING REMARKS

A numerical model for fully geometrically nonlinear solid mechanics based on rate-dependant or independant material laws is presented. The robust and consistent numerical treatment of the theoretical formulations ensures generality and future upgradability of the model. The capabilities of the model to represent physically realistic behaviours are clearly shown in the four simulations of real engineering problems. These results closely correspond to observed phenomena, confirming the model's reliability.

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Viscoplastic flow in singular regions of Mohr-Coulomb yield surface

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SUMMARY — *The paper describes an approach towards solving the difficulties of non-associated viscoplastic flow in singular regions of Mohr-Coulomb yield surface. For explicit integration of the Perzyna viscoplastic law the procedure consists of the decision whether the state of stress is in singular region at any time instance and, if this is true, the determination of the correct viscoplastic flow. Proper singularity indicators are only briefly stated. The regularisation of viscoplastic flow rule in singular regions is based on the assumption that the viscoplastic flow would take place in the direction that would eventually bring the stress point to the corner or the apex of the current yield surface, if the full relaxation process would have time to develop. The requirement for a uniform yield potential field around the whole of the yield surface is recognised when deriving the Perzyna's distance norm $\Phi(f)$ of the viscoplastic stress point from the current yield surface in the singular regions of the Mohr-Coulomb yield surface. Two numerical examples are included.*

1 Introduction

Non-regular regions of multi-surface yield criteria present a considerable computational difficulty as elastic domain is defined by a number of multiple convex yield surfaces intersecting non-smoothly and multiple yield functions may be active at the same time. If the inelastic flow in the singular regions of Mohr-Coulomb yield surface is not consistently defined, erroneous results may be obtained, which can even lead to non-realistic failure modes. Traditional approach towards solving the corner singularities has been that of "rounding" procedures, by replacing the non-smooth yield surface by the similar smooth yield surface near a singularity, e.g. Mohr-Coulomb with Drucker-Prager criterion sufficiently [4]. However, such an approach has limitations as term "sufficiently near" a singularity is pretty vaguely defined and often more singularities are introduced at the intersection of both yield surfaces.

The generalisation of plasticity when multiple yield surfaces may become active at the same time has been provided by Koiter [2] and consistently extended for various numerical algorithms by Simo *et al.* [8]. For procedures based on the elastic predictor and plastic

corrector the problem of singular yield surfaces can be split into the decision as to whether multiple yield surfaces are active (*i.e.* whether the trial state of stress is in singular region) and if this is true, the use of Koiter's generalisation to obtain correct stress return to the yield surface. This procedure can also be extended for viscoplastic analyses in a way that is presented in this paper.

Proper singularity indicators to establish whether the viscoplastic state of stress is in any of the singular regions of Mohr-Coulomb yield surface at any given time instance are derived for the case of non-associated viscoplasticity. A regularisation of the Perzyna-type non-associated viscoplastic flow in these regions is proposed in the framework of explicit integration of the differential stress-strain law.

2 Perzyna elasto-viscoplastic model

In the Perzyna viscoplasticity theory [6] the viscoplastic strain rate is defined with

$$\dot{\epsilon}_{vp} = \gamma \langle \Phi(f) \rangle \frac{\partial q_{vp}}{\partial \sigma} = \gamma \langle \Phi(f) \rangle \mathbf{b} \quad (1)$$

where q_{vp} is the viscoplastic potential function, $\mathbf{b} = \partial q_{vp} / \partial \sigma$ is the viscoplastic flow vector and γ is the fluidity parameter. $\Phi(f)$ is an arbitrary function of the flow function and the notation $\langle \cdot \rangle$ implies

$$\langle \Phi(f) \rangle = \begin{cases} \Phi(f) & \text{for } f > 0 \\ 0 & \text{for } f \leq 0 \end{cases} \quad (2)$$

Function $\Phi(f)$ most commonly takes the form of the power function

$$\Phi(f) = \left(\frac{f}{F_0} \right)^N \quad (3)$$

in which F_0 is the current reference yield function and constants N and M are chosen such as to fit the experimental data.

For viscoplastic materials under transient loading both $\Phi(f)$ and γ must reflect the real material properties and time integration of the viscous effects must be done in the context of real-time. In this paper the Euler forward scheme is applied and the time-dependent integration procedure of differential stress-strain law can be described as follows.

First a viscoplastic state of stress $\sigma^n = \mathbf{D}_e \epsilon_e^n$ is calculated at the beginning of the time step, where the elastic strains are obtained by subtracting the accumulated viscoplastic strains from the current total strains $\epsilon_e^n = \epsilon^n - \epsilon_{vp}^n$. Upon violation of the yield criterion $f^n(\sigma^n, Y^n) > 0$ the current viscoplastic strain rate $\dot{\epsilon}_{vp}^n$ is calculated using eq. (1) and the total viscoplastic strain at the end of the time step is estimated by the Euler forward scheme with

$$\epsilon_{vp}^{n+1} = \epsilon_{vp}^n + \dot{\epsilon}_{vp}^n \Delta t \quad (4)$$

The equivalent yield stress $Y^{n+1} = Y(\kappa^{n+1})$ at the end of the time step is for the case of *isotropic hardening* related to the viscoplastic deformation by means of the softening parameter κ , which can also be integrated in time with

$$\kappa^{n+1} = \kappa^n + \dot{\kappa}^n \Delta t \quad (5)$$

if the *strain hardening hypothesis* [4] is assumed, where the current softening parameter rate is proportional to the *effective viscoplastic strain rate*

$$\dot{\kappa}^n = \dot{\epsilon}_{vp}^n = \sqrt{\frac{2}{3}(\dot{\epsilon}_{vp}^n)^T \dot{\epsilon}_{vp}^n} \quad (6)$$

That way the values of ϵ_{vp}^{n+1} , Y^{n+1} and subsequently $f^{n+1}(\sigma^{n+1}, Y^{n+1})$ can always be estimated at the end of the time step Δt during which inelastic straining occurs and are fully defined at any given time instance t^n . The time integration of the governing equations of motion is undertaken in parallel and in relation to the time integration of the viscoplastic constitutive equations (see Ren [7]).

3 Mohr-Coulomb yield criterion

The Mohr-Coulomb yield criterion in principal stress space in the working sextant $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is defined as [4]

$$f(\sigma, c, \phi) = F(\sigma) - F_0(c, \phi) = \frac{1}{2}(\sigma_1 - \sigma_3) + \frac{1}{2}(\sigma_1 + \sigma_3) \sin \phi - c \cos \phi = 0 \quad (7)$$

Considering all other stress combinations which can cause yielding (e.g. $\sigma_1 \geq \sigma_3 \geq \sigma_2$, $\sigma_2 \geq \sigma_1 \geq \sigma_3$, etc.), the complete yield surface in a form of irregular hexagonal pyramid aligned with the space diagonal p (hydrostatic axis) can be obtained, which is graphically illustrated in principal stress space in Figure 1.

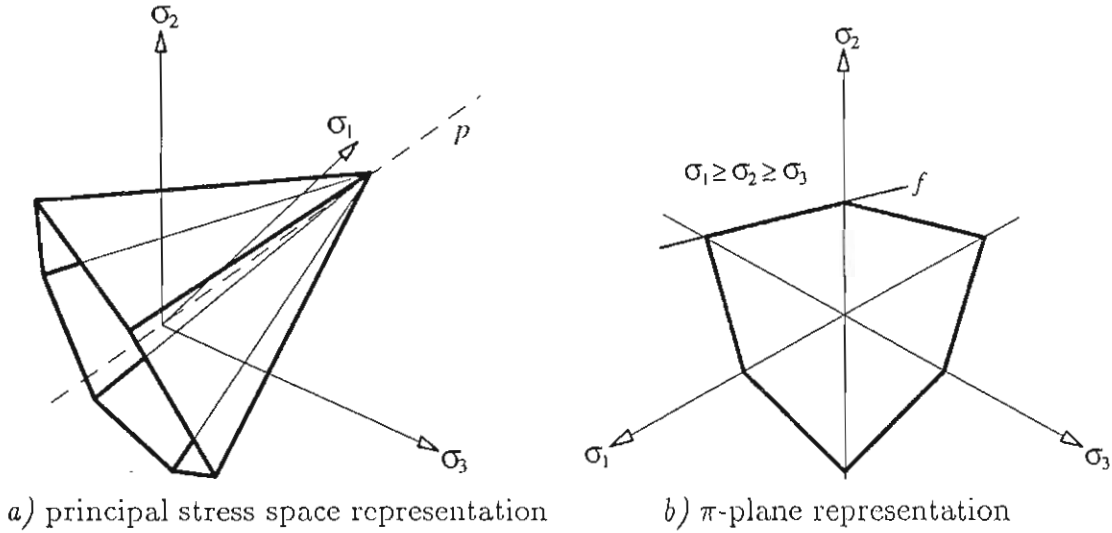


Figure 1: Representation of the Mohr-Coulomb yield surface

An excessive volumetric change during the inelastic straining can be observed, if an associated viscoplasticity case is assumed with $q_{vp} \equiv f$, where the inelastic flow is assumed to take place in the direction defined with the flow vector $\mathbf{b}^T = \mathbf{a}^T = (\partial f / \partial \sigma)^T = \frac{1}{2}(\sin \phi + 1 \quad 0 \quad \sin \phi - 1)$. Therefore a clear need for different viscoplastic potential function and with that a non-associated flow rule is present. It is customary to use the original Mohr-Coulomb yield function with the angle of dilatancy ψ substituting the angle of internal friction ϕ [3], therefore obtaining a viscoplastic potential function by rewriting eq. (7) as

$$q_{vp}(\sigma, \psi) \equiv q_{vp}(\sigma, \psi) = \frac{1}{2}(\sigma_1 - \sigma_3) + \frac{1}{2}(\sigma_1 + \sigma_3) \sin \psi - \text{constant} = 0 \quad (8)$$

with the flow vector now being equal to $\mathbf{b}^T = (\partial q_{vp} / \partial \sigma)^T = \frac{1}{2}(\sin \psi + 1 \quad 0 \quad \sin \psi - 1)$

4 On multi-surface elasto-plasticity and elasto-viscoplasticity

If a number of regular yield functions intersect in a way to cause singularities in the combined yield surface, Koiter [2] proposed that for the case of non-associated plasticity the plastic strain increment based on elastic predictor and plastic corrector algorithms can be obtained with

$$\dot{\epsilon}_p = \sum_{j=1}^m \dot{\lambda}_j \mathbf{b}_j \quad (9)$$

which returns the stress point correctly from the trial state of stress σ_T to the combined yield surface. In eq. (9) $\mathbf{b}_j = \partial q_{pj} / \partial \sigma$ are the flow vectors with q_{pj} the relevant plastic potential functions, m is the number of active yield functions f_j , and λ_j are the corresponding plastic multipliers that can be obtained from the Prager's consistency condition satisfied for all m active yield functions (cf. Pankaj [5]).

Following eq. (9) it would seem straight forward to write the viscoplastic strain rate eq. (1) for singular regions as

$$\dot{\epsilon}_{vp} = \sum_{j=1}^m \gamma_j \langle \Phi(f_j) \rangle \mathbf{b}_j \quad (10)$$

However in the limiting case, this equation does not reduce to the rate-independent form [8]. This can be explained with the reference to the gradients of the viscoplastic potential fields in regular and singular regions.

It is known that for a potential field to be uniform, the absolute value of its gradient should be equal at any point in that field. This is obviously true, if considering any of the six possible viscoplastic potential functions obtained with the permutation of indices in eq. (8) that govern the direction of viscoplastic flow in regular regions of Mohr-Coulomb yield surface as

$$|\text{grad} q_{vp}| = |\mathbf{b}_1| = |\mathbf{b}_2| = \dots |\mathbf{b}_6| = \sqrt{\frac{1}{2}(\sin^2 \psi + 1)} \quad (11)$$

In the singular region where two potential fields q_{vp1} and q_{vp2} with differently oriented gradients of the same absolute value are simply added together, if using eq. (10), which results in the field gradient in this region being equal to

$$|\text{grad} q_{vp}^*| = |\text{grad}(q_{vp1} + q_{vp2})| = |\mathbf{b}_1 + \mathbf{b}_2| = \sqrt{\frac{1}{2}(3 \sin^2 \psi - 2 \sin \psi + 3)} \quad (12)$$

which is clearly different from the absolute value of the gradient in regular regions. Therefore erroneous results obtained if using eq. (10) are the consequence of a loss of uniformity of the potential field in the singular region. Hence, eq. (10) can not be used as a generalisation method for Perzyna viscoplastic model.

In order to define objective viscoplastic strain rate by using eq. (1) in singular regions one has to find the direction of viscoplastic flow by finding the correct flow vector $\mathbf{b}^n$ and a correct norm of the state of stress potential in the yield potential field $\Phi(f^n)$.

5 Corner and apex singularity indicators

In Mohr-Coulomb yield surface two kinds of singularities exist – the edges where two different planes of the Mohr-Coulomb yield surface intersect and are usually referred to

as *corners* and the point of Mohr-Coulomb pyramid, *i.e.* *apex*, where all six planes meet (see Figure 1). Proper singularity indicators and stress return procedures for the ideal and hardening/softening associated plasticity have been originally developed by de Borst [1] and later extended by Pankaj [5]. In this paper the singularity indicators developed along the similar lines for the case of non-associated Mohr-Coulomb plasticity are only briefly stated, bearing in mind that they are only used to indicate a possible presence of the viscoplastic state of stress in any of the singular regions of the yield surface at any given time instance t^n . The interested reader should refer to Ren [7] for a complete derivation of singularity indicators.

Figure 2 shows two corner singular regions S_{12} (f_1 and f_2 active) and S_{23} (f_2 and f_3 active) in the working sextant $\sigma_1 \geq \sigma_2 \geq \sigma_3$ in π plane representation of Mohr-Coulomb yield surface, where μ_{12} and μ_{23} are singular corner region indicators respectively.

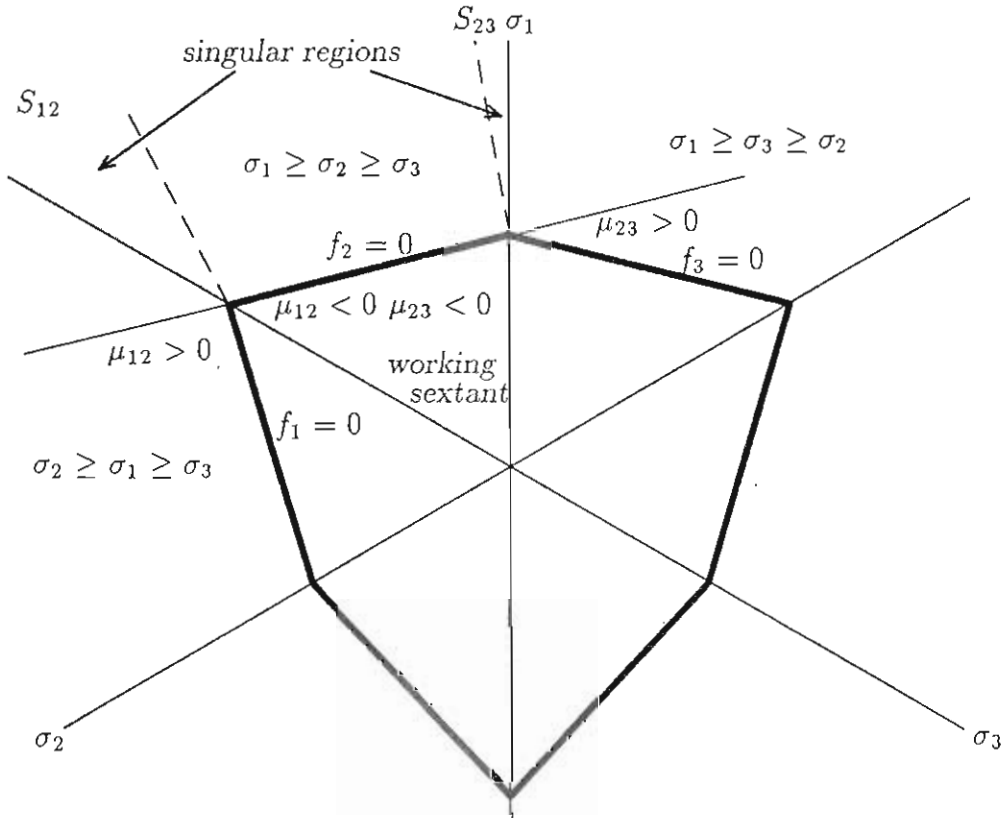


Figure 2: Singular corner regions of Mohr-Coulomb yield surface

The basic idea used to obtain simple expressions to identify whether the state of stress σ_T is in a singular corner region or not is that, if the state of stress was in any of singular corner regions, the corrected state of stress using a plastic single vector return would violate the inequality $\sigma_1 \geq \sigma_2 \geq \sigma_3$ [5]. If the trial state of stress was in a singular region S_{12} near the corner $\sigma_1 = \sigma_2$ (Figure 2), the final state of stress σ_P after a plastic single vector return would obey the inequality $\sigma_{P2} - \sigma_{P1} > 0$ which yields a singularity indicator μ_{12} [7]

$$\mu_{12} = f_2 + \frac{sp - 2\nu + 1}{(1 - 2\nu)(p + 1)} (\sigma_{T2} - \sigma_{T1}) > 0 \quad (13)$$

where $s = \sin \phi$ and $p = \sin \psi$. Similarly, if the trial stress state was in a singular region S_{23} near the corner $\sigma_2 = \sigma_3$, the corrected state of stress using a plastic single vector stress return obeys the inequality $\sigma_{P3} - \sigma_{P2} > 0$, leading to the singularity indicator μ_{23}

$$\mu_{23} = f_2 + \frac{sp - 2\nu + 1}{(1 - 2\nu)(1 - p)} (\sigma_{T3} - \sigma_{T2}) > 0 \quad (14)$$

When the state of stress is in a regular region both singularity indicators μ_{12} and μ_{23} are negative. If μ_{12} is positive the state is in a corner region S_{12} , where yield surfaces f_2 and f_1 are active. On the other hand, if μ_{23} is positive, the state of stress is in the corner region S_{23} and yield surfaces f_2 and f_3 are active.

As the Mohr-Coulomb yield surface is in the form of a pyramid in principal stress space, an additional singularity exists at the apex of the pyramid where all six planes meet. It is possible to find a set of planes, which define the apex singularity region, within which all states of stress will be required to return towards the apex. These planes also meet at the apex of the Mohr-Coulomb surface and form the *inverted pyramid* [5] as it is graphically illustrated in Figure 3.

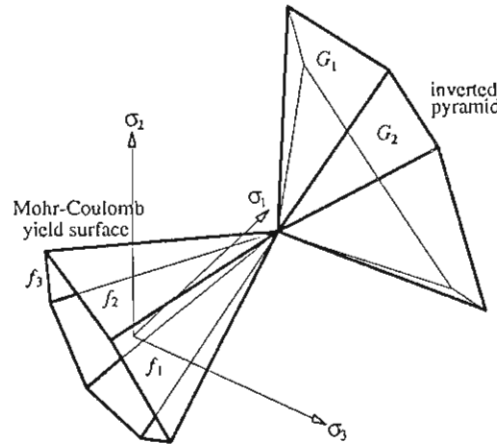


Figure 3: Mohr-Coulomb yield surface with apex singular region within the inverted pyramid

The equations of the inverted pyramid planes can be obtained by considering a state of stress in principal stress space where two yield conditions are active. If the active conditions are f_2 and f_3 , a plastic two vector stress return can be obtained by using eq. (9). By equating the components $\sigma_{P1} = \sigma_{P3}$ of the final corrected stress σ_P and by considering that if such a return is to reach the apex, one can derive the equation of the first plane to be

$$G_1 = r_1\sigma_1 + q_1(\sigma_2 + \sigma_3) - t_1c \cot \phi = 0 \quad (15)$$

where

$$\begin{aligned} r_1 &= s[(s + 2\nu - 1)(p + 2\nu - 1) - 2\nu^2s^2 - 2\nu^2p^2] \\ q_1 &= (s - 2\nu + 1)(2\nu p - 2\nu - s + 1) \\ t_1 &= r_1 + sq_1 + pq_1 \end{aligned} \quad (16)$$

Similarly, if the active yield conditions were f_1 and f_2 , one can obtain the equation of the other desired plane to be

$$G_2 = r_2(\sigma_1 + \sigma_2) + q_2\sigma_3 - t_2c \cot \phi = 0 \quad (17)$$

where

$$\begin{aligned}
r_2 &= (s + 2\nu - 1)(2\nu p + 2\nu - s - 1) \\
q_2 &= s[(s - 2\nu + 1)(p - 2\nu + 1) - 2\nu^2 s^2 - 2\nu^2 p^2] \\
t_2 &= sr_2 + pr_2 + q_2
\end{aligned} \tag{18}$$

Three of the six planes of the inverted pyramid can be obtained, if the indices of σ_1 , σ_2 and σ_3 are permuted in eq. (15). The other three planes can be obtained by interchanging indices in eq. (17). Equations (15) and (17) may be used in an analogous way to yield conditions to check if state of stress is in the apex singularity region. If $G_1 > 0$ and $G_2 > 0$ the stress is in the singular apex region and a multivector stress return is required.

In case of associated flow $q_p \equiv f$ it can be shown that the above singularity indicators correctly recover the solutions provided by Pankaj [5].

6 Direction of viscoplastic flow in singular regions

In this paper an approximation is made, that if the viscoplastic state of stress at time instance t^n is in any of the singular regions, the viscoplastic flow during a time step Δt^n would take place in the direction which would eventually bring the state of stress back to the corner or the apex of the current yield surface defined with the current active yield planes f_i^n , if the full relaxation process would have time to develop during the time step. Finding the direction of the viscoplastic flow in that way corresponds to finding an ideal plasticity solution for the active yield and viscoplastic (plastic) potential functions f_i^n and $(q_{vp})_i^n$.

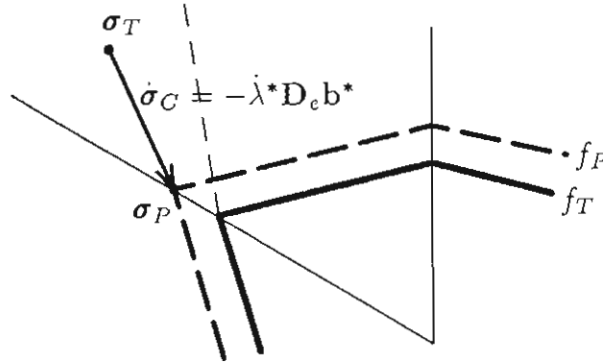


Figure 4: Ideal plastic stress return in singular region

Figure 4 shows the case where the viscoplastic stress *i.e.* elastic predictor σ_T is inside the singular corner region and $\dot{\sigma}_C$ is the plastic corrector to the corner, which can be according to eq. (9) written as

$$\dot{\sigma}_C = \sigma_P - \sigma_T = -D_e \sum_{j=1}^m \dot{\lambda}_j b_j \tag{19}$$

where D_e is the elastic constitutive matrix. One can write that

$$\sum_{j=1}^m \dot{\lambda}_j b_j = \dot{\lambda}^* \mathbf{b}^* \tag{20}$$

where the flow vector $\mathbf{b}^* = \partial q_p^* / \partial \sigma$ and the plastic multiplier $\dot{\lambda}^*$ would correspond to a single vector stress return to the corner. q_p^* now denotes the current plastic flow potential in the

singular region. Following from the requirement that the potential field should be uniform also in the singular regions, one can assume that the current flow vector can be obtained from

$$\mathbf{b}^* = \sum_{j=1}^m \dot{\lambda}_j \mathbf{b}_j \frac{|\mathbf{b}^*|}{|\sum_{j=1}^m \dot{\lambda}_j \mathbf{b}_j|} \quad (21)$$

where $|\mathbf{b}^*|$ is known and represents the absolute value of the field gradient defined with eq. (11).

Following the initial assumption the flow vector $\mathbf{b}^*$ can also be considered to define the current direction of the viscoplastic flow in the singular region. Proposed generalisation of the viscoplastic flow vector is valid both for corner and apex regions since the direction of viscoplastic straining is uniquely defined with eq. (21), which in turn depends on the type of the singularity region.

7 Consistent evaluation of $\Phi(f)$ function

The requirement for a uniform yield potential field around the whole of the yield surface can also be recognised when deriving the distance norm of the viscoplastic stress point from the current yield surface $\Phi(f^*)$ in the singular regions of the Mohr-Coulomb yield surface. This corresponds to fitting the sections of cylindrical uniform potential fields along the edges of the yield surface in the corner singular regions and the section of a spherical uniform potential field inside the inverted pyramid for the case of apex singularity. The absolute value of the gradient of the yield potential field is then constant and equal to

$$|\text{grad} f| = |\mathbf{a}_1| = |\mathbf{a}_2| = \dots = |\mathbf{a}_6| = |\mathbf{a}^*| = \sqrt{\frac{1}{2}(s^2 + 1)} \quad (22)$$

Based on the above assumption the potential of any point in the potential field is then equal to its shortest distance δ from the yield surface, multiplied by the absolute value of the field gradient

$$f^*(\boldsymbol{\sigma}_{vp}) = \delta |\text{grad} f| = \delta \sqrt{\frac{1}{2}(s^2 + 1)} \quad (23)$$

The shortest distance of the viscoplastic state of stress $\boldsymbol{\sigma}_{vp}^T = (\sigma_1, \sigma_2, \sigma_3)$ from the yield surface in the singular regions is then equal to the shortest distance of the stress point from the corresponding edge or apex, which can be easily determined [7].

Since all six equations which define six possible Mohr-Coulomb yield surfaces share the common reference value $F_0 = c \cos \phi$, it can be assumed that the same reference value can be used also in singular regions. Therefore, the viscoplastic norm function defined with eq. (3) can be evaluated simply by substituting f with f^* .

The Perzyna's viscoplastic strain rate in singular regions of Mohr-Coulomb yield surface can now be obtained simply by introducing $\mathbf{b}^*$ and f^* in the original equation for viscoplastic strain rate eq. (3.16)

$$\dot{\boldsymbol{\epsilon}}_{vp} = \gamma(\Phi(f^*)) \mathbf{b}^* \quad (24)$$

As illustrated in the following numerical examples, the presented technique is clearly a valid regularisation of viscoplastic flow rule in singular regions of the Mohr-Coulomb yield surface.

In the limiting case, if the proposed generalisation is also applied for regular regions of multi-surface Mohr-Coulomb yield criterion, it indeed recovers the solutions obtained by using the standard procedures.

8 Numerical examples and conclusions

All examples to be discussed are for viscoplastic Mohr-Coulomb material with the following hypothetical data:

$$\begin{array}{llll} E = 1.0 & \phi = 30^\circ & c \cos \phi = F_0 = 2.25 & \gamma = 10 \\ \nu = 0.2 & \psi = 10^\circ & \Phi(f^*) = f^*/F_0 & \Delta t = 0.2 \end{array}$$

8.1 Example 1: Corner singularity

If the viscoplastic stress at time t^n is equal to $(\sigma_T^n)^T = (6 \ 5 \ 1)$, singularity indicator $\mu_{12} = 1.025 > 0$ (eq. (13)) shows that the stress point is in the singular corner region S_{12} so the yield functions f_1^n and f_2^n are active ($f_1^n = 1.25$, $f_2^n = 2$). Using eq. (21) the flow vector in the singular region is obtained to be equal to $(b^*)^T = (0.4764 \ 0.2194 \ -0.4899)$ and according to eq. (23) the potential of the viscoplastic stress point is $f^* = 2.2613$, which yields the following viscoplastic strain increments $\Delta \epsilon_{vp}^n = \dot{\epsilon}_{vp}^n \Delta t$ during the time step $(\Delta \epsilon_{vp}^n)^T = (0.9576 \ 0.4409 \ -0.9847)$.

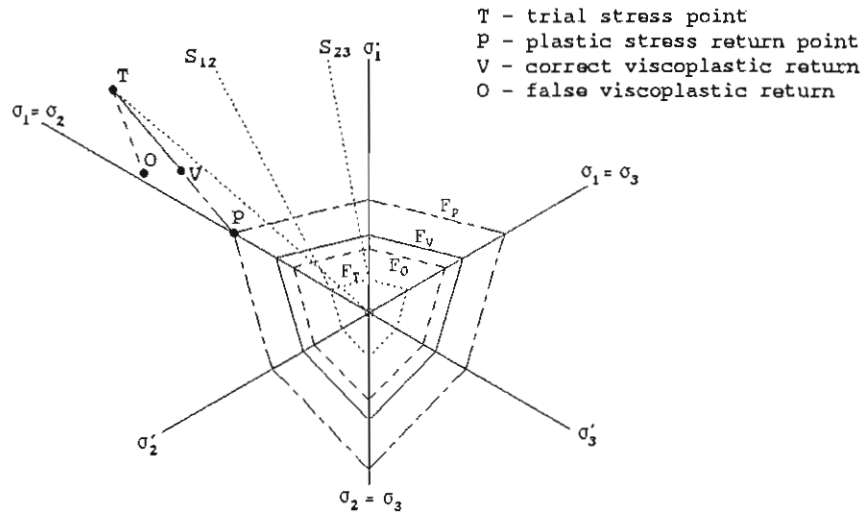


Figure 5: Viscoplastic stress return in S_{12} singular corner region

Figure 5 illustrates the example, where the viscoplastic stress point T is clearly inside the singularity region S_{12} . Point P represents a correct ideal plastic return to the corner of the yield surface, whereas points O and V represent viscoplastic returns. The correctness of viscoplastic stress return direction V using the proposed generalisation is self evident. The viscoplastic stress return direction O , based on the traditional single vector return would clearly lead to the wrong solution.

8.2 Example 2: Apex singularity

If the viscoplastic state of stress at time t^n is equal to $(\sigma_T^n)^T = (16 \ 11 \ 5)$, the singularity indicators G_1 and G_2 show that the stress predictor is in the apex singular region, so yield functions f_1^n , f_2^n and f_3^n are all active ($f_1^n = 4.75$, $f_2^n = 8.5$ and $f_3^n = 7$) and a three vector return is required to return the stress point correctly to the apex in case of the plastic solution. According to eq. (21) the flow vector in the singular region is equal to $(b^*)^T = (0.6396 \ 0.2596 \ -0.1963)$. The potential of the viscoplastic stress point is according to eq. (23) equal to $f^* = 10.4508$, which yields the following viscoplastic strain increments $\Delta \epsilon_{vp}^n = \dot{\epsilon}_{vp}^n \Delta t$ during the time step $(\Delta \epsilon_{vp}^n)^T = (5.9418 \ 2.4120 \ -1.8237)$.

T - trial stress point
P - plastic return point
V - correct viscoplastic return
O - false viscoplastic return

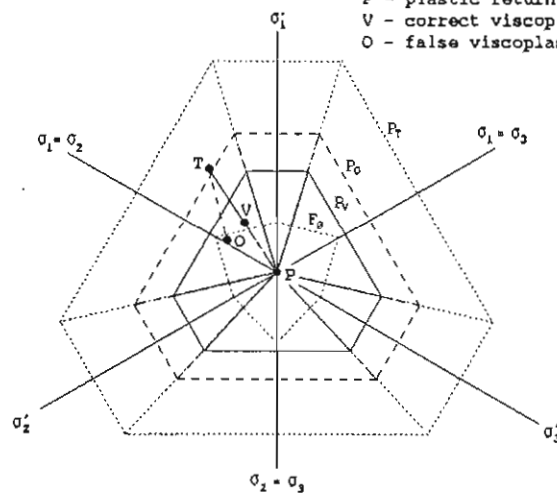


Figure 6: Viscoplastic stress return in apex singular region

Similar conclusion as for corner singularity can also be made for the apex singularity example, which is illustrated in Figure 6. The viscoplastic stress point T is inside the singular inverted pyramid (corresponding cross-section of which is shown with line P_T), where all six Mohr-Coulomb yield surfaces are active and multi-vector stress return is required to return the stress point in the direction of the apex P . Using the approach that does not recognise the need for a multi vector return in these singular regions leads to a wrong viscoplastic stress return O . Again, viscoplastic stress return V , using the proposed regularisation technique is correctly oriented towards the apex which would eventually reach, if the full relaxation process would have time to develop.

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MODELIRANJE RUBNOG TRENJA KOD ANALIZE METODOM KONAČNIH ELEMENATA U FORMULACIJI PLASTIČNOG TOKA

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SAŽETAK

Kod analize postupaka preoblikovanja metala jedan od osnovnih problema je numerički efikasno modeliranje rubnog sloja s trenjem. U ovom radu su razvijeni postupci sa umetanjem posebnog rubnog sloja elemenata uz analizu metodom konačnih elemenata u formulaciji viskoplastičnog toka. Ovi elementi generiraju dodatno opterećenje ili pak narinjavaju zakon trenja kao ograničenje preko kaznene funkcije. U oba slučaja je sila trenja modelirana u funkciji polja brzina radi numeričke stabilnosti i konvergencije ukupnog iterativnog postupka. Rezultati dobiveni s ova dva pristupa su uspoređeni kod simulacije postupka hladnog istiskivanja.

MODELLING OF BOUNDARY FRICTION WITH THE PLASTIC FLOW FORMULATION FINITE ELEMENT ANALYSIS

SUMMARY

Numerically efficient modelling of the boundary layer with friction is one of the basic problems in analysis of metal forming processes. In this paper, numerical procedures which involve inserting a layer of interface elements are developed for the flow-formulation viscoplastic FE analysis. These elements generate additional load or impose the friction law as a constraint via the penalty function. In both cases, the friction force is modelled as a function of the velocity field for numerical stability and convergence of the complete iterative process. The results obtained with both approaches are compared on the simulation of the cold extrusion process.

1. UVODNI IZRAZI

Polazi se od jednadžbe viskoplastičnog elementa formulacije toka

$$\Phi(\underline{v}) = \left(\int_V \left(\frac{\bar{\sigma}}{\bar{\epsilon}} \right) \underline{B}^T \underline{D} \underline{B} dV \right) \cdot \underline{v} + K \left(\int_V \underline{B}^T \underline{C} \underline{C}^T \underline{B} dV \right) \cdot \underline{v} - \int_{S_f} \underline{N}^T \underline{p} dS = 0 \quad (1)$$

gdje je $\underline{\dot{\epsilon}} = \underline{B}^T \underline{\dot{v}}$, $\bar{\epsilon}^2 = \underline{\dot{\epsilon}}^T \underline{D} \underline{\dot{\epsilon}}$, $\dot{\epsilon}_{ii} = \underline{C}^T \underline{\dot{\epsilon}}$, $\bar{\sigma}$ ekvivalentno naprezanje, $\bar{\epsilon}$ ekvivalentna

brzina deformacije, $\underline{v}$ vektor brzine, $\underline{p}$ vektor opterećenja, $\underline{N}$ matrica interpolacionih funkcija.

S obzirom na materijalnu nelinearnost (1) sadržanu u viskozitetu u prvom članu izraza, nakon postupka sastavljanja (asembliranja) globalnog sustava problem se mora rješavati nelinearnim postupcima. Drugi razlog za nužnost primjene iterativnih postupaka jest ovisnost viskoziteta o akumuliranoj efektivnoj deformaciji u promatranoj, točki koja nije poznata unaprijed, kao i (promjenjivost) ovisnost rubnih uvjeta o polju brzina odnosno budućem rješenju, npr. kod postojanja trenja na rubnim plohama.

Na rubnim plohama između materijala u viskoplastičnom toku i okoline dolazi uz normalne reaktivne sile i do pojave sile trenja, koja se opire relativnom gibanju između materijala i okoline. Na tom rubu je potrebno zadovoljiti uvjete trenja. Ako se sila trenja modelira kao funkcija relativne brzine, tada se dobija dodatno ograničenje na varijablama polja, odnosno vektoru brzine tog elementa.

S obzirom da se konačni nelinearni sustav prema (1) rješava iterativnim metodama, direktnim i/ili gradientnim, postupak uključivanja trenja ne smije narušiti numeričku stabilnost ni konvergenciju algoritma. Također je nepoželjno narušavanje simetrije globalnog sustava jednadžbi radi korištenja standardnog frontalnog postupka.

Relativna tangencijalna brzina u nekoj točki se definira kao razlika brzine okoline i brzine materijala u smjeru tangente na rubni brid razgraničenja

$$v_t^R = v_t^B - v_t^M \quad (2)$$

Stanja lijepljenja $v_t^R = 0$, $p_t \leq \eta \cdot p_n$ i klizanja $v_t^R \neq 0$, $p_t = \eta \cdot p_n$ između okoline i materijala definiraju se u ovisnosti o relativnoj brzini. Maksimalna sila trenja se modelira u funkciji relativne brzine, kao adhezivno trenje (funkcija σ_y , "constant shear friction")

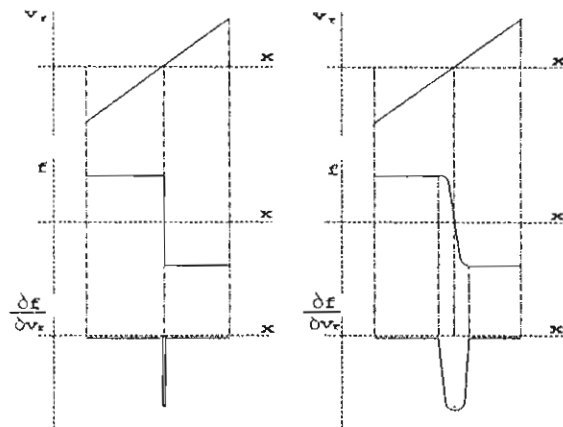
$$p_t = [\eta \cdot \sigma_y] \cdot \frac{v_t^R}{v_t^{RF}} \quad (3)$$

odnosno koheziono trenje (p_n , "Coulomb friction")

$$p_t = [\eta \cdot p_n] \cdot \left(\frac{2}{\pi} \cdot \arctg \frac{v_t^R}{a} \right) \quad (4)$$

gdje je a konstanta. Ovakva funkcionalna ovisnost o relativnoj brzini doprinosi stabilnosti iterativnog postupka, a izrazi moraju biti primjenjivi i kod gradientnih metoda rješavanja gdje su potrebne i prve derivacije izraza.

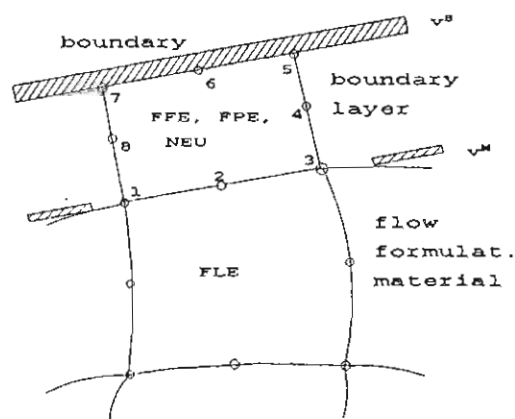
U odnosu na konstantnu silu trenja p_t , koja je numerički vrlo nepovoljna (slika 1), regularizirani izraz (4) s funkcijom arctg znatno doprinosi stabilnosti postupka.



Slika 1. Konstantna sila trenja, regularizirani arctg model
● Constant friction force and regularized arctg model

2. ELEMENT GENERATOR SILE TRENJA

Vlastiti element za generiranje sile trenja (FFE), tanki sloj uzduž ruba okolina- materijal u plastičnom toku (slika 2), postavljen je kao element sa 8 čvorova. Rub $\eta=1$ (lokalni čvorovi 7,6,5) pripada okolini, odnosno služi za definiciju relativne brzine u modelu trenja, dok rub $\eta=-1$ (čvorovi 1,2,3) pripada materijalu. Vektor relativne brzine se sada definira kao vektor čije su komponente razlike tangencijalnih brzina u trima rubnim čvorovima 7-1, 6-2, 5-3 prema (2), pri čemu α označava kut između globalne x-osi i tangente na rub s trenjem



Slika 2. Geometrija FFE / FPE elementa rubnog sloja
● Layout of FFE / FPE element for boundary layer

$$\underline{v}_c^R = \underline{T}(\alpha) \cdot \underline{v} \quad (5)$$

Element FFE ne mijenja ukupnu (globalnu) matricu krutosti sustava, već samo generira dodatno vanjsko opterećenje, silu trenja. Na temelju vektora rješenja $\underline{v}$ prethodne iteracije, koristeći (3), (4) i (5), izračuna se sila trenja za trenutno polje brzine. Dobiveni vektor generiranih vanjskih sila se superponira na izvorno vanjsko opterećenje susjednog regularnog FLE elementa u zajedničkim čvorovima (slika 2). Postupak FFE slijedi:

1. matrica krutosti FFE elementa se u prvoj fazi postavi jednakom jediničnoj matrici (matrica identiteta), $\underline{K} = \underline{I}$
2. vektor opterećenja FFE elementa se postavi jednakim vektoru (rješenja) brzina elementa u prethodnoj iteraciji, odnosno početno pretpostavljenom vektoru brzina elementa $\underline{f} = \underline{v}$
3. Za odgovarajuće lokalne stupnjeve slobode ($i = 1, 3, 5$) FFE elementa koji

pripadaju kontinuumu, elementi matrice krutosti K_{ii} te elementi vektora opterećenja elementa f_i postavljaju se na nulu, $K_{ii}=f_i=0$ ($i=1,3,5$). Ovime se implicira singularnost matrice krutosti elementa, ali unatoč tome globalna matrica krutosti ostaje nesingularna zbog doprinosa susjednog FLE elementa (slika 2). Ako tangencijalna os ruba i os x čine neki kut α , tada se kao ograničenje putem kaznene funkcije, može uvesti beskonačna krutost u smjeru normale uz

$$(v_x/v_y)_k = \tan \alpha_k, \quad \mathcal{L}(\alpha) = \underline{T}''(\alpha) \cdot \underline{y} = 0 \quad \text{i narinuti putem norme. Ovdje se}$$

oslobadjaju lokalni stupnjevi slobode $K_{ii}=f_i=0$, ($i=1, \dots, 6$).

4. Na temelju vektora brzina prethodne iteracije uz (5), dobije se koristeći modele za silu trenja (3), (4), vektor "vanjskih" sila trenja. Ove generirane vanjske sile mogu se pridijeliti (asocirati) odgovarajućim stupnjevima slobode FFE elementa ili (u zajedničkim čvorovima) susjednog FLE elementa.

Ukupno se time očuva rješenje (brzine) iz prethodne iteracije, uz istovremeno postizanje nesingularnosti FFE elementa, koji u čvorovima zajedničkim sa FLE ima nultu krutost a generira silu, dok u ostalima (okolina) prenese brzinu iz prethodne iteracije.

FFE element razvijen u ovom radu može izazvati globalnu singularnost sustava u nekim aplikacijama. Takav primjer primjene je i modeliranje nekih procesa preoblikovanja materijala (npr. valjanja), gdje se tečenje materijala fizički ostvaruje isključivo silom trenja. Tada nakon diskretizacije ne postoji ni jedan čvor, koji pripada regularnim flow elementima FLE, u kojem je kao rubni uvjet propisan pomak (brzina) u smjeru osi trenja, npr. osi x . Razvijen je numerički postupak za prevladavanje ovog problema pronalaženjem i definiranjem "neutralne" točke odnosno zone u svakoj iteraciji. Pregledom svih rubnih čvorova na bridu s trenjem pronalazi se onaj s najmanjim apsolutnim iznosom relativne brzine. Taj čvor, odnosno niz susjednih čvorova s relativnom brzinom manjom od neke usvojene tolerancije ("neutralna zona") "učvršćuje" se za okolinu, tj dobiva zadani vektor brzine odgovarajuće točke okoline, čime se relativna brzina postavlja na nulu.

S obzirom na nelinearnost problema i odgovarajući iterativni postupak, "neutralni" čvor odnosno zona nisu permanentni kroz iteracije. Stoga je u nekoj općoj iteraciji nužno prvo "osloboditi" neutralnu zonu iz prethodne iteracije narinutih veza s okolinom, odgovarajućom modifikacijom vektora opisa propisanih pomaka. Uz to je potrebno vratiti u čvorove neutralne zone njihove izvorne brzine iz iteracije prije nastanka "neutralne" zone. Tek tada je potrebno postaviti brzine i opis rubnih uvjeta za novonastalu "neutralnu" zonu tekuće iteracije. Ovaj postupak se algoritamski ostvaruje sustavnim pretraživanjem vektora brzina FFE elemenata te globalnih vektora s opisom rubnih uvjeta.

FFE numerički postupak može generiranjem dodatne vanjske sile uz nerealno veliki koeficijent trenja u nekim situacijama lokalno proizvesti izmjeničnu promjenu smjera toka u rubnom sloju u uzastopnim iteracijama.

3. ELEMENTI S KAZNENOM FUNKCIJOM TRENTA

Alternativni pristup uvođenju sile trenja na rubnim plohama u model formulacije toka jest i zapis Coulomb-ovog uvjeta trenja ili drugog odabranog modela trenja u obliku funkcije ograničenja. Tada se u varijacionoj formulaciji dobiva problem uvjetnog ekstrema s ograničenjima, koji se može riješiti pomoću Lagrange-ovih multiplikatora ili metodom kaznene funkcije. Ovdje je izvorno razvijen element (FPE "Friction Penalty Element"), gdje je ograničenje trenja preko kaznene funkcije narinuto na regularni (FLE) element formulacije toka (1).

Ograničenje trenja temeljeno na Coulomb-ovom zakonu može se pisati u obliku

$$\underline{P} = \underline{P}_t - \eta \underline{P}_n = 0 \quad (6)$$

Norma vektora ograničenja $\delta(\underline{P}^2)$, odnosno $\underline{P}^T \delta \underline{P}$ je primjerice uz (3) i (5) izvedena kao

$$\underline{P}^T \delta \underline{P} = \frac{\sigma_y}{V_{RF}} \left[\frac{\sigma_y}{V_{RF}} \cdot \underline{T}^T \cdot \underline{Y} - \underline{P}_n \right]^T \cdot \underline{T} \cdot \delta \underline{Y} \quad (7)$$

Dodatni član (7) se množi s kaznenom konstantom te dodaje u varijaciju potencijala, a cjelokupni izraz izjednačava s nulom shodno uvjetu stacionarnosti. Uz odbacivanje proizvoljne varijacije vektora brzine $\delta \underline{y}$ te transpoziciju slijedi originalni element $FPE_{(FLE)}$

$$\begin{aligned} \Phi(\underline{y}) = & \left[\int_V \left(\frac{\sigma}{\epsilon} \right) \underline{B}^T \underline{D} \underline{B} dV + K \int_V \underline{B}^T \underline{C} \underline{C}^T \underline{B} dV + K_f \left(\frac{\sigma_y}{V_{RF}} \right)^2 \cdot \underline{T}^T \cdot \underline{T} \cdot \underline{Y} \right] \\ & - \left[\int_{S_t} \underline{N}^T \underline{P} dS + K_f \left(\frac{\sigma_y}{V_{RF}} \right) \cdot \underline{T}^T \cdot \underline{P}_n \right] = 0 \end{aligned} \quad (8)$$

U nekim slučajevima je normalna sila p_n , i sama rezultat rješavanja globalnog sustava u prethodnoj iteraciji, numerički dosta nestabilna i značajno varira kroz iteracije. Stoga se stabilniji tijekom proračuna dobije korištenjem σ_y kao referentne veličine u modelu trenja. Konačni sustav se rješava iterativno uz inkrementiranje opterećenja. Kod modeliranja trenja metodom kaznene funkcije, može se pojaviti i problem zaključavanja ("locking") elementa, kada je elementu narinuto previše ograničenja za postojeći broj stupnjeva slobode, te više ne može dobro fizikalno opisati izvorni problem. Zadovoljavajući narinute uvjete ograničenja, element može dati nerealno polje brzina, što je ovdje prevladano "mekšim" narinjavanjem uvjeta trenja, odnosno ispravno doziranom iznosom kaznene konstante K_f da ne dodje do

relativne dominantnosti ni jednog člana ukupnog izraza.

Elementi (numerički postupci) FFE i FPE su umetnuti kao vrlo tanki rubni sloj elemenata sa odgovarajućim malim pripadnim volumenom, mogu imati i fizički koincidentne čvorove gornjeg i donjeg ruba. FFE postupak ne doprinosi energiji sustava jer implicira nultu potencijalnu energiju uz dodatno narinjavanje ograničenja održanja vektora brzine čvorova okoline iz prethodne iteracije. S druge strane element $FPE_{(FLE)}$ ima smisao elementa formulacije toka (1), na koji je narinuto dodatno ograničenje, te tome odgovara i njegov energetski doprinos globalnom sustavu (kroz mali pripadni volumen). Uvedeno ograničenje plod je uvođenja trenja preko kaznene funkcije, i djeluje na relativnu brzinu između okoline i materijala. Odgovarajući dodatak (7) ugrađen u (1) manifestira se i energijskim doprinosom sustavu, i to kao element toka FLE uz dodatak kaznenog člana koji je aktivan samo kada je narušeno ograničenje, inače nestaje.

Modeliranje trenja upotrebom FPE elementa "invertira" problem rubnih uvjeta; za silu trenja definiranu u funkciji normalne sile ili granice popuštanja (3), (4) te relativne brzine, element FPE narinjava na rubni sloj pripadnu razliku brzine. Ova generirana razlika brzine prema okolini unosi se mehanizmom kaznene funkcije na polje brzina. Stoga element FPE "transformira" rubni uvjet sile, odnosno zadovoljenje uvjeta trenja kao ograničenja, u odgovarajući uvjet na polju brzine (u rubnom sloju) metodom kaznene funkcije.

4. PRIMJERI

Pomoću razvijenog vlastitog programskog paketa za analizu viskoplastičnog toka metala metodom konačnih elemenata simulirani su neki tehnološki postupci preoblikovanja materijala. Primijenjeni su i vlastiti postupci modeliranja rubnog trenja.

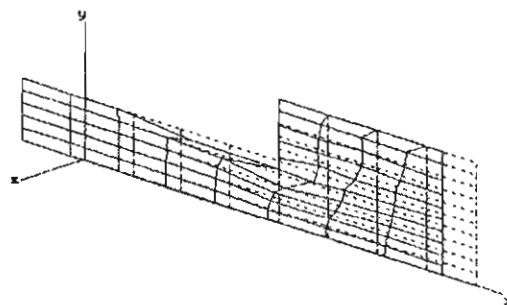
U nizu numeričkih primjera variran je koeficijent trenja u rubnom sloju, kaznena konstanta ograničenja nestišljivosti, kaznena konstanta ograničenja trenja, tip nelinearnog iterativnog postupka i broj iteracija, model rubnog trenja u ovisnosti o relativnoj brzini (linearni, apsolutni, arc tg), model rubnog trenja u funkciji normalne sile te naprezanja popuštanja, konstitutivni model materijala u kombinaciji s određivanjem povijesti deformacije točke, fiksni rubni uvjeti rubnog sloja, te broj rubnih čvorova u kojima je narinuto ograničenje trenja.

Ovdje je prikazana samo usporedba rezultata dobivenih primjenom FFE i FPE elementa na istom primjeru hladnog 2D istiskivanja metala. Korištena je kaznena konstanta nestišljivosti

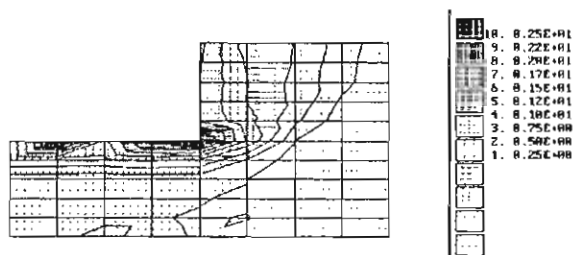
$K = 10^4$, materijal konstitucije $\bar{\sigma} = \bar{\sigma}(\bar{\epsilon}) = 0.0752 \cdot \bar{\epsilon}^{0.066}$, kombinirani postupak Newton-

Raphson i direktna iteracija.

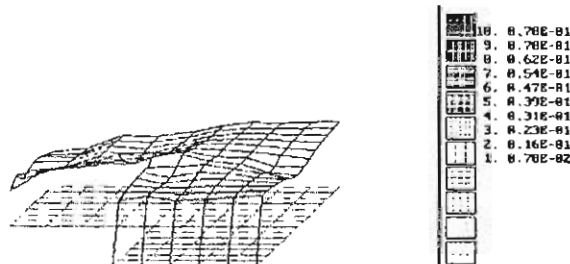
Primijenjene su mreže s 60 odnosno 98 elemenata (do 682 stupnja slobode), koeficijent trenja je zadan u iznosu 0.1. Na slikama 3,4,5 prikazani su polje brzina te raspodjela ekvivalentne brzine deformacije i ekvivalentnog napreznja za ovaj primjer, uz upotrebu FFE elementa.



Slika 3. Polje brzina, FFE element
● Velocity field with FFE element



Slika 4. Ekvivalentna brzina deformacije, FFE
● Equivalent strain rate distribution with FFE



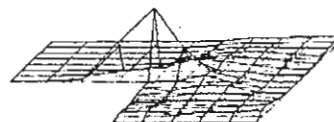
Slika 5. Ekvivalentno napreznje sa FFE na rubu
● Equivalent stress with FFE at boundary

Alternativno je uvršten i model trenja s $\arctg$ funkcijom, ali se globalni rezultati vrlo malo razlikuju. Iznos koeficijenta (faktora) trenja za rubni sloj ima relativno mali utjecaj na globalne veličine i na lokalne veličine dalje od rubnog sloja. Konvergencija je vrlo dobra, residualne vrijednosti funkcije i prirasta vektora varijable su zadovoljavajuće nakon desetak iteracija ($< 10^{-4}$).

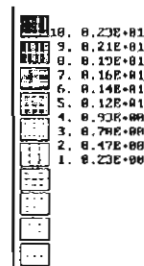
U nastavku je isti primjer simuliran uz upotrebu FPE elementa. Globalno polje brzina i efektivne brzine deformacije, efektivnog napreznja, sila istiskivanja (na klipu), te ostale relevantne veličine usporedjene su sa istim poljima dobivenim sa FFE. Takodjer su lokalno u rubnom sloju usporedjene brzine. S obzirom na veću numeričku osjetljivost pristupa s kaznenom funkcijom, kontinuirano je praćena konvergencija i residuum funkcije i prirasta varijable. Ograničenje trenja narinuto je kaznenom funkcijom s konstantom iznosa 10^1 . Na slici 6, raspodjeli brzina, dobro je uočljiva razlika u x-brzinama točaka gornjeg i donjeg ruba elemenata $FPE_{(FLE)}$ koju generira upravo element $FPE_{(FLE)}$ ugrađenom kaznenom funkcijom. Usporedba rezultata na slikama 3-5 (FFE) te 6-7 (FPE) pokazuje vrlo dobro poklapanje rezultata, čak i u rubnom sloju, unatoč iz osnove različitim modeliranjem trenja. Brzina u izlaznom presjeku se razlikuje 7%, sila istiskivanja 9%, brzina u rubnom sloju s trenjem najviše 7%.



Slika 6. Polje brzina, FPE elementi na rubu
● Velocity field with FPE elements at boundary



Slika 7. Ekvivalentna brzina deformacije sa FPE
● Equivalent strain rate distribution with FPE



ZAKLJUČAK

Razvijene metode za uključivanje rubnog trenja u analizu viskoplastičnog toka materijala metodom konačnih elemenata na konkretnim primjerima pokazuju dobru konvergenciju i opis rubnog sloja. Usprkos potpuno suprotnom načinu uvođenja trenja u numerički model rezultati se zadovoljavajuće poklapaju. Razvijene metode su, uz male numeričke probleme u određenim situacijama, vrlo pogodne za primjenu.

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Application of a Homogenisation Technique to Masonry

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Abstract:

An homogenisation technique for the computation of equivalent material properties of masonry is presented. As presented here this technique is capable of calculating the equivalent material properties of a stack of masonry units with two sets of mutually perpendicular mortar joints on the basis of their dimensions and elastic properties. Both masonry units and mortar can be either isotropic or orthotropic. Also, comments are offered on the equation for the calculation of the characteristic compressive strength of masonry as proposed in Eurocode No. 6. It is argued that this equation lacks rationale in terms of engineering mechanics and a new approach based on the analysis described is proposed. It is argued that tensile stress induced in masonry units which are part of masonry panels subjected to pressure loads are responsible for its failure. A diagram for the calculation of the tensile stresses in masonry units when masonry is subjected to compressive loads is given.

1 Introduction

Masonry is highly nonhomogenous and anisotropic material due to the presence of two sets of mortar joints. In the analysis of the masonry structures in engineering practice it is usually assumed that it is isotropic since this assumption leads to a considerable simplification of the calculations. On the other hand this simplification does not allow calculation of stresses in masonry units and joints. One way to overcome this problem is using the Finite element method and model each unit and joint. This approach, however, leads to a huge models and expensive analysis.

Here, a homogenisation technique is used to calculate equivalent material properties for masonry which ensures the same overall response to the original material. This method is quite general because it allows orthotropic properties of masonry units and joints. In term of the analysis of the masonry structures it enables calculation of stresses in both constituents material. If the finite element method is used for the analysis it is not necessary to model each unit and joint, so a model with much fewer degrees of freedom can be safely used.

Utilising this method in the analysis of stresses induced in the constituents it is shown that the usual equations for the calculation of the compressive strengths of masonry used in codes lack rationale. A new rational approach is also described.

2 Derivation of the equivalent elastic properties

An equivalent material is defined as a material whose overall mechanical response under an arbitrary stress path is the same as that of brick masonry. This approach has already been adopted in rock mechanics by Zienkiewicz and Pande [1], Gerrard [2], Gerrard and Pande [3] and in masonry by Pande et al. [4] and Kralj et al [5]. The main assumptions in developing the equivalent material properties are:

1. bed and perpend joints are continuous, mutually perpendicular and appear in regular sets with constant spacings. Mortar can be either isotropic or orthotropic.
2. masonry units are all of the same dimensions. They can be either isotropic or orthotropic.
3. there is no relative slippage between the constituents on their interfaces.

The assumption of a continuity of the perpend joints may look questionable, but it has been shown (Middleton et al. [6]) that the error involved is negligible.

Homogenisation process will be applied in two stages. In the first stage one layer of masonry units with perpend joints is considered and homogenised giving a material which is called EQ1. In the second stage a stack consisting of alternating layers of the material EQ1 and bed joints is homogenised giving the final equivalent material. In figure 1 dimensions of constituents and coordinate axes orientations are given. The elastic properties of the each constituents are defined by nine independent constants for the more general orthotropic case:

$$(E_{xm}, E_{ym}, E_{zm}, \nu_{xym}, \nu_{yzm}, \nu_{zxm}, G_{xym}, G_{yzm}, G_{zxm}) \quad (1)$$

where subscript m stands for a material (masonry unit, bed joint or perpend joint). The principal requirement imposed on the equivalent material is to have the same overall response as the parent (composite) material when subjected to an arbitrary loading condition. In terms of stresses and strains this requirement implies that stresses and strains in the equivalent material ($\{\sigma_c\}$ and $\{\epsilon_c\}$) equal the average stresses and strains in the parent material, i.e.:

$$\{\sigma_e\} = \frac{1}{V} \sum_m \int_{V_m} \{\sigma_m\} dV \quad (2)$$

$$\{\varepsilon_e\} = \frac{1}{V} \sum_m \int_{V_m} \{\varepsilon_m\} dV \quad (3)$$

Continuity requirements across layer interfaces lead to an introduction of auxiliary stresses $\{s_m\}$ and strains $\{e_m\}$ such that:

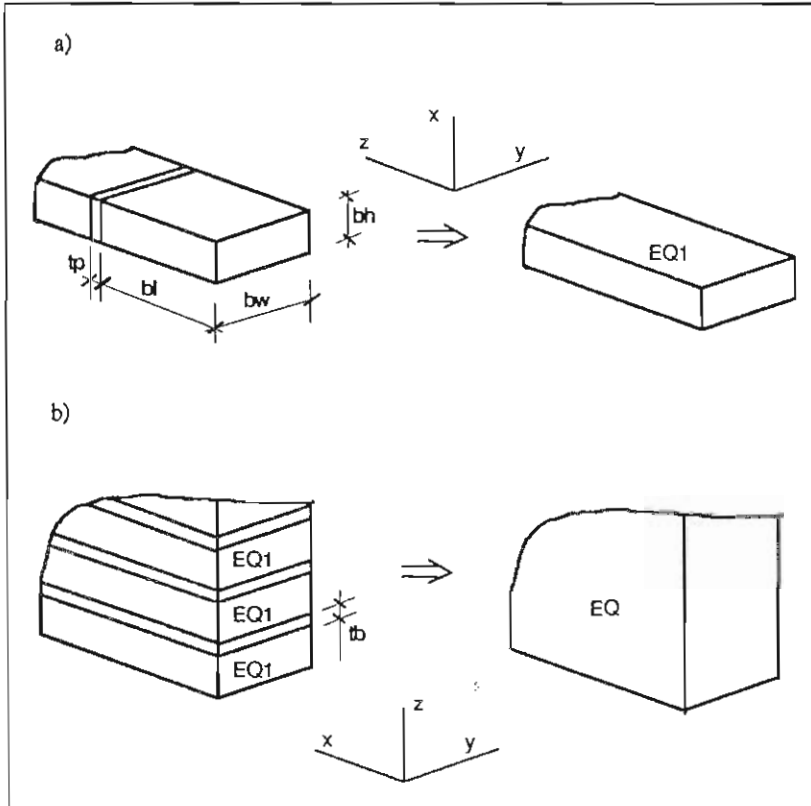


Figure 1 - Homogenisation sequence (a) stage 1, b) stage 2), dimensions of the constituents and coordinate system orientation

$$\{\sigma_m\} = \{\sigma_e\} + \{s_m\} \quad (4)$$

$$\{\varepsilon_m\} = \{\varepsilon_e\} + \{e_m\} \quad (5)$$

For this particular case of coordinate system orientation, equilibrium and compatibility conditions auxiliary stresses and strains are (note that the local z axis is always perpendicular to the interface plane, fig. 1 a) and b)):

$$\{s_i\} = \{s_{xm}, s_{ym}, 0, s_{xym}, 0, 0\} \quad (6)$$

$$\{e_m\} = \{0, 0, e_{zm}, 0, e_{xzm}, e_{zym}\} \quad (7)$$

To ensure an equivalence of the strain energies between parent and equivalent material the following relationships must be satisfied:

$$\sum_m t_m \{s_m\} = 0 \quad (8)$$

$$\sum_m t_m \{e_m\} = 0 \quad (9)$$

where t_m stands for a relative dimension of the constituent. Introducing stresses (4) and strains (5) into the elastic stress-strain relationship of the constituents:

$$\{\varepsilon_m\} = [C_m] \{\sigma_m\} \quad (10)$$

where

$$[C_m] = \begin{bmatrix} \frac{1}{E_{xm}} & \frac{1}{E_{ym}} & \frac{1}{E_{zm}} & 0 & 0 & 0 \\ -\frac{\nu_{xym}}{E_{xm}} & \frac{1}{E_{ym}} & -\frac{\nu_{xzm}}{E_{ym}} & 0 & 0 & 0 \\ -\frac{\nu_{xzm}}{E_{xm}} & -\frac{\nu_{xym}}{E_{ym}} & \frac{1}{E_{zm}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xym}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xzm}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{yzm}} \end{bmatrix} \quad (11)$$

and using the strain energy conditions (9) and (10) the auxiliary stresses and strains can be eliminated. From these relationships the equivalent elastic properties of the stack of masonry units with mortar joints (EQ1) can be expressed as:

$$\nu_{xye} = \frac{\zeta}{\alpha}; \quad \nu_{xze} = \chi - \frac{\lambda\zeta}{\alpha}; \quad \nu_{yze} = \lambda - \frac{\chi\zeta}{\alpha} \quad (12)$$

$$E_{xe} = \frac{\alpha\beta - \zeta^2}{\alpha}; \quad E_{ye} = \frac{\alpha\beta - \zeta^2}{\beta}$$

$$\frac{1}{E_{zm}} = \sum_m \frac{t_m}{E_{zm}} + \sum_m \left[\frac{\nu_{xye}}{E_{xe}} - \frac{\nu_{xzm}}{E_{xm}} \right] \chi_m + \sum_m \left[\frac{\nu_{yze}}{E_{ye}} - \frac{\nu_{yzm}}{E_{ym}} \right] \chi_m \quad (13)$$

$$\begin{aligned}
G_{xye} &= \sum_m t_m G_{xym} \\
\frac{1}{G_{xze}} &= \sum_m \frac{t_m}{G_{xzm}}; \quad \frac{1}{G_{yze}} = \sum_m \frac{t_m}{G_{yzm}}
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\alpha &= \sum_m t_m E_{ym} / \delta_m \\
\beta &= \sum_m t_m E_{xm} / \delta_m \\
\varsigma &= \sum_m t_m E_{ym} \nu_{xym} / \delta_m \\
\chi_m &= (\nu_{xzm} + \nu_{xym} \nu_{yzm}) t_m / \delta_m; \quad \chi = \sum_m \chi_m \\
\lambda_m &= (\nu_{xzm} + \nu_{xym} \nu_{yzm}) t_m / \delta_m; \quad \lambda = \sum_m \lambda_m \\
\delta_m &= 1 - \nu_{xym} \nu_{yxm}
\end{aligned} \tag{15}$$

To obtain the elastic properties for the second stage of homogenisation (i.e. to include bed joints) the equivalent material for the stack of masonry units and perpend joints (EQ1) is taken as a homogeneous layer of a new stack with bed joints as a second layer. The same procedure can now be applied to this new stack giving the overall equivalent material for masonry. To maintain the convention of coordinate system orientation it is necessary to rotate the material coordinate system of the EQ1 from orientation as in fig. 1 a) to Fig. 1 b). For the case of isotropic constituents this procedure is simplified and can be found in Pande et al. [6].

3 Stresses in constituent materials

The equivalent material derived in the previous chapter is homogeneous and orthotropic. Using the derived material properties (equations (13) and (14)) a stress field $\{\sigma_e\}$ in the structure can be calculated using the FEM or any other suitable tool. From this stress field stresses in the constituent materials can be uniquely calculated. Solving equation (4) for the auxiliary stresses leads directly to the stresses in both layers of the equivalent material. For the system of two alternating layers the following expressions can be derived:

$$\begin{aligned}
\sigma_{xm} &= \sigma_{xe} + C_m (E_{xm} A_m + \nu_{xym} E_{ym} B_m) \\
\sigma_{ym} &= \sigma_{ye} + C_m (\nu_{yxm} E_{xm} A_m + E_{ym} B_m) \\
\sigma_{zm} &= \sigma_{ze} \\
\tau_{xym} &= (G_{xzm} / G_{xze}) \tau_{xye} \\
\tau_{xzm} &= \tau_{xze} \\
\tau_{yzm} &= \tau_{yze}
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
A_m &= \left[\frac{1}{E_{xe}} - \frac{1}{E_{xm}} \right] \sigma_{xe} - \\
&\quad \left[\frac{\nu_{xye}}{E_{xe}} - \frac{\nu_{xym}}{E_{xm}} \right] \sigma_{ye} - \left[\frac{\nu_{xze}}{E_{xe}} - \frac{\nu_{xzm}}{E_{xm}} \right] \sigma_{ze} \\
B_m &= - \left[\frac{\nu_{yxe}}{E_{ye}} - \frac{\nu_{yxm}}{E_{ym}} \right] \sigma_{xe} + \\
&\quad \left[\frac{1}{E_{ye}} - \frac{1}{E_{ym}} \right] \sigma_{ye} - \left[\frac{\nu_{yze}}{E_{ye}} - \frac{\nu_{yzm}}{E_{ym}} \right] \sigma_{ze} \\
C_m &= \frac{1}{1 - \nu_{xym} \nu_{yxm}}
\end{aligned} \tag{17}$$

In this case stresses are firstly calculated in the equivalent material EQ1 and bed joints. In the subsequent step material EQ1 is taken as the equivalent material and stresses in masonry units and perpend joints are calculated.

4 Remarks on compressive strength of masonry

Equation for the characteristic strength of masonry given in Eurocode No. 6 is given as:

$$f_k = K (f_b')^\alpha (f_m)^\beta \tag{18}$$

where

- f_k - characteristic masonry compressive strength,
- f_m - average mortar compressive strength,
- f_b' - normalised unit compressive strength.
- K, α, β - constants.

This equation is mainly based on the results of physical experiments and is known to give satisfactory results for practical purposes. But, it appears it lacks rationale from the point of view of engineering

mechanics since the parameters that directly influence the compressive strength of masonry are not accounted for. Closer investigation of the equations (16) and (17) shows that for the case of a masonry panel subjected to a uniformly distributed axial pressure normal to the bed joints the stress state in masonry units is tension - tension - compression (direction of applied pressure). It can be argued that the tensile stresses normal to the direction of the applied pressure are responsible for the failure of the masonry. This argument is also supported by the experimental results where a pattern of cracks in masonry under the same loading conditions indicates these tensile stresses.

If the total pressure applied on the panel has a value P and the cross sectional area of the panel normal its direction is A then an average stress in masonry is :

$$f = \frac{P}{A} \quad (19)$$

Using equations (16) and (17) maximal tensile stress in masonry units can be calculated as:

$$\sigma_t = S_f f \quad (20)$$

where stress factor S_f can be obtained from the equations (16) and (17) and is described in more details in [6]. This factor is a function of dimensions of masonry units and thickness of mortar joints and Young's moduli and Poissons ratios of masonry units and mortar joints. In figure 2 a variation of this factor with the ratio of Young's moduli of the constituents is given for isotropic units and mortar and for two different sets of relative dimensions t_x and t_y .

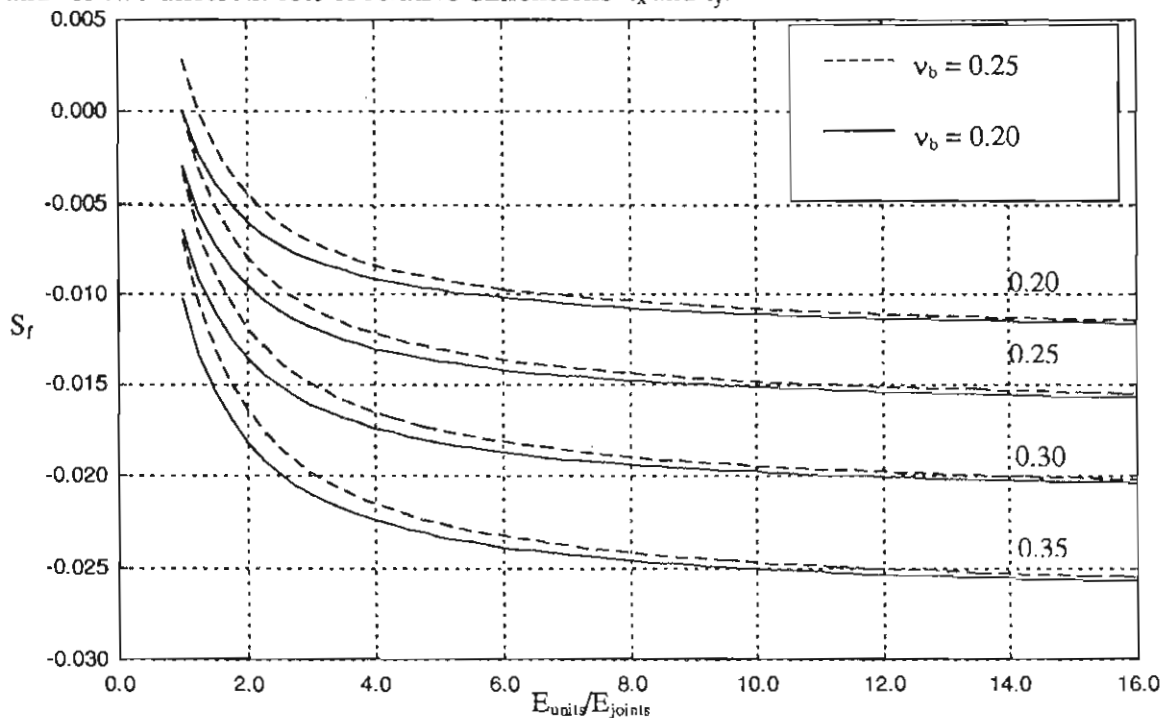


Figure 2 - Stress factor for $t_x = 0.05$ and $t_y = 0.05$

5 Conclusions

A numerical approach has been proposed to determine the orthotropic equivalent material properties for masonry material. The calculated homogenised properties can be used to analyse the response of masonry structures due to any loading conditions and the equations to calculate local stresses in constituent materials have also been developed. It has been shown that stresses induced in masonry units due to the uniform pressure on a masonry panel are tension-tension-compression. This result has been used to develop a rationale equation for the compressive strength of masonry based on fundamental engineering mechanics in contrast to the equation used in EC6 which is based on physical tests and which lacks this rationale. Finally, a sample diagram showing the relationship between tensile stresses in the masonry units due to the uniform vertical pressure on a masonry panel is given.

This approach can be used in the analysis of arbitrary masonry structures either by analytical or numerical (FEM) methods. If the finite element method is used there is no requirement to discretise each unit and joint which leads to models with far fewer degrees of freedom and greatly reduced computational cost.

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INTERACTION BETWEEN A POINT DEFECT AND A SCREW DISLOCATION IN ELASTIC CRYSTAL

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Abstract—A screw dislocation in elastic, homogeneous, isotropic crystal is subjected to the stress field of a point defect which is simulated as a center of dilatation. We analyse an interaction between both defects using the method of eigenstrains. The solution for displacements, strains, and stresses is given in the closed form.

1. Introduction

In a homogeneous linearly elastic material an eigenstrain is considered as localised initial or misfit strain which is generated by inelastic deformation, thermal expansion, phase transformation, initial strains, etc. In this work we assume that the elastic modulus of the domain with initial strains is the same as that of the surrounding crystal. A determination of the corresponding stress field named eigenstress defines the so-called eigenstrain problem. Eigenstresses can be viewed as a self-equilibrated internal stresses caused by one or several eigenstrains in a body free from any other external force and surface constraint. In this respect the eigenstresses carry resemblance with the self-equilibrated internal stresses which are commonly known as residual stresses in engineering practice. In what follows we consider two potential sources of eigenstrain. The first one pertains to the kinematical field of a straight screw dislocation. The second one describes the strain field of a center of dilatation which can be used to model a point defect in a crystal. We examine an interaction between stress fields induced by both sources and determine the force per unit length of a dislocation line caused by the stress field of the center of dilatation.

2. Basic definitions and field equations

Consider an infinite free body $\mathcal{D}$ (elastic crystal) in absence of external surface tractions and body forces. A given distribution of eigenstrain ϵ_{ij}^* is acting in the subdomain Ω of $\mathcal{D}$. We seek a corresponding displacement u_i , strain ϵ_{ij} , and stress σ_{ij} at an arbitrary point $\mathbf{x}(x_1, x_2, x_3) \in \mathcal{D}$ referring to a fixed rectangular Cartesian co-ordinate system. A summation convention on repeated indices is used throughout this paper. The total strain is defined as a sum of elastic strain e_{ij} and eigenstrain ϵ_{ij}^* :

$$\epsilon_{ij} = e_{ij} + \epsilon_{ij}^* = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.1)$$

The elastic strain is related to stress by Hooke's law,

$$\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^*), \quad C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}), \quad \nu = \lambda/2(\lambda + \mu), \quad (2.2)$$

where C_{ijkl} is the tensor of elastic moduli, λ, μ are Lamé's constants, and ν is Poisson's ratio. The equilibrium equations are

$$\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad (i = 1, 2, 3). \quad (2.3)$$

In what follows we consider $\mathcal{D}$ as a free body for which there are no surface tractions and body forces. Consequently, the corresponding natural boundary condition on the free surface of the body can be written as $n_i \sigma_{ij} = 0$. The same reasoning applies when $\mathcal{D}$ is considered as an infinitely extended body for which $\sigma_{ij} \rightarrow 0$ as $x \rightarrow \infty$ which can be considered as a corresponding boundary condition at infinity. This may be regarded as a plausible assumption since the size of a defect is small in comparison with the size of a macroscopic body. We are interested in the local spatial structure of induced stress and displacement fields in the vicinity of a dislocation line. If we substitute (2.2)₁ into (2.3) and consider the symmetry relation $C_{ijkl} = C_{ijlk}$ we arrive at a fundamental system of equations which has to be solved for a given eigenstrain ϵ_{ij}^* :

$$C_{ijkl} \frac{\partial^2 u_k}{\partial x_l \partial x_j} = C_{ijkl} \frac{\partial \epsilon_{kl}^*}{\partial x_j}. \quad (2.4)$$

In order to solve (2.4) we transform the equation in the Fourier space by employing a spatial 3-D Fourier transform. We define the Fourier transform $\bar{f}(k) = \mathcal{F}[f(x)]$ of the function $f(x)$ as

$$\bar{f}(k) = \mathcal{F}[f(x)] = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} f(x) \exp(-ik \cdot x) dx, \quad f(x) = \mathcal{F}^{-1}[\bar{f}(k)] = \int_{-\infty}^{\infty} \bar{f}(k) \exp(ik \cdot x) dk, \quad (2.5)$$

where $k = (k_1, k_2, k_3)$ is the wave vector and the integration domain in Fourier and physical space is carried out over the whole space:

$$\int_{-\infty}^{\infty} dk = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 dk_2 dk_3, \quad \int_{-\infty}^{\infty} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 dx_3. \quad (2.6)$$

The existence of the integrals involved depends on the assumption that the functions being transformed tend to zero as $r = |r| \rightarrow \infty$. In all cases surface integrals over surfaces at infinity will be taken zero as a consequence of the appropriate boundary condition at infinity. The Fourier transform of (2.4) yields

$$Q_{ik}(k) \bar{u}_k(k) = -i C_{ijkl} k_j \bar{\epsilon}_{kl}^*(k), \quad Q_{ik}(k) = C_{ijkl} k_j k_l, \quad (2.7)$$

with the solution for the displacement vector $\bar{u}(k)$ of the following form:

$$\bar{u}_i(k) = -i Q_{ir}^{-1}(k) C_{rjkl} k_j \bar{\epsilon}_{kl}^*(k) = -i N_{ij}(k) C_{jlrs} k_l \bar{\epsilon}_{rs}^*(k) D^{-1}(k), \quad (2.8)$$

$$D(k) = \mu^2 (\lambda + 2\mu) k^6, \quad N_{ij}(k) = \mu k^2 [(\lambda + 2\mu) \delta_{ij} k^2 - (\lambda + \mu) k_i k_j], \quad k^2 = k_j k_j. \quad (2.9)$$

The inverse transform of (2.8) considering (2.1-2) gives rise to the following expressions in the spatial domain:

$$u_i(x) = -i \int_{-\infty}^{\infty} C_{jlrs} \bar{\epsilon}_{rs}^*(k) k_l N_{ij}(k) D^{-1}(k) \exp(ik \cdot x) dk, \quad (2.10)$$

$$\epsilon_{ij}(\mathbf{x}) = (1/2) \int_{-\infty}^{\infty} C_{klrs} \tilde{\epsilon}_{rs}^*(\mathbf{k}) [k_j N_{ik}(\mathbf{k}) k_l + k_i N_{jk}(\mathbf{k}) k_l] D^{-1}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) d\mathbf{k}, \quad (2.11)$$

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl} \left[\int_{-\infty}^{\infty} C_{pqrs} \tilde{\epsilon}_{rs}^*(\mathbf{k}) k_i k_q N_{kp}(\mathbf{k}) D^{-1}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) d\mathbf{k} - \epsilon_{kl}^*(\mathbf{x}) \right]. \quad (2.12)$$

3. A stress field of a straight screw dislocation and a center of dilatation

A dislocation represents a linear defect in a crystal. It involves a number of atoms arranged along a line. A screw dislocation can be created by a planar cut in a crystal which according to our definition defines the domain Ω . The line AD is a dislocation line of a screw dislocation, Fig. 1. All the bonds across the surface Ω are broken and the forces of the crystal are separated so that an extra half-plane of atoms can be inserted in the slot. The region in the neighbourhood of dislocation which extends over the length of tens of ångströms undergoes a noticeable distortion relative to the vicinity of a perfect crystal. Far away from the dislocation a lattice structure of the crystal regains its regularity.

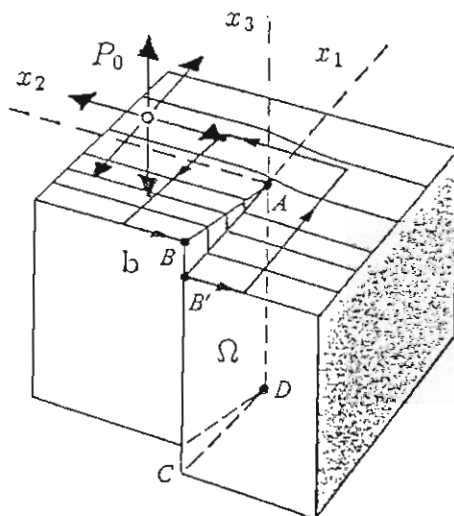


Fig. 1 A screw dislocation and a point defect located at $P_0(\mathbf{x}_0)$. Line AD marks the dislocation line surrounded by strained material. Line $B'B$ represents Burgers vector $\mathbf{b}$. Points B and B' were coincident before the dislocation was created.

In the mathematical description of a screw dislocation we adopt the following notation: Domain Ω is defined as the half plane ($x_2 = 0, x_1 < 0$). The x_3 axis coincides with the positive orientation of the dislocation line. Consequently, the tangent vector on a dislocation line can be defined as $\mathbf{t}(0, 0, 1)$. The vector required to complete the circuit which is taken over the path within the perfect crystal thus indicating the misplaced array of atoms (defect) is called a Burgers vector. The Burgers vector of a screw dislocation is parallel to the dislocation line. In our case it is given as $\mathbf{b} = (0, 0, b)$ defining the relative slip of the half plane Ω in the direction of $\mathbf{t}$. The x_3 axis represents a positive straight screw dislocation if $\epsilon_{23}^* = \epsilon_{32}^*$ and

$$\epsilon_{23}^*(\mathbf{x}) = (b/2) H(-x_1) \delta(x_2), \quad (3.1)$$

where $H(x)$ and $\delta(x)$ represent the Heaviside unit step function and Dirac's delta function with the following properties:

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad \int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0).$$

Point defects on the other hand are localised in a crystal in such a way that one or several elementary cells containing the point defect can be virtually replaced by perfect elementary cells resulting in an undisturbed crystal. They represent lattice defects of zero dimensionality and they do not possess lattice structure in any dimension. The elementary point defects are vacancies, interstitial atoms, and impurity atoms. In general, they are mobile configurations having an intrinsic energy associated with them. In homogeneous materials point defects can be produced by mechanical deformation or by irradiation under thermal equilibrium. In heterogeneous materials point defects can be produced by alloying or by diffusion of gases into the material. They move from one stable position to another by overcoming the energy barrier usually known as activation energy. Vacancies are less mobile than interstitials, but they are formed more easily. To simulate a stationary point defect we use a point source of dilatation as a limiting case of a sphere (inclusion) forced into a spherical hole of slightly different size in a matrix of elastic material. Mathematically this reads

$$\epsilon_{ij}^*(\mathbf{x}) = \epsilon^* \delta_{ij} \delta(\mathbf{x} - \mathbf{x}_0), \quad (3.2)$$

where constant ϵ^* is the strength of the center of dilatation located at the spatial position $\mathbf{x}_0$, and $\delta(\mathbf{x} - \mathbf{x}_0)$ is the 3-D Dirac delta function with the properties defined above. The total eigenstrain imposed on the system is thus defined as a sum of the eigenstrain of the screw dislocation (3.1) and eigenstrain due to the point defect (3.2). The Fourier transform of (3.1), having considered the properties of the Heaviside and Dirac function, cf. Lighthill (1964), is

$$\bar{\epsilon}_{23}^*(\mathbf{k}) = -\frac{b/2}{(2\pi)^3} \left[\frac{\delta(x_3)}{ik_1} \right]. \quad (3.3)$$

Having used (2.10-12) the corresponding non-zero displacements, strains, and stresses are given as

$$u_3(\mathbf{x}) = \frac{b}{2\pi} \arctan\left(\frac{x_2}{x_1}\right), \quad u_2(\mathbf{x}) = 0, \quad u_1(\mathbf{x}) = 0, \quad (3.4)$$

$$\epsilon_{31}(\mathbf{x}) = -\frac{b}{4\pi} \left(\frac{x_2}{x_1^2 + x_2^2} \right), \quad \epsilon_{32}(\mathbf{x}) = \frac{b}{4\pi} \left(\frac{x_1}{x_1^2 + x_2^2} \right), \quad (3.5)$$

$$\sigma_{31}(\mathbf{x}) = -\frac{\mu b}{2\pi} \left(\frac{x_2}{x_1^2 + x_2^2} \right), \quad \sigma_{32}(\mathbf{x}) = \frac{\mu b}{2\pi} \left(\frac{x_1}{x_1^2 + x_2^2} \right) - \mu b \delta(x_2) H(-x_1). \quad (3.6)$$

The displacement field around a screw dislocation in an infinite isotropic elastic medium is everywhere parallel to the dislocation line. Moreover, $\nabla^2 u_3 = 0$ is a harmonic function which changes by b when the dislocation is encircled once. The stress field exhibits complete radial symmetry. The stresses and strains are proportional to $1/r$, where r is the radial distance from the dislocation line, and diverge to infinity as r goes to zero. The results are essentially the same as known in the literature

obtained by other methods, e.g. Nabarro (1967). Since the solid cannot withstand infinite stress on the dislocation line, the solution which is based on the macroscopic theory of elasticity is not valid any more. Consequently, we have to employ non-linear atomistic model. The region where phenomenological solution breaks is called the core of a dislocation and it spreads over the range of approximately $10\text{\AA}$, Kittel (1986). Analysis of this type is outside of the scope of this paper.

To determine the displacements, strains and stress field caused by the center of dilatation we follow the same line of analysis as in the case of screw dislocation. The Fourier transform of (3.2) yields

$$\bar{\epsilon}_{ij}^*(\mathbf{k}) = \frac{\epsilon^*}{(2\pi)^3} \delta_{ij} \exp(-i\mathbf{k} \cdot \mathbf{x}_0). \quad (3.7)$$

After substituting (3.7) into (2.10-12) and performing the required inverse Fourier transforms we obtain

$$u_i(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) (x_i - x_{0i}) = -\frac{\epsilon^*}{4\pi} \left(\frac{1+\nu}{1-\nu} \right) \left[\nabla \left(\frac{1}{R} \right) \right]_i, \quad (3.8)$$

$$\epsilon_{ij}(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[\delta_{ij} - \frac{3}{R^2} (x_i - x_{0i})(x_j - x_{0j}) \right], \quad R = \sqrt{(x_k - x_{0k})(x_k - x_{0k})} \quad (3.9)$$

$$\sigma_{ij}(\mathbf{x}) = \frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[\delta_{ij} - \frac{3}{R^2} (x_i - x_{0i})(x_j - x_{0j}) \right] - 2\mu \left(\frac{1+\nu}{1-\nu} \right) \epsilon^* \delta_{ij} \delta(\mathbf{x} - \mathbf{x}_0). \quad (3.10)$$

Displacement field due to the center of dilatation is homogeneous of degree -2. It is divergence free ($\nabla \cdot \mathbf{u} = 0$) and curl free ($\nabla \times \mathbf{u} = 0$). From (3.9) it follows that $\epsilon_V = \epsilon_{kk} = 0$, so there is no volumetric strain associated with this model of point defect. The total stress and strain in the material is the sum of the contributions from the dislocation and the center of dilatation. The stress field of the system is given in the following form:

$$\sigma_{11} = \frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[1 - \frac{3}{R^2} (x_1 - x_{01})^2 \right] - 2\mu \left(\frac{1+\nu}{1-\nu} \right) \epsilon^* \delta(\mathbf{x} - \mathbf{x}_0), \quad (3.11)$$

$$\sigma_{22} = \frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[1 - \frac{3}{R^2} (x_2 - x_{02})^2 \right] - 2\mu \left(\frac{1+\nu}{1-\nu} \right) \epsilon^* \delta(\mathbf{x} - \mathbf{x}_0), \quad (3.12)$$

$$\sigma_{33} = \frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[1 - \frac{3}{R^2} (x_3 - x_{03})^2 \right] - 2\mu \left(\frac{1+\nu}{1-\nu} \right) \epsilon^* \delta(\mathbf{x} - \mathbf{x}_0), \quad (3.13)$$

$$\sigma_{12} = -\frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[\frac{3}{R^2} (x_1 - x_{01})(x_2 - x_{02}) \right], \quad (3.14)$$

$$\sigma_{13} = -\frac{\mu}{2\pi} \left(\frac{bx_2}{x_1^2 + x_2^2} \right) - \frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[\frac{3}{R^2} (x_1 - x_{01})(x_3 - x_{03}) \right], \quad (3.15)$$

$$\sigma_{23} = \frac{\mu}{2\pi} \left(\frac{bx_1}{x_1^2 + x_2^2} \right) - \frac{\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{\epsilon^*}{R^3} \right) \left[\frac{3}{R^2} (x_2 - x_{02})(x_3 - x_{03}) \right] - \mu b \delta(x_2) H(-x_1) \quad (3.16)$$

From (3.11-16) it follows that the mixing in the stress field takes place only in the stress components σ_{13} and σ_{23} . All other components exhibit the stress field which originates from the center of dilatation.

4. The force on a dislocation

To determine the force on the screw dislocation we use a general procedure by Peach and Koehler (1950) developed for the force on a dislocation caused by applied or internal stresses. In our case the

source of internal stress is the center of dilatation. The equation predicts that a stress can exert a force on a dislocation in a direction perpendicular to the slip plane. The Peach-Koehler equation for the force per unit length of the dislocation with the tangent vector $\mathbf{t}$ and Burgers vector $\mathbf{b}$ adjusted to our definition of the positive orientation of the Burgers vector and the tangent on the dislocation line reads, cf. Peach & Koehler (1950),

$$\mathbf{F}_S = -\mathbf{t} \times \mathbf{G}, \quad \mathbf{G} = \boldsymbol{\sigma} \cdot \mathbf{b}, \quad F_{Si} = -\varepsilon_{ijk} t_j \sigma_{kl} b_l, \quad (4.1)$$

where ε_{ijk} is the permutation tensor, σ_{ij} is the stress exerted on a dislocation, $\mathbf{t} = (0, 0, 1)$, and $\mathbf{b} = (0, 0, b)$. The components of $\mathbf{F}_S$ are thus given by

$$F_{S1} = \sigma_{32} b = -\frac{3\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{b\epsilon^*}{R^5} \right) (x_2 - x_{02})(x_3 - x_{03}), \quad (4.2)$$

$$F_{S2} = -\sigma_{31} b = \frac{3\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) \left(\frac{b\epsilon^*}{R^5} \right) (x_1 - x_{01})(x_3 - x_{03}), \quad (4.3)$$

$$F_{S3} = 0. \quad (4.4)$$

In (4.2-4) we have used the stress field of the center of dilatation defined in (3.10). Fig. 2 shows the normalised value $\bar{F}_{S1}$ of the x_1 component of the force per unit length of the dislocation line on the rectangle $\{x_2/b \in [-300, 300]\} \otimes \{x_3/b \in [-240, 240]\}$ in the plane $x_1 = 0$. The normalised values of the components of the vector $\mathbf{F}_S$ are defined as

$$\bar{F}_{S1} = \frac{2\pi(1-\nu)10^7}{3\mu(1+\nu)b} F_{S1}, \quad \bar{F}_{S2} = \frac{2\pi(1-\nu)10^7}{3\mu(1+\nu)b} F_{S2}$$

In both cases the strength of the center of dilatation was approximated by $\epsilon^* \approx b^3$. The center of dilatation was located at $\mathbf{x}_0 = (0, 50b, 0)$.

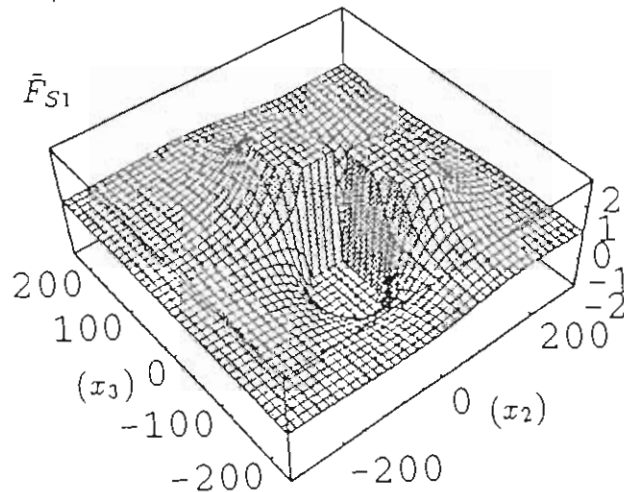


Fig. 2 The normalised value $\bar{F}_{S1}(x_2/b, x_3/b)$ in the plane $x_1 = 0$.

The normalised value $\bar{F}_{S2}$ of the x_2 component of the force per unit length of the dislocation line on the rectangle $\{x_1/b \in [-300, 300]\} \otimes \{x_2/b \in [-240, 240]\}$ in the plane $x_3/b = -20$ is displayed in Fig. 3.

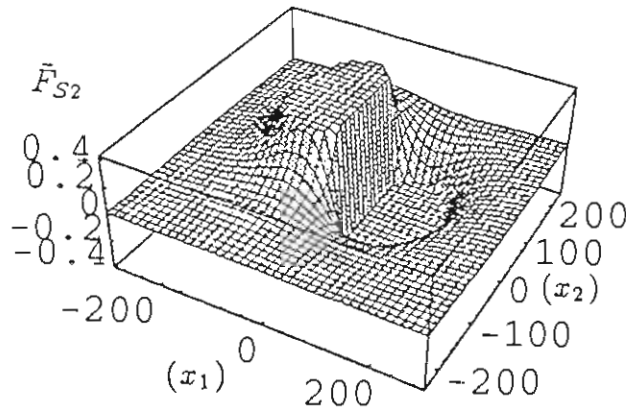


Fig. 3 The normalised value $\bar{F}_{S2}(x_1/b, x_2/b)$ in the plane $x_3/b = -20$.

To determine the total force on the straight screw dislocation we integrate (4.2) and (4.3) from $x_3 = -\infty$ to $x_3 = \infty$.

$$F_1 = -\frac{3\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) b\epsilon^* \int_{-\infty}^{\infty} \frac{(x_2 - x_{02})(x_3 - x_{03})dx_3}{R^5}, \quad (4.4)$$

$$F_2 = \frac{3\mu}{2\pi} \left(\frac{1+\nu}{1-\nu} \right) b\epsilon^* \int_{-\infty}^{\infty} \frac{(x_1 - x_{01})(x_3 - x_{03})dx_3}{R^5}. \quad (4.5)$$

It is a matter of simple exercise to show that both integrals equal to zero. Consequently, $\mathbf{F} = (0, 0, 0)$ and there is no net force on the screw dislocation. Although point defects and dislocations generally interact elastically and exert forces on each other, this is not the case in our particular example. Since the strain field around the point defect is spherically symmetric, such a defect will produce no net force on the screw dislocation because its stress field is pure shear. There is no hydrostatic stress field around the screw dislocation which can be relaxed by the presence of the point defect. However, if a point defect produces a non-spherical distortion, so that there is an interaction energy between a screw dislocation and a point defect, then the net force on a screw dislocation will not be zero in general.

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BEHAVIOUR OF FIBER-REINFORCED PLASTICS AND ESTIMATION OF COMPONENTS SERVICE STRENGTH

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ABSTRACT

The service strength of components produced from fiber reinforced plastics depends on operational stresses and environmental conditions, of which decisive are the temperature and the humidity. The composition and percentage of individual ingredients strongly influence the properties of used material and must be determined by experiments.. The estimation of the component service strength must include three decisive criteria: ***fatigue life***, ***residual strength*** and ***stiffness loss***. These criteria depend on each other and are nonlinearly related to the operational loading.

Basic data necessary for the evaluation, individual criteria and their relation as well as a new, up to now unpublished general procedure to estimate the structural service strength of such components, are presented in this paper.

1. INTRODUCTION

Most components produced from fiber reinforced plastics are loaded in service by complex loading. For the usage on cars, ships, domestic and common engineering equipment lot of components are produced from plastic material sheets (compression - and injection - molded thermoset composites like SMCs, RIMs or thermoplastics) replacing steel, aluminium, wood or other materials, because of a better usage/cost ratio and due to it a higher profit. Because these materials are used mainly for secondary components, which failure influence neither the safety nor the function of the machine or system, the approach during the design and evaluation of such components is usually highly simplified. This is the reason why so many of these components fail during relatively short operational usage. Therefore, a procedure which takes into account decisive parameters for the evaluation of service strength of such components is of importance, nevertheless that their failure by most customers is accepted as the state-of-the-art.

2. BEHAVIOUR OF FIBER REINFORCED PLASTICS

Under operational loading fiber reinforced plastics change their mechanical properties continuously. Stress-deformation (strain) relation is changed due to the stiffness change, so that the modulus of elasticity is decreasing from the initial value E_0 to E_D . This behaviour is described schematically in Fig. 1.

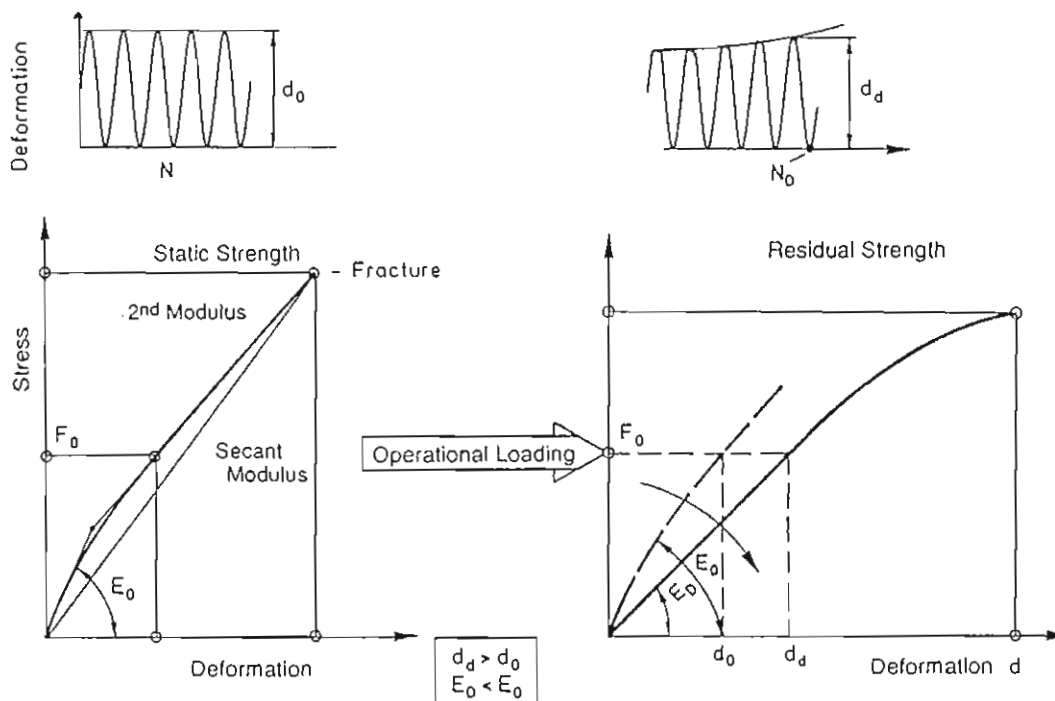


Fig. 1: DEFORMATION UNDER OPERATIONAL LOADING

Under operational load F_o after N_D cycles the deformation increases from initial value d_o to d_d , the modulus of elasticity decreases from E_o to E_d and the strength is changed from its initial static strength S_o to residual strength S_R .

An important parameter for the appropriate design is the function $E = f(N)$, which determines the stiffness change due to the operational loading cycles N . Usually after a higher change under first few cycles a stabilized state is achieved and later a continuous, slight change occurs to the value E_D used for the design, as shown on Fig. 2.

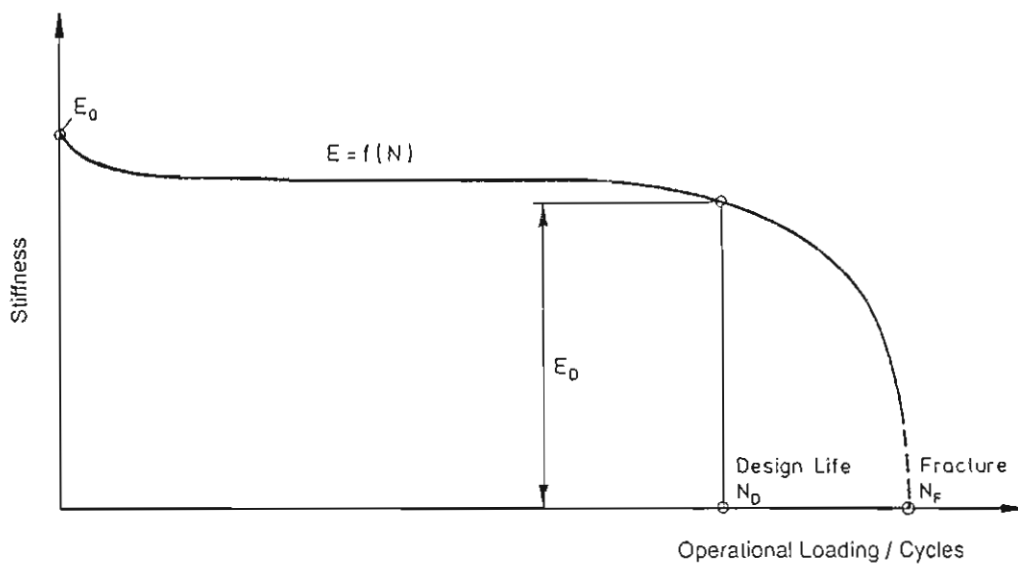


Fig. 2: STIFFNESS CHANGE DUE TO OPERATIONAL LOADING

After the number of cycles N_D , used for the design, depending on the value of allowable stiffness loss which used as failure criterium, the stiffness change progressively up to the total fracture at N_F cycles.

The mechanical properties as **static strength** and **modulus of elasticity** for plastics are strongly influenced by the stress state (tensile, compressive or flexural), temperature and the humidity. A general overview concerning these influences are presented in Fig. 3. The highest values of static strength are achieved under flexural (bending) loading, which is the most common case for plastic sheet components, and lowest under tensile stress state. The temperature increase diminishes in all cases the static strength as well as humidity, shown schematically in Fig. 3.

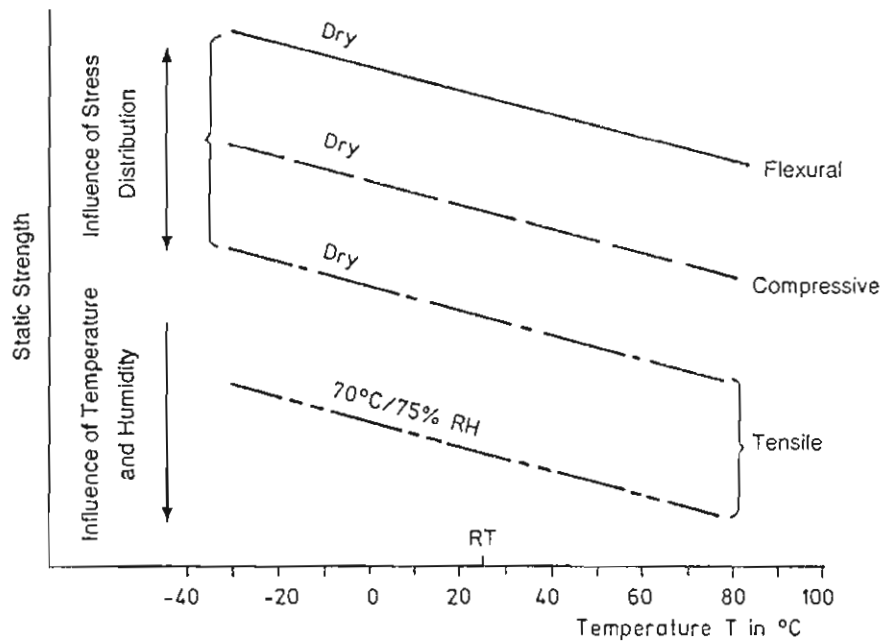


Fig. 3: INFLUENCE OF STRESS STATE, TEMPERATURE AND HUMIDITY ON THE STATIC STRENGTH

Usually for the usage under normal conditions the data of the static strength under flexural stress state at room and increased temperature (e.g. 70° C for most car components) and saturated humidity (e.g. 75 % relative humidity at room or increased temperature) could be used. Of course it depends on environmental conditions at usage which data should be chosen for the design, but usually the "worst case" should be the basis for the evaluation.

For the fatigue evaluation the Wöhler (S-N) curves are used (Fig. 4). For the presentation of these curves the fatigue life (e.g. number of cycles) up to the total fracture can't be used for design purposes, especially in the case of variable operational loading. Usually these curves should be related to certain damage (damage of matrix, delamination etc.) but for practical purposes an integral criterion like stiffness loss, which includes all individual damage modes, should be used. In Fig. 4 the Wöhler-curve with related fatigue dependent data, typical for sheet molding compounds (SMC), is presented in relation to its static strength.

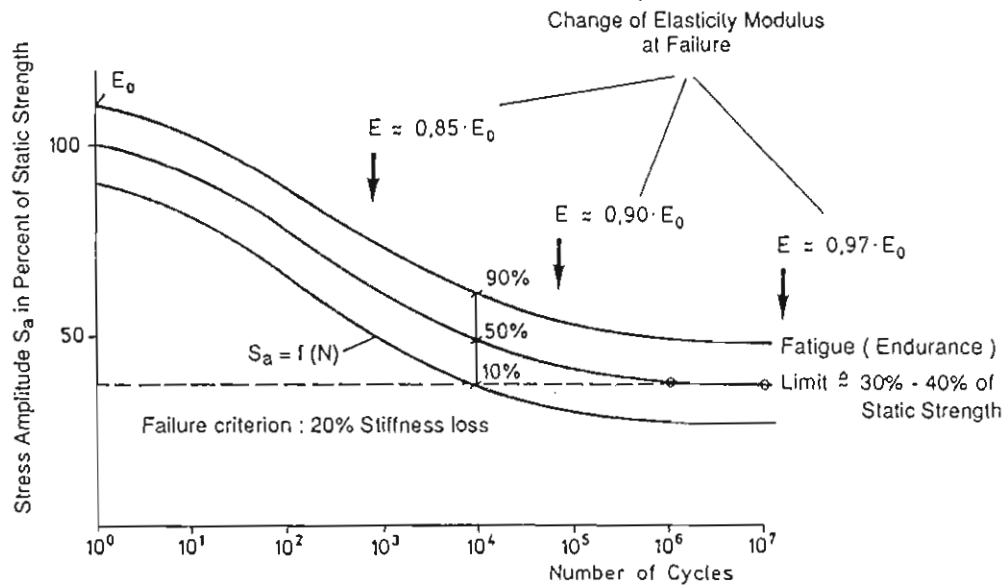


**Fig. 4: FATIGUE STRENGTH FOR SMC-MATERIALS
IN RELATION TO STATIC STRENGTH**

The Wöhler-curves $S_a = f(N)$ are presented for different probability of survival and a failure criterion of 20 per cent stiffness loss. For the low cycle fatigue to about 10^3 cycles, based on this failure criterion, elasticity modulus decreases about 15 per cent in relation to its initial value E_0 ($E \approx 0.85 E_0$). For the higher number of cycles the decrease of E is lower.

Usually the fatigue limit under constant environmental conditions achieves the value of 30 to 40 per cent of the static strength, whereby the elasticity modulus decreases in relation to initial value only by about 3 per cent.

The corresponding fatigue data are determined as well by specialized institutes as by users (mostly car industries), and some of them are published. A selected overview of the corresponding literature is given in References.

3. DESIGN CRITERIA

Design engineers need material data and detailed knowledge of the influence on the service strength to be able to fulfill their task to carry out a reliable design. The design evaluation must include three decisive criteria: **Fatigue Life**, **Residual Strength**, **Stiffness**. These criteria depend on each other and must be evaluated based on appropriate procedure. The design evaluation procedure is described in Fig. 5.

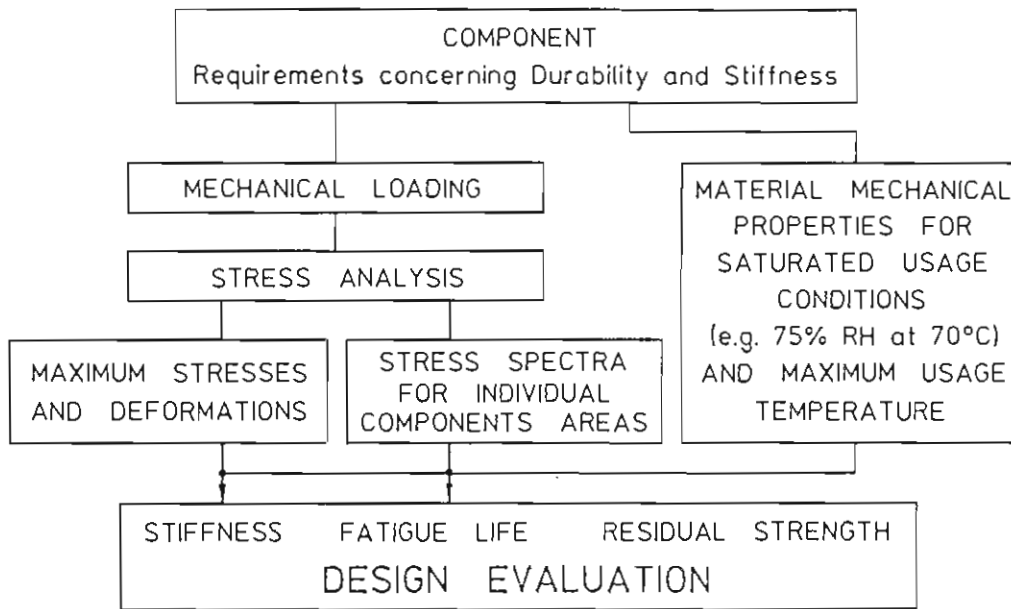


Fig. 5: DESIGN EVALUATION PROCEDURE FOR FIBER REINFORCED PLASTICS

Based on requirements for the durability life, the operational stresses (stress spectra of local stresses on highly stressed spots of component, the maximum nominal stress in the highly stressed cross section) the allowable deformations and the material properties under usage conditions, the design evaluation is carried out. An overview of the three decisive **design criteria** and their relation is given schematically in Fig. 6 and the general approach to be used for the design evaluation is presented.

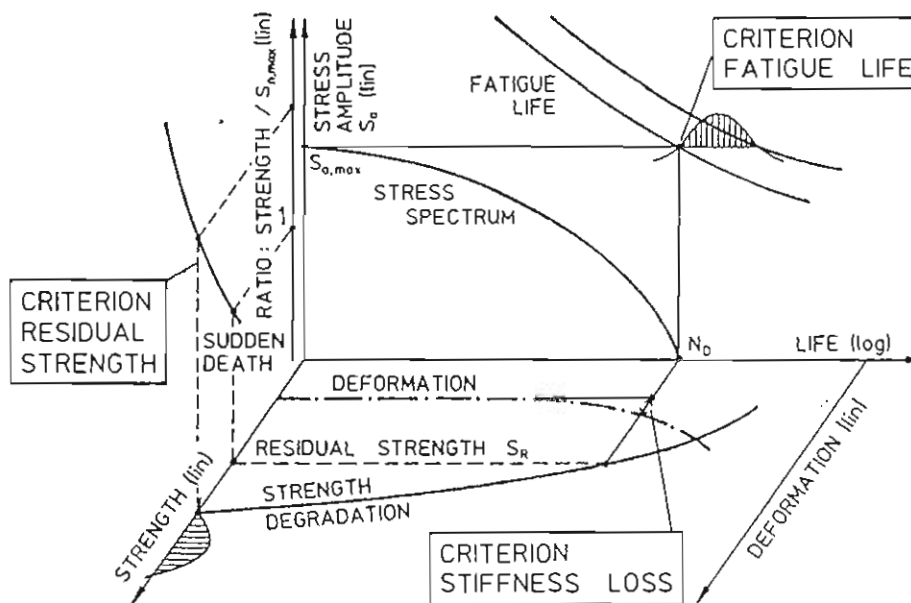


Fig. 6: CRITERIA FOR DESIGN EVALUATION

The most important **criterion** is the **fatigue life** to be achieved by the durability requirement. To prove this requirement the spectra of the "hot spot" operational stresses must be known as well as the material fatigue data in form of Wöhler- (S-N) curves. During the operational usage the component deformation increases as well as the static strength degrades. To fulfill the functional requirement it must be proved that the component deformation due to the **stiffness loss** doesn't surpasses the allowable value. Finally, also the degradation of the static strength at operational loading must be controlled; for this criterion the nominal stress $S_{n,max}$ in a section of the component must be lower than the **residual static strength**, e.g. the ratio of static strength to nominal stress must always be higher than one. If this ratio achieves the value lower than one a total fracture at maximum operational load, known as "sudden death", will occur.

For the fatigue life estimation as a state-of-the-art the modified Palmgren-Miner linear accumulative damage hypothesis is used. The life estimation based on this hypothesis is presented in the flow-diagramme on Fig. 7.

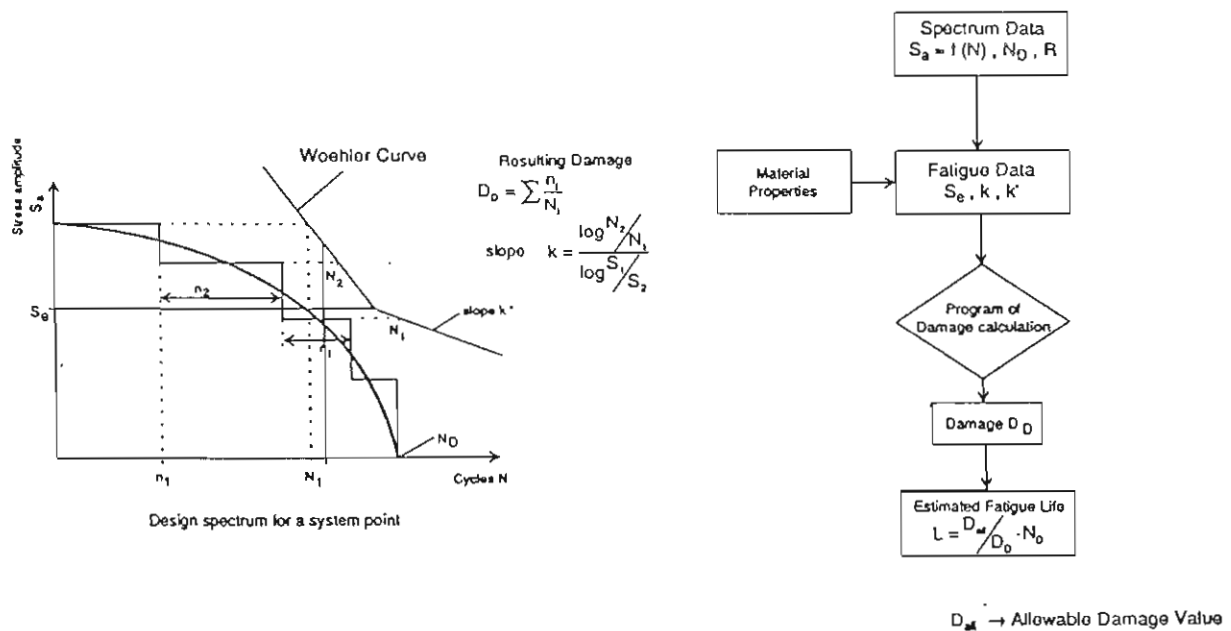


Fig. 7: PROCEDURE OF FATIGUE LIFE ESTIMATION

Due to the material behaviour under operational loading the accuracy of the life estimation by this procedure is restricted. Therefore, an allowable damage value lower than one ($D_{all} < 1$) should be used to avoid the overestimation of the fatigue life. Based on experimental results with glasfiber reinforced plastics an allowable damage value $D_{all} = 0.1$ is proposed.

4. CONCLUSION

For the design evaluation of components made from reinforced plastics the decisive criteria are: **fatigue life**, **residual strength** and **stiffness** loss. The data necessary and the procedure for design evaluation are:

NECESSARY DATA



PROCEDURE



1. CRITERION: FATIGUE LIFE

- 1.1 S-N Curve (Wöhler - Curve) or $\bar{S}-\bar{N}$ Curve (Durability Life Curve) at $P_f = 10\%$ and Failure Criterion "Stiffness loss of 20%" ($d_D / d_o = 1,2$) with $\bar{R} \approx \bar{R}_{service}$

⇒ Evaluation based on Damage Accumulation Calculation to estimate Fatigue Life :

- 1.2 Service Stress Spectra of Local Stresses

$$L_s = N_s = \frac{D_{as}}{D_d} \cdot N_D$$

$$-S_{max} \text{ and } S_{min} \rightarrow \bar{S}_a = \frac{S_{max} + S_{min}}{2}; \bar{R} = S_{min} / S_{max}$$

$L_s \approx N_s$ = Expected Fatigue Life

$$-\text{Service Stress Spectrum} \rightarrow S_a = f(N)$$

D_D = Calculated Damage for Spectrum with N_D cycles

$$-\text{Number of Cycles } N_D$$

D_{all} = Allowable Damage Value (e.g. 0,1)

Probability of occurrence for spectra $P_0 \leq 1\%$

2. CRITERION: FRACTURE STRENGTH

- 2.1 Static Strength Degradation to the Fatigue Life

⇒ Evaluation as engineering approach :

- 2.2 Maximum Nominal Service Stress $S_{n,max}$ in individual sections taking into account prestress due to assembly and rare event overloading or possible (unwished) misuse

$$S_{n,max} \leq \frac{S_o}{K_F}$$

S_o = static strength at 75% RH and maximum service temperature

K_F = factor depending on material data based on 20% stiffness loss

3. CRITERION: STIFFNESS - LOSS

- 3.1 E_i / E_o change at service loading conditions

⇒ Evaluation : For fatigue strength failure criterion "20% stiffness loss" : it is necessary only to prove that $1,2 d_o < d_{all}$

- 3.2 Requirements about allowable maximum deformations d_{all}

$$d_o = \text{initial deformation at maximum load} = S_{n,max}$$

The design engineer must use the corresponding data and procedure to carry out a reliable design evaluation.

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UPOTREBA KOMPOZITNIH OJAČANJA ZA POPRAVAK
ZAMOROM OŠTEĆENIH KONSTRUKCIJA

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THE USE OF COMPOSITE PATCHES FOR REPAIR OF CRACKED STRUCTURES

Abstract: The replacement of the cracked part in a complex structure is both costly and time-consuming. One possible alternative to replacing the part is to use adhesively bonded carbon fibre reinforced plastic (CFRP) patches. The method offers several advantages over other techniques; for example speed of repair, cost, efficient load transfer, absence of further holes in the structure which might act as crack initiator, stiffening and sealing the area around the crack and a minimum increase in weight. The effect of adhesively bonded woven CFRP patches on retarding the growth of fatigue cracks in centre cracked Al-alloy plate (10 mm thick) under constant and variable load amplitude is being investigated.

Sažetak: Zamjena zamorom oštećenih dijelova metalnih konstrukcija skupa je i dugotrajna. Umjesto nje često je moguća primjena ugljičnim vlaknima ojačanih plastika - kompozita koji se jednostavno zalijepe preko oštećenja. Prednosti ove metode su brzina popravka, niska cijena, povoljan tok naprezanja, odsustvo novih provrta u konstrukciji, ukrućenje i brtvljenje napuklog dijela te minimalno povećanje težine. U radu je pokusima ispitana učinkovitost lijepljenih kompozitnih ojačanja na usporavanje rasta zamorne pukotine kod uzoraka iz aluminijske slitine (debljine 10 mm) sa centralnom dvobridnom pukotinom pri konstantnim i promjenljivim amplitudama vlačnog opterećenja.

1 UVOD I CILJ RADA

Najčešći uzrok oštećenja i lomova metalnih konstrukcija je zamor materijala. Uz ovu pojavu vezani su veliki materijalni gubici i često opasnost po ljudske živote. Jedina sretna okolnost kod zamornih oštećenja je relativno dug rast pukotine od njene inicijalizacije do konačnog loma komponente. To omogućava otkrivanje pukotine i poduzimanje odgovarajućih postupaka sanacije. Kako su zamjene oštećenog dijela ili cijele konstrukcije dugotrajne i skupe, a uobičajena reparaturna zavarivanja ponekad neučinkovita ili pak nemoguća, u ovoj se oblasti stalno traže nova rješenja.

Jedna od mogućnosti usporavanja ili zaustavljanja rasta zamornih pukotina je i primjena kompozitnih materijala. Glavni razlog sve češće upotrebe kompozita su izuzetna mehanička svojstva punila (najčešće ugljična vlakna) kvalitetno povezanih u cijelinu matricom kao osnovnim materijalom (najčešće polimer). Primjena kompozita u svrhu sanacije oštećenih dijelova potječe iz zrakoplovne industrije, a sastoji se od pažljivo odabranog sastava, načina gradnje i odabira dimenzija ojačanja te njegovog lijepljenja na površinu koja se želi ojačati. Svojim oblikom i svojom funkcijom ova ojačanja podsjećaju na zakrpe ili flastere pa ih se u literaturi najčešće nalazi pod tim imenima:

"CFRP Patches" - Carbon fibre reinforced plastic patches

"CFK Pflaster" - Kohlenfaserverstärkter Kunststoff Pflaster

Zadatak kvalitetno dimenzioniranog i zalijepljenog ojačanja je da preuzme dio opterećenja s metalnog dijela i na taj način značajno umanji raspon naprezanja čime je odgođeno daljnje širenje postojeće pukotine, a ako do njega i dođe, znatno je sporije nego li u slučaju bez ojačanja.

Uvidom u pristupačnu literaturu o gradnji i primjeni kompozitnih

materijala u svrhu ojačanja zamorom ugroženih komponenti može se zaključiti:

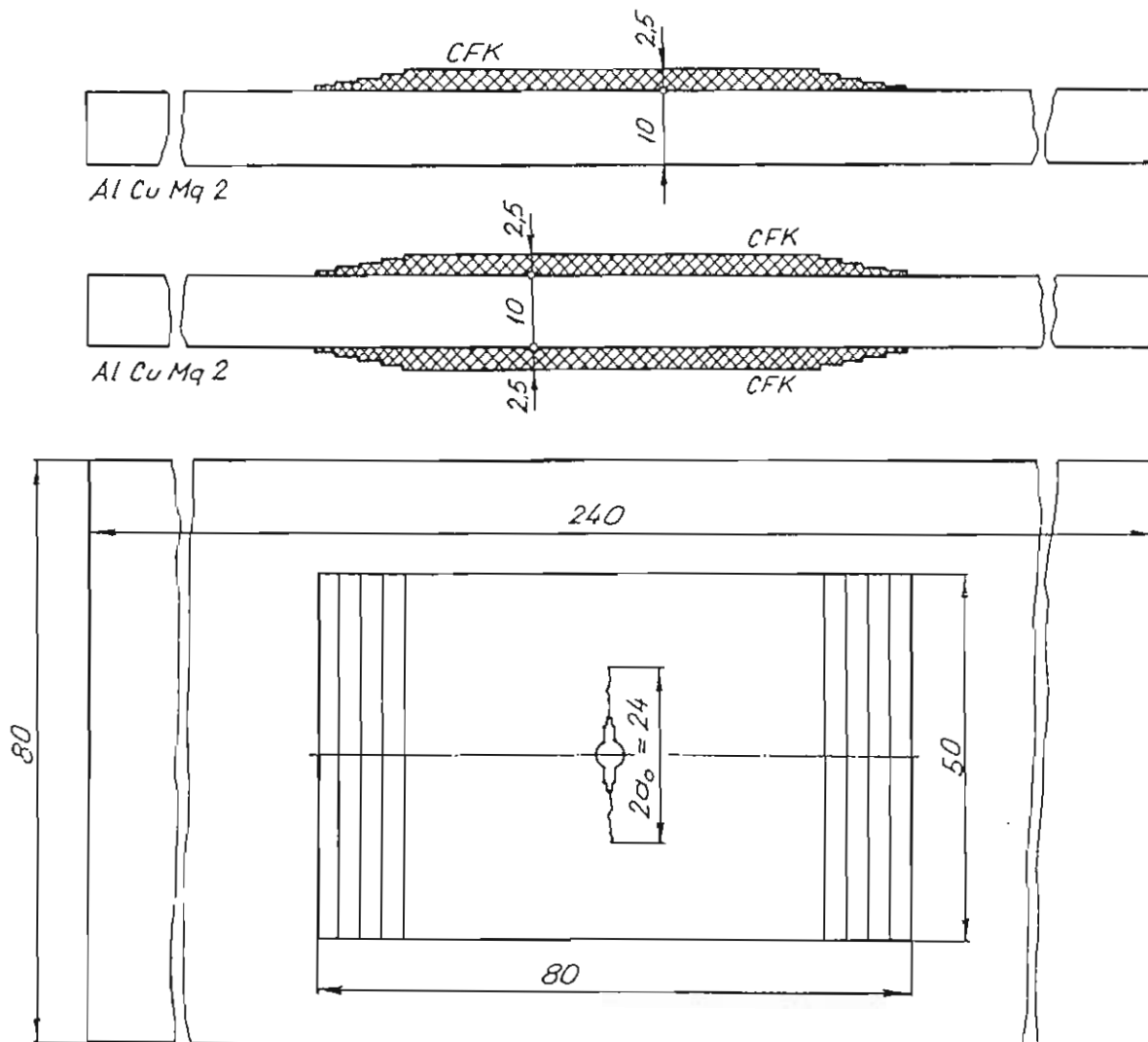
- najčešće korišteni kompozitni materijali su laminati izrađeni od polimerne matrice s "beskonačnim" ugljičnim vlaknima. Čvrstoća ovih vlakana iznosi i preko 20 GPa, no u kombinaciji sa znatno mekšom (ali i neophodnom) polimernom matricom čvrstoća laminata pada i u ovisnosti o načinu gradnje i smjeru opterećenja iznosi 1-2 GPa, što je još uvijek više od prekidne čvrstoće konstrukcijskog čelika,
- kod gradnje ojačanja treba težiti što većem modulu elastičnosti u smjeru vanjskog opterećenja i simetričnoj gradnji (smjer vlakana),
- debljina kompozitnog ojačanja (broj laminata) od presudnog je utjecaja na zamornu čvrstoću saniranog dijela,
- veličina i oblik (kvadrat, pravokutnik ili krug) ojačanja nemaju jak utjecaj na zamornu čvrstoću saniranog dijela; uobičajeno je lijepiti ojačanja širine 150% duljine pukotine,
- završetke ojačanja u smjeru opterećenja treba raditi stupnjevano radi smanjenja smičnog naprezanja na prijelazu metal-ojačanje,
- najčešće korištena su epoksidna ljepila čija je primjena jednostavna i ne zahtjeva povišene temperature ni jače pritiske,
- debljina ljepila nema većeg značaja na zamornu čvrstoću spoja, a uobičajene su debljine 0.1 - 0.2 mm,
- bolji rezultati se postižu dvostranim nego jednostranim ojačanjem,
- osim čišćenja ne zahtjeva se nikakva obrada metalne plohe,
- otkaz dijela s ojačanjem nastaje najčešće ravninom između metala i ojačanja ili delaminacijom u prvim slojevima ojačanja,
- bolji rezultati postižu se kod duljih pukotina, itd.

Cilj istraživanja je da se provjeri učinkovitost kompozitnih laminata iz ugljičnih vlakana zalijepljenih na zamorom oštećene limove veće debljine (10 mm) i pod različitim amplitudama opterećenja.

2 OPIS I IZVOĐENJE POKUSA

Ispitivanje je obavljeno na način da su metalni uzorci zamarani do određene duljine pukotine nakon čega su uzorci skidani s umaralice i sanirani ljepljenjem prethodno pripremljenim ojačanjima. Nakon toga uzorci su vraćeni u umaralice i zamarani jednakim opterećenjem do konačnog loma. Kao kriterij učinkovitosti odabran je odnos vijeka trajanja uzorka od definirane duljine pukotine do loma u slučaju s i bez sprovedene sanacije kompozitima iz ugljičnih vlakana.

2.1 Uzorci i kompozitna ojačanja

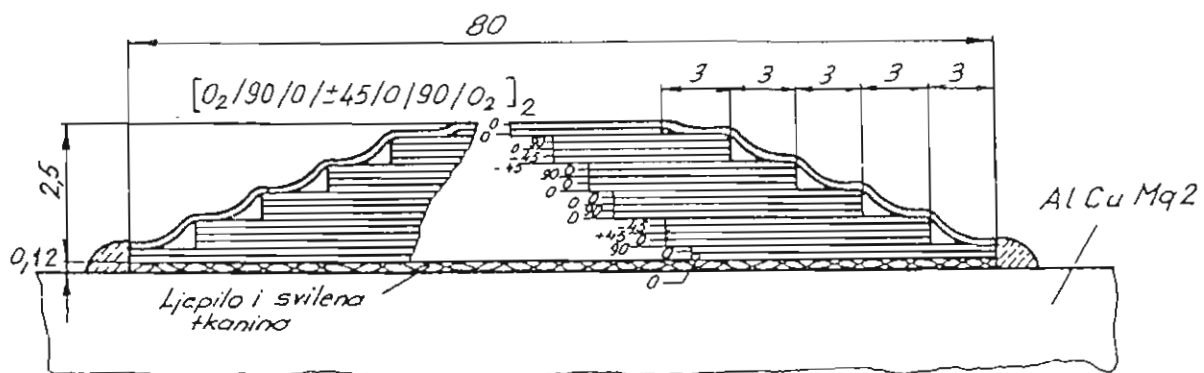


Slika 1 Jednostrana i dvostrana ojačanja kompozitnim laminatima
Figure 1 One and both face CFRP patches application

Kao uzorci za ispitivanje rasta i sanacije zamorne pukotine odabrani su plosnati štapovi iz Al-slitine AlMgCu2 (DIN 1725), sl. 1. Širina uzorka 80 mm osigurava znatnu fazu stabilnog rasta pukotine, dok je debljina višestruko veća od do sada ispitivanih limova u zrakoplovnoj industriji i iznosi 10 mm. Vlačna čvrstoća (486MPa) i granica tečenja (396MPa) ove slitine odgovaraju mehaničkim svojstvima konstrukcijskih čelika.

Kao sredstvo za sanaciju lima na mjestu s pukotinom odabran je polimer ojačan ugljičnim vlaknima (kompozit) uobičajene oznake CFK. Jednostrano i dvostrano zalijepljena ojačanja prikazana su na sl.1, a način gradnje samih ojačanja (flastera, zakrpa) prikazan je na sl.2. Odabrano ljepilo, također na temelju literature i preporuke stručnjaka Fraunhofer instituta u Darmstadtu, je tzv. "hladno ljepilo" tipa XA 9323 tvrtke "3M". Kompoziti pripremljeni iz 20 lamina debljine 0.12 mm i složenih u različitim smjerovima (sl.2) imaju slijedeća mehanička svojstva:

-modul elastičnosti u uzdužnom smjeru	$E_1 = 93200 \text{ MPa}$
-modul elastičnosti u poprečnom smjeru	$E_2 = 41900 \text{ MPa}$
-modul smika	$G = 10900 \text{ MPa}$
-Poissonov koeficijent	$\nu = 0.22$



Slika 2 Način gradnje primjenjenih kompozitnih materijala
Figure 2 Constitution of applied CFRP patches material

2.2 Umaralice i opterećenja

Svi pokusi su obavljani na Fraunhofer institutu za pogonsku čvrstoću (LBF) u Darmstadtu i to na servo-hidrauličnoj umaralici tipa "Schenck" maksimalne sile 250 kN. Umaralica je upravljana računalom, što osim generiranja željenog opterećenja omogućava stalnu kontrolu sile, pomaka i broja ciklus opterećenja te njihov zapis. Naprezanja s konstantnim amplitudama su iznosila:

$$\sigma_a = \pm 72 \text{ MPa} \quad \text{uz} \quad R = -1 \quad (\sigma_{sr} = 0 \text{ MPa}) \quad \text{i}$$

$$\sigma_a = \pm 72 \text{ MPa} \quad \text{uz} \quad R = 0 \quad (\sigma_{sr} = +72 \text{ MPa}).$$

Naprezanja s promjenljivim amplitudama (spektrima) su urađena na način da je Gaussova raspodjela podjeljena na osam razina konstantnih amplituda, veličina spektra je 100000 ciklusa, minimalna amplituda u spektru je 20 % maksimalne, a maksimalna je iznosila:

$$\bar{\sigma}_a = \pm 72 \text{ MPa} \quad \text{uz} \quad \bar{R} = 0 \quad (\sigma_{sr} = +72 \text{ MPa}).$$

Pokusi su na svim razinama obavljani s trostrukim ponavljanjem.

3 REZULTATI ISPITIVANJA

Ostvarena produljenja vijeka trajanja uzoraka saniranih lijepljenjem kompozitnih ojačanja u odnosu na preostali vijek trajanja nesaniiranih iznosila su u ovisnosti o amplitudama naprezanja:

-kod jednostrano zalijepljenog ojačanja

9.8 puta kod konstantnih amplituda i odnosa $R = -1$,

6.2 puta kod promjenljivih amplituda (spektra) i $\bar{R} = 0$,

-kod dvostrano zalijepljenih ojačanja

>61.8 puta kod konstantnih amplituda i odnosa $R = -1$

(lom uzorka van područja ojačanja),

47.7 puta kod konstantnih amplituda i odnosa $R = 0$ i

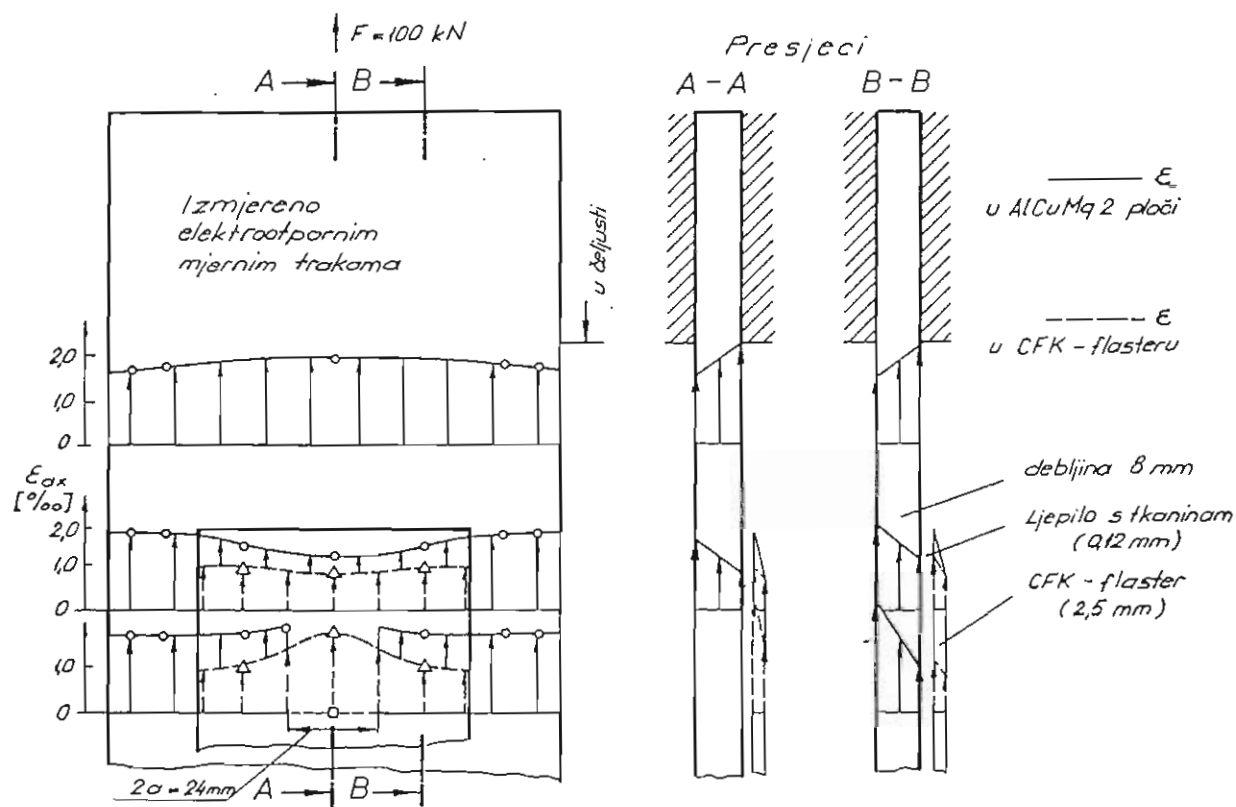
>120 puta kod promjenljivih amplituda (spektra) i $\bar{R} = 0$

(pokusi prekinuti kod $N = 6000000$ zbog dugotrajnosti).

4 ANALIZA REZULTATA

Jednostrana i dvostrana ojačanja zamorom napuklih metalnih uzoraka primjenom zalijepljenih kompozita iz ugljičnih vlakana pokazala su se izuzetno uspješnim. Razlog njihove učinkovitosti svakako je velika nosivost kompozitnih laminata, pri čemu ipak treba odvojeno analizirati zbivanja kod jednostranih i dvostranih ojačanja.

Kod ojačanja zalijepljenih samo s jedne strane manji je udio naprezanja koji se prenosi s metalnog uzorka na sam "flaster", a raspodjela tog naprezanja nepovoljnija je nego li u slučaju dvostranih ojačanja. Zbog neizbježnog savijanja javljaju se maksimalna lokalna naprezanja u vrhu pukotine s neojačane strane uzorka, sl 3.



Slika 3 Raspodjela deformacija kod jednostrano ojačanog uzorka
Figure 3 Strain distribution at one face reinforced specimen

Zbog ovakve raspodjele naprezanja pukotina se, znatno usporena, širi nepravilno do izlaska iz područja ojačanja, a zatim nastupa trenutni lom uzorka. Gubitak veze uzorak-ojačanje nastaje prvenst-

veno ravninom spoja zbog tangencijalnih naprezanja u sloju ljepila ali i delaminacijom unutar samog kompozitnog ojačanja.

Kod dvostranih ojačanja srednje naprezanje uzorka u ravnini s pukotinom reducirano je za oko 50%, što uz izostanak savijanja rezultira ogromnim produljenjem vijeka trajanja, a u nekim slučajevima opterećenja i beskonačnim (trajna dinamička čvrstoća). Povećanje dinamičke izdržljivosti uslovljeno je, pored nosivosti ojačanja, i znatnim smanjenjem raspona faktora inteziteta naprezanja i s tim u vezi veličine plastične zone u vrhu pukotine. Zbog postojeće veće plastične zone kada se pukotina širila u uzorku bez ojačanja, sada je napredovanje pukotine, ako do njega i dođe, višestruko usporeno.

5 ZAKLJUČCI

Na temelju urađenih pokusa i analize rezultata može se zaključiti:

- adekvatno dimenzioniranim i zalijepljenim kompozitnim ojačanjima iz ugljičnih vlakana može se višestruko usporiti ili pak potpuno zaustaviti rast zamorne pukotine na limovima debljine 10 i više mm,
- ohrabrujuće dobri rezultati postižu se kako kod konstantnih, tako i kod promjenljivih amplituda aksijalnog opterećenja,
- mnogo bolji rezultati ostvareni su kod dvostranih ojačanja kako zbog njihove veće nosivosti, tako i zbog nepostojanja savijanja,
- odgađanje i usporavanje rasta pukotine posljedica su velike nosivosti ojačanja ali i znatnog smanjenja plastične zone ispred vrha pukotine u odnosu na njenu veličinu prije sanacije,
- kako zbog ovih rezultata, tako i zbog u uvodu opisanih prednosti, preporuča se korištenje kompozitnih ojačanja gdje god je to moguće i uz oprez kod većih debljina limova i primjene s jedne strane,
- daljnja istraživanja trebaju potvrditi učinkovitost metode kod još debljih limova, drugih oblika i razina opterećenja, drugih vrsta kompozita i načina njihove gradnje, itd.

ZAHVALA: Autor iskreno zahvaljuje prof.dr. V. Grubišiću i suradnicima s LBF-Instituta na velikoj pomoći tijekom ovog rada.

LITERATURA: Domazet, Ž. "Metode zaustavljanja širenja pukotina", Disertacija, FSB - Sveučilište u Zagrebu, 1993.

Utjecaj dužine pukotine (a/W) na CTOD krivulje otpornosti i žilavost loma čelika povećane čvrstoće

Effect of crack depth (a/W) on CTOD - Resistance Curves and Fracture Toughness of a HSLA Steel

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CTOD krivulje otpornosti su izmjerene na epruvetama za savijanje sa različitim dužinama zamorne pukotine. Karakteristične vrijednosti otvaranja vrha pukotine (CTOD) su određene procedurom koja je propisana standardima BS 5762 i ASTM E 1290 te istovremeno s tkz. δ_5 direktnim mjerenjem u vrhu zamorne pukotine. Ispitivani materijal je mikro legirani čelik povećane čvrstoće s omjerom 0.92 između granice tečenja i zatezne čvrstoće te eksponentom deformacijskog ojačavanja 0.06. Korištena je tehnika sukcesivnog rasterećivanja za određivanje dužine pukotine tokom samog mjerenja.

CTOD - Resistance curves were measured on bend specimens with different precrack lengths. The crack tip opening displacement was determined using the standard extrapolation procedures of BS 5762 and ASTM E 1290, as well as it was directly measured at the fatigue crack tip in term of δ_5 . The material tested was a HSLA steel with the yield stress to ultimate tensile strength ratio of 0.92 and low strain hardening exponent of 0.06. The unloading compliance technique to determine the crack growth during the test was used.

1. UVOD

Metode ispitivanja pri elasto-plastičnoj mehanici loma iziskuju, u glavnom, epruvete sa dubokim zamornim pukotinama, gdje je omjer dužine pukotine prema visini epruvete (a/W) veći od 0.5. Namjena standardnih ispitivanja je odrediti karakteristične vrijednosti parametara mehanike loma za ispitivani materijal. Dosadašnja iskustva su pokazala da duboka zamorna pukotina i visok "constraint" (troosno stanje napona u vrhu pukotine) omogućuju postizanje geometrijski relativno nezavisnih mjerenja žilavosti loma. U mnogo primjera se u realnim konstrukcijama pojave plitke pukotine kod kojih je omjer dužine pukotine prema ostalim dimenzijama konstrukcije mnogo manji od 0.5, odnosno nastala je potreba za vjerodostojnijom ocijenom žilavosti loma za primjer potpuno drugačije geometrije epruvete ili dijela konstrukcije.

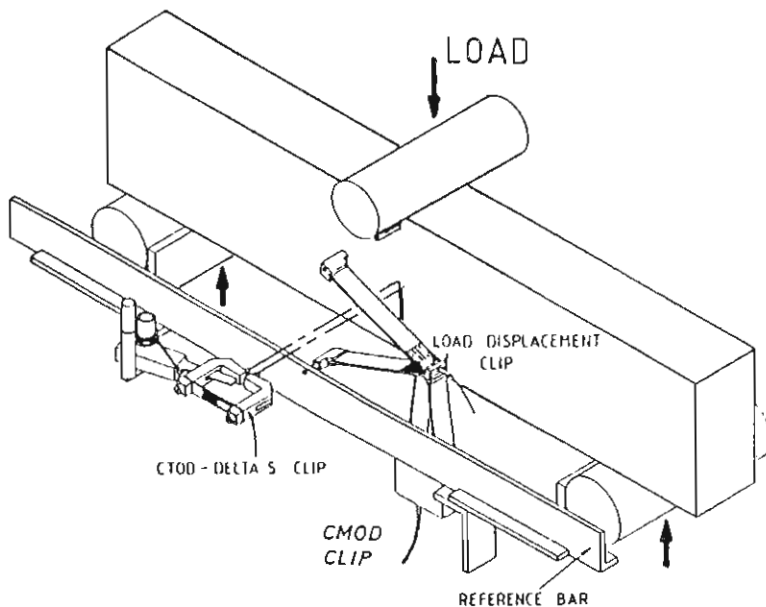
Krivulje otpornosti materijala su krivulje kod kojih se prikazuje određeni parametar mehanike loma, u ovom slučaju otvaranje vrha pukotine (CTOD - Crack Tip Opening Displacement), u zavisnosti od prirasta pukotine Δa . U početnom strmijem-linearnom dijelu krivulje se još ne pojavi razdvajanje materijala nego dolazi do zatupljivanja prvobitno oštre pukotine. S rastom opterećenja to zatupljivanje dostiže svoju kritičnu vrijednost kod koje dolazi do razdvajanja materijala i kod žilavih materijala do stabilnog širenja pukotine. S širenjem pukotine se promjeni nagib krivulje otpornosti tako da postaje položenija. Na sijecištu obiju krivulja dolazi do inicijacije stabilnog širenja pukotine (vrijednost δ_i), mada se u inženjerskoj praksi koristi vrijednost $\delta_{0.2BL}$ (slika 4), (4).

2. MATERIJALI, UZORCI I POSTUPCI ISPITIVANJA

Ispitivanja su se vršila na mikrolegiranom čeliku povećane čvrstoće NIONICRALu 70A, koji proizvodi Željezara Jesenice. To je ekonomsko zanimljiv čelik namjenjen za izradu kompleksnih konstrukcija stoga je izuzetno važno kako se ponašaju dijelovi strojeva sa plitkim površinskim pukotinama, koje praktično ne možemo izbjeći pri različitim zavarenim spojevima. Sa zatezim ispitivanjem standardnog uzorka (s promjerom 6 mm) dobili smo sljedeće rezultate: modul elastičnosti $E \approx 200$ GPa, granica tečenja $R_{p0.2}=720$ MPa, zatezna čvrstoća $R_m=780$ MPa, rastezanje pri maksimalnoj sili $e_n=7.5\%$, rastezanje pri lomu $A=22.4\%$ i kontrakcija $Z=67.8\%$. Standardne epruvete za savijanje pravokutnog presjeka 18×36 mm, s zarezom na rubu (SENB - Single Edge Notch Bend), su izrezane iz 40 mm debele ploče u smjeru valjanja. Epruvete su bile zamarane na servo-hidrauličkom stroju za ispitivanje u skladu sa standardima (1),(2), tako da smo postigli omjere između dužine pukotine i visine epruvete (a/W) od 0.1, 0.2, 0.3, 0.5 i 0.6.

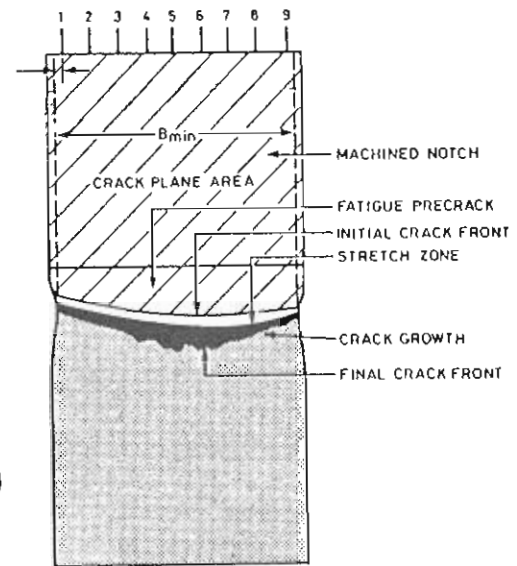
Za određivanje CTOD krivulja otpornosti je upotrebljena metoda elastične popustljivosti s djelomičnim resterećivanjem jedne epruvete (Single specimen unloading compliance method). Epruvetu za savijanje, s već urađenom zamornom pukotinom, smo opterećivali na servo-hidrauličkom stroju u kontroli pomaka (0.5 mm/min). Tokom ispitivanja smo izvršili više djelomičnih rasterećenja do približno 25% trenutne vrijednosti sile. Signali sa mjerne sonde stroja za pomak i silu, te mjerača δ_5 i mjerača otvaranja pukotine (slika 1) su preneseni prijeko analogno-

digitalnog plotera na PC-računar. Po završetku ispitivanja epruvetu ponovo zamaramo s malim amplitudam sile te tako označimo prirast pukotine pri duktilnom lomu. Epruvetu potom prelomimo i u prelomnom dijelu izmjerimo početnu dužinu pukotine a_0 te prirast pukotine Δa (slika 2). Izračunane vrijednosti dužine pukotine na osnovu popustljivosti epruvete tokom ispitivanja ne smiju odstupati za više od 10% od srednjih izmjerenih vrijednosti.



Slika 1 Epruveta za savijanje opremljena za δ_5 mjerenja

Figure 1 Instrumentation of bend specimen for δ_5 measurements



$$a = \frac{1}{8} \left(\frac{a_1 + a_9}{2} + \sum_{i=2}^8 a_i \right)$$

Slika 2. Mjerenje dužine i prirasta pukotine

Figure 2. Measurements of crack crack lengths and crack extension

Važnu ulogu pri svakom ispitivanju, koje temelji na popustljivosti epruvete kod djelomičnog rasterećivanja (slika 3), ima preciznost kalibracijske jednadžbe za dužinu pukotine. Pri epruveti za savijanje (SENB) po standardu ASTM E 1152-87 (5) koristi se polinom petog stupnja:

$$\frac{a}{W} = 0.999748 - 3.9504u + 2.9821u^2 - 3.21408u^3 + 51.51564u^4 - 113.031u^5 \quad (1)$$

gdje su:

$$u = \left(\sqrt{\frac{BWE'C}{S/4}} + 1 \right)^{-1}$$

B, W . . . debljina i visina epruvete

$E' = E(1-\mu^2)$

S . . . raspon između potpora

C . . . popustljivost epruvete

Jednadžbu je razvio Wu (6) s inverznom aproksimacijom vrijednosti, dobivenih s integracijom Tadinovog izraza (7) za linearno elastičnu intezivnost naprezanja:

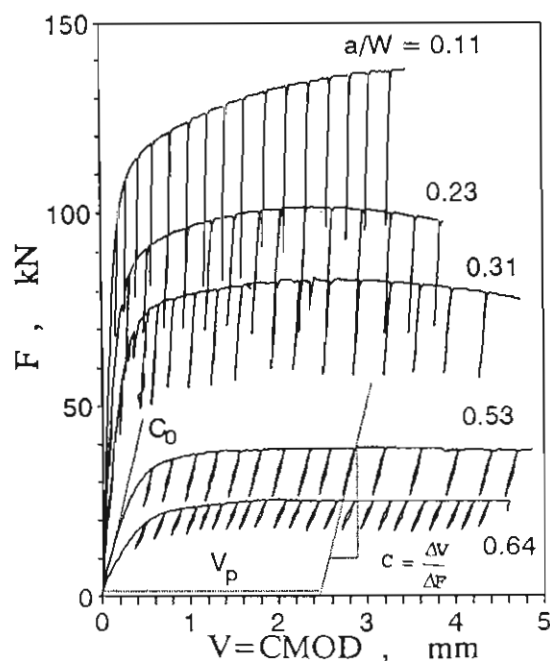
$$C\left(\frac{BWE'}{S/4}\right) = 24\left(\frac{a}{W}\right) \left[0.76 - 2.28\left(\frac{a}{W}\right) + 3.87\left(\frac{a}{W}\right)^2 - 20.4\left(\frac{a}{W}\right)^3 + \frac{0.66}{(1-a/W)^2} \right] \quad (2)$$

Poznato je da ima izraz (2) 1% točnosti za sve a/W omjere.

Jednadžba 1 se podudara s vrijednostima po jedn. 2 u području točnosti 0.04% za $a/W > 0.3$, dok za manje a/W omjere daje netočne vrijednosti na primjer za $a/W = 0.1$ greška može iznositi i 88%. Dakle ta jednadžba nije primjerna za epruvete s plitkim pukotinama. Stoga smo za plitke pukotine, po metodi najmanjih kvadrata, za omjere a/W od 0.05 do 0.5 odredili novi polinom:

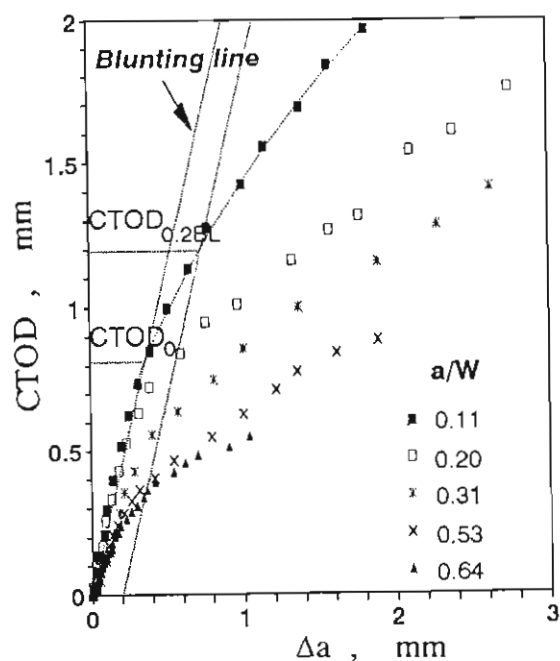
$$\frac{a}{W} = 1.01878 - 4.5367u + 9.0101u^2 - 27.333u^3 + 74.4u^4 - 71.489u^5 \quad (3)$$

koji omogućuje sukladnost s jedn. 2 u razredu 0.06%.



Slika 3. Karakteristični zapisi kod pokusa sa djelomičnim resterećivanjem epruvete

Figure 3. Typical unloading compliance records



Slika 4. CTOD krivulje otpornosti za epruvete sa različitim dubinama pukotine

Figure 4. CTOD resistance curves for test specimens with different crack depths

Za određivanje CTOD krivulja otpornosti većina ispitivanja još uvijek koristi CTOD-CMOD korelaciju propisanu standardom BS 5762 (1). Po tom standardu je otvaranje vrha pukotine ($CTOD = \delta_{BS}$) sastavljeno iz elastičnog (δ_{ssy}) i plastičnog (δ_{fpl}) dijela. U našim istraživanjima smo koristili nekoliko modificiran izraz δ_{BS} , koji uzima u obzir korekciju plastične komponente zbog prirasta pukotine:

$$CTOD = \delta_{ssy} + \delta_{fpl} = \frac{K_I^2(1-\mu^2)}{2ER_{p0.2}} + \frac{r_p(W-a)+a-a_0}{r_p(W-a)+a+z} V_p \quad (4)$$

gdje su: K_I linearni elastični faktor intenziteta napona, μ Poissonov broj, V_p plastična komponenta otvaranja na mjeraču CMOD, koji je pritvrđen na udaljenosti z od površine epruvete, i r_p faktor rotacije, koji je definiran kao udaljenost vrha pukotine od osi rotacije normiran s dužinom nepuknotog dijela epruvete - ligamentom $b=W-a$.

Kao što je prikazano na slici 1, izvršili smo i neposredno mjerenje otvaranje vrha pukotine na bočnoj strani epruvete u vrhu početne zamorne pukotine pomoću posebnog mjerača. Pošto je početna mjerna dužina tog mjerača 5 mm (po 2.5 mm sa sva-ke strane od vrha pukotine) izmjerena vrijednost otvaranja vrha pukotine označuje se δ_5 (6).

3. DISKUSIJA REZULTATA

Glavni problem pri određivanju CTOD vrijednosti za epruvete s plitkim pukotinama (npr. $a/W < 0.3$) po jednadžbi 4 je u ocijeni faktora rotacije r_p . Trenutno važeći standardi (1), (2) preporučuju za omjer $a/W > 0.45$ vrijednosti $r_p = 0.4$ odnosno 0.44 za sve materijale. Jednadžba 4 će dati točne rezultate samo u slučaju, ako se upotrijebi i prava vrijednost faktora r_p . Faktor rotacije se može eksperimentalno odrediti pomoću δ_5 direktnih mjerenja otvaranja vrha pukotine tako da se zamjene CTOD vrijednosti u BS jednadžbi s δ_5 :

$$\frac{K_I^2(1-\mu^2)}{2ER_{p0.2}} + \frac{r_p(W-a)+a-a_0}{r_p(W-a)+a+z} V_p = \delta_5 \quad (5)$$

Iz jedn. 5 se može izraziti r_p u obliku:

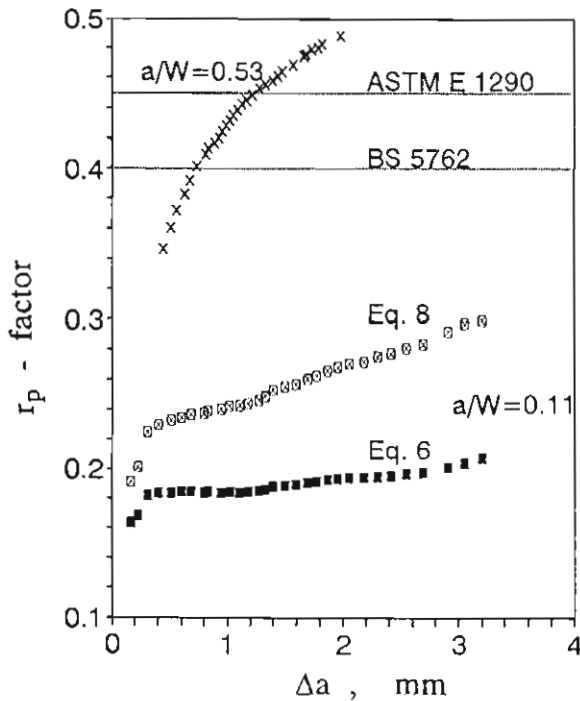
$$r_p = \frac{(\delta_s - \delta_{ssy})(a+z) - V_p(a-a_0)}{(V_p - \delta_s + \delta_{ssy})(W-a)} \quad (6)$$

Isto tako mogu se odrediti r_p vrijednosti upotrebom metode mjerenja otvaranja pukotine pomoću dvaju mjerača (Duble Clip Gage Method), koje su razvili Heyer in McCabe (8). Ukoliko uzmemo u obzir samo plastične komponente otvaranja pukotine (V_{p1} i V_{p2}) oba mjerača, koja su pričvršćena na različitim visinama (z_1 i z_2), potom možemo izračunati faktor r_p iz sličnosti trokuta:

$$r_p = \frac{V_{p1}(a+z_2) - V_{p2}(a+z_1)}{(V_{p2} - V_{p1})(W-a)} \quad (7)$$

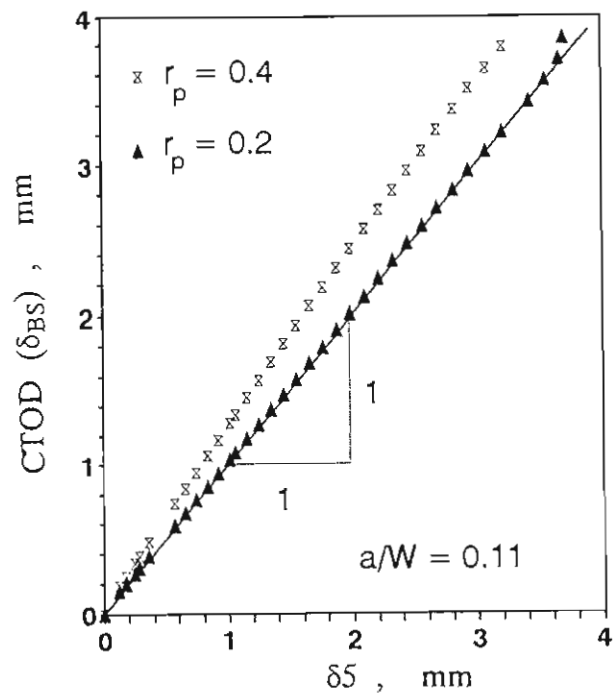
U našem slučaju je prvi mjerač δ_5 pa je $V_{p1} = \delta_{5p}$ na visini $z_1 = -a_0$, stoga možemo pri drugom mjeraču indeks 2 izostaviti:

$$r_p = \frac{\delta_{5p}(a+z) - V_p(a-a_0)}{(V_p - \delta_{5p})(W-a)} \quad (8)$$



Slika 5. Faktori rotacije za epruvete s plitkom in dubokom pukotinom

Figure 5. Rotation factors for shallow and deep cracked specimens



Slika 6. Utjecaj faktora rotacije na odnos δ_{BS} - δ_5 za epruvete s plitkim pukotinama

Figure 6. Effect of rotation factor on the δ_{BS} - δ_5 relationship for shallow cracked specimens

Na slici 5 su prikazane vrijednosti faktora r_p , koje su dobivene na oba opisana načina, u ovisnosti od prirasta pukotine Δa . Za epruvete s dubokom pukotinom s porastom pukotine vrijednost r_p raste od 0.4 do 0.5, što predstavlja nekoliko višu vrijednost od preporučene $r_p=0.4$ po BS 5762 standardu, dok je za plitke pukotine ($a/W \approx 0.1$) njegova vrijednost mnogo niža te se tokom procesa žilavog širenja pukotine bitno ne mjenja. Spomenuto, navodi na zaključak da pri epruvetama s plitkim pukotinama leži centar rotacije puno bliže vrhu pukotine.

Sa slike 6 se može razabrati da pri epruvetama s plitkim pukotinama standardne metode po BS 5762 i ASTM E 1290 daju znatno veće vrijednosti za otvaranje vrha pukotine u usporedbi s neposredno izmjerenom vrijednošću δ_5 . S smanjivanjem faktora r_p na vrijednost 0.2 odnosno 0.25 postiže se dobra skladnost između obje CTOD vrijednosti.

4. ZAKLJUČCI

U radu je diskutiran utjecaj dužine pukotine (a/W) na CTOD krivulje otpornosti te na žilavost loma mikrolegiranog čelika povećane čvrstoće. Upotrebom CTOD izraza po BS standardu u kombinaciji sa direktnim mjerenjem pomicanja otvora pukotine δ_5 , na eksperimentalni način određen je faktor rotacije r_p . Analiza eksperimentalnih rezultata, dobivenih na epruvetama za savijanje metodom elastične popustljivosti, daje sljedeće zaključke:

- Za ispitivani čelik NIONICRAL 70A je karakteristična velika mogućnost plastične deformacije i visok nivo žilavosti loma.
- Granica između plitkih i dubokih pukotina je u području omjera a/W od 0.1 do 0.2, što je vidljivo iz ponašanja CTOD krivulja otpornosti (slika 2). Vrijednost parametra CTOD pri inicijaciji žilavog loma, a još izrazitije $CTOD_{0.2BL}$ parametra se bitno povećava kod epruveta s plitkim pukotinama.
- Faktor rotacije r_p je moguće eksperimentalno odrediti korištenjem CTOD jednadžbe po BS 5762 u kombinaciji s neposrednim mjerenjem otvaranja vrha pukotine δ_5 . Za epruvete s plitkim pukotinama vrijednost r_p je između 0.2 i 0.3, dok za epruvete s dubokim pukotinama su vrijednosti nekoliko veće od

standardnih. Dakle standardne vrijednosti $r_p=0.4$ (1) i $r_p=0.44$ (2), nisu primjerne za sve materijale i dužine pukotina.

- Neposredno mjerenje otvaranja vrha pukotine δ_5 na epruvetama za savijanje s plitkom i dubokom pukotinom omogućuje brzo i jednostavno ocjenjivanje žilavosti loma pri čemu nije potrebno poznavati mehanička svojstva materijala i vršiti bilo kakve korekciju faktora rotacije.
- Predložen je novi izraz za izračun dužine pukotine na osnovu izmjerene poputljivosti epruvete, koji je dovoljno precizan i za omjere $a/W < 0.3$.

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OCJENA UTJECAJA STANJA MATERIJALA NA LOMNO PONAŠANJE ČELIKA POVEĆANE ČVRSTOĆE

ASSESSING THE EFFECTS OF MATERIAL'S CONDITION ON THE FRACTURE BEHAVIOUR OF HSLA STEEL

GUBELJAK N., LEGAT J., ALUJEVIČ A.*

U ovom članku je prikazan utjecaj različitih stanja mikro legiranog čelika na otpornost materijala prema stabilnom širenju pukotine i to na sobnoj te T_{NDT} temperaturi (temperaturi krkog loma).

Veličine mehanike loma $J_{0.2BL}$ i J_c predstavljaju primjerne parametre žilavosti loma za koje su pomoću logaritamsko-normalne razdiobe određeni intervali pouzdanosti. Dobiveni rezultati pokazuju da kod mikrolegiranih čelika povećane čvrstoće viša granica tečenja materijala može prouzrokovati i veće smanjenje otpornosti materijala na stabilno širenje pukotine s promjenom stanja materijala i temperature.

In this paper the effects of three different supply conditions of HSLA steels on ductile crack growth resistance at room and T_{NDT} temperature are shown.

The values of $J_{0.2BL}$ and J_c were chosen as suitable parameters and treated by the lognormal distribution function for determination of confidence intervals.

The results obtained show that HSLA steels with higher yield strength are more sensitive to the condition of the steel and to temperatures than lower-strength steels are.

1. UVOD

Noseći elemenat konstrukcije može otkazati ukoliko nastupi prekoračenje graničnih napetosti materijala, nestabilnost konstrukcije ili lom elementa. Prva dva kriterija su već poznata u praksi i propisana su standardima. Treći kriterij, kriterij loma, za razliku od već spomenutih dvaju kriterija, tretira materijal kao anizotropnu heterogenu strukturu, u kojoj se nalaze greške iz kojih mogu nastati pukotine kritičnih veličina.

Pri izradi konstrukcije potrebno je izvršiti plastično preoblikovanje materijala koji je potom na konstrukciji ispostavljen različitim atmosferskim utjecajima i procesu staranja materijala. Dakle mehanička svojstva materijala se tokom izrade i rada konstrukcije mijenjaju. Time se isto tako mijenja i otpornost materijala na stabilno širenje pukotine.

Problem postaje složeniji ukoliko je radna temperatura konstrukcije niska (kod utekućenih plinova) i u slučaju rasipanja rezultata žilavosti loma.

Stoga je namjena ovog članka odrediti stupanj promjene otpornosti niskolegiranih čelika na stabilno širenje pukotine pri različitim stanjima materijala s primjernom pouzdanošću.

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2. OPIS MATERIJALA

Da bi se mogao pratiti, pored utjecaja stanja, i utjecaj granice tečenja materijala te utjecaj temperature krhkog loma na otpornost materijala prema stabilnom širenju pukotine uzeta su dva mikro legirana čelika (NIOVAL 47 in Č.0562). Kemijski sastav tih dvaju čelika je prikazan u tabeli 1.

Tabela 1. Kemijski sastav (u težinskim %) čelika.

Čelik	C	Si	Mn	P	S	Cr	Ni	Mo	Nb	Al
NIOVAL 47	0.19	0.42	1.49	0.013	0.005	0.13	0.10	0.04	0.05	0.087
C.0562	0.17	0.32	1.28	0.020	0.009	0.21	0.23	0.05	0.03	0.045

Isto tako na osnovu praktičnih iskustava promatrana su tri različita stanja materijala:

- a) čelik u normaliziranom stanju (uobičajeno stanje isporuke), oznaka stanja "A",
- b) čelik u staranom stanju, po 10% plastičnoj deformaciji se zagrijava na 250°C za 30 minuta, oznaka stanja "B",
- c) deformirano stanje po 10% plastičnoj deformaciji, oznaka stanja "C".

Iz valjane ploče debljine 20mm so izrađene zatezne epruvete, epruvete za DWT (drop weight test) te kompaktne zatezne epruvete (CT-compact tension). Mehanička svojstva su određivana na sobnoj temperaturi te na T_{NDT} -temperaturi krhkog loma po ASTM E208. Rezultati tih ispitivanja su prikazani u tabeli 2.

Tabela 2. Mehanička svojstva čelika NIOVAL 47 i Č.0562 na sobnoj i T_{NDT} temperaturi.

Čelik	Oznaka uzorka	Granica tečenja	Zatezna čvrstoća	Jednako-mjerno rastezanje	T_{NDT}	Zatezna čvrstoća na T_{NDT}	eksponent očvršćenja n
		MPa	MPa	%	°C	MPa	
NIOVAL 47	VA	442	610	13.7	-72	514	0.1309
	VB	647	726	4.7	-104	780	0.0731
	VC	627	684	4.5	-132	757	0.0353
C.0562	CA	366	553	14.3	-92	475	0.1528
	CB	586	656	4.8	-112	653	0.0700
	CC	595	624	4.1	-120	675	0.0414

Na osnovu dobivenih rezultata, prikazanih u tabeli 2, možemo zaključiti da čelici u normaliziranom-dobavljenom stanju ("A") imaju višu duktilnost (veće rastezanje i veći eksponent očvršćenja) u odnosu na stanja "B" i "C", što je posljedica segregacije i deformacijske iskorišćenosti materijala. Kod oba čelika s opadanjem temperature se

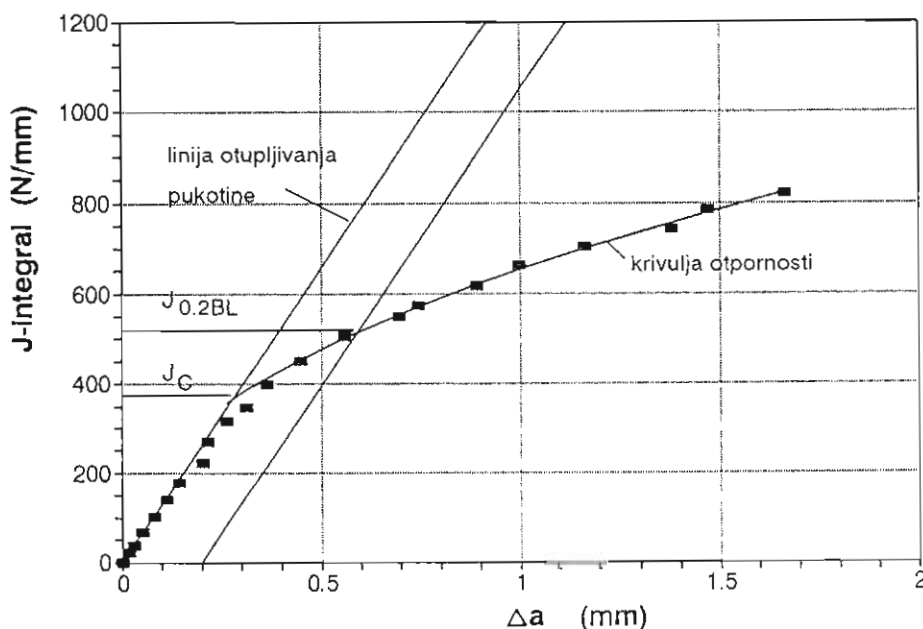
povećava granica tečenja materijala, uz istovremeno smanjivanje jednakomjernog rastezanja materijala. Uobičajeno je da s statičnim ispitivanjem zateznih epruveta dobivamo opću sliku o mehaničkim svojstvima materijala ali je još uvijek nepoznata otpornost materijala na stabilno širenje pukotine.

Na osnovu spomenutih ispitivanja odlučeno je da se naredni eksperimenti vrše na sobnoj ($+20^{\circ}\text{C}$) temperaturi i T_{NDT} temperaturi (-72°C i -92°C) obaju čelika u normaliziranom stanju.

3. ODREĐIVANJE J-R KRIVULJE OTPORNOSTI I KRITIČNE VRIJEDNOSTI J-INTEGRALA

Za opisivanje ponašanja tijela s pukotinom ima mehanika loma u elasto-plastičnom području na raspolaganju dva parametra: kritičnu vrijednost J-integrala /1/ i kritično vrijednost otvaranja vrha pukotine (CTOD) /2/.

Krivulje otpornosti materijala su krivulje kod kojih prikazujemo određeni parametar mehanike loma (u našem slučaju je to J-integral) u ovisnosti od prirasta pukotine Δa . U početnom strmijem (linearnom) dijelu krivulje još ne nastupi razdvajanje materijala te je prividni prirast posljedica zatupljivanja (zobljavanja) prvobitno oštre pukotine. S daljim rastom opterećenja dostiže otupljivanje pukotine svoju kritičnu vrijednost te dolazi do razdvajanja materijala i stabilne propagacije pukotine. S porastom pukotine krivulja otpornosti postaje položnija.



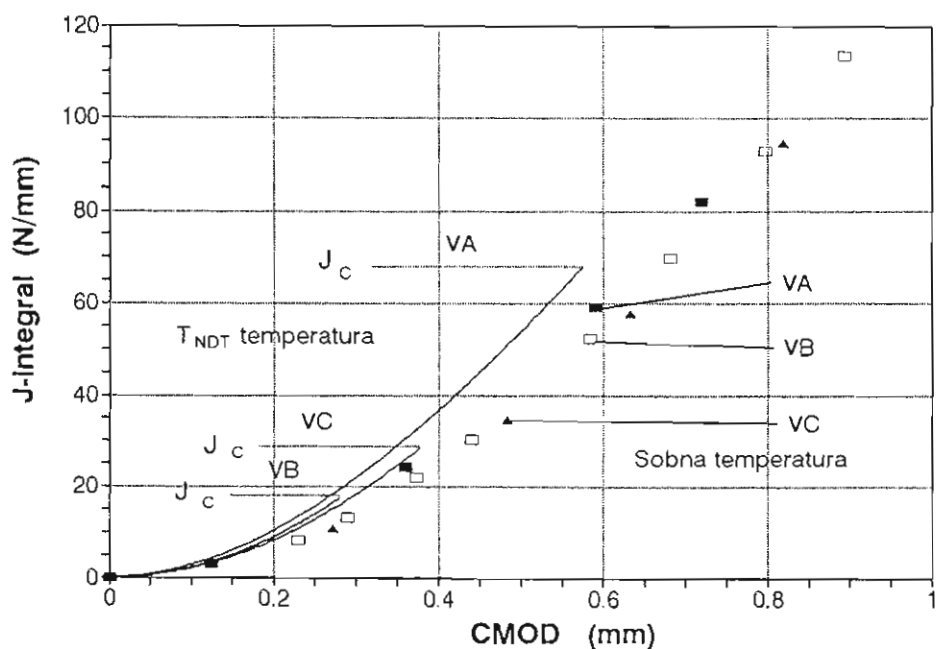
Slika 1. J-R krivulja otpornosti materijala i određivanje vrijednosti $J_{0.2BL}$.

Figure 1. $J-\Delta a$ resistance curve and the determination of $J_{0.2BL}$ value.

Smatra se da na sijecištu linije otupljivanja i krivulje otpornosti nastupa inicijacija pukotine (vrijednost J_c), dok se u inženjerskoj praksi obično koristi vrijednost $J_{0.2BL}$ (slika 1.) [3]. Pred ispitivanjem je potrebno izraditi zamornu pukotinu u vrhu mehaničkog zareza kako propisuje standard ASTM E813. Za određivanje J-R krivulja otpornosti koriste se epruvete za savijanje u tri točke (3PB - three point bend) ili zatezne kompaktne epruvete (CT) s zamornom pukotinom čija dužina a_0 iznosi oko polovice širine epruvete W ($a/W \approx 0.5$). Upotrebljena je metoda elastične popustljivosti pri kojoj se prati zavisnost F-CMOD (sila-otvaranje pukotine) te tokom opterećivanja epruvete vrši djelomično rasterećivanje. Prilikom rasterećivanja epruvete na osnovu nagiba krivulje rasterećenja određuje se krutost epruvete C_i čije je opadanje u korelaciji s porastom dužine pukotine Δa_i . Po završetku ispitivanja se epruveta rastereti i na primjeren način označi konačni prirast pukotine Δa (npr. zagrijavanjem epruvete na 300°C ili ponovnim zamaranjem s malim amplitudama opterećenja) koji se po prelomu usporedi s prirastom pukotine izračunanim pri ispitivanju Δa_k .

4. REZULTATI ISPITIVANJA

Ispitivanja su izvršena na 6 serija po 15 standardnih CT epruveta. Na osnovi F-CMOD zapisa (ovisnost sila-otvaranje pukotine) određene su J-R krivulje otpornosti na sobnoj temperaturi slika 1, iz kojih su, na već pomenuti način dobivene inženjerske vrijednosti $J_{0.2BL}$ parametra žilavosti loma.



Slika 2. J-CMOD krivulje i vrijednosti J_c na T_{NDT} temperaturi.

Figure 2. J-CMOD curves and J_c values at T_{NDT} temperatures.

Prilikom ispitivanja CT epruveta na T_{NDT} temperaturama je isto tako snimljena ovisnost F-CMOD, gdje je za razliku od ispitivanja na sobnoj temperaturi kod svih epruveta došlo do krhkog loma epruveta. Iz konačne vrijednosti u F-CMOD diagramu su izračunane vrijednosti J_C koje predstavljaju parametar žilavosti loma na T_{NDT} temperaturi, slika 2.

5. DEFINIRANJE MODELA VJEROJATNOSTI

Pri ispitivanju je dobiveno znatno rasipanje rezultata žilavosti loma ($J_{0.2BL}$ i J_C), stoga je potrebno odrediti pouzdanost rezultata pomoću odgovarajućeg modela. Korištena je funkcija gustoće vjerojatnosti logaritamsko-normalne razdiobe za definiranje intervala pouzdanosti s 99% povjerenjem:

$$f(x) = \frac{1}{V_y \cdot \sqrt{2\pi}} \cdot e^{-\frac{(\ln(x) - \mu_y)^2}{2V_y^2}} \quad (1)$$

gdje je V_y^2 varijanca logaritamsko-normalne razdiobe i μ_y matematičko očekivanje distribuirano po zakonu normalne razdiobe za $y = \ln(x)$.

Korištenjem teorije vjerojatnoće [4], pouzdanost se može definirati s izrazom:

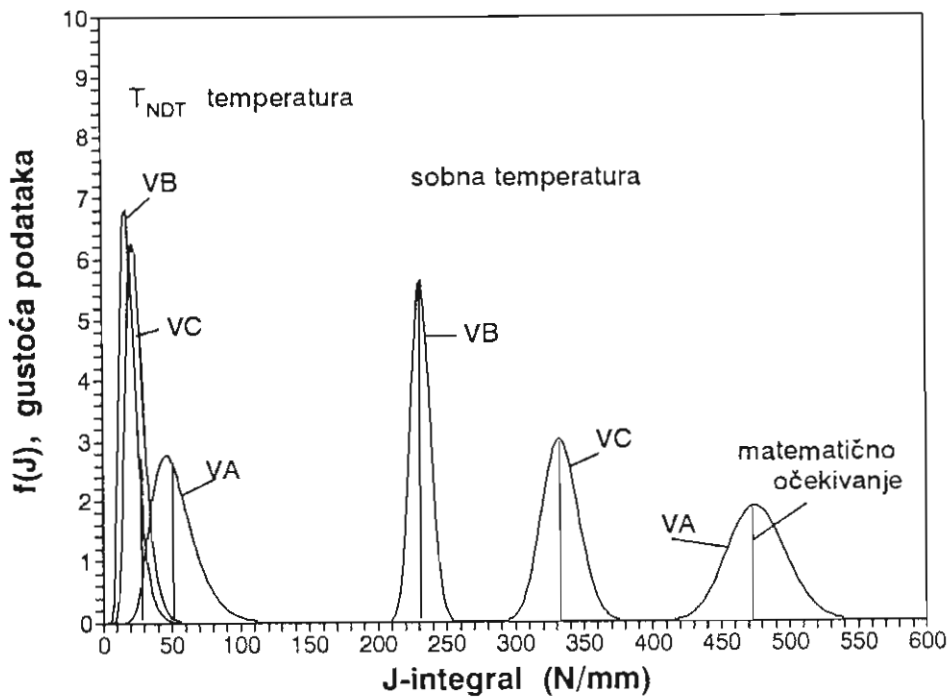
$$R(x) = 1 - F(x) = 1 - \int_{x_{inf}}^x f(x) dx \quad (2)$$

Pomoću tog izraza su definirane granične vrijednosti intervala pouzdanosti sa vjerojatnošću 99% za parametre žilavosti loma $J_{0.2BL}$ i J_C , što je prikazano u tabeli 3.

Tabela 3- Matematičko očekivanje i granične vrijednosti intervala pouzdanosti sa vjerojatnošću 99% ($J_{0.2BL}$ i J_C) na sobnoj i T_{NDT} temperaturi obaju čelika.

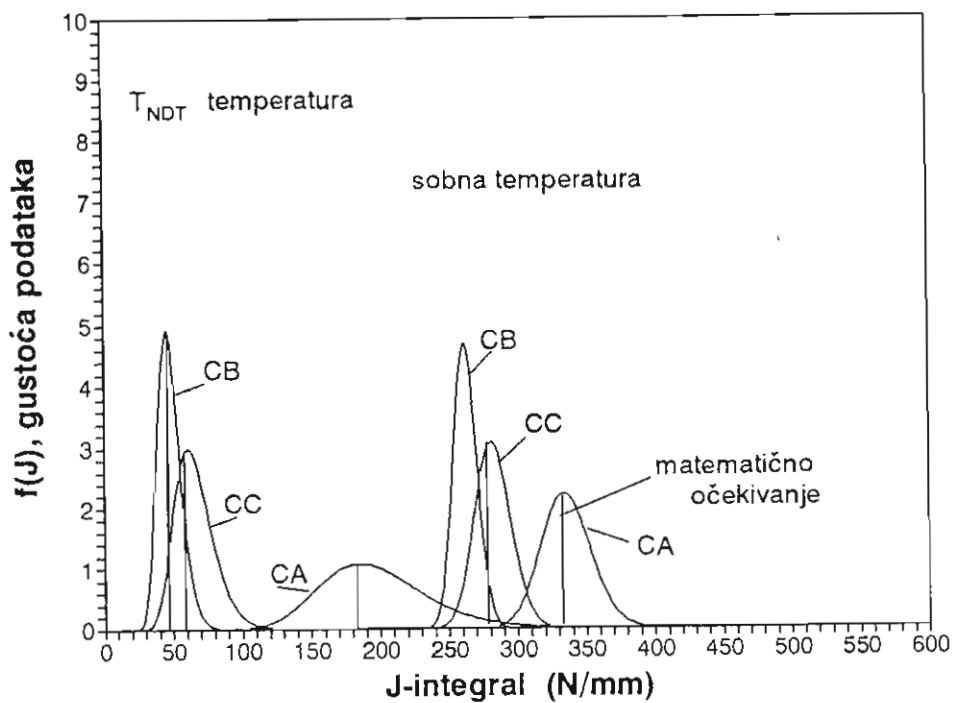
oznaka epruveta	Parametar žilavosti $J_{0.2BL}$ N/mm			Parametar žilavosti J_C N/mm		
	lijevi rub	matematičko očekivanje E	desni rub	lijevi rub	matematičko očekivanje E	desni rub
VA	420.5	476.5	532.5	3.31	52.62	102.1
VB	209.4	231.6	253.8	1.23	20.44	40.1
VC	297.9	333.2	368.4	3.85	25.02	46.2
CA	305.3	335.5	365.6	87.09	196.39	305.7
CB	240.1	262.5	285.0	24.12	47.62	71.2
CC	247.7	282.3	316.9	24.65	65.40	106.1

S ciljem jasnije interpretacije, dobivene funkcije gustoće vjerojatnosti logaritamsko-normalne razdiobe ilustrirane su na slici 3 za čelik NIOVAL 47 te na slici 4 za čelik Č.0562.



Slika 3. Funkcije vjerojatnosti logaritamsko-normalne razdiobe parametra žilavosti loma za sva tri stanja čelika NIOVAL 47 na sobnoj i T_{NDT} temperaturi.

Figure 3. Probability distribution of toughness parameters at room and T_{NDT} temperatures for all states of steel NIOVAL 47.



Slika 4. Funkcije vjerojatnosti logaritamsko-normalne razdiobe parametra žilavosti loma za sva tri stanja čelika Č.0562 na sobnoj i T_{NDT} temperaturi.

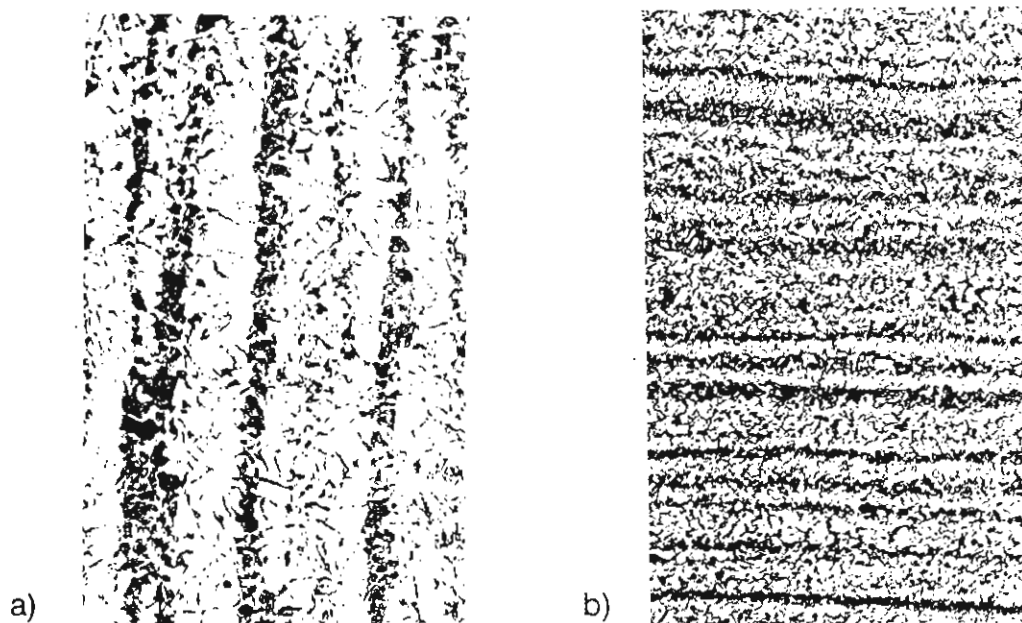
Figure 4. Probability distribution of toughness parameters at room and T_{NDT} temperatures for all states of steel Č.0562.

6. DISKUSIJA

Za potpuno razumjevanje lomnog ponašanja materijala izvršena su metalografska ispitivanja i ispitivanja udarne žilavosti loma po Charpyju. Metalografska ispitivanja su pokazala da oba čelika, NIOVAL 47 i Č.0562 imaju mikrostrukturu u obliku traka. Pri čeliku Č.0562 je struktura feritno-perlitna unutar feritne strukture (slika 5.a), dok je pri čeliku NIOVAL 47 struktura perlitno-bainitna unutar feritne strukture, (slika 5.b). U oba čelika su prisutni nemetalni uključci sulfida, aluminata, silikata i globularnih oksida.

Ispitivanja udarne žilavosti po Charpyju su pokazala, kao što su već istražili J. Vojvodič-Gvardjančić i suradnici /5/, da se s staranjem mikrolegiranog čelika prilično smanji energija potrebna za inicijaciju pukotine, dok energija propagacije pukotine ostaje nepromjenjena ili se vrlo malo smanji.

Dobiveni rezultati lomno mehaničkih ispitivanja pokazuju da čelici u normaliziranom stanju (stanje isporuke, oznake epruveta VA i CA) imaju najvišu otpornost prema stabilnom širenju pukotine na sobnoj temperaturi, slike 3 i 4, tabela 3. Čelik NIOVAL 47 (oznaka VA) ima višu otpornost zbog perlitno-bainitne strukture što daje višu zateznu čvrstoću, granicu tečenja i višu udarnu žilavost u odnosu na feritno-perlitnu strukturu čelika Č.0562. Žilavost loma obaju čelika u deformiranom stanju te na svim temperaturama je niža od žilavosti loma u normaliziranom stanju, što je posljedica deformacijske iskorišćenosti materijala.



Slika 5. a) Perlitno-feritna struktura čelika Č.0562 in b) perlitno-bainitna struktura NIOVAL 47 pri povećanju 100x

Figure 5. a) The perlite-ferritic microstructure of Č.0562 and b) the perlite-bainitic microstructure of NIOVAL 47 with initial 100x.

Vrijednosti žilavosti loma se u staranom stanju još dodatno smanje zbog segregacije nemetalnih uključaka po granicama kristala, što su potvrdila ispitivanja prelomnih površina epruveta udarne žilavosti.

Iz dobivenih rezultata, na slikama 3,4 odnosno u tabeli 3, je vidljivo da je najveće rasipanje rezultata prisutno na sobnoj temperaturi i pri normaliziranom stanju. Dok se sa deformiranjem i staranjem materijala (a još izrazitije s temperaturom) rasipanje rezultata smanjuje. Razlog je u tome da se s deformiranjem i staranjem smanjuje količina faktora koji utječu na lomno ponašanje čelika.

7. ZAKLJUČAK

Čelik sa višom granicom tečenja materijala ima u normaliziranom stanju i na sobnoj temperaturi višu otpornost spram stabilnom širenju pukotine. Dok u slučaju deformiranog, a posebno kod staranog stanja, žilavost loma može se čak preko 50% smanjiti pri čeliku NIOVAL 47 s višom granicom tečenja i postići nižu vrijednost od čelika Č.0562 s nižom granicom tečenja, gdje je smanjenje iznosilo 22%. Dakle mikrostruktura i podložnost materijala procesu staranje te duktilnost materijala trebaju imati poglavitiiji značaj za odabir materijala kojim se želi postići viša otpornost na širenje pukotine od ostalih mehaničkih svojstava (granice tečenja, zatezne čvrstoće pa čak i od udarne žilavosti) loma.

Ovim ispitivanjima se je pokazalo da ispitivanje materijala, pomoću postupka mehanike loma, predstavlja bitnu kariku u lancu karakterizacije lomnog ponašanja materijala. Ta ispitivanja pružaju jasniji uvid u utjecaj deformacijskog i staranog stanja materijala na lom i doprinose većoj projektivnoj i radnoj sigurnosti konstrukcije.

8. LITERATURA

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MEHANIČKA SVOJSTVA GRADSKOG OTPADA

SAŽETAK

U okviru rješavanja problematike zbrinjavanja gradskog otpada nameće se nužnost izgradnje sanitarnih odlagališta. Ova zadaća ima nesumnjivo interdisciplinarni karakter, ali s naglašenom ulogom geotehničkog inženjerstva. U to su, između ostalog, uključeni problemi slijeganja i stabilnosti materijala u odlagalištu. U vezi s time u članku se razmatra problematika mehaničkog ponašanja gradskog otpada. Koristi se uobičajeni pristup u mehanici tla, ali se ukazuje i na razlike u ponašanju tla i otpadnog materijala. Korištenjem objavljenih rezultata iz svjetske literature, prikazani su neki eksperimentalni postupci za određivanje parametara čvrstoće otpada. Zaključuje se da je opravdano uvođenje pojma "mekanika otpada" (waste mechanics).

SUMMARY

Within the framework of municipal solid waste management it is necessary to construct sanitary landfills. This task has by no means interdisciplinary feature but with pronounced geotechnical role. The main problems to be solved are, among the other problems, settlements and stability of waste material in the landfill. In relation to these problems the paper deals with the mechanical behaviour of municipal solid waste. Classical soil mechanics approach is adopted but the differences between the soil behaviour and waste behaviour are emphasized. Using some published data from the literature several experimental procedure for the determination of shear strength parameters are presented. It is concluded that there is every reason to introduce the term "waste mechanics".

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1. UVOD

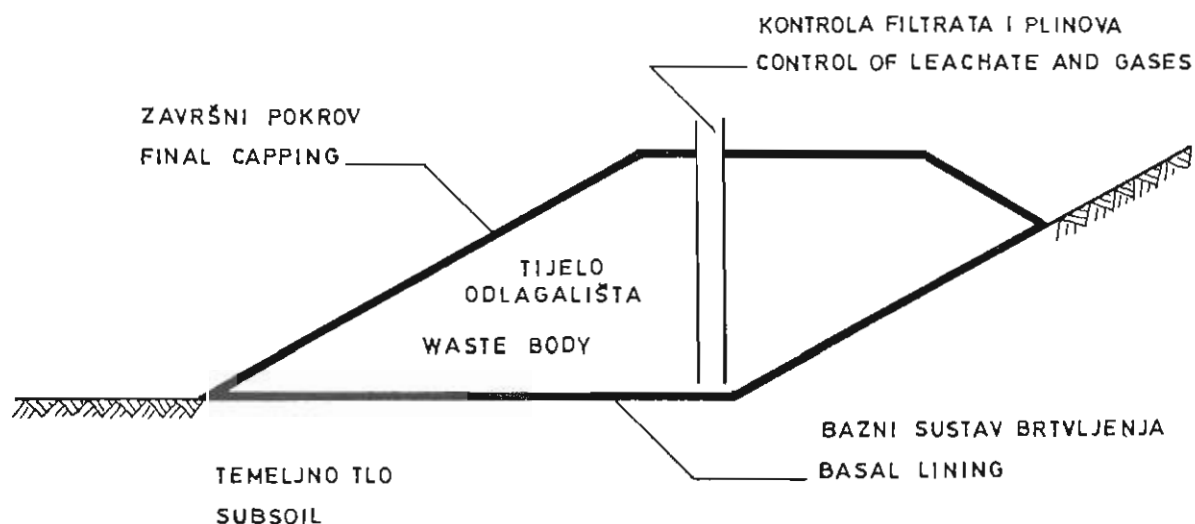
Zbrinjavanje otpada predstavlja jedan od najvećih problema današnje civilizacije. Stupanj ozbiljnosti problematike odlaganja otpada u nas dobro ilustriraju aktualni službeni podaci o količinama otpada u Hrvatskoj. Prema navodima iz Nacionalnog izvještaja o okolišu i razvitku (1992) godišnja količina otpada u Hrvatskoj procjenjuju se na cca 3.350.000 t. Od toga je cca 250.000 t opasnog otpada. Uz napomenu da je Hrvatska ipak relativno mala zemlja, može se razabrati o kakvim se količinama otpada radi u svjetskim razmjerima.

Suvremeni načini rješavanja ovog problema zasnivaju se na konceptu koji se sastoji od tri osnovna elementa (Potočnik, 1992):

- a) smanjivanje količine, tj. izbjegavanje otpada,
- b) ponovna uporaba, tj. iskorištavanje otpada,
- c) odlaganje neizbježnog ostatka otpada.

U ovisnosti o manje ili više uspješnoj realizaciji ovog koncepta, u svakoj urbanoj cjelini prisutna je veća ili manja količina otpada koja se mora negdje odlagati. Pri tome je izgradnja sanitarnih odlagališta nužan korak u konačnom rješavanju problema gradskog otpada.

Sanitarno odlagalište (slika 1.) može se definirati kao "objekt koji je izgrađen i opremljen za trajno, kontrolirano, organizirano i sigurno odlaganje otpada" (Milanović, 1992).



Slika 1. Tipično sanitarno odlagalište

Figure 1. Typical sanitary landfill

Tehnologija odlaganja otpada počela se razvijati u visoko razvijenim zemljama prije nešto više od dvadeset godina. U proteklom razdoblju u svijetu su izgrađena mnoga velika sanitarna odlagališta.

Prilikom planiranja, projektiranja, građenja, korištenja, te konačnog zatvaranja odlagališta treba riješiti cijeli niz problema. Dio tih problema je i geotehničkog karaktera, kao npr. geotehničko definiranje mjesta predviđenog za odlagalište, projektiranje i izgradnja nepropusnih barijera (sustavi brtvljenja), stabilnost pokosa tokom i nakon odlaganja, te slijeganje i nosivost materijala u odlagalištu nakon njegovog zatvaranja. Upravo ovaj geotehnički aspekt predstavlja okvir za razmatranje problema mehaničkog ponašanja gradskog otpada.

2. SANITARNO ODLAGALIŠTE KAO GEOTEHNIČKI ZADATAK

Tlo je nakupina zrnaca (čestica) različitih veličina, oblika i zbijenosti. Ova zrnca čine skelet tla, a međuprostori - pore ispunjeni su vodom i zrakom. Takav trofazni medij tretira se, u svrhu određivanja promjene naprezanja i deformacija, kao kontinuum.

Klasičnim pristupom u mehanici tla problemi promjene naprezanja i deformacija u tlu rješavaju se odvojenim promatranjem stanja hipotetskog loma i prognoze deformacija pri radnim naprezanjima. Naime, za proračun naprezanja kod sloma tla i za proračun stabilnosti primjenjuje se model kruto plastičnog materijala, dok se za procjenu deformacija pretpostavlja linearno elastično ponašanje, uz dovoljan faktor sigurnosti u odnosu na slom. Ovakav pristup redovito se primjenjuje u svim standardnim zadacima geotehničkog inženjerstva.

Sanitarno odlagalište, odnosno projektiranje, izgradnja i održavanje odlagališta, može se dakle prepoznati kao tipičan geotehnički zadatak. U tom smislu odlagalište mora imati dokazanu sigurnost za dulje vremensko razdoblje. To znači da, u skladu s usvojenim postavkama u mehanici tla, u tijelu odlagališta ne smije doći do rušenja, tj. pojave sloma u otpadnom materijalu. Osim toga deformacije temeljnog tla, brtvenih slojeva i eventualno prisutnih pomoćnih konstrukcija ne smiju prekoračiti dozvoljene vrijednosti.

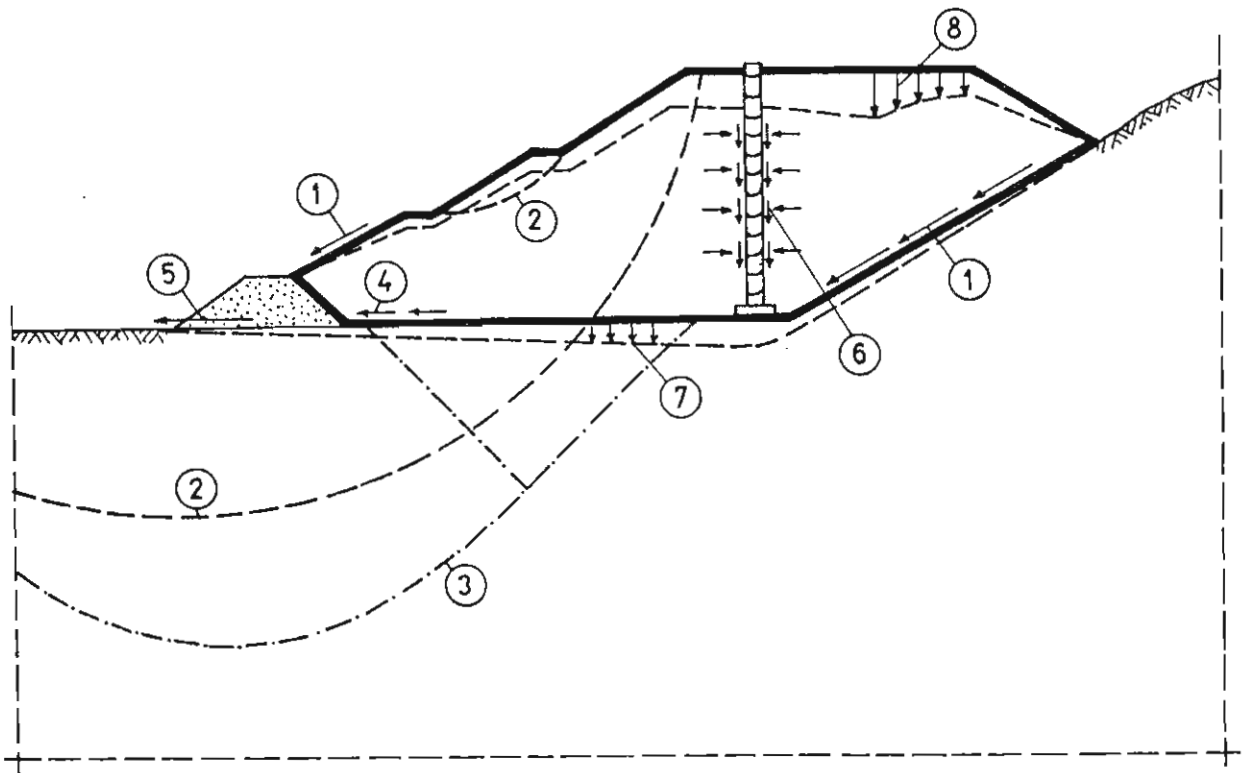
Na slici 2. nalazi se shematski prikaz mogućih problema stabilnosti i deformacija tipičnog odlagališta (Jessberger, Kockel 1993).

Od problema stabilnosti navedeno je sljedeće:

1. Otpornost na klizanje pokrovnog sloja
2. a) Lokalni slom u tijelu odlagališta po pretpostavljenoj kliznoj plohi
b) Globalni slom kroz tijelo odlagališta i kroz temeljno tlo po pretpostavljenoj kliznoj plohi
3. Nosivost temeljnog tla, uz pretpostavljen mehanizam sloma tla
4. Otpornost na klizanje između baznog sustava brtvljenja i temeljnog tla
5. Stabilnost podnožne berme
6. Stabilnost pomoćne konstrukcije

Problemi deformabilnosti su:

7. Slijeganje temeljnog tla
8. Vertikalni i horizontalni pomaci tijela odlagališta.



Slika 2. Shematski prikaz geotehničkih problema sanitarnog odlagališta (prema Jessberger , Kockel, 1993)

Figure 2. Schematic diagram of geotechnical problems in sanitary landfill (after Jessberger, Kockel, 1993)

Da bi se riješili navedeni geotehnički problemi nužno je, osim poznavanja parametara čvrstoće i deformabilnosti temeljnog tla i materijala u brtvenim slojevima, poznavanje parametara čvrstoće i deformabilnosti otpadnog materijala. No, danas se relativno malo zna o geotehničkim karakteristikama gradskog otpada. Djelomično je to zato što sve do nedavno nije niti postojala potreba za tim podacima. Značajnije su, međutim, poteškoće u dobivanju i ispitivanju pouzdanih uzoraka otpada. Problematika postaje još složenija zbog činjenice da sastav i karakteristike otpadnog materijala znatno variraju unutar samog tijela odlagališta, a osim toga i mijenjaju se tokom vremena uslijed raspadanja.

Vrijednosti parametara čvrstoće i deformabilnosti otpadnog materijala prvenstveno æe ovisiti o njegovom sastavu. U tom smislu potrebno bi bilo usvojiti sustav klasifikacije otpada, kao npr. što je to predložio Fang (1993). Sasvim pragmatičan pristup ovom problemu usvojen je u priručniku "Preporuke za geotehničko projektiranje odlagališta otpada" (1993). U njemu je usvojena sljedeća klasifikacija otpada:

- a) Otpad sličan tlu, na koji se mogu primijeniti principi mehanike tla, i
- b) Ostali otpad, na koji se ne mogu izravno primijeniti principi mehanike tla, a prognoza mehaničkog ponašanja takvog otpada može se napraviti jedino na osnovu rezultata promatranja sličnih odlagališta.

Prema tome potrebno je ustanoviti u kojoj se mjeri standardni geotehnički postupci (laboratorijski i in-situ) mogu izravno primijeniti za ispitivanje parametara čvrstoće i deformabilnosti gradskog otpada.

U nastavku ovog teksta razmatra se samo problematika mehaničke čvrstoće otpadnog materijala.

3. PARAMETRI ČVRSTOĆE OTPADNOG MATERIJALA

Primjenjujući model kruto plastičnog materijala, posmična čvrstoća tla definira se prema Mohr-Coulomb-ovom kriteriju loma. Najčešće se ovaj kriterij loma izražava preko komponenti naprezanja na plohi sloma kao

$$\tau_f = c + \sigma_n \operatorname{tg} \phi$$

gdje su

- τ_f posmična čvrstoća
- σ_n normalno naprezanje na ravnini sloma
- c kohezija
- ϕ kut čvrstoće na smicanje.

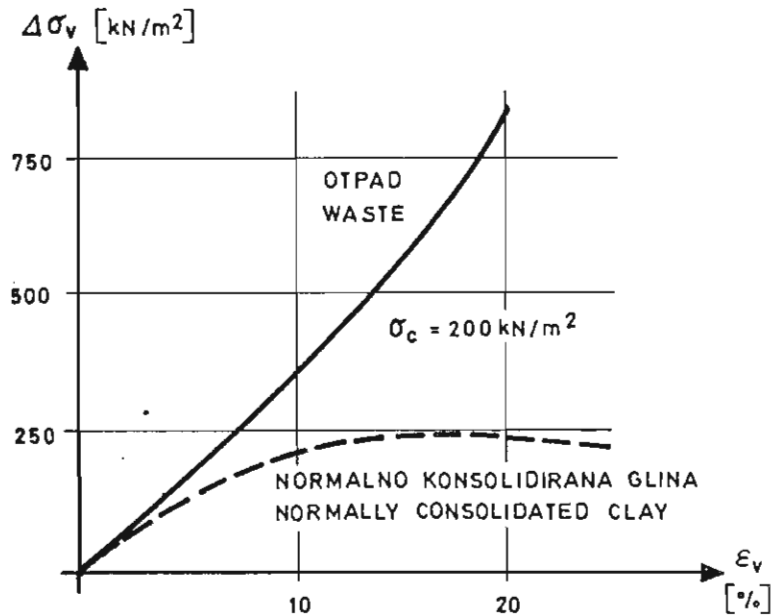
Dosadašnja ispitivanja parametara čvrstoće otpadnog materijala pokazuju da se mehaničko ponašanje otpadnog materijala može tek djelomično usporediti s ponašanjem materijala tla. Razlike u mehaničkom ponašanju tla i otpadnog materijala u sanitarnim deponijama nastaju zbog sljedećih činjenica (Sanchez-Alcitturi et al., 1993):

- otpadni materijal je znatno rahliji u odnosu na tlo, što ima za posljedicu veliku stišljivost otpadnog materijala,
- čestice otpadnog materijala su vrlo raznolike, a neke od njih i vrlo deformabilne, dok su čestice tla praktički nedeformabilne,
- u otpadnom materijalu prisutan je vremenski proces raspadanja koji uzrokuje dodatno slijeganje kao i smanjenje vrijednosti parametara čvrstoće s vremenom.

Usprkos ovim razlikama, interpretacija rezultata ispitivanja čvrstoće otpadnog materijala metodama uobičajenim u mehanici tla, ipak je prihvatljiva.

U tom svijetlu, u posljednjih nekoliko godina napravljeni su određeni pokušaji procjene čvrstoće gradskog otpada.

Jessberger i Kockel (1993) proveli su nekoliko serija ispitivanja gradskog otpada u triaksijalnom aparatu. Za razliku od standardne veličine uzorka u mehanici tla, korišteni su veliki uzorci (promjer 30cm, visina 60 cm). Iz materijala pripremljenog za ispitivanje prethodno su odstranjene komponente većih dimenzija. Provedeni su konsolidirani drenirani pokusi, a izmjereni odnos vertikalnog naprezanja $\Delta\sigma_v$ i vertikalne relativne deformacije ϵ_v prikazan je na slici 3.



Slika 3. Rezultati triaksijalnih pokusa (prema Jessberger, Kockel, 1993)

Figure 3. Results of triaxial tests (after Jessberger, Kockel, 1993)

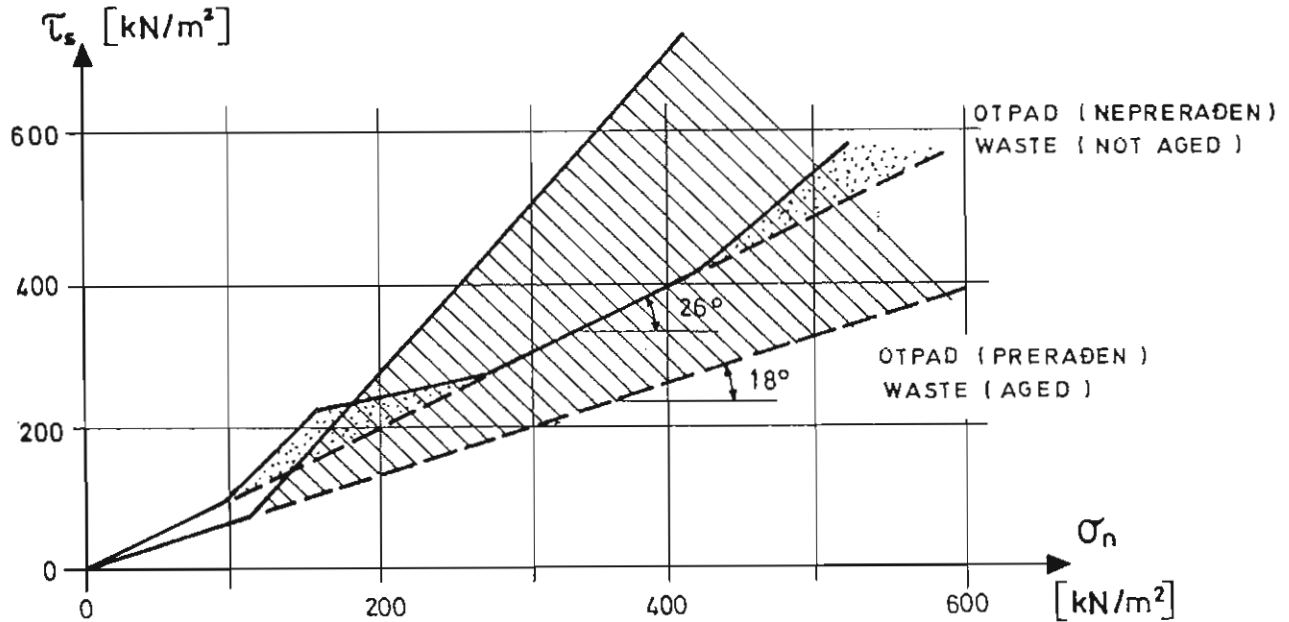
Značajke dobivenih rezultata su slijedeće:

- stanje sloma ne može se ustanoviti niti kod visokih vrijednosti vertikalnih naprezanja,
- pokus je prekinut kod veličine vertikalne relativne deformacije $\epsilon_v = 20 \%$,
- materijal pokazuje očvršćivanje s porastom deformacije.

Da bi se naglasila bitna razlika ove σ, ϵ krivulje u odnosu na uobičajeno mehaničko ponašanje tla, na istoj slici je pokazana i tipična krivulja koja se može dobiti ispitivanjem uzorka tla uz istu veličinu bočnog pritiska u ćeliji triaksijalnog aparata σ_c .

Utjecaj vlaknaste strukture gradskog otpada promatrao je Kolsch (1993) ispitivanjem uzoraka u aparatu za izravno smicanje. On je definirao stanje posmičnog loma preko veličine "kliznog" posmičnog naprezanja koje se javlja pri značajnom porastu posmične deformacije.

Odnos "kliznog" posmičnog naprezanja τ_s i normalnog naprezanja σ_n u takvom vlaknastom uzorku je diskontinuiran, tj. mehanizam sloma se tek djelomično odvija po Coulombovom zakonu (slika 4).



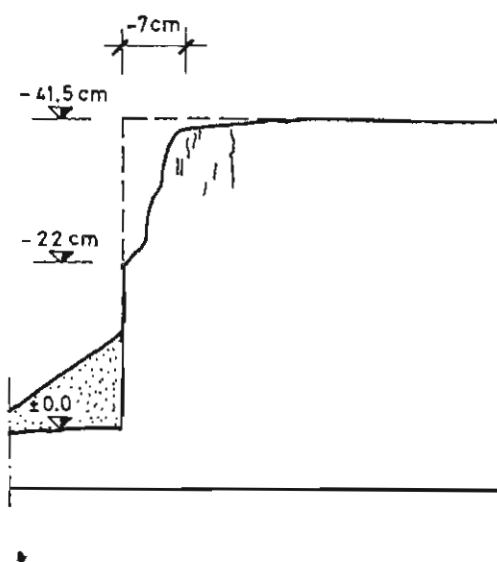
Slika 4. Pokusi izravnog posmika na uzorku otpada (prema Kolsch, 1993)

Figure 4. Direct shear tests on waste samples (after Kolsch, 1993)

Sanchez-Alciturri et al. (1993) pokazali su rezultate ispitivanja čvrstoće otpadnog materijala in-situ metodama: standardnim penetracijskim pokusom i statičkom penetracijom. Interpretacijom rezultata ispitivanja na odlagalištu Meruelo, Cantabria (sjeverna Španjolska) autori su utvrdili da ispitani otpadni materijal ima karakteristike koje odgovaraju tlu klasificiranom kao rahli pijesak, odnosno meki zaglinjeni pijesak i prah.

Određivanje vrijednosti parametara čvrstoće povratnom analizom pokazali su Cowland et al. (1991). Na jednom od odlagališta u Hong Kongu sustavom jaraka oblikovan je blok otpadnog materijala. Pomoću određenog broja repera mjereni su vertikalni i horizontalni pomaci kroz razdoblje od cca 20 dana. Premda nije došlo do zarušavanja bokova jarka, na temelju registriranih pukotina u promatranom bloku moglo se uočiti formiranje kliznog tijela u obliku klina. Računskom analizom dobivene su teoretske vrijednosti parametara čvrstoće: $c = 4$ kN/m², $\phi = 56^\circ$. Ovako visoka vrijednost za ϕ najvjerojatnije je posljedica "armiranja" otpada, tj. vlaknaste strukture otpada.

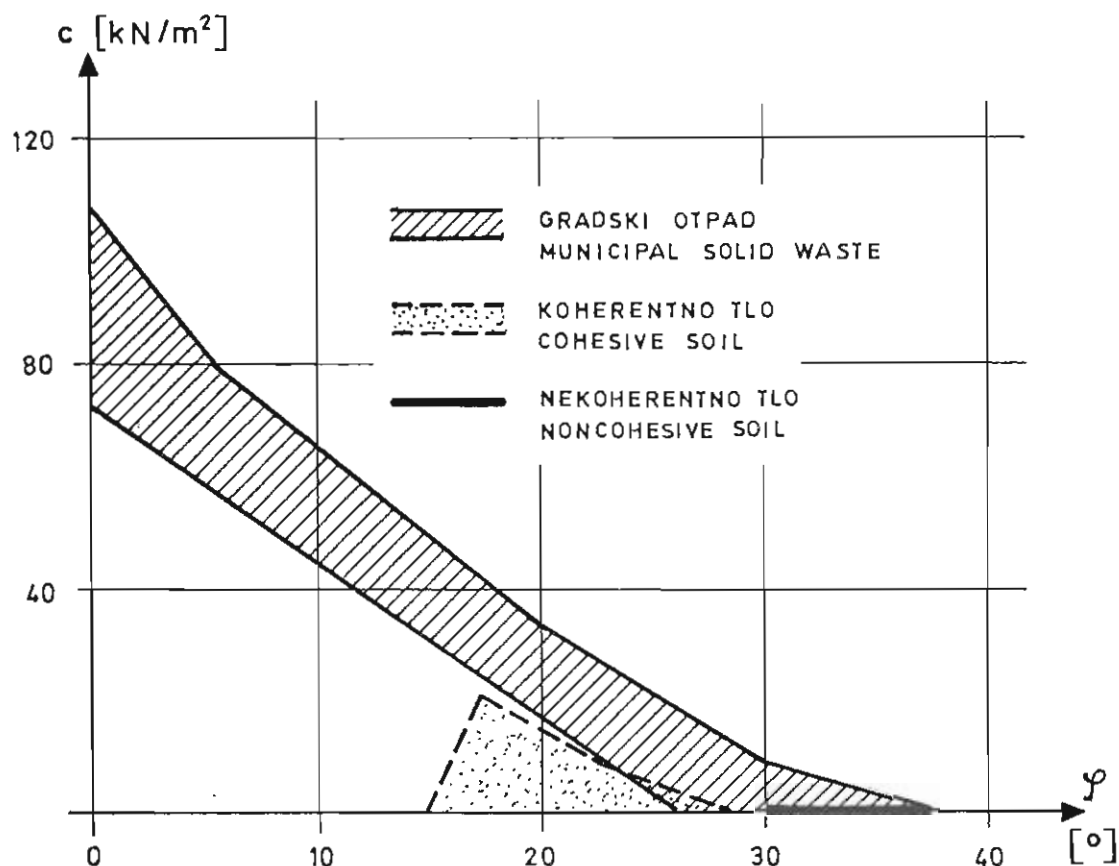
Jedan od najsuvremenijih načina određivanja mehanizma sloma otpadnog materijala kao i procjene parametara čvrstoće je primjena geotehničke centrifuge. Jessberger i Kockel (1991) pokazali su neke rezultate ispitivanja prototipa dvodimenzionalne vertikalne kosine. Tokom ispitivanja dostignuto je stanje sloma kosine uslijed vlastite težine otpadnog materijala. Iz prikaza na slici 5. jasno se vidi slom materijala u gornjem dijelu modela, odnosno vertikalne pukotine koje ukazuju na pojavu horizontalnih vlačnih naprezanja. Donji dio kosine ostao je stabilan uz pojavu samo manjih deformacija. Ovo je vjerojatno posljedica težine nadsloja koja je uzrokovala bolje međusobno uklještenje čestica otpada i povećanu čvrstoću u odnosu na gornji dio kosine.



Slika 5. Pojava sloma i pukotine u gornjem dijelu kosine (prema Jessberger, Kockel, 1993)

Figure 5. Failure mechanism and cracks in the upper slope part (after Jessberger, Kockel, 1993)

Na osnovi rezultata raznih vrsta ispitivanja, koja su provedena u SAD i Kanadi, Singh i Murphy (1990) dali su grafički prikaz računskih vrijednosti parametara čvrstoće gradskog otpada (slika 6.). Za usporedbu na istoj slici prikazane su i računске vrijednosti (prema DIN 1055) istih parametara za sve vrste tla.



Slika 6. Računske vrijednosti parametara čvrstoće gradskog otpada (prema Singh, Murphy, 1990)

Figure 6. Design values of shear parameters of municipal solid waste (after Singh, Murphy, 1990)

4. ZAKLJUČAK

U ovom prikazu napravljen je pokušaj da se, u okviru geotehničkog aspekta problematike sanitarnih odlagališta, ukaže na potrebu poznavanja mehaničkih svojstava otpadnog materijala. Prikazani su neki eksperimentalni postupci za određivanje parametara čvrstoće otpada, pri čemu su naglašene razlike u odnosu na klasične postupke u geotehničkom inženjerstvu.

U prikazu su korišteni neki objavljeni rezultati iz svjetske literature, budući da takva istraživanja kod nas još ne postoje. Međutim, obzirom na do sada znatan broj provedenih ispitivanja iz predmetnog područja, čini se sasvim opravdano uvođenje pojma "mekanika otpada" (waste mechanics) kako je to npr. navedeno u već spomenutoj publikaciji "Preporuke za geotehničko projektiranje odlagališta otpada" (1993).

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THE PHENOMENON OF FRICTION IN PRESENT – DAY
THEORY AND PRACTICE

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Summary

In this paper a development of the appearances of friction is analysed. The principal characteristics of single stages of investigation and announced theories of friction have defined. It is established that the education system, particularly the technical mechanics, behindhands with the application of new understandings of friction and the realization of new multi – scientific discipline – tribology.

1. Introduction

The intensive increase of the number of frictional techniques and faster regimes of work, which is a general trend of the development is a cause of the numerous frictional problems. In the production structures, the expenses and depreciations excessively grow and the significant influence on the environment is present.

Practical experience together with valid arguments certify, that the expenses on the basis of frictional noncontrolled growth have a substantial influence on the business effect. These expenses will increase considerably if the necessary care for the frictional appearances is not taken. The reduction of these expenses is therefore, a constant and primary task of the all productive capacities. The technical structures, always have the key role in the production activity. Therefore, they are responsible for the losses on the basis of friction too.

Although the subject of friction is a very ancient one, it remains a highly controversial one. The phenomenon of friction has been the most thoroughly investigated, since it represents on the one hand a universal attribute of matter, and thus of interested to natural scientists, whereas on the other hand it is one of the most important factors affecting the operation of mechanisms, and thus becomes the concern of engineers. This engineering aspect of friction for the theme of this paper is the most important [3].

The fact is namely, that the knowledge about friction in the predominant part of engineering is on the level of the meaning of formula: $T = fN$. Such a state is not good and needs to be changed. The condition for that is a better knowledge of friction, its' nature, and its' consequences in practice.

2. Friction – the development of cognitions

Friction is usually defined as the appearance of interacting surfaces at relative motion. Friction is physical reality whose nature is the object of constant interest. The long period of investigation of friction characterize the long stages of stagnation and the relatively short stages of the great and efficient activities. Till now the investigation of friction consists of five characteristical stages.

- 2.1. The first stage lasts the longest; more than four millennia. This is the stage of the observation and understanding of friction and its' positive and negative characteristics. Friction of wood on wood and obtainment of fire was useful for man, but the resistance by transportation of masive stone blocks for building pyramids and large stone statues, provoked great difficulties. The ingenious idea of the application of lubricant and the change of sliding friction to rolling friction

diminished these difficulties. It must be however emphasized, that the use of the lubricant and the rotary motion has caused a stand still for nearly two millennia for the discovery of the phenomenon of the nature of friction

- 2.2. The second stage is very short. It was in this period during which the versatile L. da Vinci (1508) worked on friction. His work on friction is verified today: that is the first complex understanding of the problems of friction and the first example of the connection of theory and practice i.e. the need that technical practice must be founded on the good knowledge of theory. This first contribution to the investigation of friction remained regrettably unknown and again a stage of standstill for about two centuries set in.
- 2.3. The third stage of the investigation of friction, characterizes numerous individual experimental work. New forms of driving energy are needed; the energy of men, horses, wind and water – mills, substituting the energy of new machines. The centre of interest for the technical inventions is found in West Europe, especially in France and England. In these countries we have the impressive evidence of different scientific studies of friction. The investigations of friction of Amontons (1699) and Coulomb (1781) have resulted with the known one – member formula for the calculation of friction. Their studies of friction are good examples of the high quality of the scientific work in concordance with practical daily problems. The definition of the differences between static and kinetic friction prove this too. In this stage, the acknowledgement of friction as the integral part of mechanics and the introduction of friction in teaching in the school system has been carried out. It must, however, be pointed out, that the deduction of the knowledge about friction based only on the formula $T = fN$, has meant further stagnation of the investigation of friction's nature for longer than one hundred and fifty years [2].
- 2.4. The fourth, more characteristic stage of the investigation of friction appears in the first half of the twentieth century and lasts for about four decades. The inferior quality of the one – number formula of friction has been surveyed by more researchers. They emphasize the dual nature of friction. Bowden – Moore – Tabor (1937, 1944, 1974) and Kragel'skij (1939, 1941) have contributed most to this stage of the investigations. They propose the two – number formula $T = A + fN$. It means, that the force of friction depends on two groups of factors, one which does not depend on the normal load and one which does. In this stage, which lasted only four decades, more complex work on the solution of friction's problems has been done than ever before [3].

- 2.5. The fifth stage, which is the most significant in the development of the science of friction was initiated at the beginning of the seventies and already notes impressive results. Namely, since 1966, with the known "Jost Report" (H. P. Jost) about losses on the basis of friction in industry of Great Britain (515 mil. pound/1965, 4 bil. pound/1989) began the development and activity all over world of the new multidiscipline: tribology.

The application of the tribological conception is based on this present – day definition: *"Tribology is the science and technology of interacting surfaces in relative motion and associated subjects and practices."* The basic task of tribology is that with the consistent application of tribological principles it will give the optimal solution of problems which are the consequence of frictional complex: friction, wear and lubrication. The advantage of the tribology manifest in this, that its application always means the reduction of losses on the basis of friction [10].

3. The qualification of the friction's theories

The analysis of the whole part of the investigation of friction's essence shows one common characteristic. That is the attempt to explain the mechanism of friction in the form of a general theory on the basis of a certain number of appearances, which appear under definite conditions more marked in the frictional processes. Some theories from their authors have immediately got their own names and some have been classified in typical groups on the basis of analysis. The following have been separated from numerous theories of friction as typical and the most significant: mechanical theory, molecular theory, adhesive theory, adhesive – deformed theory, deformation theory, energy theory, molecular – kinetic theory.

In the mechanical theory the standpoint of numerous researchers (L. da Vinci, Amontons, Coulomb, Euler) have been inserted which considered that friction is a mechanical appearance and its cause is the roughness of contact surfaces. The raising and the descending of one surface over another in the sliding process means that there are not the loss of energy. But, the friction is really a dissipative process and consequently the mechanical theory must be discarded as inadequate for the explanation of friction's mechanism.

The molecular theory (Desaguliers 1724) considers friction as a physical appearance which only happens due to the molecular nature of the matter. The theoretical explanation (Tomlison 1929) is based on the hypothesis of the atomic stability in the crystal grating of solids and (Derjagin 1934) on the molecular roughness of surfaces which is the consequence of a discrete matter's structure. Neither of these theoretical standpoints gives satisfactory practical

solutions [2].

The adhesive theory (Bowden, Tabor 1944, 1971) is based on the existence of the firm adhesion between the materials which are in contact due to the adhesion. The fact is, however that the adhesion is not visible when the normal force which connects the surfaces is removed. The adhesive theory, although on the most complexly worked out theories still can not have the character of general friction's theory [3].

The adhesive - deformative theory (Kragel'skij) considers that the friction appears because of the deformation of the surface layers of materials with the penetration of the surface asperities and the resistance of surface layers during the time of their destruction. This theory consequently takes into consideration the two main characteristics appearances which are always present in the friction's process: the deformation of surface layers and the adhesion which is active in the zone of contact ($T = T_d + T_a$). This theory, by the intensive research has significantly widened the understanding of friction. Since, thesis of this theory is only good for the static conditions of friction, the theory required supplementary explanations [4].

Deformation theory (Makušok 1974) states that the friction is a specific form of the load, relatively expressing of the internal friction. This theory follows the development and studies of the topography contacts, reological properties of materials, etc. in the condition of a general state of strain and deformation of the whole system. By leaving out the energy character of contact relations there is no possibility of the explanation of friction by this theory [5].

The energy theory (Kosteckij 1970) bases the explanation of friction on the dissipative character of this process. The force of friction is the consequence of spending energy in thermal, sound and elastic appearances and on the deformation and wear of layer surfaces. The problem of friction, by this theory needs to be explained by the synthesis of knowledge from adequate parts of mechanics, physics of solid bodies and the physical chemistry of surface appearances [6].

The molecular - kinetic theory is the realization of the idea of Schallamach (1903) that an essential difference exist between the friction of metals and polymers. Although Barteneo (1960) explains many experimental results, this theory can not however consider the general theory of friction for the polymers, because it cannot explain the frictional characteristics of polymers in the field of low temperatures [7].

Based on the analysis of fundamental and comparative characteristics of the existing theories of surface friction it can be concluded:

- (a) With numerous investigations of friction, different theories of friction are emerging and they continue to appear. It is evident, that there has been no recognized theory of friction.

- (b) The part conception of the creation of a general theory of friction is based on the qualifications of some of the most marked appearances in the friction process. That standpoints has neither given or can give the results, because the friction isn't a unique physical appearance. The friction's mechanism is only the explanation of the combined interaction of different processes conditioned by the characteristic relationship of frictional connections.

4. Contemporary trends of friction's investigation

Although the aspiration to the definition of general friction theory is not realized, the history of the phenomenon of friction shows that a great progress is attained on becoming acquaintance with the whole frictional complex, friction, wear and lubrication. The appearance of tribological conception and the conditions of its application trend the contemporary investigations of friction to the following problems: dependence of contact surfaces, definition of real contact surface, influence of the external load and surface roughness and state of stresses and deformations on the surface of contact.

5. The fundamental characteristics of tribology

The current period has been largely one of investigating and understanding the basic mechanisms of friction, wear and lubrication. Of course there are still many aspects that we do not understand, there are many failures and consequently enormous losses of materials, energies to industry and qualities of the environment, much of which could be saved by wise management and an enlightened approach. That makes possible the application of tribological conception, whose a principal intention is all work activities, competently and on time to ensure tolerant processes of friction and so avoid the negative consequences.

The practical application of tribology, meanwhile, understands the strict acceptance of tribological principles. This means, that in project and practice, the tribological problems must have one primary place of the validity. This fact is of fundamental importance. The application of tribological conception is not heretofore the technical question, but the acceptance of the new philosophy in relation to frictional problems. The recognition of tribology as a concept and of the economic, scientific and technological advantages that could be gained from it, lead to the rapid growth of research and development in tribology in many countries. Such research and development covered a wide spectrum and some of the results have been spectacular. The fact is meanwhile, that analysis of the present – day application of the tribology verifies its fast acceptance from the scientific structures and inadequately from the

industry. That is non logic. The explanation about the satisfactory treatment of friction in industry is very difficult to accept. The just reason is the unconfession of presence of the anarchic state and slow acceptance of the new relation to the frictional factor in present – day work conditions. That is not acceptable, because the philosophy of tribology is really very simple. Tribology is divided principally on micro and macro tribology.

The application of microtribology starts from the principle that with a small investement it is possible to realize significant savings. That is most easily achieved in two ways.

The first is, that tribology education is widely introduced into the school system.

The second is, that each work project is also obliged to have tribological study. It is unnecessary to accentuate that besides these key tasks, the application of tribological conception anticipates a permanent and manysided control of all tribological processes.

Macrotribology means a substantially higher level requirements in relation to friction and its consequences. They are primarily aimed at the owners of the industrial institutions and responsible state structures for the protection of national interests. The importance of macrotribology consists in its strategical character, because it enables the relations of very big financial savings, protection of the environment, cultural monuments and e.t.c. [8,9].

6. Conclusions

Synthesis of the achieved results in only two and half decades from the appearance and the activity of tribology, already new qualities tribology as the technology of the future. The arguments for that are the numerous reports on the national tribological conferences and world congresses of tribology. It is clear that tribology involves an interdisciplinary exercise. It demands the activities of the physicist, the chemist, the metallurgist, the material scientist and the mathematician as well as the engineer and the designer. generally it can to say “as human dependence on machines of all kinds continues to grow, and as more and more people throughout the world take advantage of modern technology, so will the value of the highest standards of tribology become more fully appreciated” [3,10].

With regard to the treatment of friction in the professional engineer's education in the future, we can finally to conclude:

1. The modern engineer must to be good able with the theoretical knowledge about the phenomenon of friction but now and with the tribological conception because the tribology is a proved science and practice which is indispensably necessary for the different losses which are the consequence of friction.
2. The conditions which must be obtained for the realization of the optimal tribological

results are:

- 2.1. Tribology must be a professional engagement of branch bussines and an indispensable part of each project and practical activity in the whole industrial structures.
- 2.2. The key contribution for the faster and more successful application of tribology has the better tribological education. Therefore the tribology must be introduced in the obligatory school system.

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INELASTIC ELEMENTS FOR INFILLED FRAME MODELLING

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SUMMARY

The proposed inelastic elements for masonry infilled reinforced concrete frames obtain dynamic analysis of infilled frame structures when they are included in a computer program for dynamic structural analysis. Elements are formulated using the results of extensive experimental investigation in the mechanism of infilled frame response to the cyclic horizontal load. Frame elements are modelled as an elastic beam element with inelastic flexural springs at both ends. Masonry infill is modelled as a pair of longitudinal inelastic springs. Mechanical properties of equivalent struts are estimated with the formulas for the effective stiffness of infilled frame and ultimate load capacity of filler wall. Properties of strut are defined in a way, that takes into account shear, flexural and axial loading, deformations of infill and interaction between infill and bounding frame.

1. INTRODUCTION

Among the various existing structural systems, the special care has to be paid to the design of mixed structural systems. They consist of materials with significantly different mechanical properties. In many countries situated in seismic regions, a commonly used structural system consists of a reinforced-concrete frame, whose inter-column spaces are, partially or completely, filled with clay-brick or concrete-block masonry walls. In the case of infilled frames, the designer has to decide whether or not to treat the filler walls as load-bearing parts of the frame. If the designer chooses the load-bearing filler walls (i.e. "integrated infill panels"), he must take into account the interactive forces that arise due to external loads, including, the horizontal loads due to earthquakes.

2. PROPERTIES OF INELASTIC ELEMENTS

The recent trends in the development of methods for the design of infilled frame structures are oriented to the experimentally supported formulation of analytical models of structural elements and their introduction in the software for the inelastic dynamic response analysis structural systems.

The herein proposed mathematical model of infilled reinforced concrete frame is composed of two types of elements: beam/column element and strut element. Their mechanical properties were derived from the observed response of tested models (Ref. [1], [2]). Inelastic elements were included in an existing and frequently used program (DRAIN-2D, Ref. [3]). Effectiveness and reliability of elements were tested using the large number of experimentally obtained responses of one-bay one-story infilled frames to the cyclic horizontal load. In the figures 4 and 5 are presented some configurations of infilled frame models with and without openings. In fig. 4 is also seen, that one may use multiple pairs of strut elements for modelling of infill itself.

2.1. Beam / column element

The same type of element can be used both for modelling of frame beam and frame column. It consists of three successively connected components: flexural springs at the ends of element and the central, elastic beam. The strength - deformation relation of springs is expressed as moment vs. curvature. This relation is defined by the spring stiffness in the every stage of structural response. It can be associated with the any set of hysteretic rules, that include experimentally or/and empirically obtained parameters.

Proposed beam/column element can be arbitrary oriented in plain. The flexural stiffness of element is associated with the flexural springs. The central, elastic part of element is axially deformable (Figure 1 (a)).

Reinforced concrete cross-section has three significant limit states: cracking (φ_{cr}, M_{cr}), yielding (φ_y, M_y) and ultimate state (φ_u, M_u). The moment vs. curvature relations can be calculated by one of the known procedures. Cracking limit can be expressed with:

$$M_{cr} = W \left[f_t + \frac{N_c - N_t}{A} \right] \quad \varphi_{cr} = \frac{M_{cr}}{EI} \quad f_t = 0,94 \sqrt{f_{cc}} \quad (1)$$

A, W and I are cross-sectional area, moment of resistance and moment of inertia, respectively, E modulus of elasticity of concrete, N_c and N_t compressive and tensile axial force, f_t flexural tensile strength of concrete and f_{cc} compressive strength of concrete.

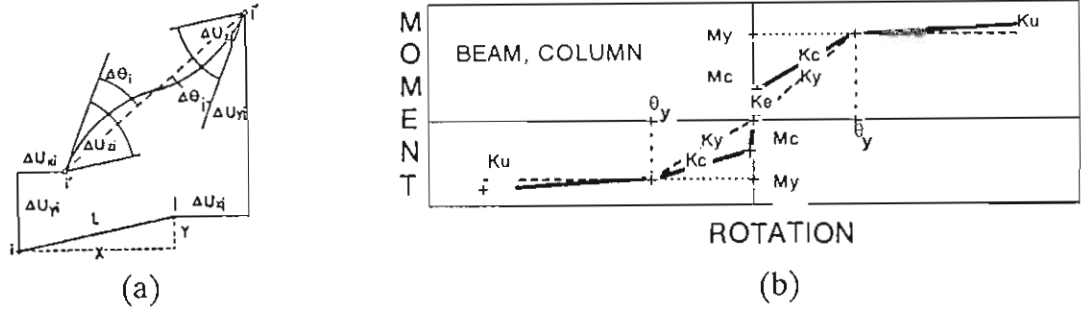


Figure 1: Joint displacements vs. deformations of liner beam relation (a) and flexural moment vs. rotation of flexural spring relation (b)

The total rotation of beam (θ) is the result of elastic rotation of beam and rotation of inelastic spring. It can be calculated from:

$$\theta = \int_0^L \phi M_v dx \quad (2)$$

where is ϕ the curvature of beam and M_v virtual flexural moments because of flexural moment equal to 1 in one of element ends.

Beams and columns can be actually modelled by one or more beam/column elements. Combining the two beam elements the actual flexural moment distribution along the beam can be simulated. At the first element, the constant distribution of flexural moment along its length is assumed. Thus the curvature (ϕ) vs. rotation (θ) relationship can be expressed as follows:

$$\theta_{cr} = \phi_{cr} \frac{L}{2} \quad \theta_y = \phi_y \frac{L}{2} \quad \theta_u = \phi_u \frac{L}{2} \quad (3)$$

At the second element the asymmetric distribution of flexural moments along its length (L) is assumed with the contraflexure point. The curvature (ϕ) vs. rotation (θ) relationship can be expressed as follows (Ref. [4]) :

$$\begin{aligned} \theta_{cr} &= \phi_{cr} \frac{L}{6} \\ \theta_y &= \frac{L}{6} \left[\alpha^2 \phi_{cr} + (1 - \alpha^3) \phi_y \right] \\ \theta_u &= \frac{L}{6} \beta^2 \phi_{cr} + \frac{L}{12} + \left[(2 + \gamma)(1 - \gamma) \left(\gamma + \frac{1 - \gamma}{p_\phi} \right) + \gamma(1 + \gamma) - 2\beta^3 \right] \frac{\phi_y}{\gamma} \\ \alpha &= \frac{M_{cr}}{M_y} \quad \beta = \frac{M_{cr}}{M_u} \quad \gamma = \frac{M_y}{M_u} \quad p_\phi = \frac{(M_u - M_y) \phi_y}{(\phi_u - \phi_y) M_y} \end{aligned} \quad (4)$$

Stiffness of envelope curve beyond the yield point is defined by coefficient (u):

$$u = \frac{(M_u - M_y)\theta_y}{(\theta_u - \theta_y)M_y} \quad (5)$$

2.2. Strut element

Strut element is modelled as a simple longitudinal inelastic spring that can be arbitrary oriented in plain. It is loaded axially. Spring is defined by axial force vs. deformation relationship (Fig 2 (a)). The strut stiffness is defined by cross-sectional area, modulus of elasticity and its length. Parameters of three-linear envelope curve are: elastic stiffness (K_e), cracking load (N_{cr}), yield load (N_y) and deformation (δ_y) and ultimate load (N_u) and deformation (δ_u):

$$\delta_{cr} = \frac{N_{cr}}{K_e} \quad K_{cr} = \frac{N_y - N_{cr}}{\delta_y - \delta_{cr}} \quad K_y = \frac{N_u - N_y}{\delta_u - \delta_y} \quad (6)$$

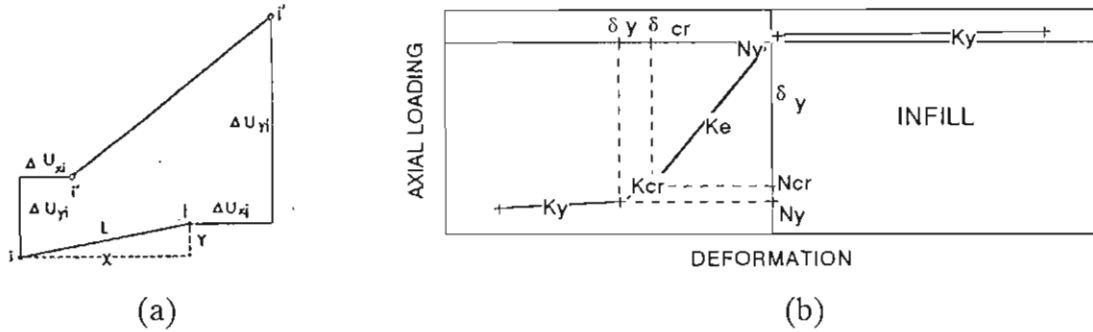


Figure 2: Joint displacements vs. deformations of strut relation (a) and axial load vs. deformation relation (b)

Definition of strut ultimate states is rather complicated. Masonry more heterogeneous than reinforced concrete. Therefore the inelastic modelling can be successful only if it is strongly supported by experimental results.

The proposed strut element represents an innovative approach to equivalent strut modelling. Until now, the element is included in DRAIN-2D program in form of new strut element. Due to this implementation, the DRAIN-2D program became more suitable for inelastic response analysis of infilled frame structures including those which are repaired and strengthened.

Elastic stiffness of equivalent strut (K_e) is calculated as flexural and shear stiffness of masonry filler panel. The frame - infill interaction is taken into account by effective cross section area (A_e) and effective moment of inertia of horizontal cross section (I_e):

$$K_e = \frac{1}{\frac{h_p^3}{3E_p I_e} + \frac{1.2h_p}{G_p A_e}}; \quad A_e = A_p + 2C_E A_f \frac{G_f}{G_p}; \quad I_e = I_p + 2C_E \frac{E_f}{E_p} \left(I_f + A_f \frac{(h_c + l_p)^2}{4} \right) \quad (7)$$

where E_f , E_p , G_f , and G_p are moduli of elasticity of frame and infill and shear moduli of frame and infill, respectively. h_c is height of column cross-section, h_p and l_p are height and length of infill, respectively. A_f and I_f are cross-sectional area and moment of inertia of column cross-section, respectively. A_p and I_p are cross-sectional area and moment of inertia of infill cross-section, respectively. C_E is coefficient of frame-infill interaction. It can be assumed the full interaction ($C_E = 1$), interaction with one column ($C_E = 0.5$) or separation between the infill and frame ($C_E = 0$). In the case of infills with openings, the interaction can be achieved only on the side of external loading action, thus C_E can be either 0.5 or 0.

Compressive strength of strut is defined as the shear strength of infill (H_{sp}). Interactive forces between the frame and infill are taken into account by the coefficient C_I .

$$H_{sp} = C_R \frac{A_p f_{tp}}{C_I b} \left[1 + \sqrt{C_I^2 \left(1 + \frac{\sigma_{oz}}{f_{tp}} \right) + 1} \right] \quad C_I = 2\alpha b \frac{l_p}{h_p} \quad (8)$$

A_p is the infill cross-section area, l_p and h_p are length and height of infill, respectively. f_{tp} is referential tensile strength of infill and σ_{oz} the average normal stress in the middle of infill due to vertical load and interactive forces. The local damages or other faults of infill can be taken into account by the reduction coefficient C_R . Coefficient b is the coefficient of the shape of shear strength distribution along the horizontal cross-section of infill. In the case of infill without opening it is assumed by value of 1.1. In the case of infill with opening, coefficient b of windward part is assumed to be 1.1 and of the opposite part 1.5. Coefficient α is a function of assumed distribution of interactive forces (Ref. [4]).

Inelastic dynamic response of structural elements is function of loading history. Therefore, the proper selection of hysteresis rules, that simulate the time relation between the loading and deformation is essential for the successful modelling. In proposed model, one of the most successful rules was used. They are expressed by hysteresis, that was proposed by the authors of IDARC program (Ref. [5]). It is the three parametric model named DDNSTH (**D**egrading, **D**eteriorating, **N**onsymmetric, **S**lip-type & **T**rilinear **H**ysteretic model).

The model has a trilinear, nonsymmetrical envelope curve, that is defined by nine independent parameters. Three additional parameters are used for the definition of the shape of hysteresis loops. They make the rule appropriate for various structural elements. The values of parameters determine the properties of stiffness degradation, strength deterioration and pinching behaviour.

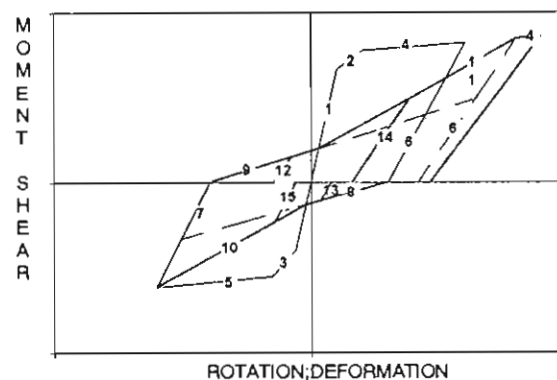


Figure 3: Hysteretic rules DDNSTH

Due to universality of DDNSTH rules, they were used in the formulation of both proposed elements. Their reliability is illustrated by the comparison of experimentally obtained and calculated responses of infilled frames.

3. VALIDATION AND VERIFICATION OF MODELS

Proposed models were validated and verified by comparison of calculated and measured response of 34 tested specimens. They were loaded with constant vertical and cyclic horizontal load up to failure. Additionally, some specimens were analytically loaded by earthquake (Fig. 6) to test all hysteresis rules, that were used in modelling of inelastic response of applied types of springs. Comparing the experimentally obtained results with the analytically obtained ones, it became clear, that proposed model is very successful on the level of global (story drift vs. shear) response of structural element (Fig. 4(a) and Fig 5). On the level of local response (strain vs. shear), the model of equivalent strut can not provide the same accuracy as the plane finite element models. This disadvantage can be reduced by using of additional pairs of equivalent struts (Fig 4 (b) and (c)). But in all cases the calculated strains in frame and infill are higher than natural ones. It means, that mathematical model gives more conservative solutions, what gives a certain additional reserve in the load carrying capacity in the inelastic range of structural response.

Modelling of masonry infilled r.c. frames on the proposed way, seem to be promising. The further research will show, how the calculated responses of two-bay two-story infilled frames match with the experimentally obtained ones. According to comparison

of test results, presented by other researchers and calculated responses obtained by our models, they seem to be enough successful and reliable. In future, the analytical models will be used in development of damage model for infilled frames that will be supported with results of further experimental investigations.

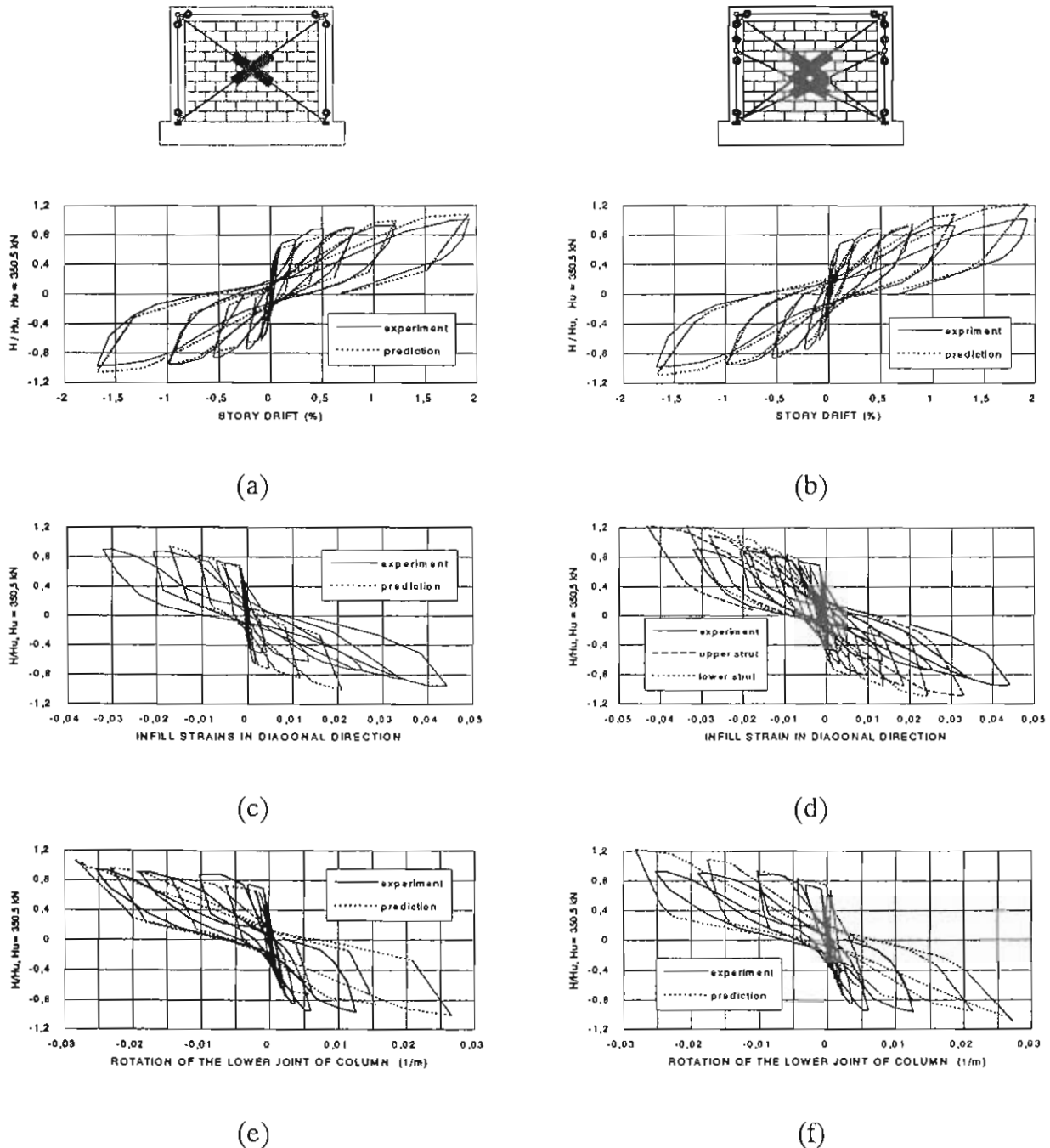


Figure 4: Comparisons of experimentally obtained and predicted responses of infilled frames: global response of the specimen modelled with one pair of struts (a) and with two pairs of struts (b), response of infill modelled with one pair of struts (c) and with two pairs of struts (d) and response of the lower joint of the frame column in the case of modelling of infill with the one pair of struts (e) and with the two pairs of struts (f).

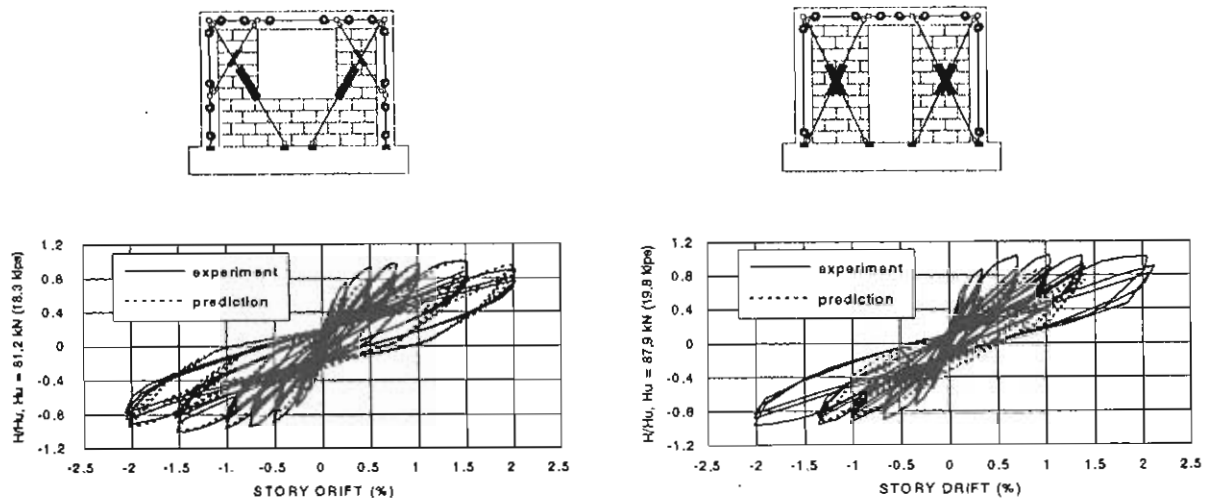


Figure 5: Comparisons of experimentally obtained and predicted global responses of infilled frames with window opening (a) and door opening (b).

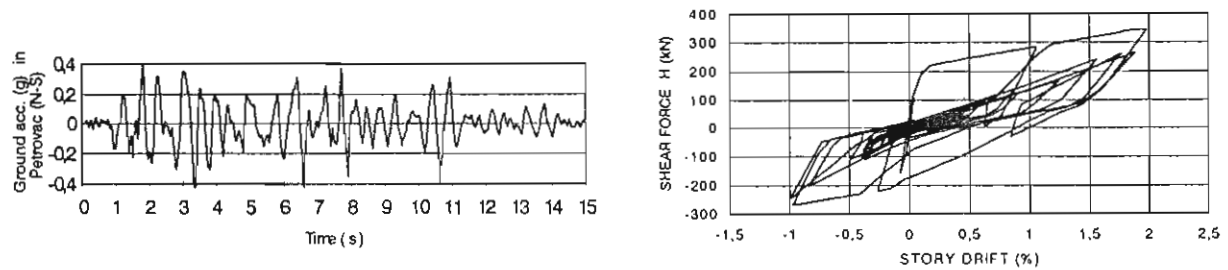


Figure 6: Calculated earthquake response of infilled frame (one pair of struts model)

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Calculation of Primary Support by Boundary Element Method

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Summary

The computation of the stresses and strains around the openings of underground rooms can be carried out by Boundary Element Method. As the basis, the Hoek-Brown's programme for the elastic continuum for plane strain was accepted. The basic loading of the massif are the forces of the own weight or horizontal and vertical pressures. The Hoek-Brown's program was extended by the effects of primary support on the stresses and strains around the opening, so that it accepts the loading with reactive forces of the support (shotcrete layers, steel arches) and with active and reactive forces of bolts. Analogously to the treatment of reinforced concrete, the ultimate bearing capacity of the concrete and steel are considered. The control of failure according to the criterion defined by Hoek and Brown is in-built in the programme. The program is extended by the pre-processor for interactive defining of the input data, as well as by post-processor for graphic presentation of results. The program enables a simple inspection into the problem of stress concentration around the openings and into the influence of different support systems.

1. Introduction

In the numerical analysis of strain and stresses around underground opening the Finite Difference Method (FDM) and Finite Element Method (FEM) are mostly used. Boundary Element Method (BEM) is a relatively new numerical method to solve the distribution of stresses and strains in plain and three-dimensional problems. The method is also applicable to nonlinear problems of plastification and in fracture mechanics. This method is also applicable to the other materials with nonlinear properties.

The basis of BEM is the solution for infinite elastic continuum under a uniform force on the finite area, i.e. with locally concentrated force (Kelvin solution). The solution of the problem is based on fulfilment of boundary conditions on the contour Γ of the area Ω (Brebbia^[1, 2]).

Boundary conditions stated by displacements μ and stresses p for the elements of the contour k and l are:

$$\mu_k = \bar{\mu}_k \quad k \in \Gamma_1 \quad (1)$$

$$p_l = \sigma_{lj} n_j = \bar{p}_l \quad l \in \Gamma_2 \quad (2)$$

$$\Gamma_1 \cup \Gamma_2 = \Gamma \quad (3)$$

The basic equation in tensor form for the application of BEM to elastic continuum is:

$$\int_{\Omega} (\sigma_{ij,j} + b_i) \cdot \mu_i d\Omega = \int_{\Gamma_1} (\bar{\mu}_k - \mu_k) \cdot p_k d\Gamma_1 - \int_{\Gamma_2} (p_l - \bar{p}_l) \cdot \mu_l d\Gamma_2 \quad (4)$$

where:

$\sigma_{ij,j}$ - derivation of stresses on coordinate j

b_i - volumetric force in i direction

Displacements, boundary loads, stresses and volumetric forces are expressed in matrix form. Unknown values of displacements and stresses can be determined by solving linear equations systems. Opposite to the FEM, the matrix of the BEM is completely filled. Diagonal elements are always positive and dominant, assuring the convergence of solution. The programme uses model with constant displacements and stresses on the length of each boundary element.

Rock massif is presumed as homogenous and elastic continuum. Primary state of stresses σ_{ij} caused by the gravity and horizontal pressure of the massif is considered as known for each point of the continuum.

By opening an underground room, primary state of stresses in rock changes to secondary state of stresses. On the boundary of the opening there appears circular stresses concentration σ_c , while shear τ_{rc} and radial stresses σ_r disappear.

For the exact solution of the condition around underground opening mechanical rock properties should be known very well. The real rock has plastic, anisotropic and viscous properties, but we should also be aware of the fact that it is by its system of joints, discontinuum rather than continuum. For the correct modelling about ten mechanical properties of the rock should be known, which can be found after opening of the particular location. The most acceptable solution and first approximation is solution of the theory of elasticity and Boundary Element Method. This kind of solution represents the state of compatibility of deformations and stresses.

In their programme Hoek and Brown^[3] use comparison of stress according to the theory of elasticity with critical stress of rock by failure criteria which is defined in the following way:

$$\sigma_1 = \sigma_3 + \sqrt{m \sigma_c \sigma_3 + s \sigma_c^2} \quad (5)$$

where:

σ_1, σ_3 - major and minor principal compressive stress

σ_c - uniaxial compressive strength of the rock

m, s - constants of the massif

For real conditions depending on massif jointing, values of the constants are within the following limits:

$$0,1 < m < 20 \quad (6)$$

$$0,001 < s < 1,0 \quad (7)$$

Checking of failure criteria makes it possible to see the areas where tensile failure and sliding are possible.

2. Computation of Primary Support

According to the New Austrian Tunnelling Method (NATM) it is usual to open the entire profile of the underground room. By means of primary support further displacement of the massif has to be stopped. Supporting is realized by shotcrete layer or by shotcrete layer reinforced with wire mesh, i.e. steel arches and/or bolts. In this way the rock massif could be stabilised, also in the cases when the rock has plastic and viscous properties. If support is effected immediately after opening, plastic and viscous deformations are marginal. Final lining of the underground room does not have almost none static function.

A computer programme for more complex calculation of stresses around underground opening supported by primary support, has been developed.

As a basis we used Hoek-Brown's programme for elastic continuum and plain strain state on the principles of BEM.

Calculations of the influences of different types of support systems on the state of stresses and strains around underground opening have been added to the main calculation. Different types of support systems, shotcrete and reinforced shotcrete, steel arches, passive and active bolts can be included in calculation.

The pressure on support depends on deformations of support system and rock. Rock deformations are increasing until stresses in the rock and reactive stress are in equilibrium and deformations on the contact of rock and support are equal. Deformations of support reach the limit of elastic behaviour of material extending to the ultimate state. System of rock and support is a statically undefined system.

The problem becomes more complicated because concrete in the support begins to function during setting and hardening, so it is not possible to define the real interaction between rock and support.

As a basis for support calculation ultimate state of strains and stresses in support has been accepted in the same way as in the calculation of reinforced concrete constructions and steel constructions.

The support with the shotcrete layer (thickness $d[m]$) is placed along the curved boundary of the profile (fig. 1). Carrying capacity of this support is:

$$S_N = \sigma_c \cdot \frac{d}{\gamma} \quad [kN/m] \quad (8)$$

where:

σ_c - compressive strength of shotcrete

$[kN/m^2]$

γ - safety coefficient ($\sim 1,3$)

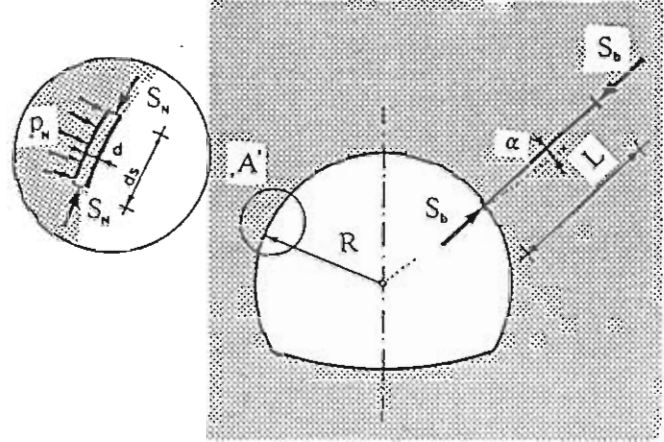


Fig. 1 Primary support systems (shotcrete layer and bolt)

The support with the layer of shotcrete causes only normal stress of the opening boundary. From the equilibrium condition of the arch element in the length of ds , radius R , rock pressure p_N for the normal boundary force S_N in the arch of the support is:

$$p_N = \frac{S_N}{R} \quad (9)$$

Limit bearing capacity of the shotcrete reinforced by wire mesh of section $A [m^2/m]$ (for one meter of the tunnel) is:

$$S_N = \frac{\sigma_c \cdot d + \sigma_y \cdot A}{\gamma} \quad (10)$$

σ_y - yield stress of steel $[kN/m^2]$

In the same way we may define the bearing capacity of steel arches support with the cross-section $A_a [m^2]$:

$$S_a = \frac{A_a \cdot \frac{\sigma_y}{e}}{\gamma} \quad (11)$$

where:

e - distance between arches

Active and passive bolts are considered as a primary support also. Bolts action is defined by two equal opposite forces, one on the boundary of the opening (intrados) and the other in the

anchoring area. Bearing capacity of the anchor rod S_b is the ultimate force:

$$S_b = A_b \cdot \frac{\sigma_y}{\gamma} \quad (12)$$

where:

A_b - area of the bolt cross-section [m^2]

The programme has been extended by preprocessor for interactive data input and by postprocessor for graphical display of results. For simpler application of this programme there is menu containing the following choices:

- data input and geometry display of opening
- calculation of strain and stresses
- writing of numerical results to the file
- graphical display of results

3. Example

As an example of the calculation there is a profile of road tunnel opening on the fig. 2. The contour of the opening contains three circle segments. For each segment, the coordinates of the centre, starting and final angles of the segment and the number of boundary elements have to be defined. The segments are divided into boundary elements of the same length. The output of results, strains and stresses are those in the middle of the boundary elements. Using this data the strains and stresses in the arbitrary points in the massif can be obtained.

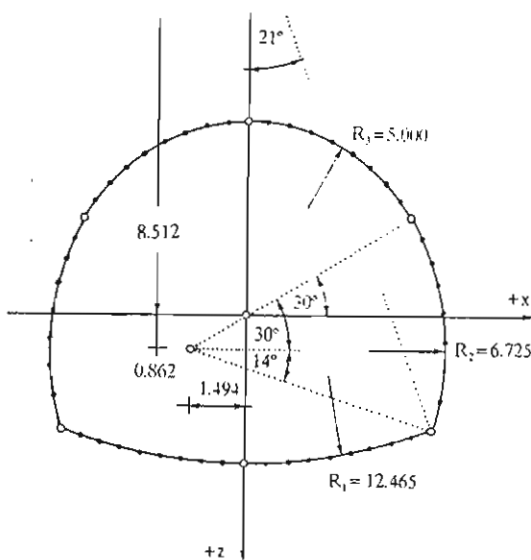


Fig. 2 Profile of road tunnel opening

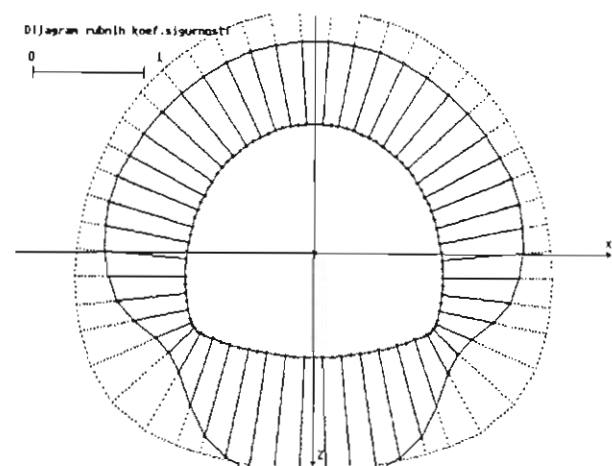


Fig. 3 Diagram of safety coefficient along the opening boundary
- without primary support

The following parameters and values are used to describe this problem:

- modulus of elasticity
- Poisson's ratio
- rock density
- uniaxial strength
- coefficient
- coefficient
- horizontal/vertical stress ratio
- depth

$$E = 3262 \text{ MN/m}^2$$

$$\nu = 0.338$$

$$\rho = 2.71 \text{ t/m}^3$$

$$\sigma_c = 50000 \text{ kN/m}^2$$

$$m = 0.575$$

$$s = 0.003$$

$$K = 0.8$$

$$h = 75 \text{ m}$$

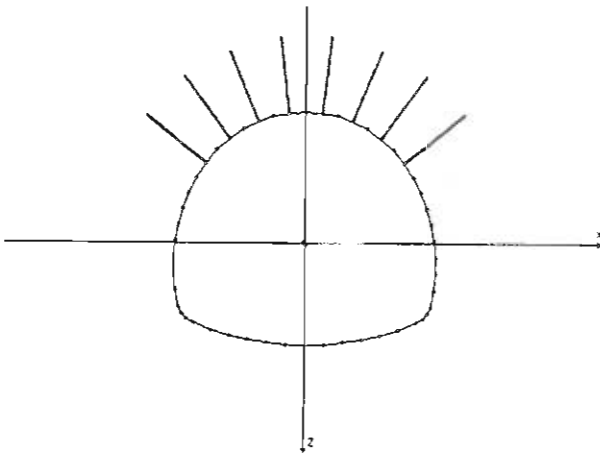


Fig. 4 Primary support by bolts on the roof

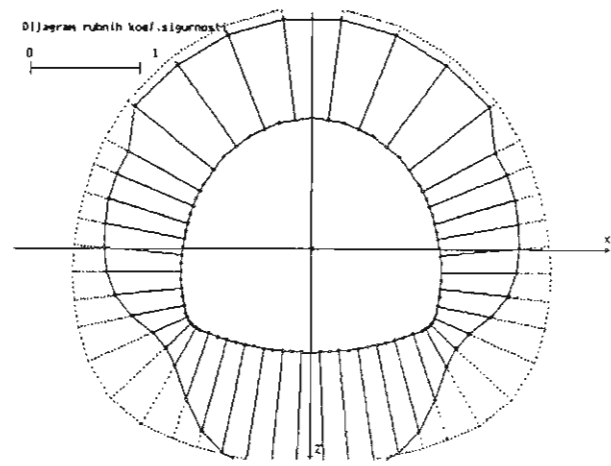


Fig. 5 Diagram of safety coefficients along the opening boundary
- support by bolts on the roof

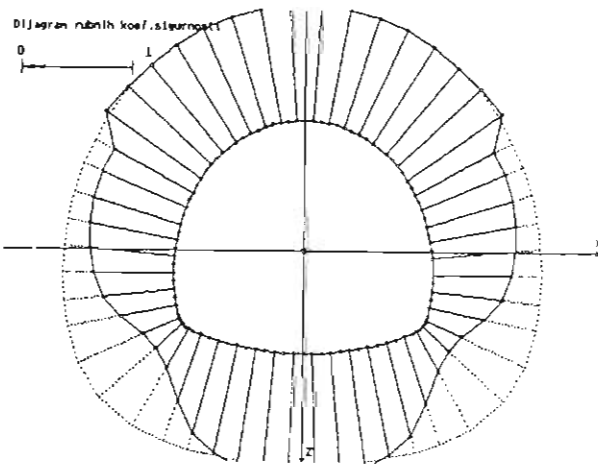


Fig. 6 Diagram of safety coefficients along the opening boundary
- shotcrete layer only on the roof

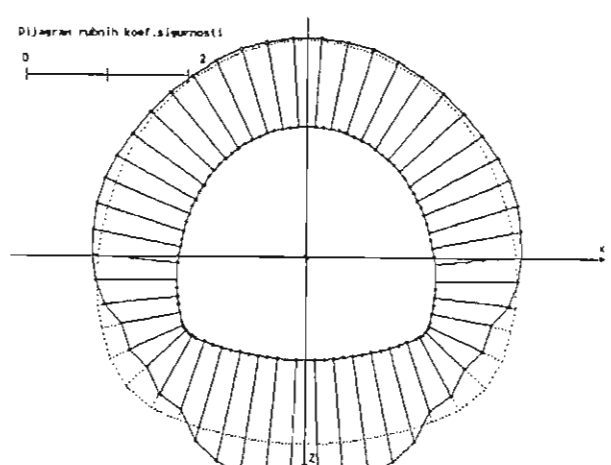


Fig. 7 Diagram of safety coefficients along the opening boundary
- support by shotcrete layer

The fig. 3 shows ratio between calculated and critical stresses along the edge of the opening by continuous line. Dotted line marks ratio of 1 of these stresses, i.e. something like "safety coefficient". Obviously the opening without support does not fulfil the conditions of Hoek-Brown's failure criteria.

The opening stability could be reached by primary support. The influence of primary support by single bolts in the opening (fig. 4) on the "safety coefficient" is shown in the figure 5. The supporting effect by shotcrete only on the roof is displayed in figure 6. In this case the opening stability can be reached by supporting the entire opening boundary by a shotcrete layer, that is visible in the fig. 7.

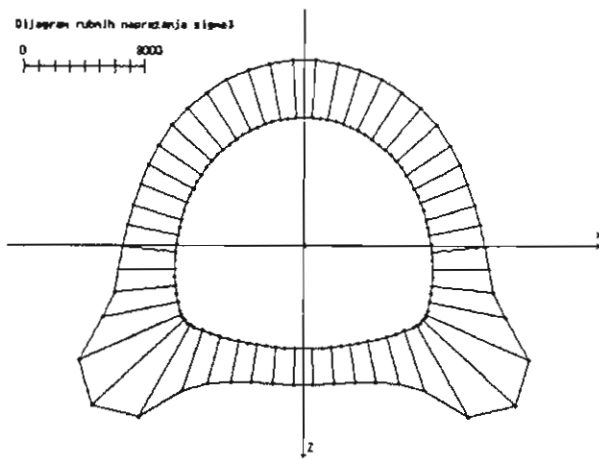


Fig. 8 Diagram of circular stresses along the opening boundary

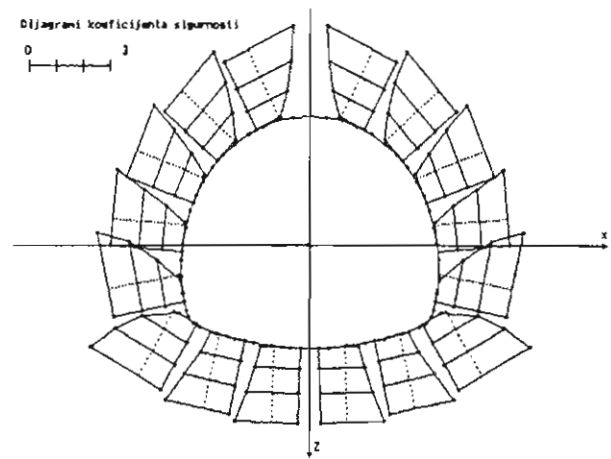


Fig. 9 Diagram of safety coefficients along radial lines

The diagram of boundary stresses (fig. 8) shows stress concentrations in the corner, that could cause a certain plastification.

Diagrams of "safety coefficients" along radial lines (fig. 9) show that the plastification zone is only on the very edge of the opening.

4. Conclusion

This programme enables a simple access into the problem of stress concentration around the openings and into the influence of different support systems. Because of the speed of calculation by the application of BEM and the simplicity of input data changes it is possible

to analyze a number of different cases of loading in a very short time as well as to observe the influence of input parameters of the massif.

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Applications of Iterative Methods in Elastostatics by the Boundary Element Method

Zoran Žunič, Djordje Žebeljan

In the present paper, application of the iterative methods in elastostatics using the Boundary Element Method is examined. Influence of the element type and number of subdomains on the solver efficiency was evaluated. Contribution of the subdomain technique as the tool for reduction of the computational time has been observed. Study is performed on the pipe loaded with the internal pressure.

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1.0 Introduction

In the elastostatic structures made of several materials as well long and thin domains often have to be evaluated. In these cases subdomain technique is used in Boundary Element Method (BEM). In the case of different materials for each material another subdomain is needed. When the long and thin domains are to be modelled such as different sorts of beams, thin pipes (fig. 3). etc., domain should be divided into several equally sized subdomains. The consequence of the sub-domain technique is only partially filled matrix of the of system of linear equations (SLE).

Usually for solving SLE in BEM elastostatics Gaussian elimination method is used. Weakness of the Gaussian method is that the number of operations needed for solving the system grows with the third power of the number of unknowns.

As the alternative to the Gauss elimination method where the number of operations is exactly determinated, iterative methods have been proposed, where the solution is reached by iterating the approximate values till the desirable accuracy of the final results is achieved. Classical iterative methods such as Jacobi and Gauss-Seidl seem to be unefficient in solving the SLE appearing in the elastostatics problems solving by BEM. Therefore improved conjugate gradient methods have been developed.

Main goal of our research was evaluation of the subdomain technique influence on the iterative solver efficiency. Beside observing the computational time needed to solve SLE as a function of subdomains (structure of the SLE matrix) and iterative method used, contribution of the subdomain technique in reducing the memory request for saving the SLE matrix was observed as well.

2.0 Boundary Element Method

Navier-Lame differential equation should be solved

$$u_{i,j} + \left(\frac{1}{1-2\nu} \right) u_{j,i} + \frac{f_i}{G} = 0 \quad (1)$$

on the domain considering the boundary conditions presented on the boundary; displacements u and tractions t

$$\begin{aligned} u &= u_0 & \text{na } \Gamma_1 \\ t &= t_0 & \text{na } \Gamma_2 \end{aligned} \quad (2)$$

Distribution of the parameters (u , t) in the domain is determined by evaluating the values on the domain boundary. Using the method of the weighted residuals differential equation eq. (1) should be transformed into integral equation. Neglecting the body forces the equation is

$$c_{ij}(\xi)u_j(\xi) + \int_{\Gamma} T_{ij}^*(\xi, S)u_j(S)d\Gamma = \int_{\Gamma} U_{ij}^*(\xi, S)t_j(S)d\Gamma \quad (3)$$

where the fundamental solution is

$$U_{ij}^*(\xi, S) = \frac{1}{8\pi G(1-\nu)} \left((3-4\nu) \ln \left(\frac{1}{r(\xi, S)} \right) \delta_{ij} + \frac{\partial r(\xi, S)}{\partial x_i} \frac{\partial r(\xi, S)}{\partial x_j} \right)$$

Integral equation is discretized, which means that the domain boundary should be divided into N_e boundary elements, unknowns u_i and t_i are approximate using polynomial approximation along the element. Discretized eq. (3) is

$$[T]\{u\} = [U]\{t\} \quad (4)$$

Taking into account the boundary conditions eq. (4) should be written

$$[A]\{x\} = \{b\} \quad (5)$$

which represents SLE which has to be solved. Matrix A is in general full, non-symmetric and it is not positive definite.

3.0 Subdomain Technique

Using subdomain technique each domain is treated separately, afterwards they are put together concerning contact conditions which are (fig. 1):

- continuity of displacements on the contact boundaries (i and j)
- stress equilibrium on the contact boundaries (i and j)



Figure 1: Example of dividing the domain into two subdomains

Prescribing these additional contact conditions the number of unknowns is equal to the number of equations and SLE has got such form as follows

$$\begin{bmatrix} [T]^{(1)} & 0 \\ 0 & [T]^{(2)} \end{bmatrix} \begin{Bmatrix} \{u\}^{(1)} \\ \{u\}^{(2)} \end{Bmatrix} = \begin{bmatrix} [U]^{(1)} & 0 \\ 0 & [U]^{(2)} \end{bmatrix} \begin{Bmatrix} \{t\}^{(1)} \\ \{t\}^{(2)} \end{Bmatrix}$$

Indices (1) and (2) are related to the first and second sub-domain (fig. 1). It is obvious that the matrixes are not full. Taking contact and boundary conditions into account SLE for five subdomains has the shape (fig. 2) where black fields are nonzero elements.

$$\begin{aligned} \{u\}_i &= \{u\}_j \\ \{t\}_i &= -\{t\}_j \end{aligned}$$

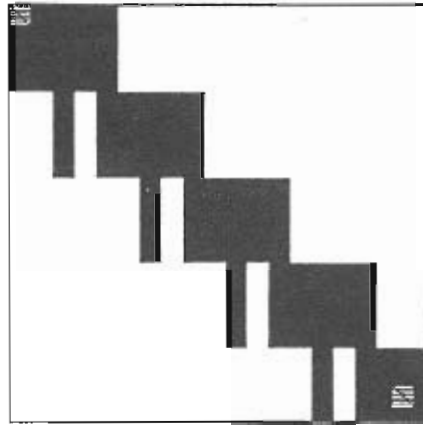


Figure 2: Example of SLE with five subdomains

4.0 Iterative Solution of System of Linear Equations

Following system should be solved

$$Ax = b \quad (6)$$

Solution of eq. (6) has been transferred into the problem of searching the minimum of the functional $F(x)$ [5] which is defined as

$$F(x) = \frac{1}{2}(x, Ax) - (b, x)$$

Minimum of the function $F(x)$ is given with the condition:

$$F'(x) = \text{grad}F(x) = Ax - b = 0$$

from which follows

$$Ax^* = b \rightarrow F'(x^*) = 0$$

In the above equation x^* represents solution of eq (6). To reach the desirable solution iterative procedure is used

$$x_{(k+1)} = x_k + \lambda_k p_k$$

where p_k is a search direction vector and λ_k is chosen in such a way that $x_{(k+1)}$ minimizes the residual

$$r = b - Ax \quad (7)$$

More detailed information about different variants of conjugate gradients methods could be found in references [1], [2], [3].

The stopping criterion is mostly defined as

$$\varepsilon = \sqrt{\frac{(r^T, r)}{(b^T, b)}} \quad (8)$$

where r is vector of residuals eq. (7) and b is the vector of the right sides

Convergence speed is strongly dependent from the magnitude and distribution of the eigenvalues of the matrix A /6/. Most convenient for the iterative methods should be that all eigenvalues will be more or less equal to one. To approach to the desirable eigenvalues both sides of eq (6) are multiplied with the so called preconditioned matrix Q .

$$Q^{-1}Ax = Q^{-1}b$$

General demand for a good preconditioner should include easy inversion, good approximation of A^{-1} and sparsity, if the system matrix possesses some kind of sparsity. In the present paper two different types of iterative solvers in combination with two preconditioners were used:

- Sonneveld CGS method /1/
- Van der Vorst bi-CGSTAB method /7/

where both methods were developed for solving nonsymmetrical problems. Van der Vorst bi-CGSTAB method is only the version of the CGS method with more uniform convergence.

Used preconditioner were:

- Jacobi type preconditioning (J) where Q is a diagonal matrix where only diagonal of A was used $q_{ii} = a_{ii}$
- incomplete factorisation preconditioning (Gauss elimination) (ILU) /5/, /8/

5.0. Index Notation of the Matrix

In the index notation the matrix is not described as two dimensional field anymore but as one real and three integer vectors /6/, /9/:

- **coA**: real vector sized NEQ, containing all nonzero elements of the matrix A ,
- **jcoA**: integer vector sized NEQ, containing column number which is related to each element of the vector **coA**,
- **jbegA**: integer vector sized NEQ + 1, containing position of the row pole elements of the matrix A in the vector **coA**,
- **jdiaA**: integer vector sized NEQ, containing position of the matrix A diagonal elements in the vector **coA**.

NEQ - number of unknowns of SLE; **NEL** - number of all from zero different elements of matrix A .

6.0 Test Example

Thin wall pipe loaded with the internal pressure (fig 3) was evaluated. Length of the pipe in comparison with the other dimensions (thickness, radius) is large, therefore plane state of strain was assumed. Concerning the geometry and boundary condition symmetry, only one fourth of the pipe was discretized using symmetry boundary condition (fixation of the circumferential displacements on the both symmetry axis (fig 3)).

Influence of the subdomains number on the efficiency of the iterative solvers were observed. They have been equally spaced around circumference ($\pi/(2n_{\text{sub}})$). Calculations have been done using linear and quadratic isoparametric elements. All test examples have the same mesh density (264 nodes). On the corners and on the contacts between the sub-domains and the boundary (different boundary conditions) semi-continuous elements of the same types have been used.

As the stopping criterion eq. (8) tolerance of $\epsilon = 10^{-8}$ was used for the Jacobi preconditioning. To reach the same accuracy of the result tolerance had to be put on $\epsilon = 10^{-12}$ for ILU preconditioning.

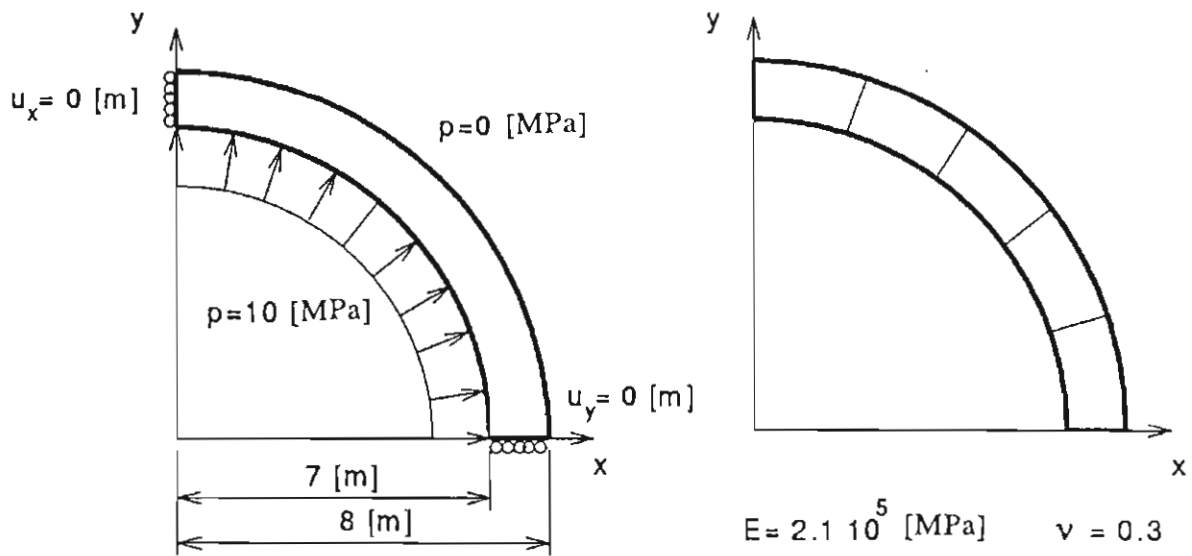


Figure 3: Sketch of the test example and example of division on sub-domains

All calculations have been performed on the Hawlett Packard 9000/710 work station. For the operations with the floating point double precision was used. Index notation was used for description of the system matrix.

7.0 Results

Comparison between the analytical and numerical solution is presented on (fig. 4).

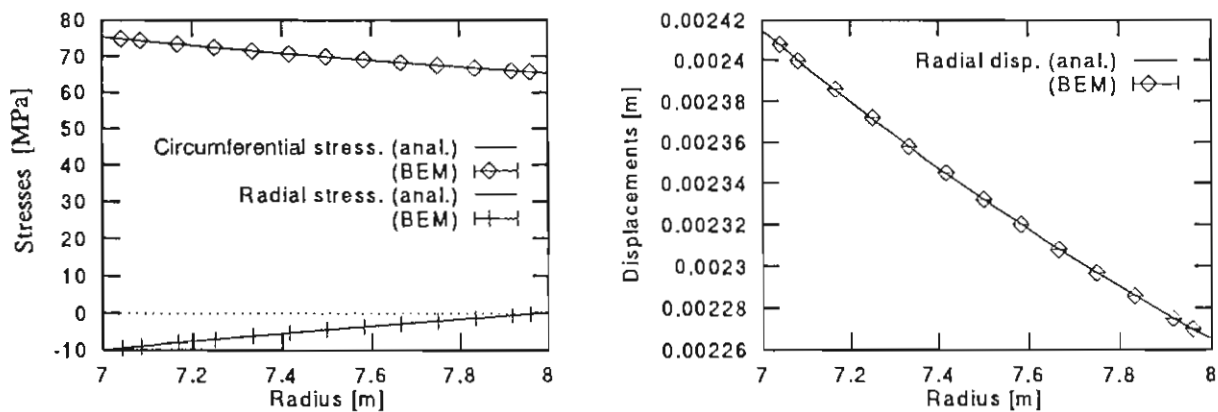


Figure 4: Comparison of the analytical and numerical results

Number of iterations as a function of preconditioner, element type as well the number of subdomains can be seen on (fig. 5). It is obvious that the total number of iterations is mainly influenced through the selection of the preconditioning matrix.

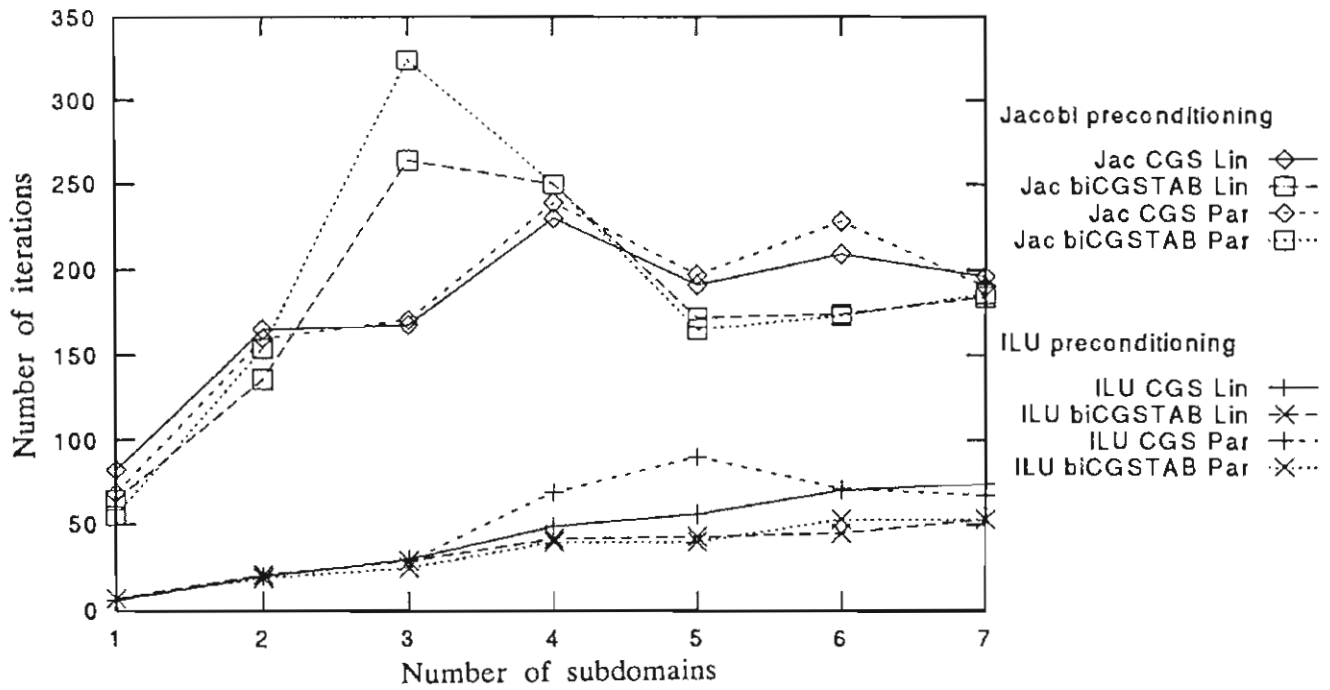


Figure 5: Iterations number as a function of subdomains

Diagram from (fig. 5) has been translated in CPU time used for particular problem as the function of subdomains on (fig. 6).

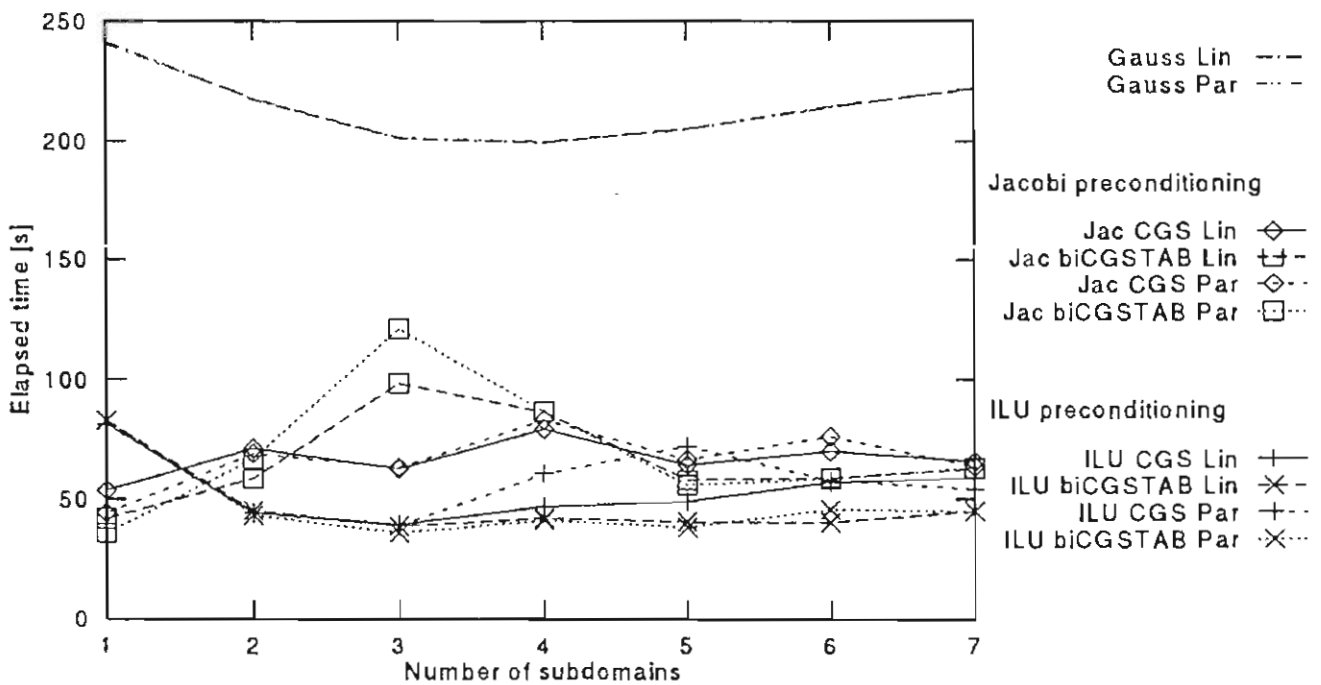
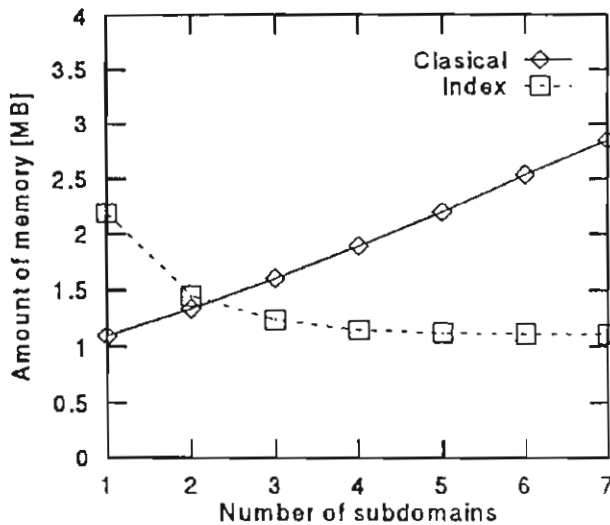


Figure 6: CPU time as a function of sub-domains

Influence of the index notation of the sister matrix on the memory needed for solving the problem in comparison with the classical notation as a function of subdomains number is displayed on (fig. 7).



Število podobmočij	Število enačb
1	536
2	592
3	648
4	704
5	760
6	816
7	864

Figure 7: Memory used as a function of subdomains

From the (fig. 7) it is obvious, that with the size of the system matrix which is connected with the number of subdomains, the memory amount is linearly increased in the classical notation and exponentially decreased in the index notation. Mesh density on the boundary was not influenced from the subdomains, additional nodes were introduced through the generation of new contact surfaces between the subdomains.

8.0 Conclusion

From the given results it can be concluded, that the iterative solvers are much faster than the Gauss solver. Most influenced factor on the iterative solver speed is the selection of the preconditioning matrix. Much lower influence has the selection between the CGS or bi-CGSTAB solver. For the fully populated matrix (one subdomain) the selection of Jacobi preconditioner should be made, for non fully populated matrix which is the result of more subdomains ILU preconditioner is preferred. Index notation of the system matrix is reasonably to be used at higher number of sub-domains (banded matrix).

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OPTIMIZACIJA KONSTRUKCIJA METODOM DEKOMPOZICIJE

OPTIMIZATION OF STRUCTURES BY DECOMPOSITION METHOD

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Sažetak

U članku je prikazan algoritam optimizacije nosivih konstrukcija uz primjenu metode dekompozicije konstrukcije. Metoda se temelji na principima dinamičkog programiranja i uz određene preinake pogodna je i za optimiziranje konstrukcija. Osnova je metode razdioba kompleksnog sistema (konstrukcije) na manje funkcionalne sklopove te daljnje razbijanje tih sklopova na sastavne dijelove. Tako razrađen sastav sistema sastoji se od više razina s različitim stupnjem detaljiranosti. Svaki sklop ili detalj optimizira se nezavisno od drugih sklopova u istoj razini, ali se pri tome uzimaju u obzir uvjeti i ograničenja dana u prethodnoj razini. Prednost je te metode pred direktnom, detaljnom optimizacijom u bržem rješavanju više manjih problema umjesto jednog većeg, te je zbog toga metoda primjerena za optimizaciju velikih sistema ili konstrukcija. Zbog razinskog pristupa rješavanju problema prijenos podataka između razina dosta je složen, na što treba obratiti pažnju pri izradi matematičkog algoritma i kompjuterskog programa.

Abstract

An algorithm for structural optimization, using the decomposition of a structure to be optimized, is considered. Based on the principles of dynamic programming the method is suitable, with small corrections and enhancements, for the structural optimization. The computational strategy involves a decomposition of the complex structure into an assembly of functional substructures which can be additionally further disassembled as many times as appropriate, the highest or the base level representing the original structure itself, while the lowest level is yielding the elementary structural cells usually specified with a set of primary design variables. Usually, this multilevel decomposition is characterized by different rate of detailing within particular levels. The optimization of each substructure or detail within a level is limited to the correspondent substructure, and is performed fully uncoupled with regards to the remaining structure in the level, but taking limitations and conditions resulting from the preceding and subsequent level analyses. Though the multilevel approach to the optimization problems involves rather complicated data transfer between the individual levels it is advantageous for large structural optimization. This advantage in respect to the direct method without a decomposition is due to the reduction of the computation time, as a consequence of solving a set of much smaller problems, rather than solving a large one met in optimization of non decomposed structures.

1. UVOD

Optimizacijom s podstrukturiranjem velik optimizacijski problem, koji je zbog velikog broja optimizacijskih varijabli i funkcijskih ograničenja vrlo nepogodan za rješavanje, pretvara

se u seriju manjih, pojedinačnih problema, koji su na neki način međusobno povezani i koji se daju razmjerno lako riješiti. Takav princip rada uobičajen je i u konstruiranju u tehničkoj praksi, gdje se najprije odredi i približno proračuna cijela konstrukcija, a potom se detaljnije razrađuju pojedini sastavni dijelovi. Isti princip upotrebljava se i u metodi konačnih elemenata (MKE) primjenom tzv. superelemenata. Dinamičko programiranje, koje potječe od Bellmana [1], najpoznatija je matematička metoda koja je primjerena za upotrebu u optimizaciji s podstrukturiranjem. Primjenu te metode na modeliranje konstrukcija u MKE usavršili su različiti istraživači, npr.: Noor [2], Aaraldsen [3]. Značajan prinos optimizaciji konstrukcija pomoću dekompozicije dali su Sobieszczanski-Sobieski [4], Kirsch [5] i Haftka [6].

Iako se sistem teoretski može rastaviti na neograničen broj razina, u praksi se on obično rastavi na dvije do tri razine. Pri tome se slijedi hijerarhija između razina, koja već postoji u sistemu ili konstrukciji (npr.: razina 1: most, razina 2: nosači i stupovi mosta, razina 3: pojasovi i rebro stupa, razina 4: ukrućenja rebra). Optimizacija pomoću dekompozicije pogodna je i za sisteme koji se mogu hijerarhično rastaviti na takav način da se u svakoj razini optimira fizikalno različita količina (npr.: razina 1: manevarske sposobnosti vozila, razina 2: masa vozila).

2. DEKOMPOZICIJA SISTEMA

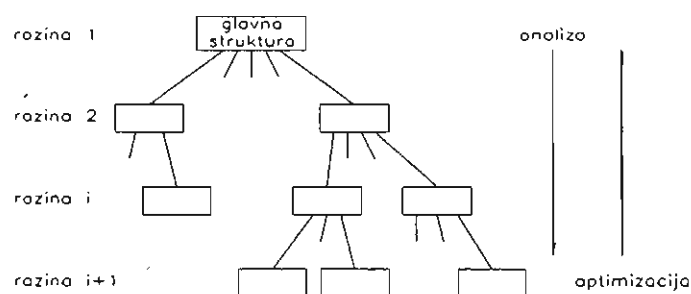
Shema sistema ili konstrukcije razrađene po principima dinamičkog programiranja prikazana je na slici 1. Pojedini okvir na slici predstavlja podstrukturu ili sklop. Podstrukture su razvrstane u razine u hijerarhičnom redu. Pri tome vrh piramide, tj. razina 1, predstavlja cijelu, sastavljenu konstrukciju, koja se razdjeljuje (grana) prema dnu piramide, gdje je razina s najviše detalja. Broj razina ne mora biti jednak u različitim granama. Takva kompozicija proizlazi iz hijerarhične ovisnosti nižih razina o informacijama koje dolaze s viših razina.

Prijenos informacija moguć je samo u vertikalnom smjeru, tj. između dva sklopa koja se nalaze jedan iznad drugog na istoj grani, dok između sklopova na istoj razini nema nikakve komunikacije. Analiza sistema izvrši se najprije na gornjoj razini i širi se prema nižim. U istom smjeru prenose se informacije o ponašanju sistema i o

stanju sistema (sistemske varijable), pomoću kojih je moguća analiza na nižim razinama. Redoslijed je optimizacije obratan: počinje od razine detalja prema višim razinama, a u tome smjeru prenose se informacije o senzitivnosti optimalnog rješenja. Te povratne informacije omogućuju da se napravi optimizacija na gornjoj razini, uzimajući u obzir osjetljivost optimalnog rješenja nižeg sklopa na promjenu sistemskih varijabli iz višeg sklopa.

Proces optimizacije sistema započinje analizom sistema od gornje razine prema donjim. Rezultati analize u razini i , R_i , služe kao ulazni podaci u razini $i+1$ (slika 2). Za analizu u pojedinom sklopu u razini $i+1$ dovoljno je poznavati samo dio rezultata iz prethodne razine. Tek svi sklopovi na razini $i+1$ koji se račvaju iz jednoga sklopa na razini i mogu, ali ne moraju upotrijebiti sve rezultate R_i . Uz rezultate analize na nižu se razinu prenose i podaci o sistemskim varijablama Y_i . Slično kao kod rezultata analize R_i , pojedini sklop na razini $i+1$ ne treba poznavati sve sistemske promjenljive Y_i , već samo njemu potrebne.

Optimizacija se najprije obavi na najnižoj razini, koju označimo indeksom $i+1$. Uzimaju se u obzir svi zahtjevi iz dotičnog sklopa u najnižoj razini i podaci o sistemskim varijablama Y_i iz sklopa koji se nalazi na istoj grani u prethodnoj, višoj razini (razina i). Cilj je optimizacije na



Slika 1: Podstrukturiranje

Substructuring

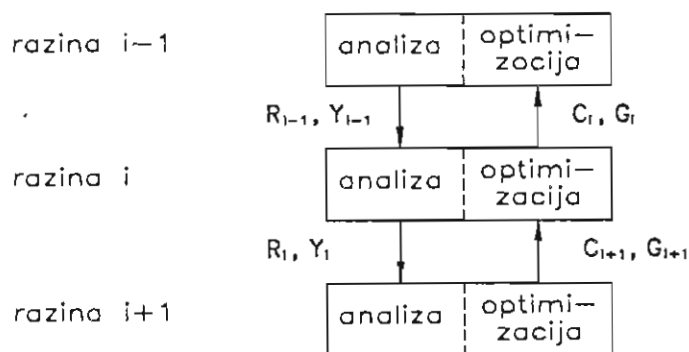
svim razinama, osim na najvišoj (razina 1) smanjivanje vrijednosti lokalne funkcije cilja C_{i+1} , koju definiramo kao razliku između sistemskih varijabli Y_i , određenih na prethodnoj razini (razina i) i istih sistemskih varijabli Y_{i+1} , izračunatih u razini $i+1$. Na višu razinu vrte se informacije o iznosu lokalne ciljne funkcije C_{i+1} i o osjetljivosti lokalnoga optimalnoga rješenja na promjenu veličine sistemskih varijabli G_{i+1} . Oznaka G_{i+1} služi samo za skraćeni zapis skupa derivacija lokalne ciljne funkcije C_{i+1} po sistemskim varijablama Y_i .

Na razini i proces optimizacije izvodi se na isti način kao na nižoj (razina $i+1$), osim što se upotrijebe i podaci o osjetljivosti G_{i+1} svih sklopova iz niže razine koji su na istoj grani u piramidi. S uvođenjem dodatnih podataka s niže razine postiže se da su rezultati optimizacije na višoj razini usklađeni s mogućnostima optimizacije na nižoj razini i da kasnije ne nastane prekid procesa optimizacije zbog toga što su zahtjevi razine i prema razini $i+1$ preveliki.

Na takav se način proces optimizacije kondenzira prema vrhu piramide. S time se završava jedan ciklus u optimizaciji cijelog sistema. Ako je pravilo zaustavljanja algoritma zadovoljeno i ako su iznosi svih lokalnih ciljnih funkcija C manji od dopuštenih, proces se prekida. Inače započnemo novu iteraciju s analizom sistema na gornjoj razini.

Za analizu sistema u određenoj razini upotrijebimo bilo koju metodu za rješavanje tog problema. Primjer: za statičku analizu nosive konstrukcije vjerojatno ćemo upotrijebiti MKE, a za izračun naprezanja u pojasovima I nosača poslužiti ćemo se analitičkim jednadžbama iz teorije linijskih nosača. Izbor metode za analizu pojedinih sklopova ne utječe na algoritam optimizacije pomoću dekompozicije, ali utječe na brzinu svladavanja problema.

Optimizacija sklopa izvodi se jednom od metoda matematičkog programiranja. Te metode zahtijevaju neprekidnost svih funkcija koje sudjeluju u optimizaciji i njihovih prvih derivacija. Zbog linearizacije obično nelinearnog problema, kakav je npr. optimizacija nosivih konstrukcija, u problem se unosi greška, čiju je veličinu teško odrediti. Najveći dio greške proizlazi iz upotrebe derivacija G_{i+1} , koje su izračunate u razini $i+1$, u traženju optimuma na razini i .



Slika 2: Tok informacija

Flow of information

3. DVORAZINSKA OPTIMIZACIJA LINIJSKIH KONSTRUKCIJA

3.1. Optimizacija u jednoj razini

Metoda optimizacije bez dekompozicije služi kao osnova iz koje je razvijena optimizacija u pojedinim sklopovima sistema. Osim toga, svi su referenčni primjeri u točki 4 izračunati tom metodom. Definirajmo najprije varijable i funkcije potrebne u optimiziranju:

- optimizacijske varijable t_i , $i=1...n$ jesu varijable čiju veličinu tražimo u procesu optimiziranja. U optimizaciji konstrukcija to su dimenzije presjeka pojedinih nosača i stupova. Sve varijable složimo u vektor, koji označimo s t .
- funkcija cilja $W(t)$ jest bilo koja izračunljiva funkcija varijabli t . Obično je to masa ili težina konstrukcije.
- ograničenja $g_j(t)$, $j=1...m$ postavljena su da se dobije funkcionalno i konstrukcijsko izvodljivo optimalno rješenje. Ograničenja određuju npr. najveća dopuštena naprezanja

i deformacije konstrukcije, najveće i najmanje dimenzije presjeka...

Standardna je formulacija problema optimizacije:

$$\begin{aligned} &\text{treba minimizirati:} && W(t) \\ &\text{p. o. (pri ograničenjima):} && g_j(t) \leq 0, \quad j=1 \dots m. \end{aligned} \quad (1)$$

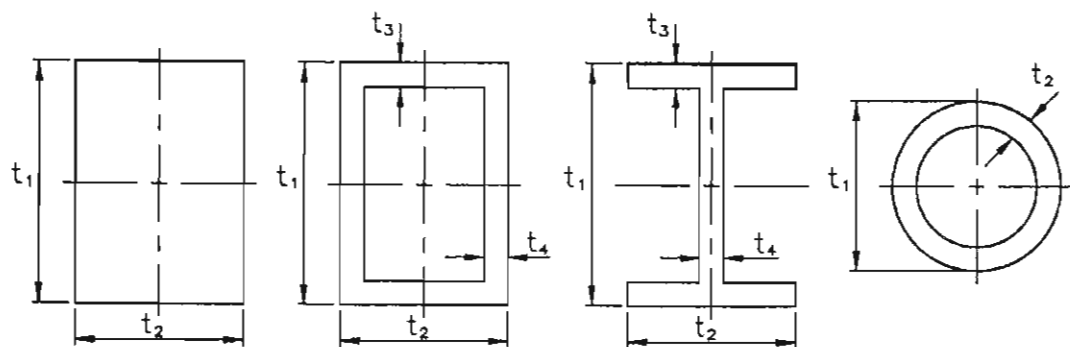
Pretraživanje optimalnog rješenja izvodi se u optimizacijskom hiperprostoru dimenzije n , i to za sve optimizacijske varijable i funkcije ograničenja istodobno. Uz povećavanje broja varijabli (dimenzije n), vrlo se izrazito povećaju potrebe za kapacitetom računala i vrijeme potrošeno za optimizaciju.

3.2. Optimizacija u dvije razine

Algoritam opisan u ovoj točki skraćen je oblik općeg algoritma optimizacije pomoću dekompozicije, opisanog u točki 2, i namijenjen je za optimizaciju linijskih konstrukcija u ravni. Osnovna je ideja algoritma rastavljanje velikog problema optimizacije, koji uključuje istodobno traženje dimenzija u presjecima svih elemenata konstrukcije, na više manjih problema koji se mogu brže riješiti. Piramida iz slike 1 reducira se na samo dvije razine, u kojima se nalaze ove varijable i funkcije:

razina 1: sastoji se od elemenata, kao što su npr. nosači, stupovi i štapovi, koji određuju oblik i veličinu konstrukcije sa statičkog gledišta. Ako je svaki element konstantnog presjeka po cijeloj svojoj dužini, ima samo dvije optimizacijske varijable, a to su površina poprečnog presjeka A i moment tromosti I . Funkcija cilja jest težina konstrukcije, a funkcije ograničenja jesu ograničenja pomaka konstrukcije u određenim točkama i veličine lokalnih funkcija ciljeva iz razine 2.

razina 2: broj sklopova (grana) u drugoj razini jednak je broju elemenata u prvoj razini. Optimizacijske varijable prve razine sad su sistemske varijable Y u drugoj razini. Optimizacijske varijable druge razine različite su dimenzije presjeka, označene s t_i , $i=1 \dots 4$. Značenje tih oznaka ovisi o obliku presjeka, a prikazuje ih slika 3. Funkcija je cilja neka proizvoljna funkcija koja je dovoljno osjetljiva na male razlike između sistemskih varijabli A i I i izračunatih vrijednosti za isti A i I , uz upotrebu varijabli t_i . Funkcije ograničenja jesu dopuštena naprezanja u konstrukciji i ograničenja dimenzija t_i .



Slika 3: Oblici presjeka

Cross-sectional shapes

3.3. Optimizacija donje razine

Optimizacija se najprije izvede na razini 2 tako da se optimira svaki sklop posebno. Sistemske varijable Y već su određene u prvoj razini, te su nepromjenljive u drugoj razini. Vektor sistemskih varijabli zapisan je kao

$$Y = (A_1, I_1, A_2, I_2, \dots, A_e, I_e)^T. \quad (2)$$

Pritom e znači broj sklopova u drugoj razini, a to je jednako broju elemenata konstrukcije na prvoj razini. Znak T znači transponiranje, tj. vektor Y zapiše se kao stupac. U proizvoljnom sklopu, npr. sklopu S ; $1 \leq S \leq e$, potrebne su samo dvije sistemske varijable: A_S i I_S . Optimizacijske varijable jesu dimenzije presjeka t_i , koje su uređene u vektor

$$t = (t_1, t_2, t_3, t_4)^T. \quad (3)$$

Funkcija cilja može biti različitih oblika, normirana ili ne, linearna ili nelinearna. Pravilno odabrana funkcija cilja može mnogo pridonijeti konvergenciji optimizacijskog problema. Funkcija cilja zapiše se u obliku

$$C_S = \sqrt{\left(\frac{A - A_S}{A_S}\right)^2 + \left(\frac{I - I_S}{I_S}\right)^2}. \quad (4)$$

Varijable A i I izračunaju se na razini 2, uz poznavanje dimenzija t , za sve presjeke sa slike 3. Funkcije ograničenja naprežanja zapisuju se u obliku

$$g_j(t) = \frac{\sigma}{\sigma_{dop}} - 1, \quad j = 1 \dots p, \quad (5)$$

gdje σ znači ekvivalentno naprežanje u točki j elementa, a σ_{dop} jest dopustivo naprežanje za tu točku. Ekvivalentno je naprežanje implicitna funkcija vektora t , a on se izračuna u skladu s teorijom nauke o čvrstoći. Funkcije ograničenja dimenzija zapisuju se u obliku gdje se ograničuju apsolutne dimenzije presjeka (minimalne i maksimalne)

$$g_j(t) = \frac{t_i}{\underline{t}_i} - 1, \quad g_k(t) = \frac{t_i}{\bar{t}_i} - 1, \quad i = 1 \dots 4, \quad j = p+1 \dots p+4, \quad k = p+5 \dots p+8, \quad (6)$$

i/ili se ograničuje razmjer pojedinih dimenzija presjeka zato da se onemoguće suviše vitki presjeci i s time povezano izbočavanje pojasova i rebara u nosaču

$$g_j(t) = \frac{t_i}{r t_k} - 1, \quad i = 1 \dots 4, \quad k = 1 \dots 4, \quad i \neq k, \quad (7)$$

gdje $\underline{t}$ i $\bar{t}$ znače minimalne i maksimalne dopustive dimenzije, a r maksimalno dopustiv omjer dviju dimenzija t_i i t_k .

Iz jednadžbi (1) i (4-7) može se formulirati problem optimizacije na drugoj razini:

$$\min. C(Y, t), \quad \text{p. o. } g_j(t) \leq 0, \quad j = 1 \dots m. \quad (8)$$

Broj funkcija ograničenja m redovito je veći od broja optimizacijskih varijabli. Minimizacija (8) može se riješiti upotrebom bilo koje optimizacijske metode. U ovom članku upotrijebljena metoda jest metoda kvadratnog programiranja, koja dopušta da se pri traženju optimalnog rješenja zaluta i izvan dopustivog optimizacijskog prostora. Derivacije funkcija cilja i ograničenja izračunate su uz pomoć diferencijske metode. Rezultat su optimizacije nove dimenzije t i vrijednost funkcije cilja C_S . Funkcija cilja osjetljiva je na promjenu sistemskih varijabli A_S i I_S , što se prikazuje u obliku prvih derivacija dC_S/dA_S i dC_S/dI_S . Te se derivacije za funkciju cilja zapisanu u jednadžbi (4) mogu izračunati analitičkim putem.

Optimizacija opisana u ovoj točki ponavlja se za svaki sklop u drugoj razini. Za upotrebu na prvoj razini treba sačuvati vrijednost funkcije cilja C_s te derivacije dC_s/dA_s i dC_s/dI_s . Te su funkcije i derivacije, slično kao optimizacijske varijable, grupirane u vektore koji su u točki 2 označeni sa C i G .

3.4. Optimizacija gornje razine

Optimizacijske varijable u prvoj razini bivše su sistemske varijable druge razine, tj. vektor Y (jednadžba 2). Funkcija je cilja masa cijele konstrukcije, a zapiše se u obliku

$$W(Y) = \sum_{s=1}^e \rho_s L_s A_s, \quad (9)$$

gdje ρ_s , L_s i A_s znače specifičnu gustoću, dužinu i površinu presjeka elementa s . Funkcije ograničenja pomaka određenih točaka u konstrukciji jesu

$$g_j(Y) = \frac{u}{u_{dop}} - 1, \quad j = 1 \dots p, \quad (10)$$

gdje u znači izračunati pomak točke j , a u_{dop} dopustivi je pomak za tu točku. Zbog bolje stabilnosti optimizacije, na gornjoj razini postave se još i ograničenja veličine optimizacijskih varijabli. Zapis tih ograničenja sličan je zapisu (6) i stoga ih nećemo ponavljati. Da bi se postigla bolja koordinacija između obje razine, uvode se dodatna ograničenja na gornjoj razini, a to su vrijednosti lokalnih funkcija ciljeva C_s . Sve lokalne funkcije ciljeva koje imaju preveliku vrijednost vode se kao ograničenja i njihove derivacije sudjeluju u procesu optimizacije na gornjoj razini.

Iz jednadžbi (9) i (10) te komentara u prijašnjem odlomku formulira se problem optimizacije na prvoj razini:

$$\min. W(Y), \quad p. o. \quad g_j(t, C) \leq 0, \quad j = 1 \dots m. \quad (11)$$

Minimizacija (11) rješava se upotrebom bilo koje optimizacijske metode. Zbog unifikacije preporuča se ista metoda optimizacije kao na drugoj razini. Analiza konstrukcije izvedena je metodom konačnih elemenata. Derivacije funkcija cilja i ograničenja izračunate su diferencijskom metodom, te se zbog toga MKE u jednoj iteraciji mora pozivati $2e+1$ puta, gdje e znači broj linijskih elemenata konstrukcije. To je dio procesa optimizacije koji potroši najviše računalnog vremena. Rezultat su optimizacije nove optimizacijske varijable, vektor Y i nova vrijednost mase konstrukcije W .

Sa završetkom optimizacije na gornjoj razini cijela konstrukcija ima nov raspored mase i krutosti. Novi ciklus započne se s analizom na gornjoj, a potom i na donjoj razini. Zatim slijedi optimizacija, kao što je već opisano. Iteracije se ponavljaju ako bilo koje ograničenje na prvoj ili drugoj razini nije ispunjeno i sve dokle nije postignuta zadovoljavajuća konvergencija rezultata.

4. PRIMJER

Algoritam optimizacije metodom dekompozicije testiran je s numeričkim primjerom iz literature [7]. Rezultati koji služe kao referenca postignuti su metodom bez dekompozicije, kao što je to pokazano u točki 3.1.

Nesimetrični portalni okvir na slici 4 sastavljen je od tri kruto spojena nosača na koje djeluju ova opterećenja:

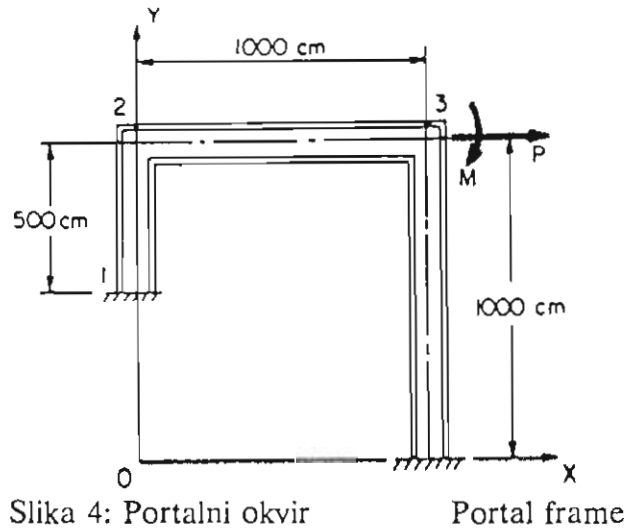
1. slučaj $P=50\,000\text{ N}$, $M=0$
2. slučaj $P=0$, $M=2,0\cdot 10^8\text{ Nmm}$

Podaci o materijalu jesu: $E=0,7\cdot 10^5\text{ N/mm}^2$, $G=0,27\cdot 10^5\text{ N/mm}^2$.

Dopustivo je naprezanje $200,0\text{ N/mm}^2$, a progib portala na mjestu gdje djeluje opterećenje, ne smije biti veći od $\pm 40,0\text{ mm}$ i rotacija veća od $\pm 0,15$ radijana. Ograničeni su također omjeri visina prema debljinama limova u nosačima, ali ti omjeri ovise o obliku presjeka i u referenci [7] nisu dani. U ovom članku ti su omjeri određeni na ovaj način:

I profil: $t_1/t_4 \leq 1000$ $t_2/t_3 \leq 1000$
cijev: $t_1/t_2 \leq 1000$.

Minimalne su dopuštene debljine limova nosača $0,10\text{ mm}$. Cilj je optimizacije postići što manju masu portala, s tim da se uvažavaju sva prije nabrojena ograničenja.



Slika 4: Portalni okvir

Portal frame

Optimizacija okvira izvršena je u ref. [7] za ove oblike presjeka: profile I, Z, U, T i okruglu cijev. Metodom dekompozicije prikazanom u članku optimiziran okvir sastavljen je od I nosača i od okruglih cijevi. U tabeli 1 izloženi su ukupni rezultati optimizacije. U tabeli 2 jesu vrijednosti optimizacijskih varijabli za različite presjeke.

Optimizacijske su varijable na prvoj razini A_1 , A_2 , A_3 , I_1 , I_2 i I_3 , a na drugoj razini t_1 i t_2 ako se optimizira cijev ili t_1 , t_2 , t_3 i t_4 ako se optimizira I nosač. Početni izbor optimizacijskih varijabli t u literaturi [7] nije poznat. U našem slučaju odabrane su takve varijable t da je početna masa dosta veća od početne mase u [7].

Elementi portala	Masa [kg]	Iteracija	Izračun MKE
I profil [7]	539.11	15	?
I profil	549.60	10	42
okrugla cijev [7]	711.91	13	?
okrugla cijev	725.94	6	36

Tabela 1: Usporedba rezultata optimizacije portalnog okvira

		Dimenzije presjeka [mm]			
Elem.	t	I profil [7]	I profil	Cijev [7]	Cijev
1	1	100.000	97.100	99.888	94.969
	2	12.424	54.437	0.115	0.100
	3	0.397	0.100		
	4	0.175	0.100		
2	1	100.000	85.096	80.110	94.171
	2	10.831	69.192	0.118	0.100
	3	0.341	0.110		
	4	0.129	0.100		
3	1	94.262	89.213	72.075	89.630
	2	8.816	52.240	0.104	0.100
	3	0.279	0.116		
	4	0.162	0.100		

Tabela 2: Portalni okvir - vrijednosti optimizacijskih varijabli

5. ZAKLJUČAK

Metoda optimizacije pomoću dekompozicije daje, kao što se vidi iz numeričkih primjera, nešto veću konačnu masu nego direktne metode optimizacije, ali je ta razlika vrlo mala u usporedbi s masom konstrukcije. Do sličnog zaključka došli su i drugi istraživači [4]. Potreban broj analiza konstrukcije sa MKE mnogo je manji, te je zbog toga vrijeme rješavanja optimizacijskog problema smanjeno. S time se otvara mogućnost rješavanja velikih problema (više stotina optimizacijskih varijabli) u prihvatljivom vremenu i na PC računarima. Rukovanje sa podacima dosta je složeno, naročito kod dekompozicije na više od dvije razine.

Metoda dekompozicije dobro se uklapa u trendove paralelnog procesiranja u novoj računarskoj tehnologiji.

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Pavao Marović and Goran Gusić

FEME - One Way of Meshing and Generating within the CAD Environment

S U M M A R Y

This paper presents the programme package **FEME** (Finite Element Modelling Extension) as one approach to **FEM** (Finite Element Method) pre- and post- processing by means of **CAD** (Computer Aided Drawing) most popular program **AutoCAD**.

The programme package FEME consists of two programs, one inside AutoCAD which converts a geometry model into its numerical correspondent and the other one which generates the finite element mesh outside the AutoCAD based on the produced numerical model.

Object oriented paradigm in defining structure geometry enables a high level of data abstraction in the pre-processing phase. Mesh generation, as well as generation of boundary conditions and loads which are defined over the geometrical entities, are performed using the so-called block mapping technique. A finite element mesh created in this way is well-conditioned and allows the use of different finite element types which is one of the best features of mapping techniques. The wide open architecture of AutoCAD also enables fast definition of the input data.

The usage and performance of the programme package FEME is illustrated by the pre- and post- processing phases of a few simple illustrative examples.

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1. INTRODUCTION

In recent years large physical problems have been solved by means of numerical methods. Huge amounts of data, concerned with these problems and their appropriate handling have arisen as objective difficulties. The need for the visualization of solutions and simpler data input became a necessity. These facts give rise to two new phases in numerical modelling by computer: pre-processing (definition of input data) and post-processing (visualization of obtained results) [1].

The finite element method (FEM), as the most common method in the area of structure numerical simulation, appeared during 1950-s, when the solutions of numerical simulations appeared as vast listings of the numerical data, presenting stresses and displacements in the discrete points of the model. The process of converting a real structure model into numerical data has also resulted in data increase. This phenomenon resulted in many errors in the data input phase. More and more complicated geometry and increasing demands for complex structure description made it impossible to properly use the results produced by the FE model. Thus, the problem of data visualization arose in the domain, first in the data input and later in the data output. For this purpose many graphical packages have been used. However, today the most widely used package is AutoCAD [2].

According to the presented facts, this paper briefly describes the FEME (Finite Element Modelling Extension) programme package [3, 4]. FEME is at the same time a pre-processor and post-processor for any finite element program and is based on the CAD program package AutoCAD. It has a built-in mesh generator for preprocessing.

2. MESH GENERATION TECHNIQUES

The adoption of the geometry based definition of the problem and automatic production of a valid finite element discretization without user intervention is the major property of the mesh generators which represents a part of the reliable application of the finite element technique. This results in the reduction of input data, mainly topology data of finite elements and their coordinates. This also reduces the possibility of producing errors.

Most of the research in the area of mesh generation is focused on techniques such as 2D-quadtrees [5], 3D-octrees [6], triangulation, paving [7] or on the different mapping techniques. The mapping technique is used in this paper as well as in papers [3, 4]. The main advantages of mapping techniques (block decomposition, transfinite mapping) are: (a) **boundary sensitivity**: mesh contours follow the contours of the boundary; (b) **orientation insensitivity**: rotation or translation of the given domain does not change the mesh topology; (c) **element versatility**: these techniques can be developed for different finite element types.

Geometrically based finite element mesh generators [5, 6, 8, 9] are, in comparison to the block mapping technique, at a higher automation level. Despite this property these techniques are usually used for only one finite element type.

The quadtree mesh generator [5] is based on a spatial decomposition procedure of the 2D domain into a set of disjointed squares. These squares are referred to as quadrants that are stored in a hierarchic tree. The root of the tree is given as a square that encloses the observed domain. Within the discretization defined by the quadtree, the mesh topology is defined on a quadrant-by-quadrant basis, using the information stored in the quadtree. The octree technique [6] is very similar but only for 3D domains disjointing domain in octaedars.

The paving technique [7] is based on iteratively layering rows of elements within the interior of a domain's boundaries. As rows begin to overlap or coincide in the interior of the geometry, they are carefully connected together to produce valid topology of the finite element mesh.

3. BLOCK MAPPING TECHNIQUE

3.1 INTRODUCTION

The block mapping technique was introduced in 2D planar domains in 1971 [10]. The domain, according to this technique, is subdivided into blocks which are compatible in the same way in which the finite elements are; then, following the subdivision parameters for each block, the blocks are mapped and subdivided into a valid finite element mesh.

The tedious geometry of the finite element model can be the source of errors and of the generation of non-valid finite element meshes. This problem can be avoided by the use of the geometry based definition of the blocks by means of CAD modellers. In this way, errors are minimized even for manifold objects of arbitrary geometric complexity.

The object oriented approach (OOP), which has been used quite often in programming, transfers the geometry definition of the domain into its numerical representation in this way.

Geometrical entities, such as points, lines, arcs, bezier curves are carriers of the properties of the finite element model. For each of these entities different properties can be assigned like kinematic boundary conditions, dynamic boundary conditions (loads), thickness, etc. Four of these entities, each of them defining a block edge, have to be selected to create a block which is then mapped and subdivided to produce a finite element mesh.

Data hiding, which is one of the common OOP properties, is used here to enable invisible correction between the mathematical model and geometrical model. Although the mathematical model has no significance for the user it is hidden behind its geometrical representative in both pre-processing and post-processing phase.

The user performs block discretization by a CAD modeller without taking into account its topology and nodal coordinates which makes FEME a completely geometrically based pre-processor.

3.2 MATHEMATICAL MODEL

Details of the mathematical model can be found in Refs. [3, 4, 15, 16], and here will be given only its practical explanations.

The aforementioned is performed using the **MESHGEN** programme which is included in the FEME programme package. Accordingly, the **MESHGEN** programme performs finite element mesh generation, evaluates coordinates of all nodes, performs all necessary numerations, and involves boundary conditions and material properties for all elements.

In the first phase, superelement topology generation is performed with the help of subroutine **BUILDTOP**. In the second phase, in subroutine **PROPAGATE**, the subdivision of superelement on a given number of elements $m \times n$ is performed, for chosen parameters m and n , while simultaneously checking the connection compatibility between the superelements. After the transformation of the geometrical model of the domain into a correspondent mathematical model, in subroutine **MESH**, the generation of finite element mesh is performed maintaining the boundary conditions and material properties. In the fourth phase, in subroutine **REMDOUBLES**, the double nodes are eliminated at superelement contacts.

Overall results of the **MESHGEN** programme are obtained in the databases with extensions: ***.CIF** - database for communication with programme AutoCAD; ***.ELM** - database with all numerical data which can be modified in accordance with the needs of the used finite element solver; and ***.OUT** - control database. Details of these procedures can be found in [3].

The evaluated FEME programme package was tested on several examples considering the speed performances. Some results concerning speed of the meshing procedure obtained for two superelements, by varying the number of nodes and elements are shown in Figure 1.

That figure and practical experience show what improvements in working speed, cost reduction and accuracy can be obtained using the FEME programme package.

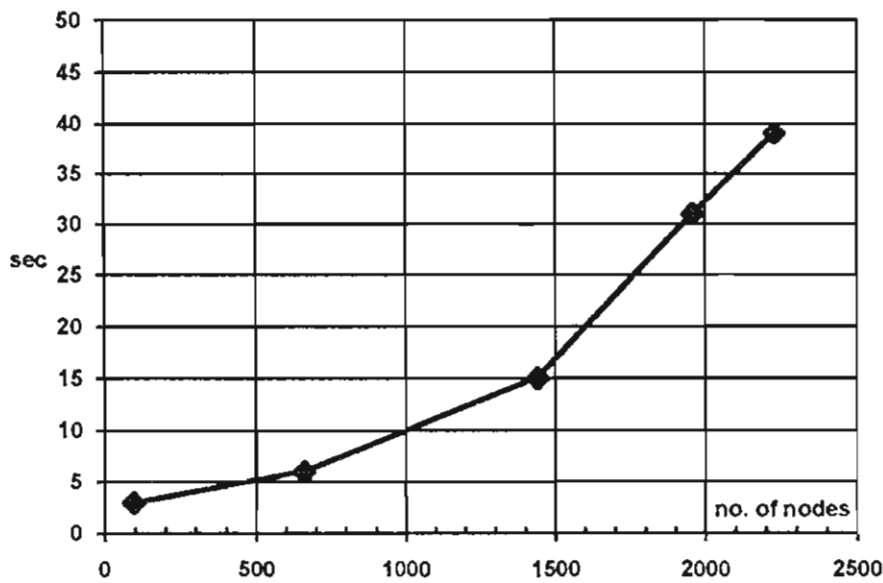


Fig. 1 The speed of the programme MESHGEN for different number of nodes and elements for 2 superelements

4. CAD SOFTWARE AND ITS APPLICATION

In the 1960-s, together with hardware development, a new group of computer applications was born, CAD - Computer Aided Design. These programs became the tools for geometry modelling both in 2D and 3D spaces.

FEM was adopted in the 1950-s. These ten years and the fact that design was put away from numerical simulation of the structure behaviour, has set apart FEM from CAD. Indeed, the analysis is not integrated in the design for good reasons, since, the design analysis is usually performed in different departments with different tools; thus, the gap between geometry modelling and structure analysis is wider than necessary [1].

Model building must be based on a geometric model. Thus, it is time consuming to build a model consisting of nodes and elements, node by node and element by element. The CAD modeller can use geometric operations in both 2D and 3D spaces, such as union, subtraction and intersection

A variety of functions, incorporated in the CAD modeller, also enables the development of the hierarchy of geometrical entities which define arbitrary structure geometry. Functions which have to be developed for a particular CAD software to connect CAD with FEM can be grouped, as follows:

- ♦ transfer; geometry model, built up by AutoCAD can be directly used selecting the entities which define the model, thus filtering necessary information about model geometry without picking up the rest of the data defined in the designing phase (dimensions, hatch, text, etc.), which are needed for a proper representation of the geometry model.
- ♦ transformation; geometry completed in this way, together with parameters for mesh density and relevant data for FEM model such as material properties over the region is transformed into a valid numerical representation of the geometrical model. Associating the loads and the boundary conditions with geometrical entities, the checking and connection of the geometry model are performed in this phase.
- ♦ completion; formatting the data for use by FEM solvers.

The first two phases are performed in AutoCAD [2], one of the most popular CAD softwares for PC compatible machines.

FEME, which stands for Finite Element Modelling Extension, was developed for this purpose using AutoCAD's open architecture. Opening the architecture of this CAD tool is performed by the included AutoLISP interpreter [11, 12], which enables the extensive use of all built-in CAD functions and the manipulation of the graphical objects existing in the current drawings data base.

AutoLISP is the part of Common LISP, a programming language developed in the 1950-s by McCarthy for the sake of artificial intelligence research which today represents one of the oldest programming languages. Its flexibility and the fact that it was wide open for user intervention have made possible its continual application.

Graphical objects and lists of graphical objects through AutoLISP by using encapsulation, data hiding and filtering are easily accessed and manipulated.

AutoLISP, as the first AutoCAD interface language, is suitable for engineering applications because: (a) it is the chosen language for research and development of artificial intelligence and expert systems; (b) it has an exceedingly simple syntax; (c) it is ideally suited for the unstructured interaction that characterizes the design process; and (d) it excels at collecting the heterogeneous objects in various sized groups which is precisely the type of information CAD systems manipulate.

Two special AutoLISP data types are implemented to provide access to AutoCAD graphical entities: (a) entity name; (b) selection set. According to the oo paradigm these data types are manipulated only by functions that act upon them and their internal structure is irrelevant to the programmer. An entity name is actually a pointer to the entity's database record and its vectors. A selection set is simply a collection of entity names.

5. ILLUSTRATIONS

What is the practical application of the aforementioned?

On the basis of a known geometrical model, its contour lines are drawn with the help of the AutoCAD. The obtained sketch is then divided in accordance with material and geometrical properties on blocks - superelements. Subelement division parameters are then introduced for each block. The obtained finite elements take over all the properties of the superelement. So, the generated finite element mesh is obtained visually on screen in AutoCAD and alphanumerically in database *.ELM. This database can be further used by an appropriate finite element solver which is connected with the FEME programme package.

After performing the numerical calculation the obtained results - displacement and deformation data - can be returned into the AutoCAD programme in order to be visualized.

The use of the FEME programme package as a pre-processor and also as a post-processor is illustrated by two examples.

EXAMPLE No. 1

The first example chosen to illustrate the possibilities of the FEME programme package is a dam structure with surrounding soil. Figure 2 shows the geometrical model of the structure according to which the superelement model, shown in Figure 3, is evaluated. The result of the generation of finite element mesh is shown in Figure 4. The obtained mesh is discretized with 175 elements and 598 nodes.

The finite element mesh generated for this example together with all data (node coordinates, element topology, all numerical data concerning nodes and elements, material properties, etc.) is obtained in about 6 seconds.

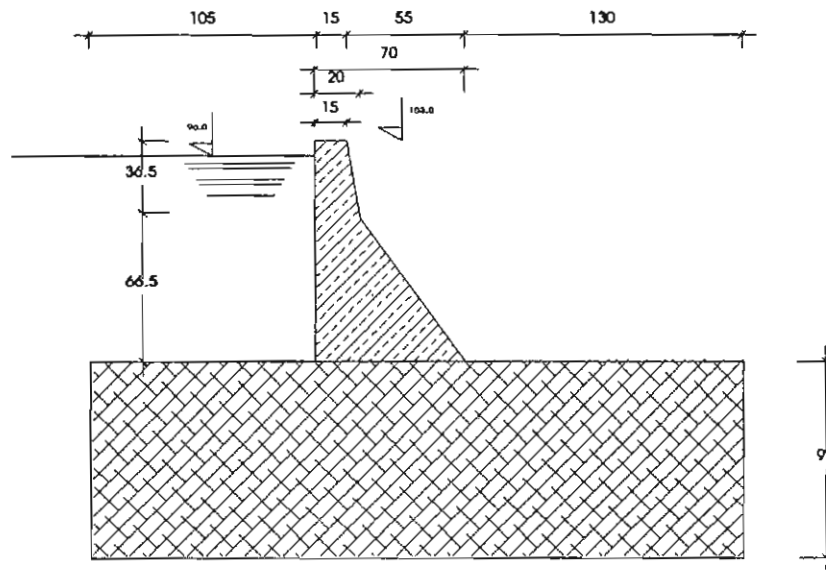


Fig. 2 Geometrical model

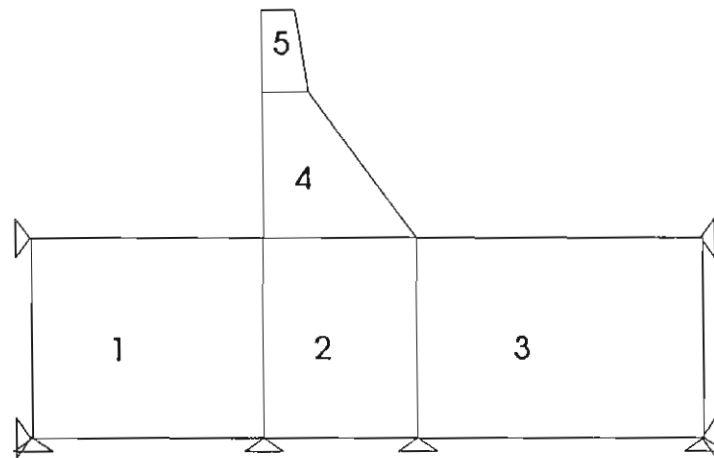


Fig. 3 Superelements model

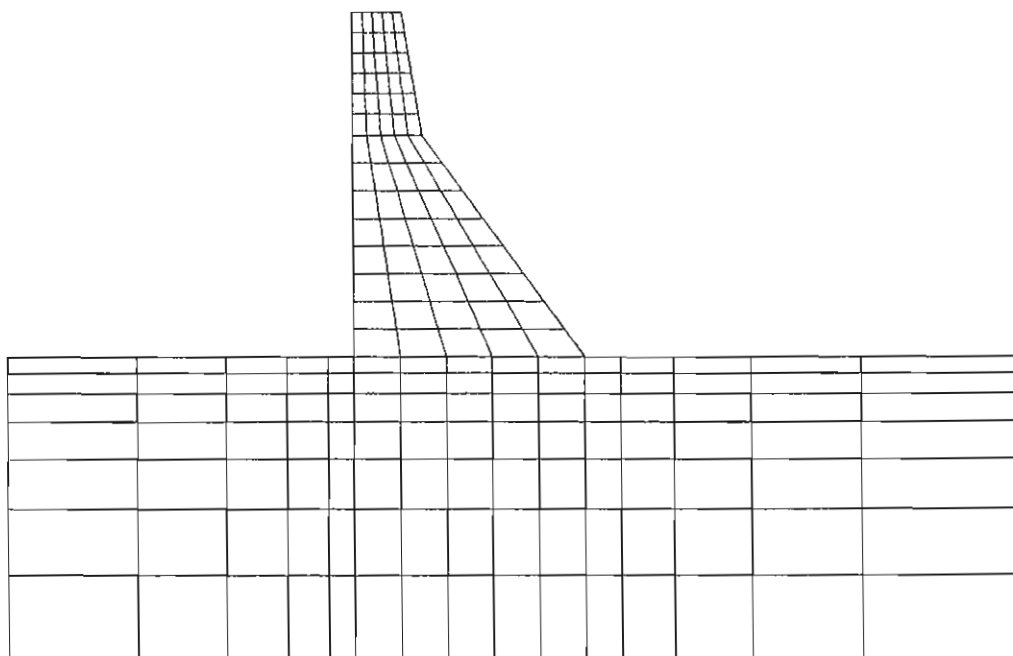


Fig. 4 Generated finite element mesh

EXAMPLE No. 2

The second example which is used to illustrate both pre- and post-processor possibilities of the FEME programme package is a straight-curved cantilever. Its geometry is shown in Figure 5 together with the superelement model and the obtained generated finite element mesh. The deformed mesh obtained by one finite element solver for the linear stress and strain analysis [13] is evaluated in the same figure.

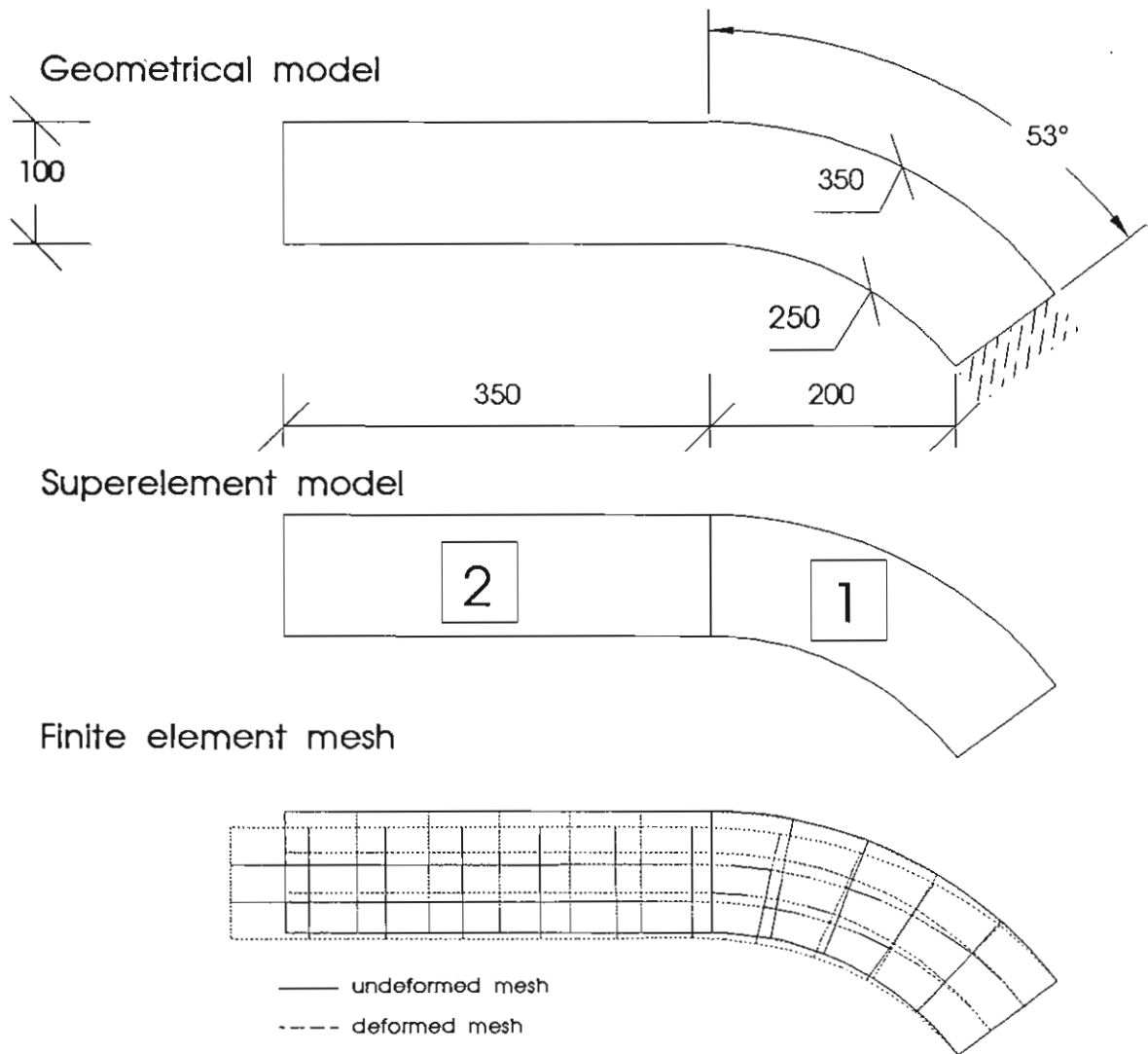


Fig. 5 Straight-curved cantilever: from geometrical model, over superelement model to finite element mesh - undeformed and deformed

6. CONCLUSIONS

This paper presents the possibilities of the **FEME** (Finite Element Modelling Extension) programme package, evaluated for pre- and post- processing of input and output data for any finite element solver by means of **CAD** programme **AutoCAD**. It is worth mentioning that the preparation of input data by drawing instead of typing all alphanumerical data is much quicker and more precise.

The **FEME** programme package is being tested now. It should be improved by fulfilling all the requirements requested by appropriate benchmark tests [14] before it can be professionally used.

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PRIMJER KORIŠTENJA EKSPERTNIH SUSTAVA ZA ANALIZU SPREGNUTIH KONSTRUKCIJA ČELIKA I BETONA

Damir Markulak, Zvonimir Žagar

Sažetak :

Prikazan je primjer preliminarnog određivanja potrebne visine čeličnog presjeka spregnute međukatne konstrukcije korištenjem ekspertnih sustava (E.S.) zasnovanih na pravilima (GURU ljuska). Prvo se definiraju bazne varijable koje bitno utječu na dimenzije spregnutog presjeka. Varirajući te varijable kroz karakteristično područje njihovih vrijednosti dobije se "training" set primjera za "učenje" ekspertnog sustava. Pritom je važno da spregnuti presjek ima visok stupanj iskorištenja. Dajući konkretne ulazne vrijednosti za bazne varijable, ekspertni sustav daje jednu ili više varijanti izbora čeličnog presjeka sa odgovarajućom vjerojatnosti.

AN EXAMPLE OF USING EXPERT SYSTEMS FOR ANALYSE COMPOSITE STRUCTURES OF STEEL AND CONCRETE

Summary :

The paper presents a method, choosing necessary steel sections height of composite structures using a rule based expert systems (GURU shell). It's necessary to define base parameters for dimensions of composite sections. Changing this parameters into characteristic domain values we get "training set" for "learning" expert system. It is important that composite sections have height level of serviceability. Giving input values for characteristic parameters, expert system gives one or more variant choosing steel section with particular likelihood.

1. Uvod

Sprezanje je konstruktivno spajanje čelika i betona kod nosivih elemenata građevinskih konstrukcija. Ove konstrukcije su ušle u primjenu pri gradnji mostova i međukatnih konstrukcija i to prvenstveno iz težnje da se smanji težina betonskih ploča koja je ograničavala raspon. Tako se od 1920. godine prelazi na primjenu relativno tankih armirano-betonskih ploča umjesto teških ispuna od nearmiranog betona.

U Kanadi su 1922. godine izvršena detaljna ispitivanja međukatne konstrukcije sa ubetoniranim I nosačima. Ispitivanjem su uočene znatne razlike u računskim i mjerenim deformacijama i naprezanjima, što je upućivalo na to da postoji "suradnja" dva različita materijala u preuzimanju korisnog opterećenja.

Treba spomenuti da je i kod nas izvršeno detaljno ispitivanje mosta preko rijeke Save u Zagrebu 1939. godine.

U SAD su od 1936. do 1944. godine obavljena kompleksna ispitivanja koja su omogućila donošenje prvih propisa za spregnute konstrukcije u mostogradnji.

Najšire područje primjene spregnutih konstrukcija su svakako mostovi i međukatne konstrukcije višetažnih zgrada, jer tu najviše dolaze do izražaja prednosti spregnutih konstrukcija u pogledu utroška materijala, veličine raspona i brzine građenja. U visokogradnji je posebno karakteristična primjena hladnooblikovanog profiliranog lima kao oplata i dijela betonske ploče u zajedničkom spregnutom presjeku.

Kad je riječ o statičkim sistemima, sprezanje se primjenjuje kod statičkih sistema proste i kontinuirane grede, te okvirnih konstrukcija za punostijene i rešetkaste nosače.

2. Vrste sprezanja

A) Prema opsegu sprezanja:

- a) Sprezanje samo za korisno opterećenje
- b) Sprezanje za korisno i dio stalnog opterećenja
- c) Sprezanje za korisno i ukupno stalno opterećenje
- d) Sprezanje uz prethodno ili naknadno prednapinjanje

B) Prema vrsti sprezanja

- a) Kruto sprezanje
- b) Elastično sprezanje
- c) Diskontinuirano (prekidno) sprezanje

3. Osnove proračuna i dokaza sigurnosti spregnutih nosača čelika i betona

Osnovu ovog rada čini jedan jednostavan program napisan u programskom jeziku Basic, a čija je svrha preliminarno određivanje dimenzija spregnutog nosača međukatne konstrukcije. Na setu primjera riješenih pomoću ovog programa, gdje je izvršeno variranje određenih parametara vršiti će se "treniranje" jednog jednostavnog ekspertnog sustava (zasnovanog na pravilima).

Pošto je postavljeni cilj u ovoj radnji preliminarno određivanje dimenzija spregnutog presjeka, proračun je vršen po teoriji elastičnosti (otpor konstrukcije se definira dostignutim graničnim naprezanjima na rubovima presjeka). Po pretpostavkama teorije graničnih stanja, ovakav način proračuna propisuje se za poprečne presjeke klase 3 i 4 - klase presjeka se određuju prema kriteriju ograničenosti rotacionog kapaciteta spregnutog presjeka, lokalnog izbočavanja metalnog dijela spregnutog presjeka i drobljenja betona. Osnovni kriterij koji (prema teoriji graničnih stanja) treba biti zadovoljen, glasi (za ovaj primjer):

$$RR/GAMAJRR \geq GG * GAMAJGG + QQJ1 * GAMAJQQ1$$

gdje je:

RR	otpornost konstrukcije
GAMAJRR	parcijalni koeficijent sigurnosti za otpornost konstrukcije prema propisima EUROCODE 4, za ovaj primjer uzet je globalni strukturni koeficijent za čelik u vrijednosti 1,1
GG	akcija uslijed djelovanja stalnog tereta i vlastite težine nosača
GAMAJGG	parcijalni koeficijent sigurnosti za akciju od stalnog opterećenja -prema EUROCODE4 za ovaj se koeficijent uzima vrijednost 1,35
QQJ1	akcija uslijed korisnog opterećenja koje se često pojavljuje
GAMAJQQ1	parcijalni koeficijent sigurnosti za akciju QQ1 - uzima se 1,5

Kao granična otpornost za betonske dijelove spregnutog presjeka, prema istim normama, uzima se 60% vrijednosti tlačne čvrstoće betonske kocke nakon 28 dana.

Utjecaj puzanja i skupljanja betona na spregnuti presjek određuje se prema normama DIN 4227, koje zasad najkompaktnije rješavaju ovaj problem. Prema tim standardima koeficijent puzanja, odnosno skupljanja betona određuje se iz određenih dijagrama ovisno o uvjetima sredine u kojoj se promatrani element nalazi, efektivnom promjeru betonskog presjeka, kao i vremenu u kojem se betonski presjek počinje opterećivati. Ovako dobivenim koeficijentom puzanja reducira se modul elastičnosti betona koji onda takav ulazi u proračun spregnutog presjeka, a čime samo zapravo uzeli u obzir puzanje betona pod dugotrajnim opterećenjem.

Kod skupljanja betona, osim redukcije modula elastičnosti, analizira se kakve posljedice ova pojava ima na jedinstveni spregnuti presjek. Poznato je da skupljanje betona ne ovisi o vanjskom

opterećenju, međutim deformacijama betonske ploče koje nastaju zbog skupljanja suprostaavlja se metalni nosač, preko moždanika. Zbog toga se u betonskom dijelu nosača javlja vlačna sila, a koja opet reaktivno djeluje na spregnuti nosač. Navedeni utjecaji i ovdje se uzimaju u obzir pri analizi stanja naprezanja za vrijeme t . Pregled osnovnih parametara spregnutog presjeka:

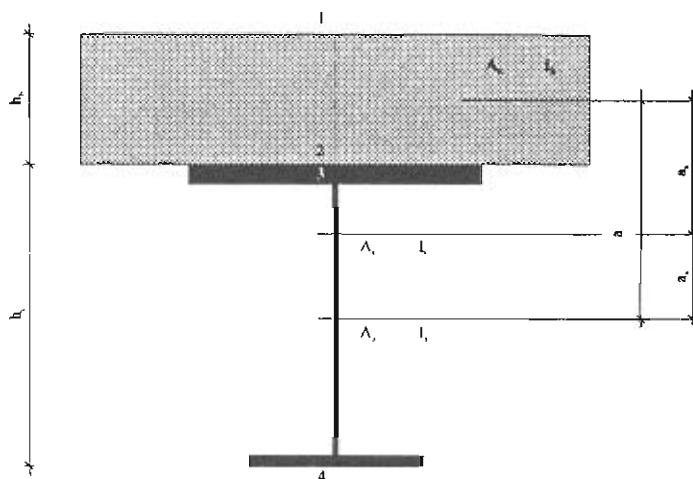
$$n = \frac{E_a}{E_b} \quad (1)$$

$$A_i = A_a + \frac{A_b}{n} \quad (2)$$

$$a_a = A_b \cdot \frac{a}{nA_i} \quad (3)$$

$$a_b = a - a_a \quad (4)$$

$$I_i = I_a + \frac{I_b}{n} + A_i \cdot a_a \cdot a_b \quad (5)$$



gdje su:

Sl. 1. Spregnuti nosač (Composite structure)

- n - omjer modula elastičnosti betona i čelika
- A_a - površina poprečnog presjeka metalnog nosača
- A_b - površina betonskog pojasa
- a - udaljenost težišta betonskog i metalnog dijela presjeka
- a_a - udaljenost težišta metalnog dijela od težišta cijelog spregnutog presjeka
- a_b - udaljenost težišta betonskog dijela presjeka od težišta ukupnog spregnutog presjeka
- I_a - moment inercije metalnog profila
- I_b - moment inercije betonskog dijela presjeka
- I_i - moment inercije idealnog spregnutog presjeka

(1), (2), (3), (4) - točke u kojima se kontrolira naprezanje spregnutog nosača

3.1. Pretpostavke sistema i opterećenja za koje se radi proračun

- a) Spregnuti nosač se prvenstveno koristi u visokogradnji, za izradu međukatnih konstrukcija
- b) Sprezanje se vrši za korisno i dio stalnog opterećenja
- c) Betonska ploča betonira se na hladnooblikovanom valovitom limu
- d) Statički sustav nosača je prosta greda i kontrola naprezanja vrši se za maksimalni moment savijanja
- e) Metalni dio presjeka čine valjani IPE nosači.

Kao bazne varijable spregnutog presjeka, čijim variranjem dobivamo niz različitih praktičnih situacija pri izvođenju ovakvih elemenata, uzete su:

1. b - razinak spregnutih nosača (varira u intervalu 180 - 300 cm)

2. L - raspon spregnutih nosača (varira u intervalu 600 - 1200 cm)
3. h_p - debljina betonske ploče (varira u intervalu 10 - 14 cm)
4. Kvaliteta čelika (sa granicom razvlačenja 240 N/mm² i 360 N/mm²)
5. g - dio stalnog opterećenja za koji se vrši sprezanje (varira od 1,0 - 2,5 kN/m²)
6. p - korisno opterećenje (varira od 2,0 - 6,0 kN/m²)
7. IPE nosač (varira od IPE 200 do IPE 500)

Kritična veličina prema kojoj se proračun valorizira je rubno naprezanje u donjem vlačnom pojasu metalnog nosača (točka (4) na slici 1.). Varijacije baznih parametara su birane tako da ova kritična veličina iznosi 80 - 90% granične vrijednosti. Na Sl. 3. prikazan je tok proračuna spregnutog nosača programom napisanim uz pridržavanje gornjih uvjeta.

4. Modeliranje i metode rješavanja problema ekspertnim sustavom

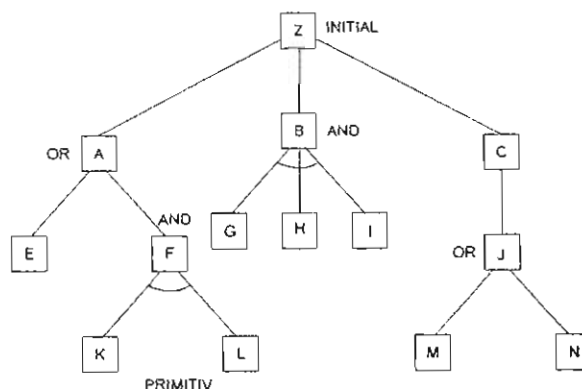
Poznato je da se pri rješavanju problema E.S. problem mora "modelirati" određenim simbolima, kojih će kasnije analiziranje morati dovesti do pojedinih zaključaka. Simbolima predstavljamo naše znanje, činjenice koje su nam poznate u svezi sa problemima. Raznještanjem simbola međusobno opet modeliramo određene odnose koje znamo da postoje među poznatim činjenicama. U modeliranju problema za rješavanje E.S. obično se koriste dva modela:

A) Položaj - prostor (State - space)

B) Raščlanjivanje problema (Reduction representation)

U modelu raščlanjivanja problema starta se sa globalnim (potpunim) prikazom problema, prema određenim uvjetima se onda problem "razbija" na manje podcjelove. Dakle, podproblemi se razrađuju do te mjere da njihovo kombiniranje može kasnije dovesti do rješenja. Raščlanjivanje problema najbolje se može ilustrirati i/ili grafom (and/or), uz neke konvencije (Sl. 2.):

- a) Primitivni (najjednostavniji) dijelovi problema su predstavljeni krajnjim (rubnim) čvorom u dijagramu
- b) Ako se promatra čvor sa nekoliko veza koje su međusobno povezane lukom, onda je to AND čvor.
- c) Čvor sa nekoliko veza koje nisu međusobno povezane lukom je OR (ILI) čvor.



Sl. 2. Model raščlanjivanja problema (Problem reduction representation)

 ** PRELIMINARNI PRORACUN SPREGNUTOG PRESJEKA **
 **
 ** DAMIR MARKULAK 1/4/94/ **

ULAZNI PARAMETRI SPREGNUTOG PRESJEKA

RASPON SPREGNUTOG NOSACA.....: l = 600
 RAZMAK SPREGNUTIH NOSACA.....: b = 180
 DEBLJINA BETONSKE PLOCE.....: hp = 10
 GRANICA RAZVLACENJA CELIKA.....: Vi = 36
 VISINA CELICNOG PROFILA.....: hc = 20
 POVRšina CELICNOG PROFILA.....: Ac = 28.5
 MOMENT INERCIJE CELICNOG PROFILA.: Ic = 1940

Sudjelujuća sirina betonskog presjeka : bs (cm) 200.3081

Analiza opterećenja :

1.Vlastita težina nosaca :gv (kN/m) 6.15
 2.Stalno opterećenje :g (kN/m) 1.8
 3.Korisno opterećenje :p (kN/m) 3.6

Geometrijske karakteristike presjeka u to

Omjer modula elastičnosti no= 5.833333
 Površina betonskog dijela Ab (cm²)= 2003.081
 Moment inercije bet. dijela Ib (cm⁴)= 16692.34
 Površina spregnutog presjeka Ai (cm²)= 371.8852
 Moment inercije spreg. presj. Ii (cm⁴)= 16406.84
 Udaljenost težišta bet. i cel. a (cm) = 21
 ab (cm) = 1.609367
 ac (cm) = 19.39063
 Položaj karakterističnih tocaka presjeka :
 ycd(cm) = 29.39063
 ycg(cm) = 9.390633
 ybg(cm) = -6.609367
 ybd(cm) = 3.390633

Omjer modula elastičnosti np= 25.725
 Površina spregnutog presjeka Ai(cm²)= 106.3651
 Moment inercije spreg. pre. Ii(cm⁴)= 11789.71
 Položaj karakterističnih tocaka presjeka :
 ycd(cm)= 25.37316
 ycg(cm)= 5.373156
 ybg(cm)= -10.62684
 ybd(cm)= -6.268444

Geometrijske karakteristike spr. pr.- skupljanje betona

Sila na nosac zbog skupljanja P(kN)= 888.9629
 Omjer modula elastičnosti ns= 15.23667
 Površina spregnutog presjeka Ai(cm²)= 159.9645
 Moment inercije spr. presj. Ii(cm⁴)= 13364.78
 Položaj karakterističnih tocaka presjeka :
 ycd(cm)= 27.25854
 ycg(cm)= 7.258545
 ybg(cm)= -8.741455
 ybd(cm)= 1.258545

Statičke veličine

1.Moment savijanja od vlastite težine nosaca
 Mg(kNcm)= 3736.125
 2.Moment savijanja od ostalog korisnog opterećenja
 Mq(kNcm)= 1093.5
 3.Moment savijanja od korisnog opterećenja
 Mp(kNcm)= 2430
 4.Ukupni moment od opterećenja 2+3
 Mu(kNcm)= 3523.5

Kontrola naprezanja nosaca u vrijeme to :

0.Naprezanja u celiku (svjezi beton,vl.tez...)
 Sigma cg(celik gore) (kN/cm²)= -19.25838
 Sigma cd = 19.25838

1.Utjecaj stalnog i korisnog opterećenja :
 Sigma bg (kN/cm²)= -2433282
 Sigma bd = 1248284
 Sigma cg = 2.016714
 Sigma cd = 6.311873

2.Ukupno naprezanje nosaca (0+1)
 *** Sigma bg (kN/cm²)= -2433282
 *** Sigma bd = 1248284
 *** Sigma cg (kN/cm²)= -17.24166
 *** Sigma cd = 25.57025

Kontrola naprezanja nosaca u vrijeme t

1. Utjecaj skupljanja betona :
 Sigma bg (kN/cm²)= -6.370731E-02
 Sigma bd = 5.851291E-02
 Sigma cg = -3.750859
 Sigma cd = 1.226426

2.Utjecaj stalnog i korisnog opterećenja:
 Sigma bg (kN/cm²)= -1234582
 Sigma bd = -7.282416E-03
 Sigma cg = 1.605834
 Sigma cd = 7.583079

3.Ukupno naprezanje nosaca (0+1+2):
 **** Sigma bg (kN/cm²)= -1871656
 **** Sigma bd = 5.123049E-02
 **** Sigma cg = -21.4034
 **** Sigma cd = 28.06788

KG Software Known Causes data Base 04/18/94

System : Spregnuti nosac 97/110 PgUp/PgDn

Cause : IPE7 400

Symptoms associated with the cause

A & Debljina A.B. ploce	10 cm to 12 cm
A & Korisno opterecenje	2.0 kN/m ² to 2.5 kN/m ²
A & Kvalitet celika	24 kN/cm ²
A & Raspon nosaca	800 cm to 900 cm
A & Razmak nosaca	180 cm to 200 cm
A & Stalno opterecenje	1.0 kN/m ² to 2.5 kN/m ²
S Add C	

Kvalitet celika

36 kN/cm²

24 kN/cm²

S Add C

Select Esc Quit

KG Software GURU Solveur 04/18/94

Describe known causes

System : Spregnuti nosac

Update the symptoms in this system

Select a symptom

Debljina A.B. ploce

Korisno opterecenje

Kvalitet celika

Raspon nosaca

Razmak nosaca

Stalno opterecenje

S Add C

F1 Help F2 Rename F3 Cut F4 Copy F5 Paste Select

KG Software Update Symptoms 04/18/94

System : Spregnuti nosac 4/6 PgUp/PgDn

Symptom name : Raspon nosaca

Question : Give the value of Raspon nosaca (min. 600 cm max. 1000cm)?

Values

600 cm

600 cm to 650 cm

600 cm to 700 cm

650 cm to 750 cm

700 cm to 800 cm

750 cm to 850 cm

800 cm to 900 cm

850 cm to 950 cm

900 cm to 1000 cm

Help: Edit F1 Help F8 Print Esc Quit Importance: Major

KG Software GURU Solveur 04/18/94

Diagnose problems

Spregnuti nosac

Select a cause, to display its remedy

Cause	Likelihood
IPE1 300	95
IPE4 220	67
IPE1 200	52

Your previous answers are

Debljina A.B. ploce	10 cm to 12 cm
Korisno opterecenje	2.0 kN/m ²
Kvalitet celika	36 kN/cm ²
Raspon nosaca	200 cm to 800 cm
Razmak nosaca	200 cm to 250 cm
Stalno opterecenje	1.0 kN/m ² to 1.5 kN/m ²

Continue

Sl. 4. Radni ekran ekspertnog sustava
Working screen expert system

6. Rješavanje problema ekspertnim sustavom

Kad je problem simbolički opisan sa reprezentirajućim modelom, kompjuter treba metodiku pomoću koje će koristiti konstrukciju modela i tako pronaći rješenje.

Pretraživanje modela je ključ za rješenje problema. Trik za efikasno rješenje modela je usklađivanje tehnike koja će se koristiti za rješavanje sa tipom problema. Pod pretraživanjem se podrazumijeva izbor puta kroz bazu znanja (model!) te pronalaženje rješenja. Ako je izabrani put bezuspješan, pokušava se sa alternativnim. Kao što se vidi, metoda rješavanja problema pomoću E.S. je slična rješavanju labirinta. Primjeri tehnika pretraživanja korištenih u E.S. su: Depth first (dubina prvo), Breadth first (širina prvo), Best first (najbolje prvo), Hill climbing ("penjanje uz brijeg"), Branch and bound ("grane i granice").

Za rješavanje konkretnog primjera spregnutog nosača koristi se GURU - solver.

Na Sl. 4. prikazuju se rezultati analize ekspertnim sustavom nakon što su mu učitane prethodno generirane vrijednosti.

7. Zaključak

Izabiranjem (preko ponuđenih opcija) određene kombinacije ulaznih parametara, sistem vrši pretraživanje bez znanja. Kao rezultat pretraživanja prezentira se izlaz kao jedna ili više mogućih varijanti sa pripadnim vjerojatnostima (likelihood) nastupa svake od njih. Prihvati li se neko od ponuđenih rješenja ta je sesija završena, dok odbijanjem svih varijanti sistem traži objašnjenja kojima će generirati novo pravilo. Dakle, svakim daljnjim korištenjem E.S. stvara se novo znanje iz tog područja.

8. Literatura

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FINITE ELEMENTS COMMUNICATION METHOD

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Abstract

The paper describes a method utilizing the pretrained run time neural network communication capability to communicate, send and receive messages to and from surrounding FE nodes and receive messages caused by foreign disturbances as well as boundary conditions. Probably this method could be utilized on parallel machines. A library of trained runtime network FE behaviors should be established prior to the use of thus method.

Sažetak

Opisuje se ideja o mogućnosti korištenja međusobnog komuniciranja čvorova prethodno trenirane neuralne mreže konačnih elemenata. Čvorovi mreže KE mogu mogu međusobno komunicirati razmjenjujući informacije o naponskom i deformacionom stanju susjednih elemenata mreže te priloženim poremećajnim silama i rubnim uvjetima. Stvarna aplikacija tog postupka primjenjiva je na paralelnim strojevima, a preduvjet je stvaranje biblioteke treniranih run time verzija KE.

1. Introduction

There could be a link among the R. V. Southwells (6,7) and the H. Cross relaxation method, as well as their offspring's, and the McCulloh-Pitts (2) neuron model development (11). Both methods are offspring of the same time (about 1930 - 1950). The computer technology was not developed at that time (11). But in the bases of that method there was a clear idea of communication among links, nodes, and that could be accomplished by continuous information exchange - node relaxation.

Today nearly all of the structural analysis are based on the FEM (10). There is a tremendous developing going on utilizing the wide use of ever faster personal computers.

As was the relaxation method pushed away for ever, so was nearly for two decades the neural network application (11). Before few years the idea of the neural network was revived, and to day the NN are used in various fields performing various tasks: pattern recognition's, language and speech recognition, prognosis making, process controlling and management's as well as other fields (9, 11). The primary ingredient in the success of the various projects seems to be the domain knowledge. We have even applied the, bay human expert trained NN, to the fields of variable energy control input to dynamically loaded structures (12, 13, 14). It was just a computer simulation, but it shows that it could be done, using computer model simulations and game playing experiences (12,13). Another application of the NN was the implementation of the run-time NN in the HALLOPTexpert system (15) for 3D glulam & timber structure design (by D. Delic). We have gained enough experience only in the very recent past, and not enough time has elapsed yet for large numbers of application to be developed. For such applications another types of computers or neuroboards are needed.

As far as possible we to day use expert systems in engineering applications (1).

Then the object oriented programming and the object oriented expert systems are developments of classical ES based on rule forming. The objects have a number of aspects or attributes. Each aspect may have many facets that are sets of associated items. An object may send and receive messages, and may behave in a proscribed way, triggering or inhibiting some actions, when receiving some messages.

There is a critic of the contemporary FE method's application. It is clear to everyone, that the structures (whatever we assume it may be) does not behave in the way we assume it do. Inside a strained structure there are no mathematical calculations going on. There is just a flow of information until the equilibrium state is reached. The material just knows how to behave because the matter has it "inherited" through its atomic, crystallographic and molecular structure, by the way it is incorporated in the structure and surrounded by other forms, various loadings and boundary conditions. In the interaction of the crystallographic structure of the matter there is no neglecting of any phenomena and there are no model calculation simplifications, as we make them in structural analysis and calculations. All of the inherited effects of the matter are always present.

Therefore we are not far from the idea to imitate the recollection of discrete nodes in a FE discrete mesh by the way of object oriented expert system to which we may delegate all of the natural behaviors (as we know them to day) by the way of trained neural network. That is in the way of thought and experiments done and published by J. Ghaboussi and J. H. Garrett Jr. who have taught a neural network to behave as the materials are dynamically behaving, in a test machine, by the way of transferring the real material behaviors (obtained by lab. Testing) to neural networks (3,4). We than may in a broader way transfer all of the FE, as well as the material, behavior knowledge to a trained NN. In the Ghaboussi-Garrett model of biaxial material behavior of concrete (3, 4) the input NN layer consists of

$\sigma_1, \sigma_2, \epsilon_1, \epsilon_2, \Delta\sigma_1, \Delta\sigma_2$, and the output layer consists of only two neurons $\Delta\epsilon_1$ and $\Delta\epsilon_2$ describing the stress and strain space. In the model there are two layers of hidden neurons each of 40 neurons. This model shows that the material behavior could be adequately represented by a trained NN.

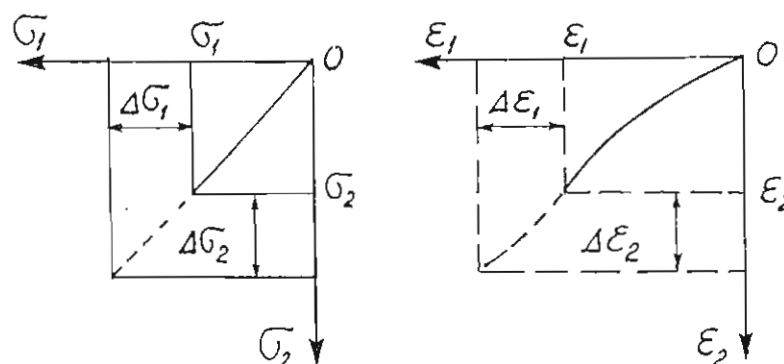


Fig. 1. The stress and deformation space in the Ghaboussi-Garet biaxial material behavior model (4).

2. The Neural Network.

The neural network is a parallel, distributed information processing structure consisting of processing elements (which may possess local memory and can carry out localized information processing operations) interconnected via unidirectional signal channels called connections. Each PE has a single output connection that branches into as many collateral connections as desired (11). Each carries the same processing element output signal. The processing element output signal may be of any mathematical type desired. We may say that the behavior of the neural network can be found in the processing of the biological neural cell as we today see it works. But we have to be careful to compare a neural processing element of a network cell to the real neuron. The neurons are much more complicated than are the PE used in neurocomputing, and their functions are not yet understood (5,9,11).

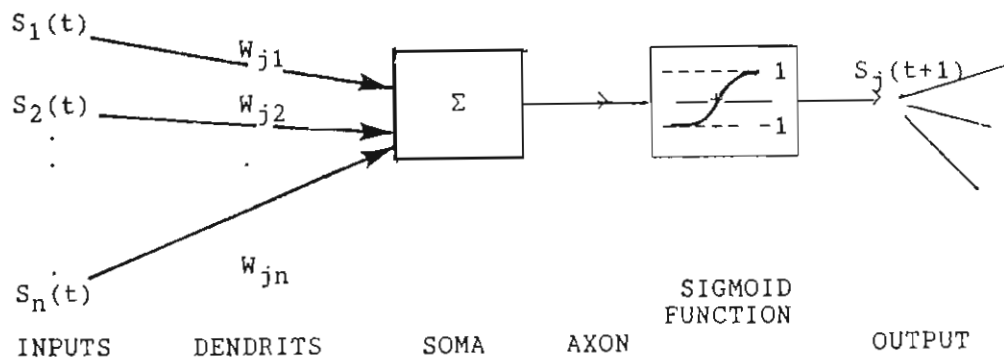


Fig. 2. The processing element (PE) of the neural network (NN).

We have used until now the sigmoid function.

$$f(x) = 1 / (1 + e^{-x}) \quad (1)$$

$$z = \sum \{W_i \cdot S_i(t) - \theta_i\} \quad (2)$$

$$S_i(t+1) = f(x) \quad (3)$$

where:

- $f(x)$ the sigmoid function. The Ward System Neuroshell uses this one.
- W_i weights (or ponderers) of neuron connections.
- $t, t+1$ two consecutive time steps in weight calculations.
- θ_i the neuron output threshold.
- S_i Inputs

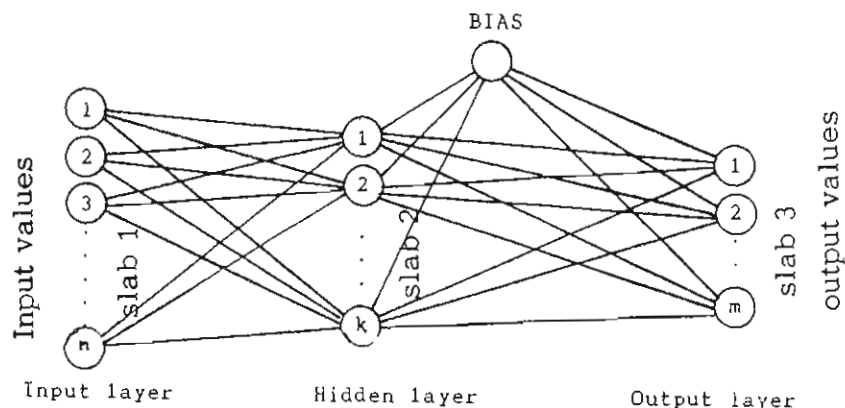


Fig. 3. A neural network with one hidden layer.

In the NN shell we have used until now (the backpropagation Wards NeuroShell), we have only one hidden layer, the input and output slabs connected to the correcting BIAS hidden node (8). Using this shell we have obtained some interesting NN, for timber classification (15, 16), two rooms dwelling architecture analysis (still in work by B. Baletic as a PhD thesis), the advisory ES for house damage classification and sanations (17, 18), and the already mentioned NN for variable energy input control of bridges loaded by overpassing vehicles, where we have encapsulated the knowledge of the dynamical structure behavior and the human reactions to sensorial information in structure stability control (9, 12, 13, 14). In meanwhile we have obtained the BRAINMAKER and the NeuroShell2 for Windows, but we have not enough time to use them.

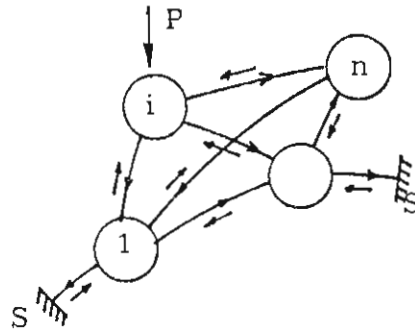


Fig. 4. The communication network of interlinked finite nodes.

The idea of FE node communication among nodes, was applied to a simple node, using a Cross relaxation program, and a trained runtime NN instead of the use of classical node distributing factors. This works as expected well, because there is no doubt that a trained NN may simply be switched to distribute the unbalanced node moments. And it is obvious that the NN can be trained on samples.

There was just difficulties, concerning the one way information path (from input layer to the output layer) of the Ward NeuroShell. To overcome that, the input and the output nodes are merged together as input and output slabs allowing both way communications.

The way to train this NN is by forming a character set of generalized forces and corresponding displacements and a set of generalized displacements and forces. The symbol " $\longleftrightarrow$ " symbolizes an active set of input and corresponding output value:

Set 1:	$\longleftrightarrow$ set:	forces
	inactive:	displacements
	$\longleftrightarrow$ set:	displacements
	inactive:	forces
Set 2:	inactive:	forces
	$\longleftrightarrow$ set:	displacements
	inactive:	displacements
	$\longleftrightarrow$ set:	forces

In the training the zero values (0.00) of forces (moments, stresses) and displacement (deformations, strains, distribution coefficients) should be included as corresponding numeric values.

The input vector, for the moment distribution example, consists of scaled moment and a number of distribution coefficients $k(I)$ and distributed scaled moment $M(I)$. The output vector consists of distributed scaled moment $M(I)$, the scaled moment M and the distribution coefficients $k(I)$. The net was trained with a randomly generated corresponding set of values, each corresponding to another.

The input vector $I(I)$ is:

[scaled_moment, distribution_coefficients: k_1, k_2, k_3 , distributed_scaled_moments: M_1, M_2, M_3]

The output vector $O(I)$ is:

[distributed_scaled_moments: M_1, M_2, M_3 , scaled_moment, distribution coefficients: k_1, k_2, k_3]

In the sense of training analog neural networks, the scaling and coding of presented values should be done to hold the upper and lower limits in a proscribed range: from 0.00 to 1.00. Shown is just a section of the graphic presentation of the training set and the learned answers: there is no difference for unseen distribution coefficients k_1 .

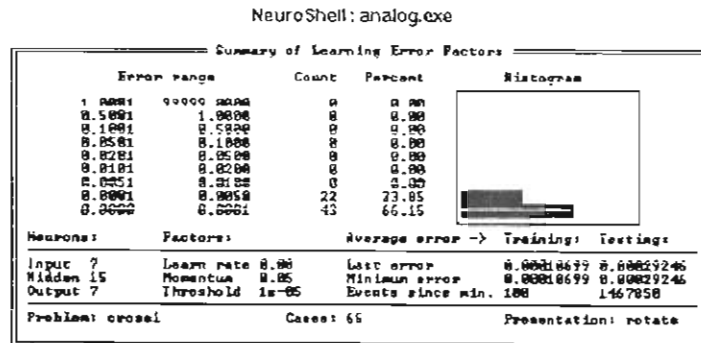


Fig. 5. The learning NeuroShell screen.

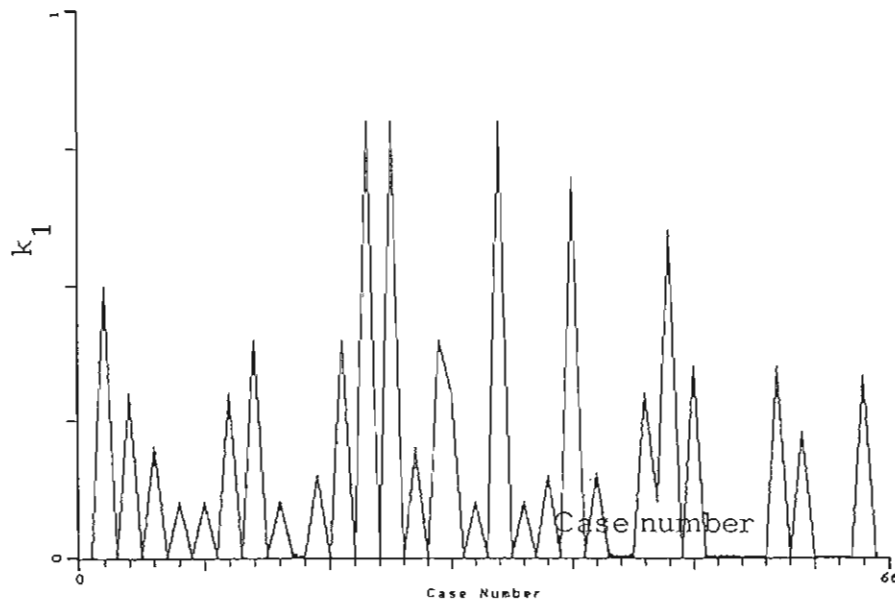


Fig. 7. The graphic representation of the training set (whole line), and the learned answers (dotted) to unseen values for distribution coefficient k_1 . The whole and the dotted line are indistinguishable.

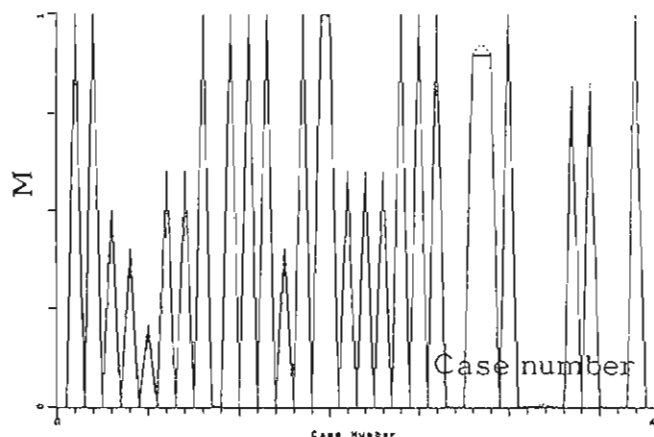


Fig. 8. The graphic representation for the value of M : the learned values from training sets (whole line) and the answers on the seen values (dotted) are nearly indistinguishable.

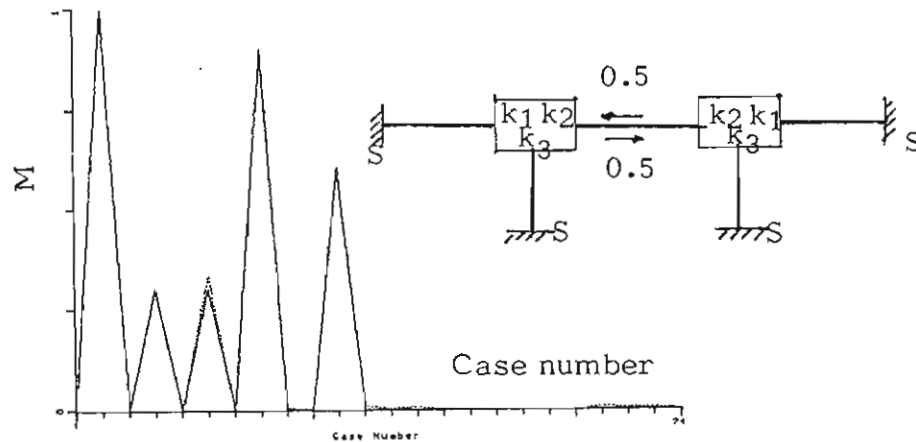


Fig. 9. The graphical display of the NN answer to an unseen set of randomly chosen input and output values: the true values (whole line) and the obtained answers (dotted). The both lines are nearly indistinguishable.

The NN architecture for the moment node distribution, had only 7 input and therefore 7 output nodes with 15 hidden nodes in the slabs. This is obvious a very primitive network.

Now the same idea may be applied to the NN for inputting forces and displacements and outputting displacement and forces to a triangular planar FE, and later on to a broader number of other FE. Here is shown the idea of a Plane Stress, or Plane Strain triangular FE of isotropic material with a constant thickness, and a constant E and G values and an inclination φ to the global coordinate system. We may write the vectors in the input and output slab of a both way passable NN.

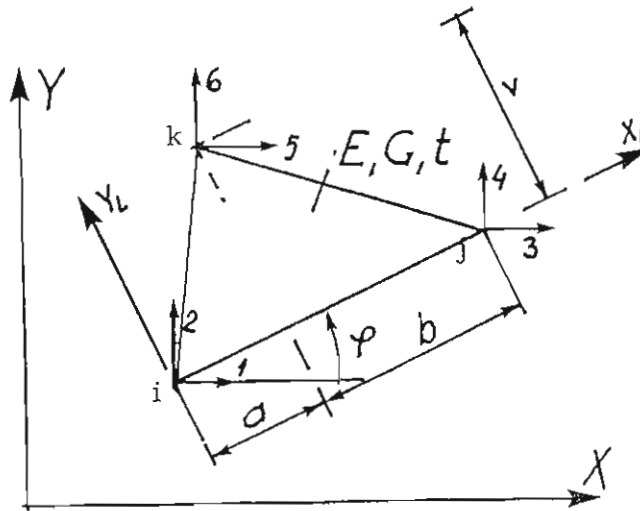


Fig. 10. A triangular planar FE.

The input vector can be organized as:

$$[I(i)] = [EX, EY, t, \varphi, a/b, v/a, DXi, DYi, DXj, DYj, DXk, DYk, exi, eyi, exj, eyj, exk, eyk, \sigma_x, \sigma_y, \tau_{xy}, P1, P2]$$

The output vector is then:

$$[O(i)] = [DXi, DYi, DXj, DYj, DXk, DYk, exi, eyi, exj, eyj, exk, eyk, \sigma_x, \sigma_y, \tau_{xy}, P1, P2].$$

This FE NN element will have after training all the attributes of the stiffness matrix as well as element transformation matrix behavior implemented.

The training is done by using a FE program which may generate a large number of sets preferably a parametric input/output generator. We plan to use the parametric input/output of the COSMOS/M or M-STRUDL capabilities to generate the training set. The other way around is to write a program which will generate the training set. The training set consists of a large number of various sets in which each of the parameter is changed gradually, holding the other as constant values. With this training set the NN is trained until it is performing reasonably well. When the NN is formed and trained the runtime version is formed in one of the higher computer languages, using the run-time program generator. The run-time FE NN is then able to behave in a precisely proscribed (trained) way. The level of rupture, large deformation, and the nonlinear behavior can be included as well as other considerations: creep, shrinking, fatigue etc.

The communication among nodes of FE mesh is obtained by a information exchange communication program which starts from the loaded elements and are spreading and receiving information toward the boundary nodes, and from them in a returning wave toward the information spreading start, until a state of equilibrium is reached.

In this way a data bank is obtained as a runtime version of a FE type. Other RT version for other types of FE may be obtained in a similar way, a whole RT library of FE node behavior, which may then be included in a information network exchange. But without special computers or neuroboards any attempt to do this on a PC is doomed to failure due to the waste computer time consuming in the NN training process on ordinary PC machines.

The NN run-time version of a FE behaviors does not calculate any theoretical calculations, nor does it use matrix formulation methods to calculate its behaviors. It just know, by inheritance obtained by training on examples, how to behave in various exposed conditions and surroundings. Its experience is shared via the communication network with other FE.

3. Conclusion

It is still too early to make any conclusion. Then obviously there is a question why this should be done. At this stage it is shown that it could be done. The second question concerning the WHY question is more difficult to answer. It seems to me that it is more in the way of process simulation on parallel machines. It may be just another approach to the FE method, where the FE are processing elements itself.

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Object Oriented Programming and Finite Element Method

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Abstract

A new approach for designing finite element programs is presented in this paper. Today, the engineering analysis system is usually integrated into the design process, which requires a considerable effort to be spent on non-traditional areas of programming the numerical analysis programs, *e.g.* human-machine interface, expert systems, data base, and visualisation. Most of the existing finite element systems are written in FORTRAN by numerical analysts. Such codes are very difficult to maintain and integrate. Here the alternative way of constructing finite element programs using the object oriented programming is presented. Object oriented programming techniques are discussed, followed by some examples of applying these techniques to the finite element program.

1 Introduction

Systems for engineering numerical analysis have been traditionally written by numerical analysts rather than software engineers, because the analyst was the one who really understood the necessary mathematical abstractions. While the software engineer could not move into the analyst's domain, the analyst could understand sufficient software techniques to assemble a working program. Over the years the analysts have been able to extend their expertise to other areas, such as computer graphics. However, the amount of knowledge that can be commanded by a single human, no matter how intelligent, is limited, and today the analysts have reached the limits with respect to the development of the next generation of analysis software [3]. Therefore the problem appears to be the organisation of the development of relevant systems, which requires the co-operative efforts from experts from several fields. Approach called the Object Oriented Programming could be solution to that problem.

2 Object oriented programming

In traditional, procedure-oriented way of programming, a program describes a series of steps to be performed, *i.e.* an algorithm. In the object-oriented way of programming, a program describes a system of interacting objects [2]. Object oriented programming provides for the description of "what" must be done, instead of "how" the computer should perform the task. Instructions "how" are usually hidden in objects. It is known what the object should do on the outside, but it does not matter how it is done inside the object.

Abstraction is the process of ignoring details in order to concentrate on essential tasks. A programming language is traditionally considered high-level, if it supports a high degree of abstraction. If the program written in low-level language, *e.g.* assembler is compared to the same program written in high-level language, *e.g.* C, FORTRAN or Pascal, it is obvious that many statements directing the computer to perform certain tasks are absent from the high-level languages. Usually the programmers are not concerned with what happens on the lower level, since the low level commands are included automatically included by a compiler. The high-level program gives a much more abstract description of what computer does, which makes program clearer and easier to understand. While traditional languages support abstraction, the object oriented languages provide more powerful abstraction mechanisms.

There are different types of abstraction in object oriented programming. The most common form of abstraction is procedural abstraction. This type of abstraction is similar to the abstraction of high-level languages. It enables the substitution of a set of commands with a single command. Another type of abstraction is the data abstraction. Data abstraction means that it is no longer important how a data type is represented by computer, *i.e.* as binary numbers. Since most programmers prefer to think in terms of decimal numbers, most high-level languages support integer and floating point data types. Different levels of data abstraction can be found in traditional high-level programming languages, where the level of abstraction varies, *e.g.* lower level of data abstraction in FORTRAN 77 and higher level in C or Pascal with the use of structures, unions, *etc.*

Object oriented languages like C++ [2] combine the procedural and the data abstraction in a form of classes. A class describes the high-level entity, which is called an object. By designing a program around abstract entities that have their own set of operations, rather than using data structures with build-in types, a program is more independent from the implementation details.

Beside of abstraction some other techniques like encapsulation, class hierarchy, inheritance and polymorphism are available in object oriented languages. The most important issue in object oriented programming is the design of a computer program. Object oriented design is not a top-down technique where one would start with the description of the major steps and then refine that description at lower levels of program. Nor is it bottom-up technique, where one would start with small classes and then build a program around them. Object oriented design involves working at both high and low levels of abstraction at all stages. The most important questions that need to be answered when designing an object oriented program are "*What are the objects?*" or "*What are the active entities in this program?*". Object oriented design of computer programs requires the following steps to be performed:

- Identification of the classes
- Assignment of attributes and behaviour
- Search for relationships between the classes
- Arrangement of the classes into hierarchies

Object oriented programming is an iterative process. Therefore, if each step is performed just once in the process, without regard to the later steps, it is unlikely to get a useful set of classes. A good initial design is always desirable and will speed up the development process.

3 Finite element method programming

Considering a simple finite element analysis program [4] and using the object oriented programming one would first try to answer the question "*What are the objects?*". First object is obviously the finite element. Finite element is defined with nodes, but nodes are far too important entity to be considered only as a part of the finite element object. Therefore finite element nodes are the second object one can identify. There is a lot of mathematics involved in the finite element programs, usually on the vector and matrix level. Therefore it is very useful to have classes identifying objects like a vector or matrix. Beside that one would need classes defining physical properties data, load data, constraint data and general data needed in the analysis. Decision must be made whether or not an entity is an active object, which interacts with other objects with its procedures and data or just an ordinary data structure, which only provides for data storage.

In the next step one would need to assign the attributes and define the behaviour of objects. Finite element can be defined with the finite element number, number of nodes, list of nodes and physical property number, *etc.* Any of this data can be treated either as private or public data in the finite element object. Private data means that no one can access this data except for the object itself, while public data identifies the data accessible to everyone outside the object. The behaviour of the object must be defined next. The finite element object should be able to calculate its own shape functions, stiffness matrix, *etc.* It should read or write finite element data or even draw a picture of the finite element.

The third step is establishing the relationships between objects. Interfaces can be build between objects so that they can pass the data from one to the another. Interfaces are useful because there is no need for the object A to know how data is represented in the object B. That enables the changing of data representation in object B without any consequences for the object A. Example 1 illustrates a simple finite element class written in C++ programming language:

```

class FiniteElement
{
    public:
        FiniteElement(const int en); // Constructor
        FiniteElement();             // Anonymous constructor
        int elNo();                  // Interface for the element number
        int matNo();                // Interface for the material number
        void matNo(const int ma);    // Interface for storing the material number
        int nnodes();               // Interface for the number of nodes
        int lnode(int index);       // Interface for the specific node number
        void lnode(int *nod);       // Interface for storing node numbers
        void sfr(double *exisp, double *etasp, \ // Shape function of the element
                double *shape, double *deriv);
        void elastic(General &gd, \ // Stiffness of the element
                FiniteElementNode **nodlis, Material **mat, Vector &p, Matrix &s);
        FiniteElement~();           // Destructor
    private:
        int ElNo;                   // Element number
        int Nnodes;                 // Number of nodes
        int Nodes[2];              // Set of nodes
        int Matr;                  // Material number
};

```

Example 1: Simple finite element class

As can be observed from Example 1, the class mainly consists of interfaces for data exchange with other objects. Only the essential data is stored in the finite element class, while everything else is derived through class functions. Class in C++ always begins with the constructor, which allocates the computer memory and initialises the data. Class usually ends with the destructor, which releases the computer memory and so destroys the object data.

The last step is the arrangement of classes into hierarchies. Finite elements are of different types. The same mesh may consist of quadrilateral and triangular finite elements. In traditional programming languages it is usual to use the conditional statements in order to determine what type of finite element is the current element and then use the appropriate procedures. In the object oriented programming classes of finite elements can be hierarchically arranged so that every type of finite element has its own class. Addressing a finite element automatically invokes the specific finite element object procedure.

Example 2 illustrates the finite element class with two types of finite elements and any type of the finite element can be added at the later stage without any change in the existing classes.

```

/* Parent finite elements class */
class FiniteElement
{
    public:
        FiniteElement(const int en);
        FiniteElement();
        int elNo();
        int matNo();
        void matNo(const int ma);
        virtual int nnodes();
        virtual int lnode(int index);
        virtual void lnode(int *nod);
        virtual void sfr(double *exisp, double *etasp, \
                        double *shape, double *deriv);
        virtual void elastic(General &gd, FiniteElementNode **nodlis, \
                            Material **mat, Vector &p, Matrix &s);

    private:
        int ElNo;
        int Matr;
};

/* Child finite element class for linear beam finite element */
class LB2 : public FiniteElement
{
    public:
        LB2(const int en);
        int nnodes();
        int lnode(int index);
        void lnode(int *nod);
        void sfr(double *exisp, double *etasp, \
                double *shape, double *deriv);
        void elastic(General &gd, FiniteElementNode **nodlis, \
                    Material **mat, Vector &p, Matrix &s);

    private:
        int Nnodes;
        int Nodes[2];
};

/* Child finite element class for quadratic beam finite element */
class QB3 : public FiniteElement
{
    public:
        QB3(const int en);
        int nnodes();
        int lnode(int index);
        void lnode(int *nod);
        void sfr(double *exisp, double *etasp, \
                double *shape, double *deriv);
        void elastic(General &gd, FiniteElementNode **nodlis, \
                    Material **mat, Vector &p, Matrix &s);

    private:
        int Nnodes;
        int Nodes[3];
};

```

Example 2: Hierarchical finite element class

Examples 1 and 2 illustrate just the definition of classes and every function and interface of the class should be defined with appropriate program coding. It is wrong to assume that the object oriented programming would reduce the extent of computer programming of numerical methods [1]. Numerical methods are generally independent of programming languages. However, the usage of special classes, like vector and matrix class makes the code much clearer and easier to understand and maintain. As an example one should consider a simple program shown in Example 3, where the vectors behave as any other numerical type.

```
Vector a(3)={1.,2.,3.};
Vector b(3)={4.,5.,6.};
Vector c(3);
float scalar;
c=a+b;
scalar=a*b;
```

Example 3: Overloading of operators in C++

Similar operations as those shown in example 3 are also possible between matrices, vector and matrix, scalar and matrix, *etc.* Furthermore, the matrix object should be able to assemble the stiffness matrix and to solve the system of equations in a way as it is shown in Example 4.

```
const int N;           // N - number of unknowns
Vector result(N);      // vector for displacements
Vector load(N);        // vector for external load
Matrix stiffness(N,N); // stiffness matrix
FiniteElement *elements; // array of finite elements
stiffness.assemble(elements); // assembling of stiffness matrix
result=stiffness.gauss(load); // Gaussian elimination
```

Example 4: Simple finite element system solution program

Even higher level of abstraction is possible with the definition of the class mesh, which consists of the finite element objects, finite element nodes, physical properties data, external load data, constraints data, *etc.* Then a complete finite element analysis program can be formulated as it is shown in Example 5.


```

#include "mainprog.h"
void main()
{
    FiniteElementMesh mesh("input.dat"); // Definition of the mesh object
    mesh.read();                        // Reading the mesh data
    mesh.solve();                       // Solving the mesh
    mesh.write();                       // Writing the mesh with results
}

```

Example 5: The complete finite element program

4 Conclusions

Object oriented programming can not be viewed as an easy way of writing numerical method programs. No less effort is required in order to build a complete finite element analysis program using the object oriented programming than when writing it in FORTRAN. But it is much easier to add new features and maintain the existing program after the skeleton of the program has been build. Because of that, the object oriented programming is very desirable for team work, where everyone can concentrate only on his part of the program, since there is no need for the consideration of personal programming tricks of co-workers.

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TRANSITION FROM REGULAR TO CHAOTIC OSCILLATIONS OF CUBIC DYNAMICAL SYSTEMS

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Abstract

The improved harmonic balance method and the harmonic acceleration method for solution of differential equation of motion of cubic dynamical systems in the frequency and time domain are described. Since the former method is almost exact and the latter approximate, in order to evaluate reliability of the latter method a correlation analysis is performed. Response of a cubic system in the symmetry breaking region, as the first step in the transition to chaos, where the both methods are operable, is determined. The harmonic acceleration method is found to be a sophisticated tool for the time integration.

1. INTRODUCTION

In order to design an optimal and safe nonlinear dynamical system, it is necessary to analyse its nonlinear phenomena, which does not occur in case of a linear system. This problem is usually studied by considering a single-degree-of-freedom system with a cubic damping and restoring force, subjected to mono-harmonic excitation. The nonlinear phenomena are the shift of natural frequency, multiple response with jumps, subharmonic, superharmonic and super-subharmonic resonances, symmetry breaking, period multiplying bifurcations, transition to chaos through cascades and escape, strange attractor of chaotic motion, fractal escape boundary, etc. /1-8/.

Response of a cubic nonlinear system may be determined in the frequency and time domain. In the former case the method of multiple time scaling, the Galerkin method, and the harmonic balance method are often in use, /1-7/. Time integration of the governing differential equation of motion is generally performed by the Runge-Kutta method, /9/. The advantage of the methods operating in the frequency domain, comparing to the time integration methods, is their accuracy and possibility of theoretical analysis of nonlinear phenomena. However, only a few basic nonlinear problems may be exactly solved in the frequency domain. Therefore, the time integration can not be avoided.

Due to the above reason, the classical harmonic balance method is improved in such a way that the exact solution for regular and symmetry breaking response may be exactly determined, /10/. Furthermore, a more sophisticated method for

time integration, i.e. the harmonic acceleration method is described, /11/. Accuracy of the methods is analysed in case of symmetry breaking response of a nonlinear system with cubic restoring force. The aim is to show that the time integration may be successfully performed in cases where the frequency solution is not possible to be achieved.

2. HARMONIC BALANCE METHOD

The differential equation of motion of a single-degree-of-freedom system with cubic restoring force, imposed to monoharmonic excitation, reads

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2x + \kappa x^3 = a_0 + c_k \cos(k\tilde{\omega}t - \vartheta_k) \quad (1)$$

where x is displacement, ω_0 natural frequency, ξ damping coefficient, κ restoring coefficient, a_0 static load, c_k and $k\tilde{\omega}$ excitation amplitude and frequency respectively, and ϑ_k is a phase angle.

The steady-state solution of (1) is represented by the particular integral. Due to its periodic nature, it is assumed in the form of Fourier series,

$$x = A_0 + \sum_{n=1}^{\infty} A_n \cos n\tilde{\omega}t + \sum_{n=1}^{\infty} B_n \sin n\tilde{\omega}t \quad (2)$$

The unknown coefficients A_0 , A_n and B_n are determined by the improved harmonic balance method. For this purpose it is necessary to extend the cubic displacement function into the Fourier series

$$\begin{aligned} x^3 = & A_0^3 + 3A_0^2 \left(\sum_{n=1}^{\infty} A_n \cos n\tilde{\omega}t + \sum_{n=1}^{\infty} B_n \sin n\tilde{\omega}t \right) + \\ & + 3A_0 \left(S_{cn}^{(2)} + \sum_{n=1}^{\infty} S_{cn}^{(2)} \cos n\tilde{\omega}t + \sum_{n=1}^{\infty} S_{sn}^{(2)} \sin n\tilde{\omega}t \right) + \\ & + S_{cn}^{(3)} + \sum_{n=1}^{\infty} S_{cn}^{(3)} \cos n\tilde{\omega}t + \sum_{n=1}^{\infty} S_{sn}^{(3)} \sin n\tilde{\omega}t \end{aligned} \quad (3)$$

where the coefficients S are sums of products of two and three coefficients A_i and B_i , $i=1,2,\dots$, respectively, which are derived in the Appendix.

Substituting (2) and (3) into (1) and equating harmonics of the same order, the following system of nonlinear algebraic equations is obtained, /10/

$$n=0: \quad (\omega_0^2 + 3\kappa S_{cn}^{(2)})A_0 + \kappa A_0^3 + \kappa S_{cn}^{(3)} = a_0 \quad (4)$$

$$\begin{aligned} 1 \leq n \neq k: \quad & (\omega_0^2 - n^2\tilde{\omega}^2 + 3\kappa A_0^2)A_n + 2\xi\omega_0 n\tilde{\omega}B_n + 3\kappa A_0 S_{cn}^{(2)} + \kappa S_{cn}^{(3)} = 0 \\ & - 2\xi\omega_0 n\tilde{\omega}A_n + (\omega_0^2 - n^2\tilde{\omega}^2 + 3\kappa A_0^2)B_n + 3\kappa A_0 S_{sn}^{(2)} + \kappa S_{sn}^{(3)} = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} n=k: \quad & (\omega_0^2 - k^2\tilde{\omega}^2 + 3\kappa A_0^2)A_k + 3\kappa A_0 S_{ck}^{(2)} + \kappa S_{ck}^{(3)} = c_k \cos \vartheta_k \\ & - 2\xi\omega_0 k\tilde{\omega}A_k + 3\kappa A_0 S_{sk}^{(2)} + \kappa S_{sk}^{(3)} = c_k \sin \vartheta_k \end{aligned} \quad (6)$$

In (6) $B_k=0$ is taken into account since the new variable θ_k is introduced. The equations (4), (5) and (6) formulate the problem completely in the frequency domain. Since each equation represents a cubic polynomial of variables A_0 , A_n and B_n , three roots for the variables are possible, with large number of combinations due to coupling.

Thus, the coefficient A_0 is determined from (4) by an iteration procedure as

$$A_0 = \frac{a_0 - \kappa S_0^{(3)}}{\omega_0^2 + \kappa A_0^2 + 3 \kappa S_0^{(2)}} \quad (7a)$$

or

$$A_0' = \pm \left[\frac{a_0}{\kappa A_0} - \frac{\omega_0^2}{\kappa} - 3 S_0^{(2)} - \frac{S_0^{(3)}}{A_0} \right]^{1/2} \quad (7b)$$

Physical meaning of different solutions is discussed within the numerical example.

To solve the coupled equations (5) it is necessary to extract coefficients A_n and B_n from sums S . According to (27) the explicit expressions for the square sums read

$$\begin{aligned} S_{cn}^{(2)} &= A_n A_{2n} + B_n B_{2n} + S_{cn}^{(2)}(A_m, B_m) \\ S_{sn}^{(2)} &= A_n B_{2n} - B_n A_{2n} + S_{sn}^{(2)}(A_m, B_m) \end{aligned} \quad (8)$$

where $m \neq n$ and

$$\begin{aligned} S_{cn}^{(2)}(A_m, B_m) &= \frac{1}{2} \sum_{j=1}^{n-1} [A_j (A_{j+n} + A_{n-j}) + B_j (B_{j+n} - B_{n-j})] + \\ &+ \frac{1}{2} \sum_{j=n+1}^{2n-1} (A_j A_{j-n} + B_j B_{j-n}) + \frac{1}{2} \sum_{j=2n+1}^{\infty} (A_j A_{j-n} + B_j B_{j-n}) + \\ &+ \frac{1}{2} \sum_{j=n+1}^{\infty} (A_j A_{j+n} + B_j B_{j+n}) \\ S_{sn}^{(2)}(A_m, B_m) &= \sum_{j=1}^{n-1} A_j (B_{n-j} + B_{j+n}) - \sum_{j=n+1}^{2n-1} A_j B_{j-n} - \sum_{j=2n+1}^{\infty} A_j B_{j-n} + \\ &+ \sum_{j=n+1}^{\infty} A_j B_{j+n} \end{aligned} \quad (9)$$

In a similar way, the cubic sums may be presented in the form

$$\begin{aligned} S_{cn}^{(3)} &= S_{cn}^{(3)}(A_n) + S_{cn}^{(3)}(B_n) + S_{cn}^{(3)}(A_n B_n) + S_{cn}^{(3)}(A_n^2 B_n) + S_{cn}^{(3)}(A_n B_n^2) + S_{cn}^{(3)}(A_m, B_m) \\ S_{sn}^{(3)} &= S_{sn}^{(3)}(A_n) + S_{sn}^{(3)}(B_n) + S_{sn}^{(3)}(A_n B_n) + S_{sn}^{(3)}(A_n^2 B_n) + S_{sn}^{(3)}(A_n B_n^2) + S_{sn}^{(3)}(A_m, B_m) \end{aligned} \quad (10)$$

that is matter of programming. It is important to point out that A_0 does not occur in sums S .

Substituting (8) and (10) into (5) the following system of pseudo-linear

equations is obtained in the explicit form

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \begin{Bmatrix} P_n \\ Q_n \end{Bmatrix} \quad (11)$$

where

$$\begin{aligned} K_{11} &= \omega_0^2 - n^2 \omega^2 + 3 \kappa A_0^2 + 3 \kappa A_0 A_{2n} + k_{11} \\ K_{12} &= 2 \xi \omega_0 n \tilde{\omega} + 3 \kappa A_0 B_{2n} + k_{12} \\ K_{21} &= -2 \xi \omega_0 n \tilde{\omega} + 3 \kappa A_0 B_{2n} + k_{21} \\ K_{22} &= \omega_0^2 - n^2 \omega^2 + 3 \kappa A_0^2 - 3 \kappa A_0 A_{2n} + k_{22} \\ k_{11} &= \frac{\kappa}{A_n} [S_{cn}^{(3)}(A_n) + S_{cn}^{(3)}(A_n^2 B_n) + S_{cn}^{(3)}(A_n B_n^2)] \\ k_{12} &= \frac{\kappa}{B_n} [S_{cn}^{(3)}(B_n) + S_{cn}^{(3)}(A_n B_n)] \\ k_{21} &= \frac{\kappa}{A_n} [S_{sn}^{(3)}(A_n) + S_{sn}^{(3)}(A_n B_n)] \\ k_{22} &= \frac{\kappa}{B_n} [S_{sn}^{(3)}(B_n) + S_{sn}^{(3)}(A_n^2 B_n) + S_{sn}^{(3)}(A_n B_n^2)] \\ P_n &= -3 \kappa A_0 S_{cn}^{(2)}(A_m, B_m) - \kappa S_{cn}^{(3)}(A_m, B_m) \\ Q_n &= -3 \kappa A_0 S_{sn}^{(2)}(A_m, B_m) - \kappa S_{sn}^{(3)}(A_m, B_m) \end{aligned} \quad (12)$$

The solution of (11) reads

$$A_n = \frac{D_A}{D}, \quad B_n = \frac{D_B}{D} \quad (13)$$

where

$$\begin{aligned} D_A &= P_n K_{22} - Q_n K_{12} \\ D_B &= Q_n K_{11} - P_n K_{21} \\ D &= K_{11} K_{22} - K_{12} K_{21} \end{aligned} \quad (14)$$

The above solution is the first one of the possible three roots. The remaining two roots have not yet been determined exactly. Their physical meaning is discussed in the numerical example.

The equations (6) may be reduced to one equation by eliminating the phase angle ϑ_k . Thus, the third order polynomial for A_k is obtained

$$\begin{aligned} P_k(A_k) &= [(\omega_0^2 - k^2 \omega^2 + 3 \kappa A_0^2) A_k + 3 \kappa A_0 S_{ck}^{(2)} + \kappa S_{ck}^{(3)}]^2 + \\ &+ [-2 \xi \omega_0 k \tilde{\omega} A_k + 3 \kappa A_0 S_{sk}^{(2)} + \kappa S_{sk}^{(3)}]^2 - c_k^2 = 0 \end{aligned} \quad (15)$$

If three values for A_k exist, the well-known jumping phenomenon occurs, /2,3/. The phase angle ϑ_k is determined from (6).

The system of nonlinear algebraic equations (7), (11) and (15) is solved successively. The value for A_k is assumed and monotonously increased. In each

step A_0 and A_n , B_n are determined from (7) and (11) respectively, and the satisfaction of (15) is checked. When the value of A_k is bounded, to find out the exact result the bisection method is employed.

3. HARMONIC ACCELERATION METHOD

In order to perform the time integration of differential equation (1) by the harmonic acceleration method, it is necessary to present (1) in a pseudo-linear form. Also, convergence of the solution is improved, if the excitation frequency as the integration frequency is used. Thus, a convenient form of (1) reads

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = q(t) \quad (16)$$

where $\omega = k\tilde{\omega}$, $\theta_k = 0$, and

$$q(t) = a_0 + c_k \cos \omega t + 2\xi(\omega - \omega_0)\dot{x} + (\omega^2 - \omega_0^2)x - \kappa x^3 \quad (17)$$

The problem is solved iteratively for each time sequence by the algorithm, /11/

$$\{Z\}_{i+1}^{(k+1)} = [T]\{Z\}_i + \{L\} q(t)_{i+1}^{(k)} \quad (18)$$

where i and k are the time and iteration step respectively. The above quantities are specified as follows:

$$\begin{aligned} \{Z\} &= \begin{Bmatrix} Z \\ \dot{Z} \\ \ddot{Z} \end{Bmatrix}, \quad \{L\} = \frac{1}{S} \cdot \begin{Bmatrix} 1 \\ \frac{a}{\Delta t} \\ \frac{b}{\Delta t^2} \end{Bmatrix}, \quad \Delta t = t_{i+1} - t_i \\ [T] &= \begin{bmatrix} \frac{P}{S} & \frac{Q}{S} & \frac{R}{S} \\ \frac{a}{\Delta t}(\frac{P}{S} - 1) & \frac{a}{\Delta t}\frac{Q}{S} - c & \frac{a}{\Delta t}\frac{R}{S} - d\Delta t \\ \frac{b}{\Delta t^2}(\frac{P}{S} - 1) & \frac{b}{\Delta t^2}\frac{Q}{S} - \frac{b}{\Delta t} & \frac{b}{\Delta t^2}\frac{R}{S} - c \end{bmatrix} \\ P &= 2\xi\omega\frac{a}{\Delta t} + \frac{b}{\Delta t^2} \\ Q &= 2\xi\omega c + \frac{b}{\Delta t} \\ R &= 2\xi\omega d\Delta t + c \\ S &= \omega^2 + 2\xi\omega\frac{a}{\Delta t} + \frac{b}{\Delta t^2} \\ a &= \frac{\omega\Delta t}{w}(1 - \cos \omega\Delta t) \\ b &= \frac{\omega^2\Delta t^2}{w} \sin \omega\Delta t \\ c &= \frac{1}{w}(\sin \omega\Delta t - \omega\Delta t \cos \omega\Delta t) \\ d &= \frac{1}{\omega\Delta t w}(2 - 2\cos \omega\Delta t - \omega\Delta t \sin \omega\Delta t) \\ w &= \omega\Delta t - \sin \omega\Delta t \end{aligned} \quad (19)$$

The transfer matrix $[T]$ and vector $\{L\}$ are the integration and load operators respectively. The integration is performed for given initial conditions $\{\tilde{z}\}_0$.

4. ILLUSTRATIVE EXAMPLE

Transition from regular to chaotic oscillations is considered in case of the following system taken from [12]:

$$\tilde{\omega}/\omega_0 = 0.4525, \quad \xi = 0.022275, \quad \kappa/\omega_0^2 = -1, \quad a_0 = 0, \quad k = 2.$$

The response amplitude x_0 , depending on the excitation amplitude c_2 in the bifurcation region is shown in Fig. 1. The regular steady-state response is determined in the frequency domain by the harmonic balance method. The first solution (7a) for A_0 is employed. Since the static load $a_0 = 0$, the response is symmetric, i.e. $A_0 = 0$, and $A_n = B_n = 0$ for $n \neq (2i-1)k$, where $i=1,2,\dots$. Stability of the harmonic response is checked by the Floquet theory, [3]. Instability occurs for $c_2/\omega_0^2 = 0.35458815$, point I_1 in Fig. 1. Therefore, employing (7b) another solution for A_0 is found out. Now, $A \neq 0$ and $A_n = B_n = 0$ for $n \neq (ik-1)$, $i=1,2,\dots$. The response becomes unsymmetric. The symmetry breaking occurs in the point E_1 . The stability of response is again checked. It is found out, that it is lost for $c_2/\omega_0^2 = 0.378382$, point I_2 in Fig. 1.

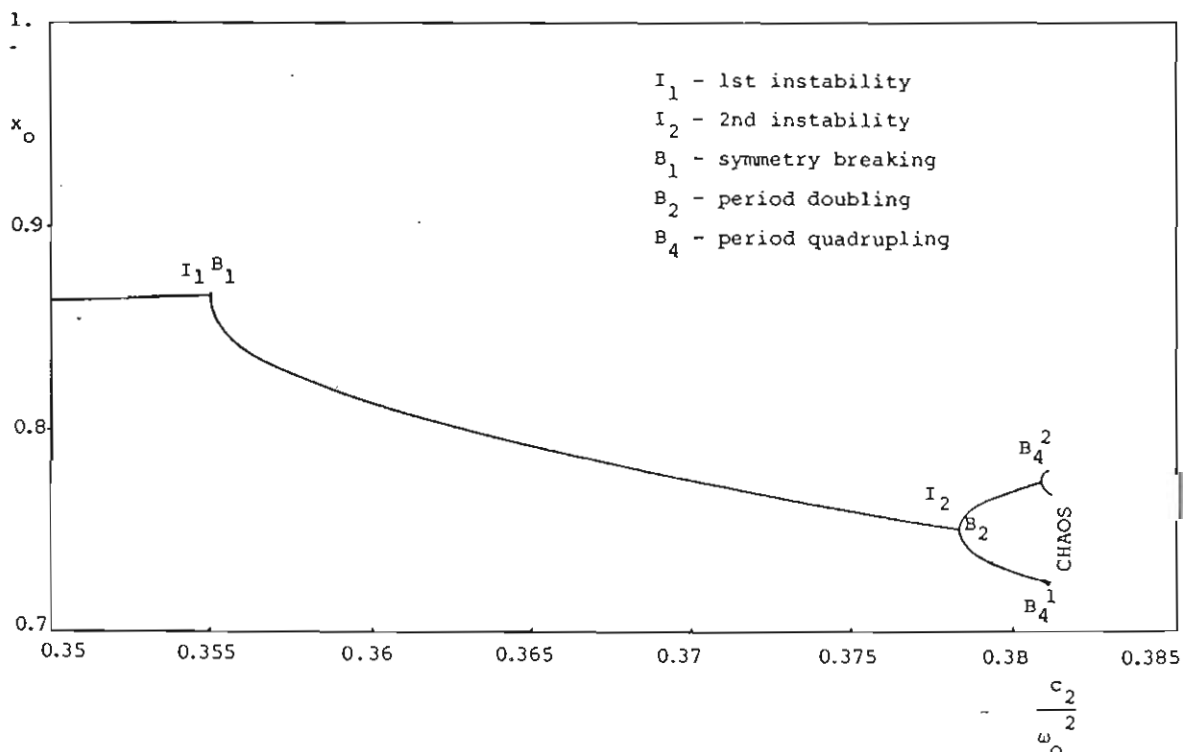


Fig. 1. Dependence of response amplitude on excitation amplitude in bifurcation region

To determine another stable solution in the frequency domain, it is necessary to find out the second roots of system (11) for $n=1$, so that $A_1 \neq 0$ and/or $B_1 \neq 0$. Since the system is homogeneous, the nontrivial solution is based on the

condition that its determinant $D=0$, equation (14). However, this is a rather difficult task which has not yet been solved exactly. Only the first approximation for the first subharmonic oscillations is achieved, /13/.

Due to the above reason the response for higher excitation is determined in the time domain, employing the harmonic acceleration method. The initial conditions $x(0)$ and $\dot{x}(0)$ are taken from the unstable steady-state solution which is defined previously in the frequency domain. The response is periodic with double excitation periods, and two different amplitudes as shown in Fig. 1. This phenomenon, known as the period doubling bifurcation, starts in point B_2 which is very closed to the instability point I_2 .

If the excitation amplitude is further increased, the response period is again doubled, and each amplitude is split into two different values, Fig. 1. This is the transition from regular to chaotic motion.

The problem is solved in the frequency domain by the improved harmonic balance method prescribing the error tolerance $\epsilon_c \leq 10^{-12}$ for the Fourier coefficients. Accuracy of the result is evaluated by checking the satisfaction of differential equation (1) in 100 instants within one excitation period. The root mean square of differences between the left and right hand side is specified as an error, /10/. It is of order $\epsilon_e \leq 10^{-7}$.

Accuracy of the time integration depends on the time step. A reasonable time step for the harmonic acceleration method is $\Delta t = T/200$, where $T = 2\pi/\tilde{k\omega}$ is the excitation period. The convergence criterion $\epsilon_c \leq 10^{-12}$ is prescribed. The steady-state response is achieved after 100 excitation periods, and then it is converted to the frequency domain by the Fourier transform. The Fourier coefficients $C_n = (A_n^2 + B_n^2)^{1/2}$ for the excitation $c_2/\omega_o^2 = 0.37$, determined directly by the harmonic balance method (HBM) and indirectly by the harmonic acceleration method (HAM), are compared in Table 1. Since the first four digits are mostly the same in the both results, the final error of the harmonic acceleration method and the Fourier transform is of order $\epsilon_e \leq 10^{-4}$. This is satisfactory from the engineering point of view.

In Table 1 the coefficients of regular response and period doubling response, determined for the excitation amplitude $c_2/\omega_o^2 = 0.35$ and 0.37 respectively, are also included to illustrate their increase in number in the transition to chaos. This problem is analysed in details in /14/.

Table 1. Fourier coefficients of response in transition to chaos

Coefficient	Regular response $c_2/\omega_o^2=0.35$	Symmetry breaking $c_2/\omega_o^2=0.37$		Period doubling $c_2/\omega_o^2=0.38$
	HBM	HBM	HAM	HAM
A_o		0.1210740927	0.1211355403	0.1512920111
C_1				0.0264590326
C_2	0.8858422210	0.8797874630	0.8797625899	0.8751252890
C_3				0.0031774884
C_4		0.0390469093	0.0390553958	0.0482287183
C_5				0.0033530953
C_6	0.0230661179	0.0210568910	0.0210400261	0.0196416862
C_7				0.0002297291
C_8		0.0021001060	0.0020987524	0.0023465186
C_9				0.0000238090
C_{10}	0.0006406208	0.0004877638	0.0004867463	0.0005511963

5. CONCLUSION

Nonlinear oscillations of cubic dynamical systems may be analysed in the frequency and time domain by the improved harmonic balance method and the harmonic acceleration method respectively. The former method is almost exact, while the latter is approximate. However, all nonlinear phenomena can not be accurately analysed in the frequency domain. Therefore, time integration is requisite.

The correlation analysis of nonlinear oscillations determined in the frequency and time domain by the above two methods shows, that the harmonic acceleration method is a sophisticated and reliable tool. It makes possible to analyse all nonlinear phenomena specified in the introduction. This is illustrated in /14/, where, for instance, the chaotic attractor, fractal escape boundary and probability density function are determined.

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APPENDIX

EXPANSION OF CUBIC DISPLACEMENT FUNCTION INTO FOURIER SERIES

Displacement function expressed by the complete Fourier series (2), may be represented in a more convenient form

$$x = A_0 + X \quad (20)$$

where

$$X = \sum_{n=1}^{\infty} A_n \cos n \tilde{\omega} t + \sum_{n=1}^{\infty} B_n \sin n \tilde{\omega} t \quad (21)$$

Cubic function of (21) reads

$$x^3 = A_0^3 + 3 A_0^2 X + 3 A_0 X^2 + X^3 \quad (22)$$

Quadratic function in (22) takes the form

$$X^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (A_j A_k \cos j \tilde{\omega} t \cos k \tilde{\omega} t + 2 A_j B_k \cos j \tilde{\omega} t \sin k \tilde{\omega} t + B_j B_k \sin j \tilde{\omega} t \sin k \tilde{\omega} t) \quad (23)$$

Employing the fundamental trigonometric identities, one obtains

$$X^2 = \sum_C^{(2)} + \sum_S^{(2)} \quad (24)$$

where

$$\sum_C^{(2)} = \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} [(A_j A_k - B_j B_k) \cos(j+k) \tilde{\omega} t + (A_j A_k + B_j B_k) \cos(j-k) \tilde{\omega} t]$$

$$\Sigma_s^{(2)} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} [A_j B_k \sin(j+k)\tilde{\omega}t - A_j B_k \sin(j-k)\tilde{\omega}t] \quad (25)$$

Furthermore, it is possible to expand (24) into the Fourier series putting $j+k=\pm n$, and further $k=\pm j \pm n > 0$. Thus,

$$X^2 = S_o^{(2)} + \sum_{n=1}^{\infty} S_{cn}^{(2)} \cos n \tilde{\omega}t + \sum_{n=1}^{\infty} S_{sn}^{(2)} \sin n \tilde{\omega}t \quad (26)$$

where

$$\begin{aligned} S_o^{(2)} &= \frac{1}{2} \sum_{j=1}^{\infty} (A_j^2 + B_j^2) = \frac{1}{2} \sum_{j=1}^{\infty} C_j^2 \\ S_{cn}^{(2)} &= \frac{1}{2} \sum_{j=1}^{n-1} (A_j A_{n-j} - B_j B_{n-j}) + \frac{1}{2} \sum_{j=1}^{\infty} (A_j A_{j-n} + A_j A_{j+n} + B_j B_{j-n} + B_j B_{j+n}) \\ S_{sn}^{(2)} &= \sum_{j=1}^{n-1} A_j B_{n-j} - \sum_{j=1}^{\infty} (A_j B_{j-n} - A_j B_{j+n}) \end{aligned} \quad (27)$$

Cubic function in (23) reads

$$\begin{aligned} X^3 &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (A_j A_k A_l \cos j \tilde{\omega}t \cos k \tilde{\omega}t \cos l \tilde{\omega}t + \\ &\quad + 3 A_j B_k B_l \cos j \tilde{\omega}t \sin k \tilde{\omega}t \sin l \tilde{\omega}t + \\ &\quad + 3 A_j A_k B_l \cos j \tilde{\omega}t \cos k \tilde{\omega}t \sin l \tilde{\omega}t + \\ &\quad + B_j B_k B_l \sin j \tilde{\omega}t \sin k \tilde{\omega}t \sin l \tilde{\omega}t) \end{aligned} \quad (28)$$

The application of the fundamental trigonometric identities leads to

$$X^3 = \Sigma_c^{(3)} + \Sigma_s^{(3)} \quad (29)$$

where

$$\begin{aligned} \Sigma_c^{(3)} &= \frac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [(A_j A_k A_l - 3 A_j B_k B_l) \cos(j+k+l)\tilde{\omega}t + \\ &\quad + (A_j A_k A_l + 3 A_j B_k B_l) \cos(j+k-l)\tilde{\omega}t + \\ &\quad + (A_j A_k A_l + 3 A_j B_k B_l) \cos(j-k+l)\tilde{\omega}t + \\ &\quad + (A_j A_k A_l - 3 A_j B_k B_l) \cos(j-k-l)\tilde{\omega}t] \end{aligned} \quad (30)$$

$$\begin{aligned} \Sigma_s^{(3)} &= \frac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [(-B_j B_k B_l + 3 A_j A_k B_l) \sin(j+k+l)\tilde{\omega}t + \\ &\quad + (B_j B_k B_l - 3 A_j A_k B_l) \sin(j+k-l)\tilde{\omega}t + \\ &\quad + (B_j B_k B_l + 3 A_j A_k B_l) \sin(j-k+l)\tilde{\omega}t + \\ &\quad + (-B_j B_k B_l - 3 A_j A_k B_l) \sin(j-k-l)\tilde{\omega}t] \end{aligned}$$

Furthermore, taking $j+k+l=\pm n$, and further $l=\pm j \pm k \pm n > 0$ follows

$$X^3 = S_o^{(3)} + \sum_{n=1}^{\infty} S_{cn}^{(3)} \cos n \tilde{\omega}t + \sum_{n=1}^{\infty} S_{sn}^{(3)} \sin n \tilde{\omega}t \quad (31)$$

where

$$\begin{aligned}
 S_o^{(3)} &= \frac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_j A_k (A_{j+k}^{+A} A_{-j+k}^{+A} A_{j-k}^{+A}) + \\
 &+ \frac{3}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_j B_k (B_{j+k}^{+B} B_{-j+k}^{+B} B_{j-k}^{+B}) \\
 S_{cn}^{(3)} &= \frac{1}{4} \sum_j^{n-1} \sum_k^{n-j} (A_j A_k A_{n-j-k}^{-3} A_j B_k B_{n-j-k}) + \\
 &+ \frac{1}{4} \sum_j^{\infty} \sum_k^{\infty} A_j A_k (A_{j+k-n}^{+A} A_{j+k+n}^{+A} A_{-j+k+n}^{+A} A_{-j+k-n}^{+A} A_{j-k-n}^{+A} A_{j-k+n}^{+A}) + \\
 &+ \frac{3}{4} \sum_j^{\infty} \sum_k^{\infty} A_j B_k (B_{j+k-n}^{+B} B_{j+k+n}^{+B} B_{-j+k+n}^{+B} B_{-j+k-n}^{+B} B_{j-k-n}^{+B} B_{j-k+n}^{+B}) \\
 S_{sn}^{(3)} &= \frac{1}{4} \sum_j^{n-1} \sum_k^{n-j} (-B_j B_k B_{n-j-k}^{+3} A_j A_k B_{n-j-k}) + \\
 &+ \frac{1}{4} \sum_j^{\infty} \sum_k^{\infty} B_j B_k (B_{j+k-n}^{-B} B_{j+k+n}^{+B} B_{-j+k+n}^{-B} B_{-j+k-n}^{-B} B_{j-k-n}^{+B} B_{j-k+n}^{+B}) + \\
 &+ \frac{3}{4} \sum_j^{\infty} \sum_k^{\infty} A_j A_k (-B_{j+k-n}^{+B} B_{j+k+n}^{+B} B_{-j+k+n}^{-B} B_{-j+k-n}^{-B} B_{j-k-n}^{+B} B_{j-k+n}^{+B}) \quad (32)
 \end{aligned}$$

DINAMIČKA STABILNOST GREDA I OKVIRNIH NOSAČA U LINEARNOM PODRUČJU

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SAŽETAK

Dan je kratki pregled linearne dinamičke stabilnosti ilustriran jednostavnim primjerima. Obraduje se odnos između kritične sile i frekvencije prisilnih vibracija. Prikazani su primjeri primjene na stupove koji poprečno vibriraju i okvirne nosače.

SUMMARY

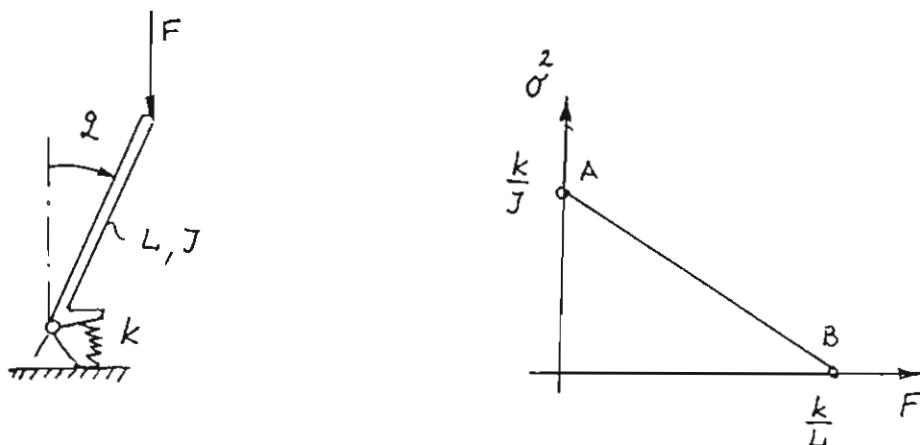
Short recovery of linear dynamic stability problem is followed by simple examples. Discussion on problem includes correlation between forced vibration frequency and critical load. Application on stability of lateral vibrating columns and frames is discussed. Examples.

UVOD

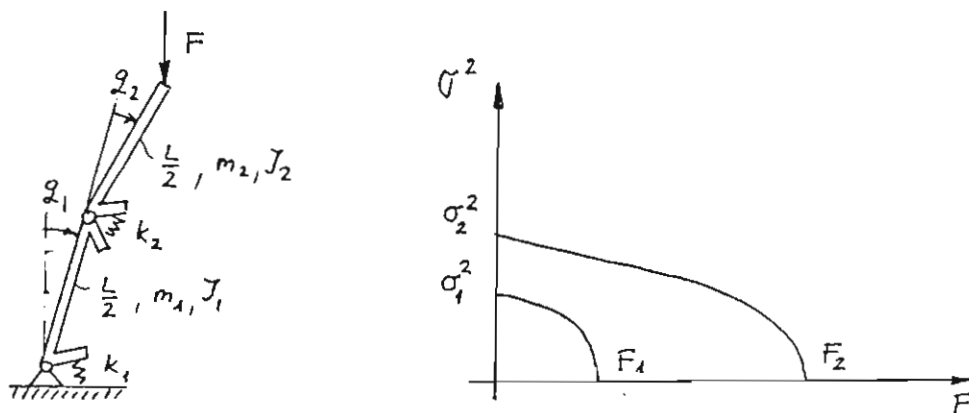
Teorija konstrukcija kao i teorija elastičnosti općenito podrazumijevaju idealizacije realnih konstrukcija a zatim računanje odziva za dobivene matematičke modele, tako da idealizirani matematički modeli koji obuhvaćaju geometriju i ponašanje materijala te proučavani fenomen predstavljaju stvarne konstrukcije. Klasična razmatranja problema koji obuhvaćaju dinamičke pojave i elastičnu stabilnost istovremeno, tj. utjecaj dinamičkih pojava na elastičnu stabilnost vode do složenih nelinearnih matematičkih modela što je često u opreci sa potrebama projektanata za brzom procjenom stabilnosti ravnoteže. U ovom radu je analizirana elastična stabilnost vitkih, uzdužno pritisnutih komponenti konstrukcija koje poprečno vibriraju malim amplitudama a mogu se modelirati štapovima. Uspoređena su analitička i numerička rješenja. Dan je i primjer praktične primjene: dinamička stabilnost okvirnog nosača podvrgnutog vibracijama.

DISKRETNI PROBLEMI

Da se uvede osnovni koncept i terminologija razmatraju se na početku dva diskretna problema (sustavi krutih tijela): prvi sa jednim stupnjem slobode i drugi sa dva stupnja slobode. Problemi su prikazani na slikama 1 i 2.



SLIKA 1: Problem s jednim stupnjem slobode



SLIKA 2: Problem s dva stupnja slobode

Uvjet ravnoteže za problem s jednim stupnjem slobode bez prigušenja, za slučaj malih pomaka, glasi:

$$kq - Jq'' - FLq = 0 \quad (1)$$

gdje su k , J and FL (elastična) krutost, inercijska (dinamička) i geometrijska krutost, a F i q uzdužna sila i kutni pomak. Za harmonijske vibracije i netrivialno rješenje, se dobiva:

$$F = K/L - J/L \sigma^2 \quad (2)$$

gdje su F i σ kritična sila i frekvencija prisilnih harmonijskih vibracija. U problemu s jednim stupnjem slobode kritična sila je linearno ovisna o kvadratu frekvencije prisilnih vibracija, kako je to prikazano na slici 1, gdje su prikazane samo pozitivne vrijednosti F i σ , koje imaju fizikalni smisao. Dvije točke su posebno zanimljive: točka A koja predstavlja stanje rezonancije (nestabilnost bez uzdužne tlačne sile) i točka B koja predstavlja stanje statičke elastične nestabilnosti.

Jednadžba problema s dva stupnja slobode, prema slici 2, za male harmonijske pomake, u matričnom obliku glasi:

$$[k]\{q\} - [J]\{q\} - [L]\{q\} = 0 \quad (3)$$

gdje su $[k]$, $[J]$ i $[L]$ matrica (elastična) krutosti, matrica inercijske i matrica geometrijske krutosti, za koje vrijedi:

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 \end{bmatrix}$$

$$[J] = \begin{bmatrix} L/2 & 0 \\ 0 & L/2 \end{bmatrix}$$

$$[M] = \begin{bmatrix} J_1 + 5mL^2/16 & \frac{1}{8}mL^2 \\ 1/8mL^2 & J_2 + mL^2/16 \end{bmatrix}$$

i gdje su m , J_1 , J_2 i L masa, moment tromosti i dužina članova sustava a k_1 , k_2 krutost opruga u zglobovima, slika 2. Kvadrati frekvencija prisilnih vibracija (σ) i koeficijent intenzivnosti kritičnog opterećenja (F) odnose se kako slijedi:

$$F_{1,2} = \frac{3(8k - L^2 m \sigma^2) \pm \sqrt{8(40k^2 + 4kL^2 m \sigma^2 + L^4 m^2 \sigma^4) - 16J\sigma^2}}{8L}$$

$$\sigma_{1,2}^2 = \frac{8\sqrt{8F^2 L^6 m^2 - 4FkL^3 m(16J + 15L^2 m) + 1280J^2 k^2 + 672Jk^2 L^2 m + 117k^2 L^4 m^2}}{256J^2 + 96JL^2 m + L^4 m^2} + \frac{FL(16J + 3L^2 m) - k(48J + 11L^2 m)}{256J^2 + 96JL^2 m + L^4 m^2}$$

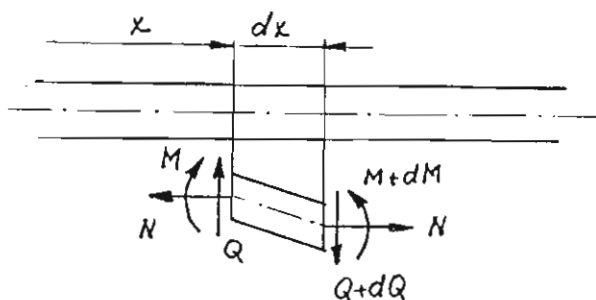
Posljednji odnos je grafički prikazan na slici 2, gdje se vidi da većina zaključaka za problem s jednim stupnjem slobode vrijedi i ovdje. Krivulja koja povezuje kvadrate frekvencije prisilnih vibracija i kritičnu silu je konveksna. U praktičnim slučajevima od najveće je važnosti najmanja kritična sila, što će biti diskutirano kasnije.

KONTINUIRANI PROBLEMI

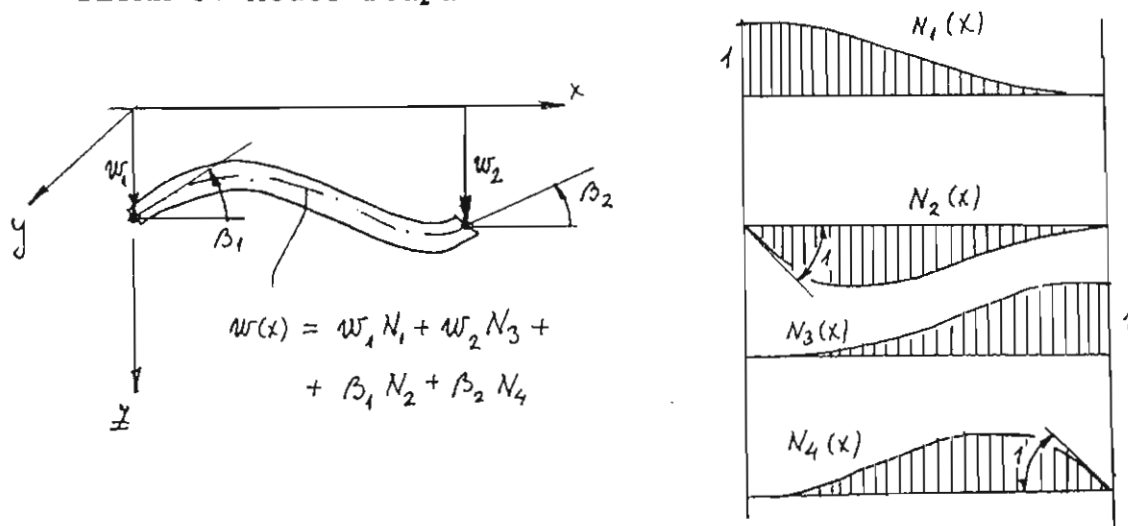
Diskretni modeli, razumljivi i jednostavni u primjeni, mogu obuhvatiti relativno mali broj realnih situacija pa se prelazi na kontinuirani model uzdužno pritisnutog štapa. Usvajaju se uobičajene pretpostavke za male pomake i deformacije, pa jednadžba uzdužno pritisnutog štapa koji i poprečno vibrira harmonijski, slika 3, glasi:

$$EI_y \frac{d^4 w(x)}{dx^4} - N \frac{d^2 w(x)}{dx^2} + \rho A w(x) = 0$$

gdje su EI_y savojna krutost, ρA masa po jedinici dužine, N uzdužna sila (negativna), $w(x,t)$ pomak u smjeru osi z a x i t su prostorna koordinata i vrijeme.



SLIKA 3: Model štapa



SLIKA 4: Probne funkcije i aproksimacija

Rješenje ove jednadžbe koje podrazumijeva separaciju vremena i prostorne varijable te uvođenje koordinate $\xi = x/L$, glasi:

$$w(x, t) = W(x) \cos(\sigma t)$$

$$W(x) = A \sinh(M\xi) + B \cosh(M\xi) + C \sin(N\xi) + D \cos(N\xi)$$

$$M = \sqrt{-U + \sqrt{U^2 + 1}} \quad N = \sqrt{U + \sqrt{U^2 + \omega^2}}$$

$$U = TL^2 / 2EI_y \quad \omega = \sigma L^2 / \alpha \quad \alpha = \sqrt{EI_y / \rho A}$$

gdje koeficijente A, B, C i D treba odrediti iz rubnih uvjeta. Rješenja za niz varijanti rubnih uvjeta je moguće pronaći u literaturi [1-2]. U radu [3] je analizirana greška Razleigheve i Glafove kada se primjenjuju na ovaj tip problema. Specijalni, ali vrlo važan slučaj konzole opterećene uzdužnom silom s masom na kraju, obrađen je u literaturi [4-8].

DISKRETIZACIJA RUBNIM ELEMENTIMA

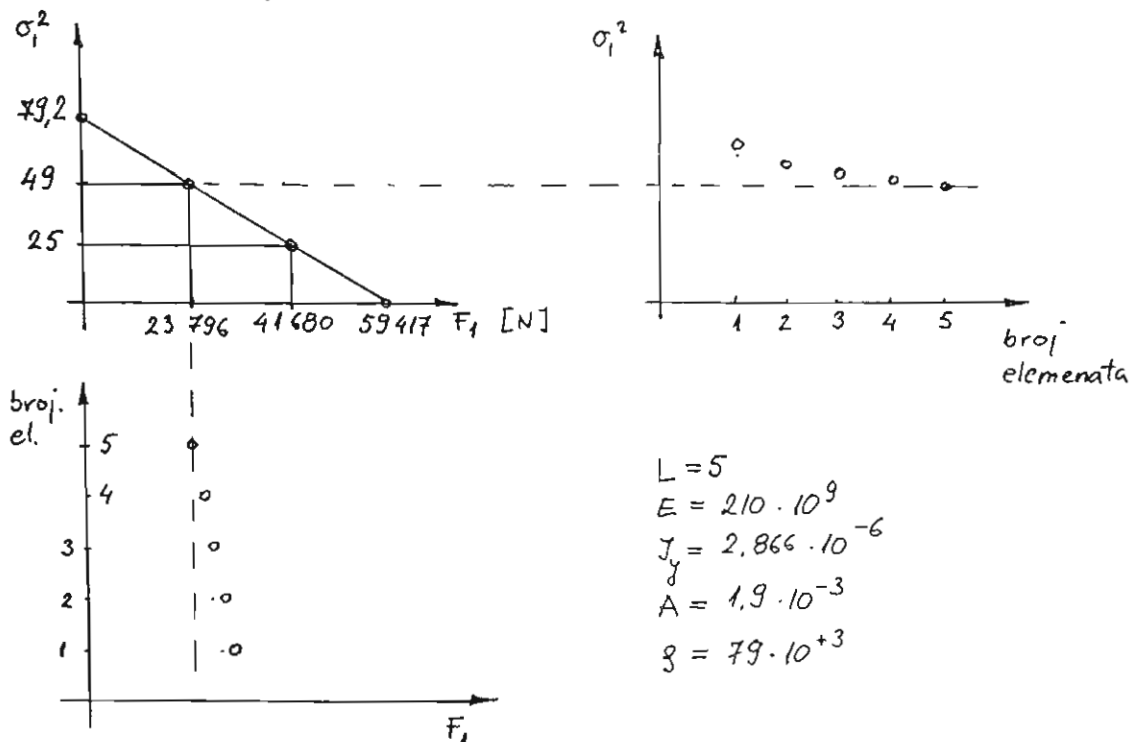
Diferencijalna jednačina (9), poslije prevođenja u slabu formulaciju i diskretizacije dobro poznatim lokaliziranim, slika 4, prelazi u jednačinu oblika (3) gdje su:

$$[k] = EI_y/L \begin{bmatrix} 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{bmatrix}$$

$$[J] = \rho AL/420 \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

$$[L] = EI_y/L \begin{bmatrix} 6/5L & 1/10 & -6/5L & 1/10 \\ 1/10 & 2L/15 & -1/10 & -1/30 \\ -6/5L & -1/10 & 6/5L & -1/10 \\ 1/10 & -1/30 & -1/10 & 2L/15 \end{bmatrix}$$

a ρ , A, L i EI_y kako je već navedeno.

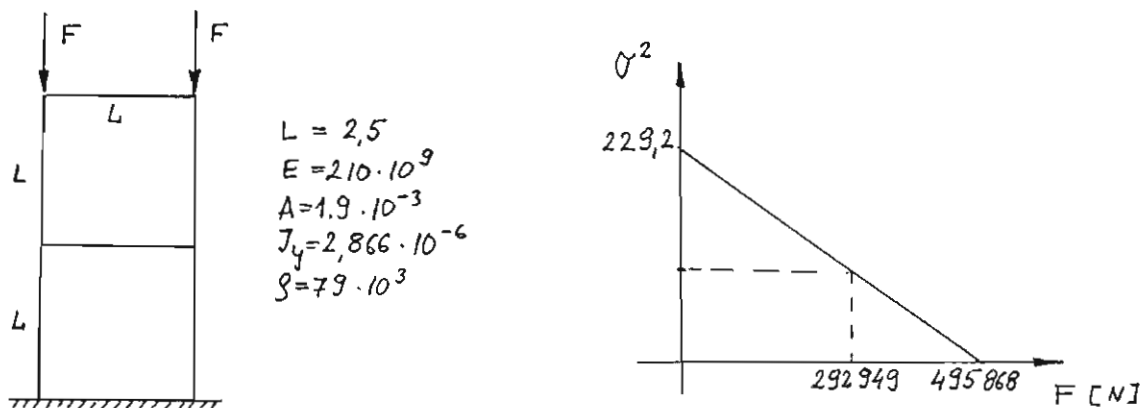


KA 5: Kvadrati frekvencija i kritična sila, konvergencija

Na slici 5 je prikazana kritična sila za odabranu frekvenciju poprečnih vibracija stupa za različite diskretizacije problema pritisknutog štapa. Zatim je model sa 5 konačnih elemenata korišten da se izračunaju kritične sile za različite frekvencije prisilnih vibracija. Na istoj slici je prikazan odnos kvadrata frekvencija i intenzivnosti opterećenja za posljednji slučaj.

Pokazuje se da je za praktične proračune, za grubu procjenu stabilnosti dovoljno izračunati slučaj izvijanja bez dinamičkih sila i slučaj prisilnih vibracija, a da se područje sigurnosti može odrediti interpolacijom u analogiji s jednostavnijim modelima. Računanje pojedinih mješovitih slučajeva uzdužno opterećenje i poprečne vibracije odabranom frekvencijom (stabilnost uz dinamičke sile), popravljja procjenu sigurnog područja.

Koristeći se transformacijom koordinata lako se dolazi do prostorno orjentiranih štapova koji se kombiniraju na uobičajen način da se dobije model ravninskih i prostornih okvirnih nosača. Procjenu stabilnog područja za okvirni nosač napravimo na isti način kao za ravni štap, kako je gore opisano. Na slici 6 je prikazan primjer okvirnog nosača i dani su rezultati proračuna.



SLIKA 6: Linearna dinamička stabilnost okvirog nosača

Općenito, ravnoteža može biti nestabilna zbog geometrijskih ili dinamičkih razloga. Kažemo da krutost ima elastični, geometrijski i inercijski dio. U slučaju kada krutost postaje jednaka nuli, ravnoteža je nestabilna.

ZAKLJUČAK

U radu je na elementarnim primjerima analizirana dinamička stabilnost u linearnom području i predloženo je skraćeno crtanje dijagrama koeficijenta intenzivnosti opterećenja naspram kvadrata frekvencije prisilnih vibracija za procjenu sigurnog područja.

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THE RELIABILITY OF ANALYSIS OF FORCED VIBRATIONS OF A MECHANICAL SYSTEM WITH CLEARANCES BY USING THE METHOD OF HARMONIC BALANCE

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SUMMARY

Forced vibrations of a mechanical system with a piecewise linear force-displacement relationship is analyzed. Frequency response characteristics are approximatively obtained by applying the method of harmonic balance. The results are valuated by using a piecewise linear technique. By comparing the obtained results, the frequency ranges in which the method of harmonic balance gives satisfactory accurate results are established.

1. INTRODUCTION

Clearances exist in many mechanical systems either by design or due to manufacturing errors. Forced vibration of a system with clearances can result in relative motion across the clearance space and impacting between the components, what very often lead to noise and large dynamic loads. The solutions of these problems (called vibro-impact problems) are usually obtained by using a piecewise linear technique [1-5], digital simulation [2, 3, 6-10] and analog simulation [9]. These methods provide a relatively easy way of studying the response of the vibro-impact system by using a computer. But, parametric analysis and obtaining the whole frequency range response are cumbersome. The obtained response is the actual displacement time history, and it can be very complicated and may include higher harmonics and chaotic motion. For many engineering applications more meaningful description of the vibro-impact problem is given by the r.m.s. values. Consequently one of the most useful tools for a steady state analysis is the frequency response function. The frequency response functions of the systems with clearances are usually obtained by using the method of harmonic balance, because this method is well applicable for systems with large non-linearities and non-analytical force-displacement relationship. Mechanical systems with clearances are non-linear systems and corresponding frequency response functions can be determined only approximatively. Because of that, it is very important to investigate the accuracy and reliability of the results obtained by using the method of harmonic balance.

2. MODEL OF A MECHANICAL SYSTEM WITH A CLEARANCE

Model of a mechanical system with a clearance shown in Figure 1. consists of two inertia elements m_1 and m_2 , a non-linear elastic element $c g(x)$, and an energy dissipation element k .

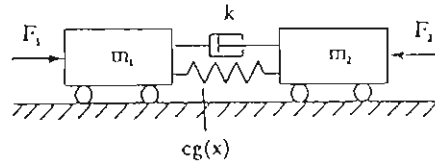


Figure 1. Model of a mechanical system with a clearance

The non-linear elastic element is defined by the function $g(x)$ shown in Figure 2.

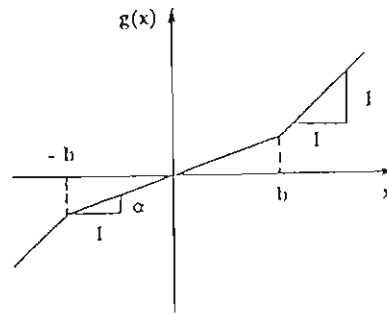


Figure 2. Non-linear force displacement relationship $g(x)$

b is one half of the clearance space and α is the ratio of the second stage stiffness to the first stage stiffness (a measure of the strength of the non-linearity). The excitation has the form:

$$F_1 = F_m + F_p \cos(\Omega t + \Phi_F), \quad (1a)$$

$$F_2 = F_m, \quad (1b)$$

where F_m represents mean transmitted force, F_p is the amplitude of the vibratory component at frequency Ω , and Φ_F is the phase angle. This two-degree of freedom system is semi-definite and, therefore, can be reduced by one, what gives

$$m \ddot{x} + k \dot{x} + c g(x) = F_m + \frac{m}{m_1} F_p \cos(\Omega t + \Phi_F), \quad (2)$$

where

$$x = x_1 - x_2, \quad m = \frac{m_1 m_2}{m_1 + m_2}, \quad (3)$$

$$g(x) = \begin{cases} x - (1 - \alpha) b, & x > b \\ \alpha x, & -b \leq x \leq b \\ x + (1 - \alpha) b, & -b > x \end{cases} \quad (4)$$

The differential equation (2) can be written in non-dimensional form:

$$\bar{x}'' + 2\zeta \bar{x}' + g(\bar{x}) = \bar{F}_m + \bar{F}_p \cos(\eta \tau + \Phi_F), \quad (5)$$

where

$$\bar{x} = \frac{x}{l}, \quad \bar{b} = \frac{b}{l}, \quad \tau = \omega_0 t, \quad \omega_0 = \sqrt{\frac{c}{m}}, \quad \zeta = \frac{k}{2m\omega_0},$$

$$\bar{F}_m = \frac{F_m}{m l \omega_0^2}, \quad \bar{F}_p = \frac{F_p}{m_1 l \omega_0^2}, \quad \eta = \frac{\Omega}{\omega_0}.$$

Derivation with respect to non-dimensional time $d/d\tau = (1/\omega_0)d/dt$ is signed by comma "'". The value l is any characteristic dimension of the system. For $l=b$, $\bar{b}=1$.

3. USING OF THE METHOD OF HARMONIC BALANCE

Because the excitation is defined as

$$\bar{F} = \bar{F}_m + \bar{F}_p \cos(\eta \tau + \Phi_F), \quad (6)$$

the approximate solution is also assumed in the same form:

$$\bar{x} = \bar{x}_m + \bar{x}_p \cos(\eta \tau - \Phi). \quad (7)$$

The constant term $\bar{x}_m$ is steady state offset introduced by $\bar{F}_m$. The positive constant $\bar{x}_p$ represents the amplitude of the forced response due to $\bar{F}_p$, and Φ is the phase angle. The approximate solution (7) is based on the assumption that the forced response is dominated by the first harmonic (the higher harmonics are neglected).

For the study of primary resonances [11] the non-linear function $g(\bar{x})$ can be expressed as

$$g(\bar{x}) = N_m \bar{x}_m + N_p \bar{x}_p \cos \varphi, \quad (8)$$

where

$$\varphi = \eta \tau - \Phi. \quad (9)$$

The coefficients N_m and N_p are called describing functions, and they are defined as:

$$N_m(\bar{x}_m, \bar{x}_p) = \frac{1}{\pi \bar{x}_m} \int_0^\pi g(\bar{x}) d\varphi, \quad (10 a)$$

$$N_p(\bar{x}_m, \bar{x}_p) = \frac{2}{\pi \bar{x}_p} \int_0^\pi g(\bar{x}) \cos \varphi d\varphi. \quad (10 b)$$

By substituting (8) and (7) into (5), and equating the coefficients of harmonics which are the same, one obtains:

$$\bar{x}_m = \frac{\bar{F}_m}{N_m(\bar{x}_m, \bar{x}_p)}, \quad (11 a)$$

$$\bar{x}_p = \frac{\bar{F}_p}{\sqrt{[N_p(\bar{x}_m, \bar{x}_p) - \eta^2]^2 + (2\zeta\eta)^2}}, \quad (11 b)$$

$$\Phi = \arctan \frac{2\zeta\eta}{N_p(\bar{x}_m, \bar{x}_p) - \eta^2} - \Phi_F. \quad (11 \text{ c})$$

Once the describing functions $N_m(\bar{x}_m, \bar{x}_p)$ and $N_p(\bar{x}_m, \bar{x}_p)$ have been derived, the problem of solving non-linear differential equation (5) is reduced to solving a set of non-linear coupled algebraic equations (11).

Frequency response functions, in dependence of excitation and system parameters, can be divided into three regions:

1. a no impact region,
2. a single-sided impact region,
3. a double-sided impact region.

Describing functions N_m and N_p are different inside each of that frequency ranges, and must be derived separately for each of them.

The stability of the solution obtained by (11) is determined by perturbing the steady state solution and studying the resulting motion [11]. If the perturbed oscillating motion decays with time, the solution is dynamically stable, and if it grows it is dynamically unstable.

4. USING OF A PIECEWISE LINEAR TECHNIQUE

Force-displacement relationship $g(\bar{x})$ (Figure 2.) is piecewise linear. Local solutions of differential equation (5) are known explicitly inside each of the stage stiffness. Local solutions can be repeatedly matched at $\bar{x}=\bar{b}$ and $\bar{x}=-\bar{b}$, to obtain a global solution of (5). Piecing together of these known local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in closed form. But, the matching of local solutions can be done easily numerically for example by using Newton-Raphson method.

Obtained response is the actual displacement time history, and it provides the study of higher harmonics and chaotic motion. For many engineering applications, the actual displacement time history is not so important as the information of mean $\bar{x}_m$ and standard deviation $\bar{x}_p/\sqrt{2}$ of the time domain response, according to (12)

$$\bar{x}_m = \frac{1}{T} \int_{-T/2}^{T/2} \bar{x}(\tau) d\tau, \quad (12 \text{ a})$$

$$\bar{x}_p = \left[\frac{1}{T} \int_{-T/2}^{T/2} (\bar{x}(\tau) - \bar{x}_m)^2 d\tau \right]^{1/2} \cdot \sqrt{2}, \quad (12 \text{ b})$$

5. COMPARISON OF THE RESULTS

By using the method of harmonic balance (independently of initial conditions) the frequency response functions can be obtained. A piecewise linear technique (for certain initial conditions) gives the actual displacement time history. In order to make the comparison of results obtained by these two methods possible the r.m.s. values of the time domain responses are calculated. The comparison of the results for $\bar{x}_p$ obtained by method of harmonic balance and a piecewise linear technique are shown in Figure 3. and 4.

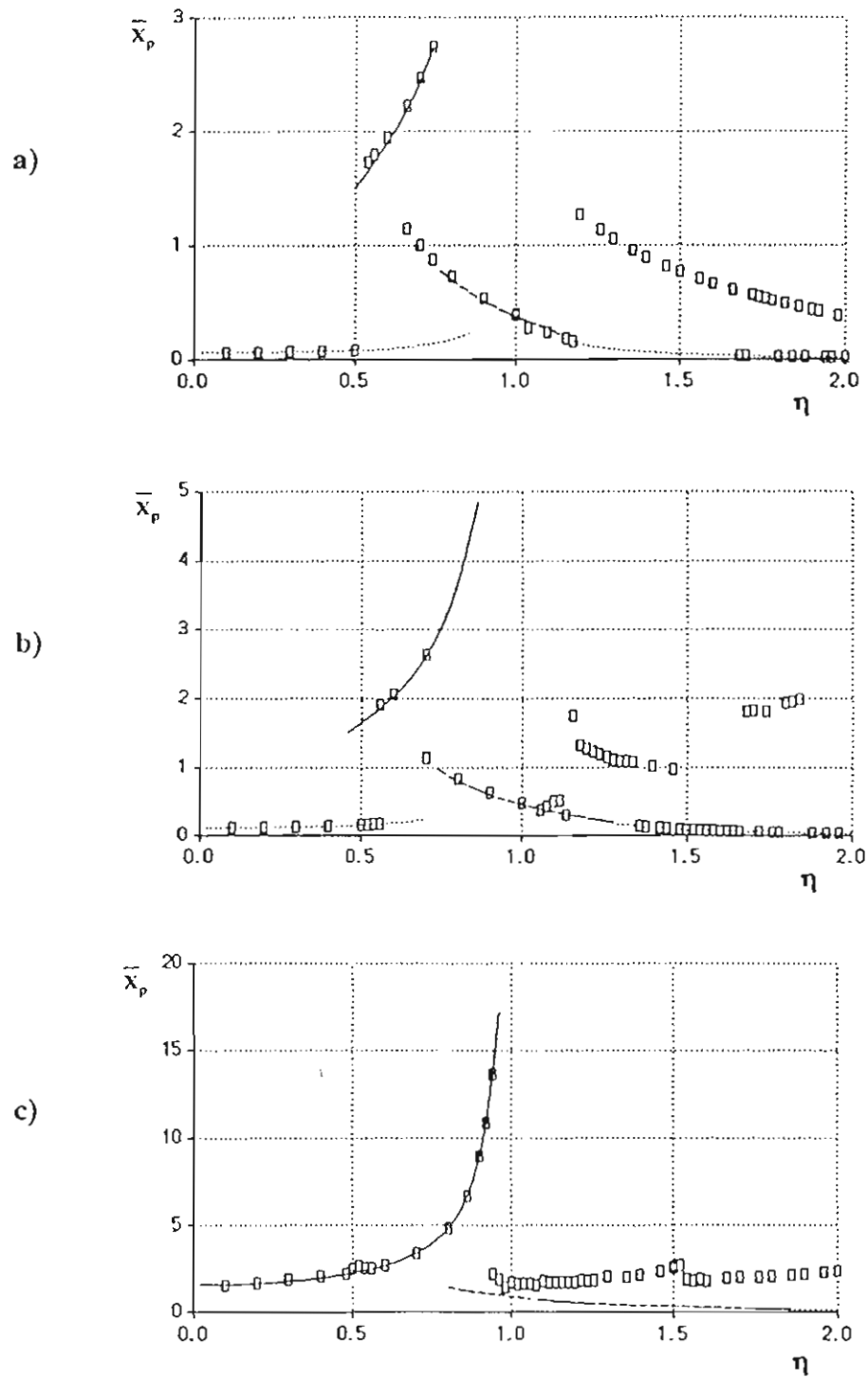


Figure 3. Frequency response $\bar{x}_p = \bar{x}_p(\eta)$ for different alternating loads $\bar{F}_p$ ($\bar{b} = 1$; $\alpha = 0$; $\zeta = 0,015$; $\bar{F}_m = 0,25$):
 a) $\bar{F}_p = 0,0625$; b) $\bar{F}_p = 0,125$; c) $\bar{F}_p = 0,5$;
 $\square$, results obtained by piecewise linear technique;
 —, two-sided impact case obtained by harmonic balance method;
 ----, single-sided impact case obtained by harmonic balance method;
 , no-impact case obtained by harmonic balance method.

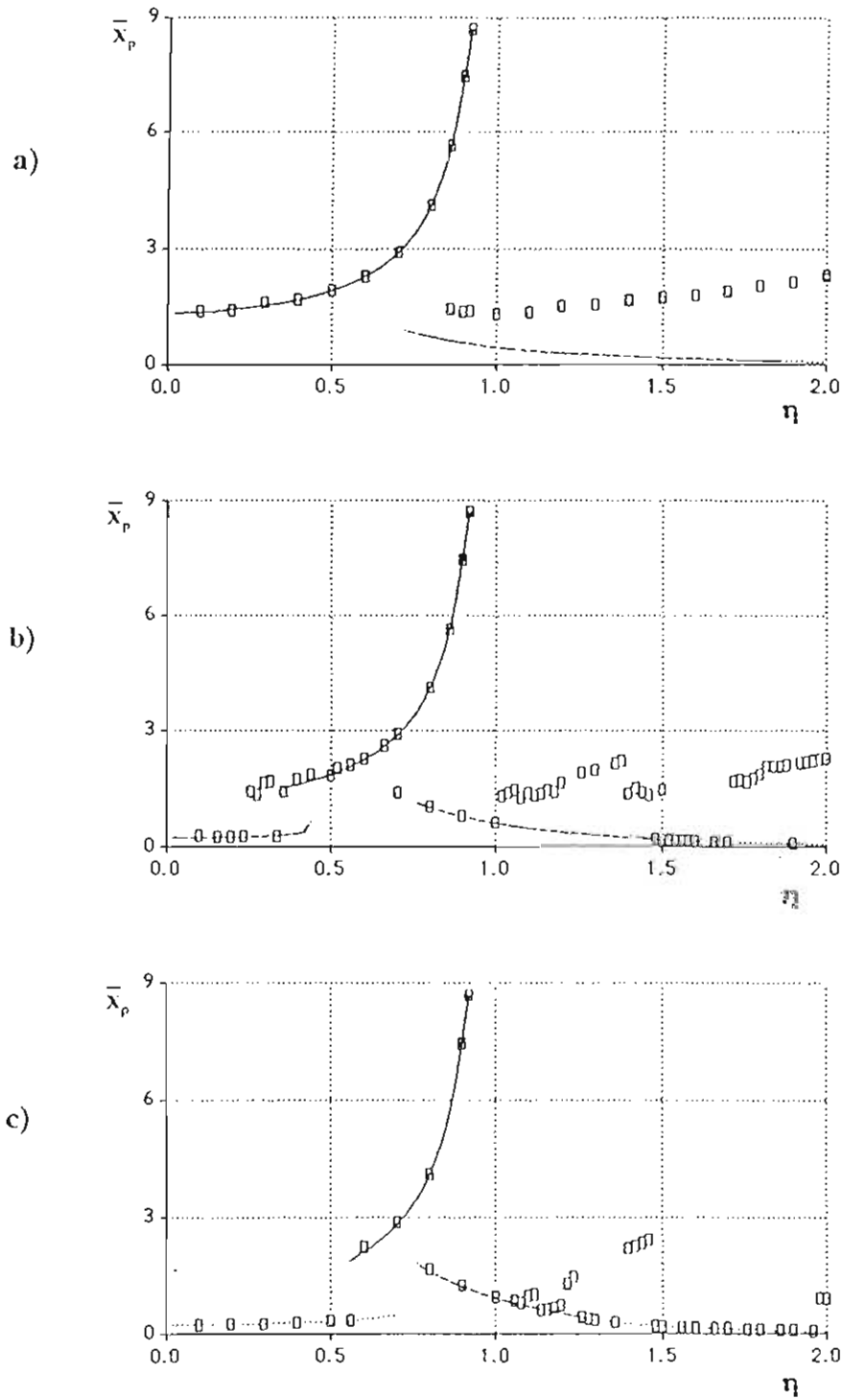


Figure 4. Frequency response $\bar{x}_p = \bar{x}_p(\eta)$ for different mean loads $\bar{F}_m$
 ($\bar{b}=1$; $\alpha=0$; $\zeta=0,015$; $\bar{F}_p=0,25$):

a) $\bar{F}_m=0,0625$; b) $\bar{F}_m=0,125$; c) $\bar{F}_m=0,5$;

$\square$, results obtained by piecewise linear technique;

—, two-sided impact case obtained by harmonic balance method;

- - - -, single-sided impact case obtained by harmonic balance method;

....., no-impact case obtained by harmonic balance method.

6. CONCLUSION

The method of harmonic balance is approximative and limited to consideration of steady-state frequency response for primary resonance only. This method enables deriving frequency response functions, and thus obtaining complete frequency response with a relatively little request for numerical calculations.

By comparing the results obtained by these methods, it is established that both methods give very similar results in the double-sided frequency ranges, which were predicted by the method of harmonic balance. By analyzing the time-domain responses in these frequency ranges, it is established that the assumption of dominance of primary resonance in response is satisfied, under which the use of the method of harmonic balance is based.

In the single-sided and no-impact frequency ranges predicted by the method of harmonic balance, the presence of superharmonics and subharmonics can have a significant effect on the system response. Therefore, in the single-sided and no-impact frequency ranges, results that are obtained by the method of harmonic balance may be inaccurate. The satisfactory results in the overall frequency range can be obtained by combining the method of harmonic balance and a piecewise linear technique.

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RAČUNANJE DINAMIČKIH KOEFICIJENATA KRUTOSTI I PRIGUŠENJA ULJNOG FILMA GALERKINOVOM METODOM

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Sažetak

U radu je prikazan numerički postupak računanja dinamičkih koeficijenata krutosti i prigušenja uljnog filma u kliznom ležaju. Reynoldsove diferencijalne jednačbe riješene su Galerkinovom metodom, pri čemu je primijenjen integralni (kvazi Reynoldsov) rubni uvjet za određivanje mjesta prekida uljnog filma. Dinamički koeficijenti uljnog filma dobivaju se pomoću malog broja računskih operacija, što znatno smanjuje vrijeme računanja spram metode konačnih diferencija i metode konačnih elemenata. Slaganje rezultata s onim dobivenim metodom konačnih diferencija je vrlo dobro.

A NUMERICAL PROCEDURE FOR CALCULATING THE DYNAMIC STIFFNESS AND DAMPING COEFFICIENTS OF OIL-FILM BY GALERKIN'S METHOD

Summary

This paper shows a numerical procedure for calculating the dynamic stiffness and damping coefficients of hydrodynamic journal bearings by Galerkin's method with quasi-Reynolds boundary condition for determination of film rupture position. Dynamic coefficients are obtained by small number of vector calculations, which significantly reduces time required for their calculation compared with finite difference method or finite element method. Good agreement was established between calculated coefficients and those obtained by finite difference method.

1. UVOD

Dinamičke koeficijente krutosti i prigušenja uljnog filma u kliznom ležaju najčešće se računa metodom konačnih diferencija [1] ili metodom konačnih elemenata [2].

Međutim, algoritmi utemeljeni na tim metodama zahtijevaju dosta vremena za računanje tih koeficijenata. Stoga su razvijene metode kod kojih se raspodjela tlaka

na rukavcu osovine pretpostavlja u obliku reda trigonometrijskih funkcija, pri čemu se primjenjuje integralni Reynoldsov rubni uvjet, kod kojeg se pretpostavlja da je linija prekida uljnog filma pravac. Uzimanjem konačnog broja članova trigonometrijskog reda i primjenom neke od metoda težinskog reziduua dobiva se približna raspodjela tlaka u uljnom filmu u stacionarnom stanju. Nakon toga se deriviranjem komponentata sile u uljnom filmu po komponentama pomaka i brzina središta rukavca dobivaju dinamički koeficijenti krutosti i prigušenja kliznog ležaja. Takav postupak zahtijeva relativno mali broj računskih operacija, te je vrijeme računanja oko sto puta kraće spram onog s metodom konačnih diferencija [3].

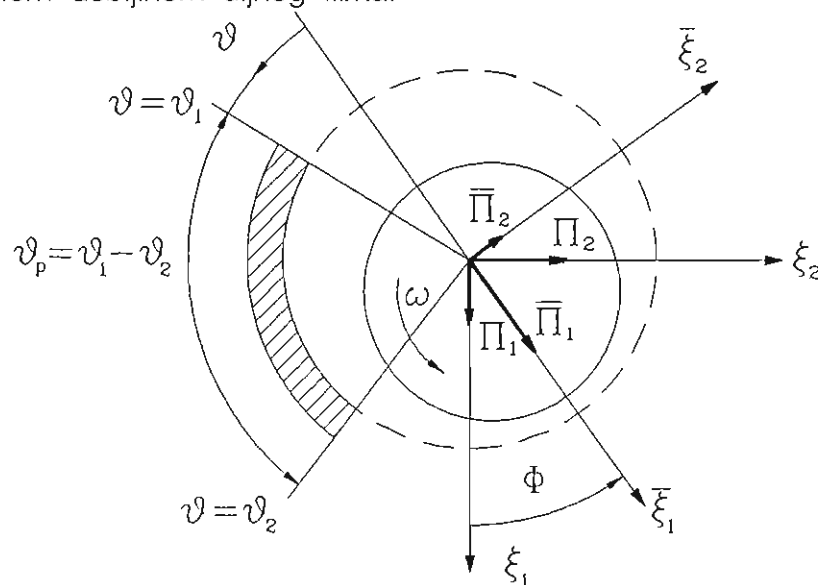
2. RJEŠAVANJE REYNOLDSOVE DIFERENCIJALNE JEDNADŽBE GALERKINOVOM METODOM

Reynoldsova diferencijalna jednačina se može napisati u bezdimenzijskom obliku (prema [3]):

$$\frac{\partial}{\partial \theta} \left(H^3 \frac{\partial P}{\partial \vartheta} \right) + \frac{1}{(2\lambda)^2} \frac{\partial}{\partial \zeta} \left(H^3 \frac{\partial P}{\partial \zeta} \right) = 2 \frac{d\varepsilon}{d\tau} \cos \vartheta - \varepsilon \left(1 - 2 \frac{d\Phi}{d\tau} \right) \sin \vartheta. \quad (1)$$

U segmentnom ležaju razmatra se svaki segment posebno, a rezultati se superponiraju. Dio ležajne košuljice smatra se dijelom zamišljenog punog kružnog ležaja (vidjeti sl. 1.).

Koordinata ϑ mjeri se od položaja koji je određen kutom zakreta rukavca, a poklapa se s maksimalnom debljinom uljnog filma.



Slika 1. Definicija koordinatnih sustava i koordinata
Figure 1. Definition of coordinate systems and coordinates

Aproksimacija raspodjele tlaka po obodu je provedena skupom funkcija:

$$\begin{aligned}\varphi_j(\vartheta) &= \sqrt{\frac{2}{\vartheta_p}} \sin \frac{j \pi (\vartheta - \vartheta_1)}{\vartheta_p} & j &= 1(1) \dots \\ \vartheta_p &= \vartheta_2 - \vartheta_1\end{aligned}\quad (2)$$

Taj skup funkcija zadovoljava rubne uvjete:

$$P = 0 \quad \text{za} \quad \vartheta = \vartheta_1 \quad \text{i} \quad \vartheta = \vartheta_2, \quad (3)$$

pri čemu je $\vartheta_1 = 0$ ako je položaj maksimalne debljine uljnog filma unutar segmenta, dok se, u slučaju kada uljni film ne doseže svoju maksimalnu debljinu unutar promatranog dijela ležajne košuljice, za ϑ_1 uzima kut koji predočava početak segmenta. Ako ne dolazi do prekida uljnog filma onda se za ϑ_2 uzima kut koji predočava kraj segmenta, a kod prekida uljnog filma kut ϑ_1 jednak kutu γ koji se dobiva iz integralnog Reynoldsovog rubnog uvjeta:

$$\int_0^1 \left(\frac{dP}{d\vartheta} \right)_{\vartheta=\gamma} = 0. \quad (4)$$

U osnom smjeru je potrebno zadovoljiti slijedeće rubne uvjete:

$$P = 0 \quad \text{za} \quad \zeta = 0 \quad \text{i} \quad \zeta = 1, \quad (5)$$

pa se za aproksimaciju tlaka uzima skup funkcija:

$$\begin{aligned}\psi_i &= \sqrt{2} \sin \omega_i \zeta \\ \omega_i &= (2i - 1)\pi & i &= 1(1) \dots\end{aligned}\quad (6)$$

Ako odabrane funkcije zadovoljavaju rubne uvjete u obodnom i osnom smjeru raspodjela se tlaka može pretpostaviti u obliku:

$$P(\vartheta, \zeta) = \sum_{i=1}^m \sum_{j=1}^n b_{ij} \varphi_j(\vartheta) \psi_i(\zeta), \quad (7)$$

gdje nepoznate koeficijente b_{ij} treba odrediti. Funkcije φ_j i ψ_i su ortonormirane.

Uvođenjem operatora:

$$\mathcal{L} = \frac{\partial}{\partial \vartheta} \left(H^3 \frac{\partial}{\partial \vartheta} \right) + \frac{1}{(2\lambda)^2} \frac{\partial}{\partial \zeta} \left(H^3 \frac{\partial}{\partial \zeta} \right) \quad (8)$$

Reynoldsova se jednažba može napisati u obliku:

$$\mathcal{L}(P) - f = 0, \quad (9)$$

gdje je

$$f = 2 \frac{d\varepsilon}{d\tau} \cos \vartheta - \varepsilon \left(1 - 2 \frac{d\Phi}{d\tau} \right) \sin \vartheta. \quad (10)$$

Uvrštavanjem (7) u (9) dobiva se

$$\sum_{i=1}^m \sum_{j=1}^n b_{ij} \psi_i \mathcal{L}_j(\varphi_j) - f = 0, \quad (11)$$

gdje je

$$\mathcal{L}_i = \frac{\partial}{\partial \theta} \left(H^3 \frac{\partial}{\partial \vartheta} \right) + \frac{H^3 \omega_i^2}{(2\lambda)^2}. \quad (12)$$

Prema Galerkinovoj metodi mora biti zadovoljena jednačba:

$$\int_A \delta P [\mathcal{L}(P) - f] dA = 0, \quad (13)$$

što, zajedno sa (7) i (11), daje:

$$\int_A \psi_k \varphi_l \left[\sum_{i=1}^m \sum_{j=1}^n b_{ij} \psi_i \mathcal{L}_j(\varphi_j) - f \right] dA = 0 \quad k = 1(1)m \quad l = 1(1)n \quad (14)$$

odnosno, budući da su funkcije ψ_i ortonormirane:

$$\sum_{j=1}^n b_{kj} \int_A \varphi_l \mathcal{L}_j(\varphi_j) dA = \int_A \psi_k \varphi_l f dA \quad k = 1(1)m \quad l = 1(1)n \quad (15)$$

ili u matričnom obliku:

$$\mathbf{L}_k \mathbf{b}_k = \mathbf{p}_k \quad k = 1(1)m \quad (16)$$

gdje su: $\mathbf{b}_k$ vektor (dimenzije n) nepoznatih koeficijenata, $\mathbf{L}_k$ kvadratna matrica (dimenzije n) s elementima

$$L_{kij} = \int_A \varphi_l \mathcal{L}_j(\varphi_j) dA, \quad (17)$$

a $\mathbf{p}_k$ vektor (dimenzije n) s elementima

$$p_{kl} = \int_A \psi_k \varphi_l f dA. \quad (18)$$

Stacionarni položaj rukavca određuje se iterativno. Prvo se za zadanu vrijednost relativne ekcentričnosti ε pretpostavi početna vrijednost kuta zakreta rukavca Φ , a

zatim se, za svaki segment ležaja, provjerava dolazi li do prekida uljnog filma. Kut prekida uljnog filma dobiva se kao nul-točka funkcije:

$$f(\gamma) = \int_0^1 \left(\frac{dP}{d\vartheta} \right)_{\vartheta=\gamma} d\zeta = \frac{4\pi}{\gamma^{3/2}} \sum_{i=1}^m \sum_{j=1}^n \frac{j b_{ij} (-1)^j}{\omega_i} \quad (19)$$

Rješavanjem sustava matričnih jednadžbi (16) dobivaju se nepoznati koeficijenti b_{ij} . Pri tome u jednadžbi (10) otpadaju derivacije po bezdimenzijskom vremenu τ .

Komponente bezdimenzijske sile u uljnom filmu, za svaki segment, računaju se u koordinatnom sustavu $O\bar{\xi}_1\bar{\xi}_2$ (vidjeti sl. 1):

$$\bar{\Pi}_1 = \int_A P \cos \vartheta dA = \sum_{k=1}^m c_k b_k \quad (20)$$

$$\bar{\Pi}_2 = \int_A P \sin \vartheta dA = \sum_{k=1}^m s_k b_k, \quad (21)$$

gdje su:

$$c_{kl} = \int_A \psi_k \varphi_l \cos \vartheta dA \quad (22)$$

$$s_{kl} = \int_A \psi_k \varphi_l \sin \vartheta dA. \quad (23)$$

Iste komponente u koordinatnom sustavu $O\xi_1\xi_2$ su:

$$\Pi_1 = \bar{\Pi}_1 \cos \Phi - \bar{\Pi}_2 \sin \Phi \quad (24)$$

$$\Pi_2 = \bar{\Pi}_1 \sin \Phi + \bar{\Pi}_2 \cos \Phi. \quad (25)$$

nakon čega se provjerava da li je zbroj komponenata u smjeru ξ_2 (za sve segmente) podjeljen s ukupnom silom (za sve segmente) manji od neke unaprijed zadane vrijednosti (npr. 10^{-3}). Ukoliko je taj uvjet zadovoljen vrijednost kuta zakreta rukavca Φ je dobro pretpostavljena, inače je potrebno promijeniti kut zakreta rukavca i cijeli postupak ponavljati sve dok se ne zadovolji odabrani uvjet.

3. DINAMIČKI KOEFICIJENTI KRUTOSTI I PRIGUŠENJA

Bezdimenzijski koeficijenti krutosti i prigušenja dobivaju se iz:

$$K_{ij} = \left(\frac{\partial \Pi_i}{\partial \xi_j} \right)_0 \quad i, j = 1, 2 \quad (26)$$

$$C_{ij} = \left(\frac{\partial \Pi_j}{\partial \xi_j'} \right)_o \quad i, j = 1, 2 \quad (27)$$

gdje je ξ_j' derivacija po bezdimenzijskom vremenu, a $_o$ predodžava stacionarni položaj.

Parcijalne derivacije (26) i (27) mogu se dobiti transformacijom odgovarajućih parcijalnih derivacija u koordinatnom sustavu $O\bar{\xi}_1\bar{\xi}_2$. Parcijalne derivacije po ξ_j se dobivaju iz jednadžbi (20) i (21):

$$\left(\frac{\partial \bar{\Pi}_1}{\partial \xi_j} \right)_o = \sum_{k=1}^m \left[\mathbf{s}_{ko}^T \left(\frac{\partial \mathbf{b}_{ko}}{\partial \xi_j} \right)_o + \left(\frac{\partial \mathbf{s}_{ko}^T}{\partial \xi_j} \right) \mathbf{b}_{ko} \right]_o \quad j = 1, 2 \quad (28)$$

$$\left(\frac{\partial \bar{\Pi}_2}{\partial \xi_j} \right)_o = \sum_{k=1}^m \left[\mathbf{c}_{ko}^T \left(\frac{\partial \mathbf{b}_{ko}}{\partial \xi_j} \right)_o + \left(\frac{\partial \mathbf{c}_{ko}^T}{\partial \xi_j} \right) \mathbf{b}_{ko} \right]_o \quad j = 1, 2 \quad (29)$$

a slično i derivacije po ξ_j' . Parcijalne derivacije vektora $\mathbf{b}_{ko}$ mogu se dobiti iz jednadžbe (16), dok se derivacije vektora $\mathbf{s}_{ko}$, $\mathbf{c}_{ko}$ i $\mathbf{p}_{ko}$ dobivaju iz jednadžbi (22), (23) i (18). Dinamički koeficijenti krutosti i prigušenja, u koordinatnom sustavu $O\xi_1\xi_2$, dobivaju su primjenom izraza za transformaciju (vidjeti u [3]), pa je:

$$\mathbf{K} = -(\mathbf{T}_o \mathbf{U}_o + \frac{\partial \mathbf{T}_o}{\partial \Phi_o} \mathbf{G}_o) \mathbf{E}_o^{-1} \mathbf{T}_o^{-1} \quad (30)$$

$$\mathbf{C} = -\mathbf{T}_o \mathbf{V}_o \mathbf{E}_o^{-1} \mathbf{T}_o^{-1}, \quad (31)$$

gdje su:

$$\mathbf{U}_o = 3 \begin{bmatrix} \left(\frac{\partial \bar{\Pi}_1}{\partial \xi_1} \right)_o & \left(\frac{\partial \bar{\Pi}_1}{\partial \xi_2} \right)_o \\ \left(\frac{\partial \bar{\Pi}_2}{\partial \xi_1} \right)_o & \left(\frac{\partial \bar{\Pi}_2}{\partial \xi_2} \right)_o \end{bmatrix} \quad \mathbf{V}_o = 3 \begin{bmatrix} \left(\frac{\partial \bar{\Pi}_1}{\partial \xi_1'} \right)_o & \left(\frac{\partial \bar{\Pi}_1}{\partial \xi_2'} \right)_o \\ \left(\frac{\partial \bar{\Pi}_2}{\partial \xi_1'} \right)_o & \left(\frac{\partial \bar{\Pi}_2}{\partial \xi_2'} \right)_o \end{bmatrix}$$

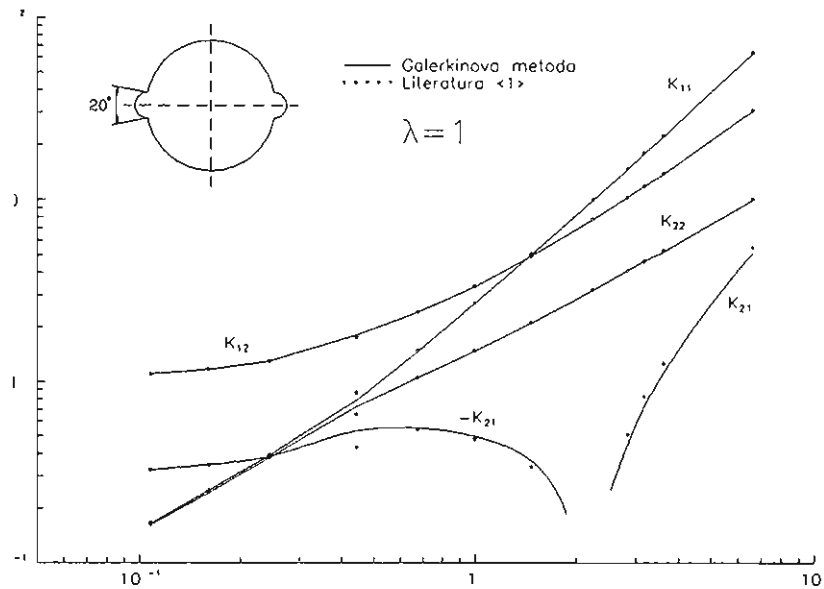
$$\mathbf{T}_o = \begin{bmatrix} \cos \Phi_o & -\sin \Phi_o \\ \sin \Phi_o & \cos \Phi_o \end{bmatrix} \quad \frac{\partial \mathbf{T}_o}{\partial \Phi_o} = \begin{bmatrix} -\sin \Phi_o & -\cos \Phi_o \\ \cos \Phi_o & \sin \Phi_o \end{bmatrix}$$

$$\mathbf{G}_o = 3 \begin{bmatrix} 0 & \Pi_{10} \\ 0 & \Pi_{20} \end{bmatrix} \quad \mathbf{E}_o = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon_o \end{bmatrix}$$

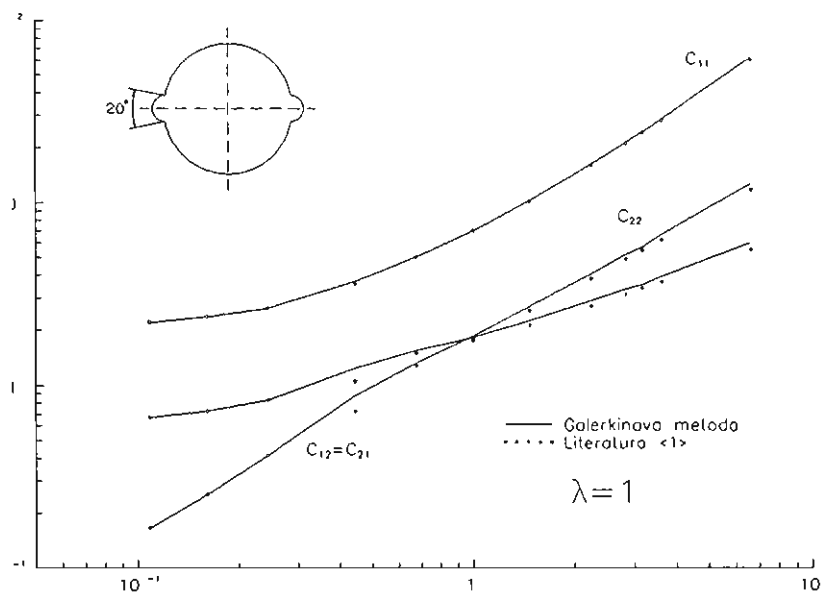
4. PRIMJER

Primjena metode data je na kružnom ležaju s dva osna žlijeba. Budući da se radi o dvosegmentnom ležaju računa se svaki segment posebno, te se rezultati

superponiraju. Na slikama 2. i 3. predloženi su dinamički koeficijenti krutosti i prigušenja u ovisnosti o Sommerfeldovom broju. Rezultati su uspoređeni s onima u [1] dobivenim metodom konačnih diferencija i pokazuju vrlo dobro podudaranje.



Slika 2. Usporedba dinamičkih koeficijenata krutosti
Figure 2. Comparison of dynamic stiffness coefficients



Slika 3. Usporedba dinamičkih koeficijenata prigušenja
Figure 3. Comparison of dynamic damping coefficients

5. ZAKLJUČAK

Izložena je metoda za računanje dinamičkih koeficijenata krutosti i prigušenja kliznog ležaja, koja se temelji na rješavanju Reynoldsove diferencijalne jednadžbe Galerkinovom metodom. Primijenjen je integralni Reynoldsov rubni uvjet koji

omogućava odabir jednostavnih trigonometrijskih funkcija za aproksimaciju raspodjele tlaka po obodu. Dinamički koeficijenti krutosti i prigušenja uljnog filma dobivaju se s relativno malim brojem računskih operacija, pa je vrijeme računanja znatno kraće spram metode konačnih diferencija i metode konačnih elemenata. To znači da se računanje dinamičkih koeficijenata može obaviti u toku samog proračuna rotora uležištenog u kliznim ležajima, umjesto njihovog prethodnog računanja i spremanja u bazu podataka.

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PREGLED VAŽNIJIH OZNAKA

- H - bezdimenzijska debljina uljnog filma
- P - bezdimenzijski pritisak
- λ - konstrukcijska značajka ležaja
- ϑ - obodna koordinata
- ζ - bezdimenzijska osna koordinata
- Φ -kut zakreta rukavca
- ϵ - relativna ekscentričnost rukavca
- γ - kut prekida uljnog filma
- Π - bezdimenzijska sila u uljnom filmu
- K_{ij} -bezdimenzijski koeficijenti krutosti
- C_{ij} -bezdimenzijski koeficijenti prigušenja

A dynamic of the non-linear Coulomb damping oscillator

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Abstract

The effect of the dry friction in dynamics is important in many areas of technology such as mechanism design, engine design, pin-jointed space structures design etceteras. Because of this, there is much motivation for research of this phenomenon. In this paper the response of a harmonically excited single degree of freedom non-linear system is investigated. The dry friction within the system is modelled in the Coulomb damping term. Time histories, phase plane diagrams and Lyapunov spectrum are computed for some range of system parameters. It is shown, that for some parameter values the system exhibits chaotic vibrations.

1. Mechanical model

As a subject of this research the system shown in Fig.1 (Narayanan, 1991) is considered. The system consists of mass having relative motion subject to the belt. The mass is clamped with linear and non-linear cubic spring. Also linear and non-linear cubic damping elements are used. The mass is subject to external excitation.

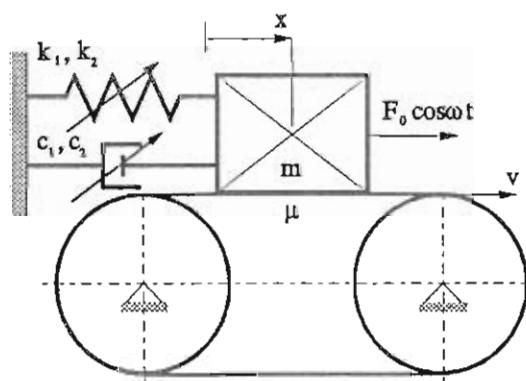


Fig.1

The dry friction within the belt and mass is modelled in the Coulomb damping term. Thus the friction force is calculated as (Feeny, 1992):

$$F_{fr} = -n(x)f(\dot{x} - v) \quad (1)$$

where the normal force is constant

$$n(x) = mg \quad (2)$$

and the friction term is

$$\begin{aligned} f(\dot{x} - v) &= -\mu_k, & \dot{x} - v < 0 \\ -\mu_s &\leq f(\dot{x} - v) \leq \mu_s, & \dot{x} - v = 0 \\ f(\dot{x} - v) &= \mu_k, & \dot{x} - v > 0. \end{aligned} \quad (3)$$

In (3) μ_s is a static coefficient of friction and μ_k is kinetic coefficient of friction. In this particular research we look the case when $\mu_s = \mu_k = \mu$. Thus it is possible to simplify (3) as follows

$$f(\dot{x} - v) = \mu \operatorname{sgn}(\dot{x} - v). \quad (4)$$

2. Mathematical model

Mathematical model is in the form of non-linear second order differential equation

$$m\ddot{x} + c_1(\dot{x} - v) + c_2(\dot{x} - v)^3 + k_1x + k_2x^3 + mg\mu \operatorname{sgn}(\dot{x} - v) = F_0 \cos \omega t. \quad (5)$$

$$x(0) = 0.001, \dot{x}(0) = 0.000$$

It is convenient to rewrite (5) as two first order equations

$$\begin{aligned} \dot{u}_1 &= u_2 \\ \dot{u}_2 &= -g \left[\mu \operatorname{sgn}(u_2 - v) + \alpha(u_2 - v) + \beta(u_2 - v)^3 \right] - \gamma_1 u_1 - \gamma_2 u_1^3 + \frac{F_0}{m} \cos \omega t \\ u_1(0) &= 0.001, u_2(0) = 0.000 \end{aligned} \quad (6)$$

where

$$\alpha = \frac{c_1}{mg}, \beta = \frac{c_2}{mg}, \gamma_1 = \frac{k_1}{m}, \gamma_2 = \frac{k_2}{m}. \quad (7)$$

3. Lyapunov exponent

The Lyapunov exponents provide a computable quantitative measure of degree of stochasticity for a trajectory. In fact the Lyapunov exponents of a given trajectory characterise the mean exponential rate of convergence or divergence of trajectories surrounding this particular trajectory. If Lyapunov exponents are positive we talk about divergence. In this case the dynamic system is unstable and its behaviour tends to be chaotic (Wolf, 1985 and Wiggins, 1990). We define the Lyapunov exponents for the trajectory generated with flow

$$\frac{du_i}{dt} = f_i(\mathbf{u}), i = 1, 2, \dots, k \quad (8)$$

with initial conditions $u_i(t=0) = u_i^0, i = 1, 2, \dots, k$, where $\mathbf{f}(\mathbf{u}) \in C^r, (r \geq 1)$ is r times continuous differentiable. If we consider trajectories with initial conditions $\mathbf{u}^0$ and $\mathbf{u}^0 + \Delta \mathbf{u}$ these trajectories evolve with time, yielding the tangent vector $\Delta \mathbf{u}(\mathbf{u}^0, t)$ with Euclidean norm

$$d(\mathbf{u}^0, t) = \|\Delta \mathbf{u}(\mathbf{u}^0, t)\|. \quad (9)$$

If we define

$$\mathbf{w} = \Delta \mathbf{u}, \quad (10)$$

the time evolution for $\mathbf{w}$ is found by linearizing (8)

$$\frac{d\mathbf{w}}{dt} = M(\mathbf{u}(t))\mathbf{w}, \quad (11)$$

where

$$M = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \quad (12)$$

is Jacobian matrix. Now we define the mean exponential rate of divergence of two initially close trajectories

$$\lambda(\mathbf{u}^0, \mathbf{w}) = \lim_{\substack{t \rightarrow \infty \\ d(0) \rightarrow 0}} \left(\frac{1}{t} \right) \ln \frac{d(\mathbf{u}^0, t)}{d(\mathbf{u}^0, 0)}. \quad (13)$$

There is an k -dimensional basis $\{\mathbf{e}_i\}$, $i = 1, 2, \dots, k$ of $\mathbf{w}$ such that for any $\mathbf{w}$, λ takes on one of the k values

$$\lambda_i(\mathbf{u}^0) = \lambda(\mathbf{u}^0, \mathbf{e}_i), \quad i = 1, 2, \dots, k \quad (14)$$

which are Lyapunov characteristic exponents. Usually these are ordered by size

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k. \quad (15)$$

4. Numerical example

The equation (6) has been solved numerically for different ω values and for parameters

$$\begin{aligned} \alpha &= 0.05 [sm^{-1}] \quad , \beta = 0.02 [sm^{-3}], \\ \gamma_1 &= 0.00 [s^{-2}] \quad , \gamma_2 = 10000 [m^{-2}s^{-2}], \\ \frac{F_0}{m} &= 10.5 [ms^{-2}], \mu = 0.6, \nu = 1 [ms^{-1}] \end{aligned} \quad (16)$$

The solutions have been obtained with fourth order Runge-Kutta method. Fifth-order coefficients are used to yield an error estimate, which is used for step size control within the chosen time step (Forsythe, 1977). Tests have been shown, that the magnitude of chosen time step did not alter the solutions significantly in this case. Also there has been difficulties with sgn term in (6). As stated in (8) the function $\mathbf{f}(\mathbf{u}) \in C^r$ has to be r times continuously differentiable. Because of this the sgn term has been approximated as

$$\text{sgn}(u_2 - \nu) := \frac{2}{\pi} \arctan[\gamma(u_2 - \nu)]. \quad (17)$$

For $\gamma = 10^{12}$ the approximation is quite accurate. During the integration there was no problem even when the mass velocity was equal to belt velocity. For Lyapunov exponents calculation the well-known algorithm by Wolf et al (Wolf, 1985) has been used. The Jacobian (12) for this particular problem is calculated in appendix. The

time history, phase plane orbit and the convergence of Lyapunov exponents for various values of forcing frequencies are shown on Fig.2-Fig.16

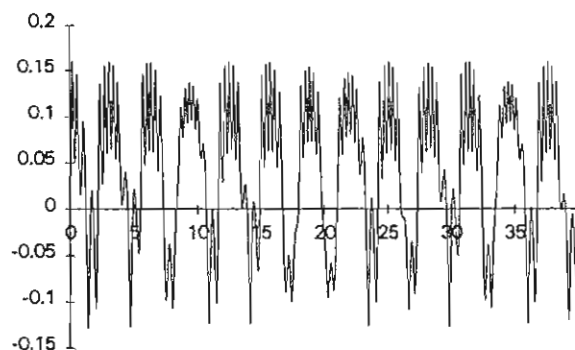


Fig.2 $\omega = 2 s^{-1}$

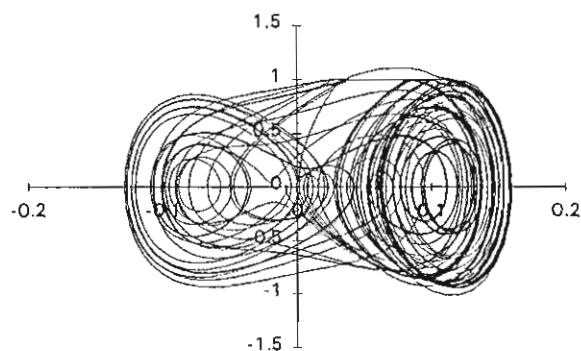
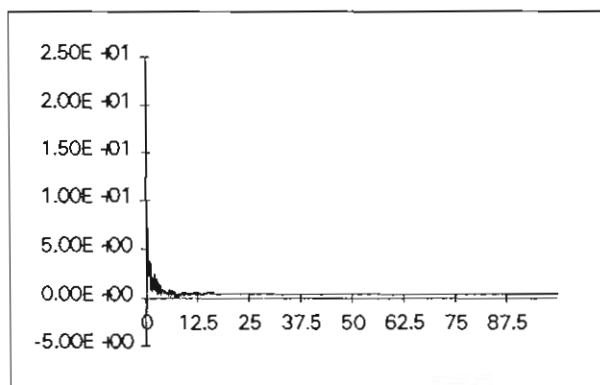


Fig.3 $\omega = 2 s^{-1}$



$\omega = 2 s^{-1}, \quad \lambda_1(t) \xrightarrow{t \rightarrow \infty} 0.43$
Fig.4

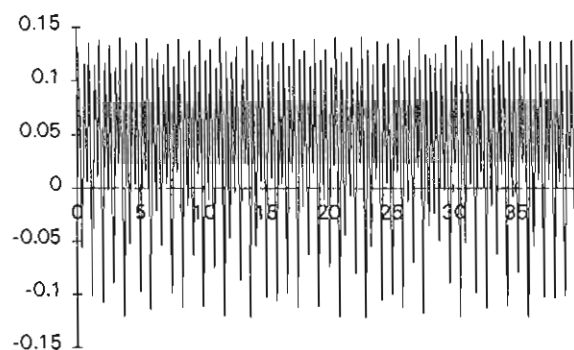


Fig.5 $\omega = 15 s^{-1}$

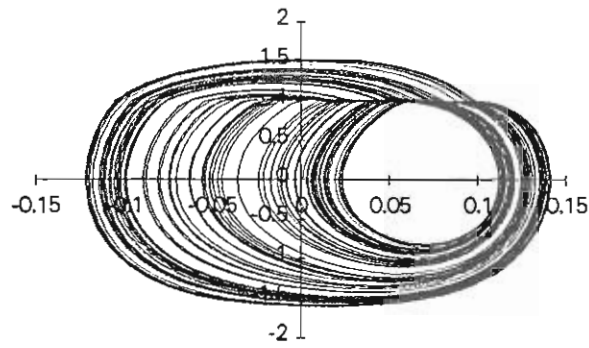
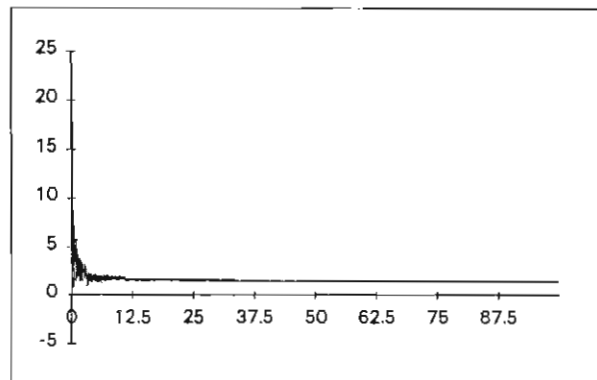


Fig.6 $\omega = 15s^{-1}$



$\omega = 15s^{-1}$, $\lambda_1(t) \xrightarrow{t \rightarrow \infty} 1.36$

Fig.7

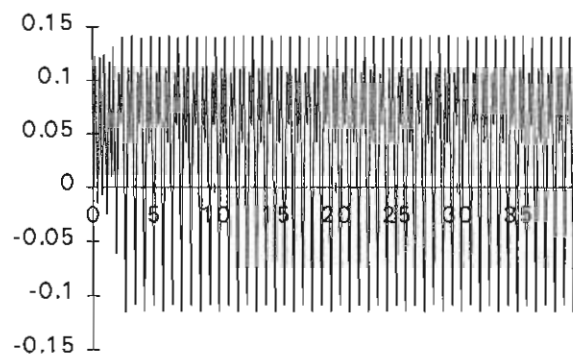


Fig.8 $\omega = 16.35s^{-1}$

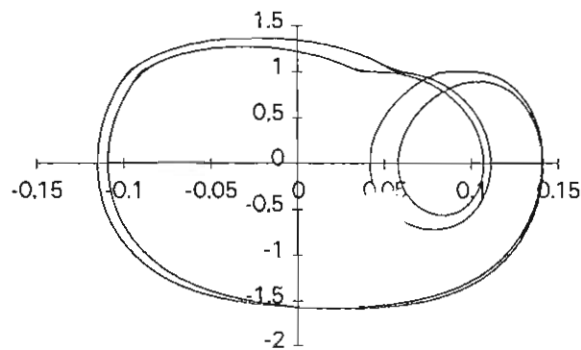
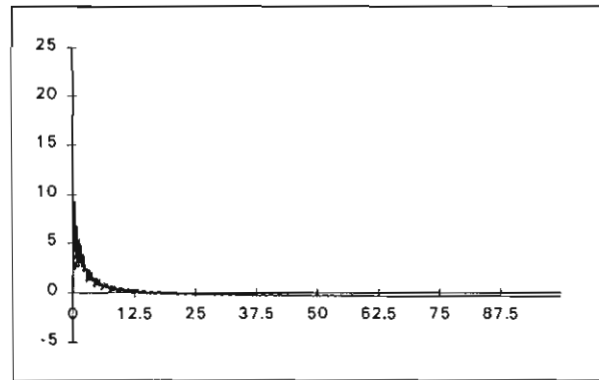


Fig.9 $\omega = 16.35s^{-1}$



$$\omega = 16.35 s^{-1}, \quad \lambda_1(t) \xrightarrow{t \rightarrow \infty} -0.36$$

Fig.10

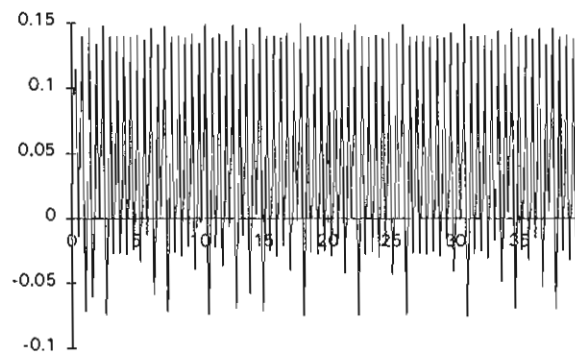


Fig.11 $\omega = 23.6 s^{-1}$

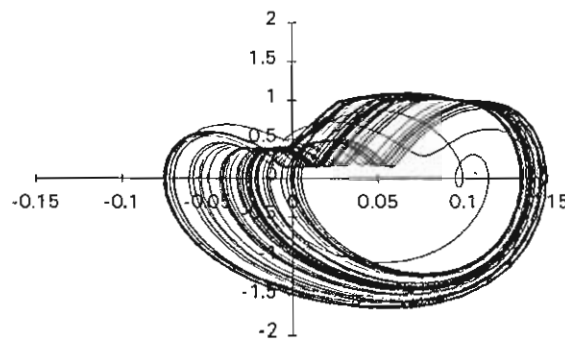
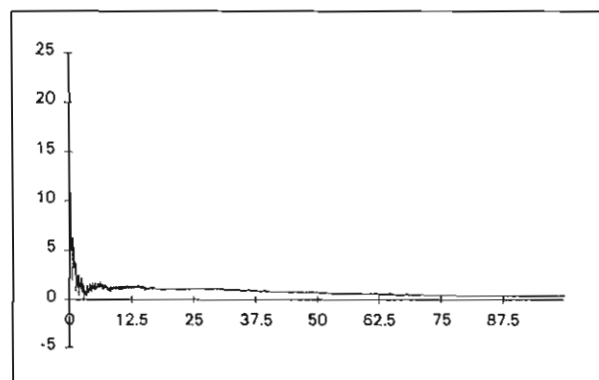
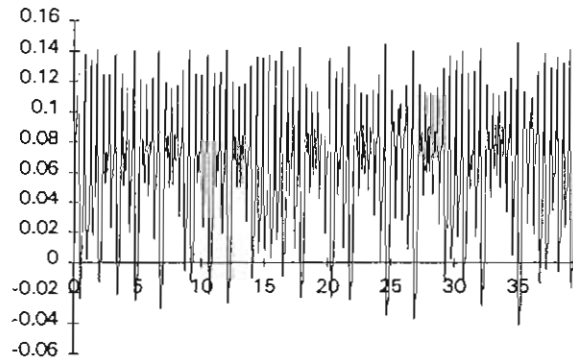
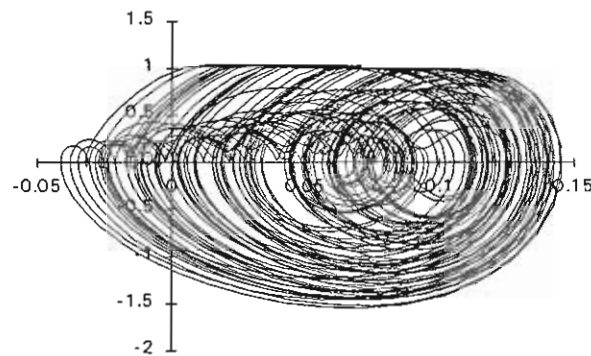
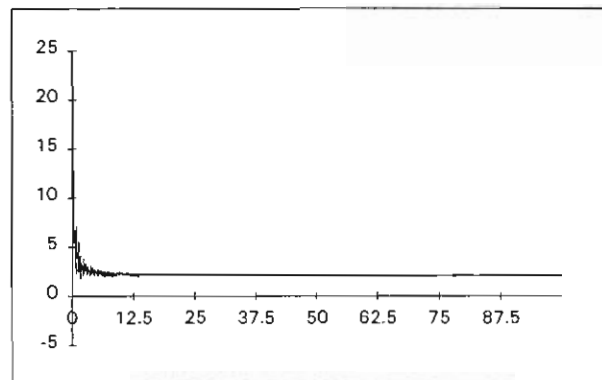


Fig.12 $\omega = 23.6 s^{-1}$



$$\omega = 23.6 s^{-1}, \quad \lambda_1(t) \xrightarrow{t \rightarrow \infty} 0.37$$

Fig.13

Fig.14 $\omega = 26s^{-1}$ Fig.15 $\omega = 26s^{-1}$ 
 $\omega = 26s^{-1}, \quad \lambda_1(t) \xrightarrow{t \rightarrow \infty} 2.05$
 Fig.16

5. Conclusions

Results show that for parameter values (16) and for different values of forcing frequency, the solutions of equation (6) could differ a lot. The time evaluations of the largest Lyapunov exponent show, that the solution of (6) could be periodic or chaotic.

Comparing results with Narayanan, 1991 it is obvious that for lower values of forcing frequency solutions are similar. However for higher values of forcing frequency the solutions for same parameters and forcing frequency values are different. For $\omega=23.6s^{-1}$ we obtain the solution that is obvious chaotic, while Narayanan obtain periodic solution.

The algorithm presented by Wolf 1985 estimate Lyapunov exponents quite accurately, however the results are dependent by numerical technique, used for integration. The accuracy of Runge-Kutta numerical scheme used in this research is

good, but the search for an optimal step size in some cases leads to significantly large computational time. It would be worth to use some other numerical scheme for integration (Sinha, 1993).

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APPENDIX

It is convenient to rewrite equation (5) in the third order autonomous form for Lyapunov exponents calculation. In this case we define $x := u_1$, $\dot{x} := u_2$, $\omega t := u_3$ and the Jacobian matrix is:

$$M = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} 0 & 1 & 0 \\ -\gamma_1 - 3\gamma_1 u_1^2 & -g \left[\alpha + 3\beta(u_2 - \nu)^2 + \frac{2\mu\gamma}{\pi(1 + \gamma^2(u_2 - \nu)^2)} \right] & -\frac{F_0}{m} \sin u_3 \\ 0 & 0 & 0 \end{bmatrix}$$

Evaluation of Vehicle Passive Safety Systems at Low Speeds by the Finite Element Method

Djordje Žebeljan

In the paper possibilities of absorbing the energy of the car collision at low speeds (up to 8 km/h) using bumpers that are parts of the car passive safety system are evaluated. For the simulations of the car collisions and to get the reference results, standard pendulum test in the middle of the bumper was used. Finite element method was applied as a tool for evaluation of the designed bumper options. Three possible options of the bumper systems are presented: two self-supported bumpers and one dash-pot + bumper beam system. In all three systems using the structural modifications have been tried to improve efficiency of the system concerning the energy absorption (J/kg).

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1.0 Introduction

Main goal of the presented work is to evaluate possibilities of absorbing the energy of the car collision at low speeds (up to 8 km/h) using bumpers which are parts of the car passive safety system. For the simulations of the car collisions and to get the reference results, the standard pendulum test in the middle of the bumper was used. A detailed description of the test realisation is given in the European standards for car safety (ECE-R4 Nr. 42. /9/). Possible collision simulations are presented in fig 1.

The finite element method was used as a tool for evaluation of the designed bumpers. Three possible options of the bumper systems are given: two self-supported bumpers and one dash-pot + bumper beam system. In all three systems using the design modifications it was tried to improve efficiency of the system concerning the energy absorption (J/kg).

2.0 Finite Element Approach

Material and the geometrical non-linear stress analysis using finite element method /4/, /5/ have been applied in a sense to approach the real conditions during the pendulum test. A bumper could also be modelled as a system of springs and dash-pots (fig. 2), where some coefficients (K_v , C , K_c) could be constant or a function of velocity, and/or deformation speed /6/, /7/.

An interesting value in comparing the efficiency of two different systems is coefficient η between minimal displacement at the maximal allowable reaction force and real displacement of the system needed to absorb prescribed amount of energy. Systems with the different coefficients are presented on (fig. 3).

5. Thermoplastic material properties have been represented as elastic-plastic material model with the von Mises yield criteria [1/, /4/, /5/, /8/, /10/.

6. In the modelling of the thermoplastic-composites orthotropic material model was used [4/, /5/, /8/, /10/.

7. Load was introduced into the system as the incremental displacement of the pendulum.

8. Calculation results were presented as the amount of energy on the displacement unit of the pendulum (fig. 1. - surface under the curve) and pendulum reaction force on the displacement unit of the pendulum (fig 2. - curve itself).

In general there are four factors having influence on the construction properties (maximal displacement, absorbed energy): material selection, loading mode, geometrical properties and fixation of the structure.

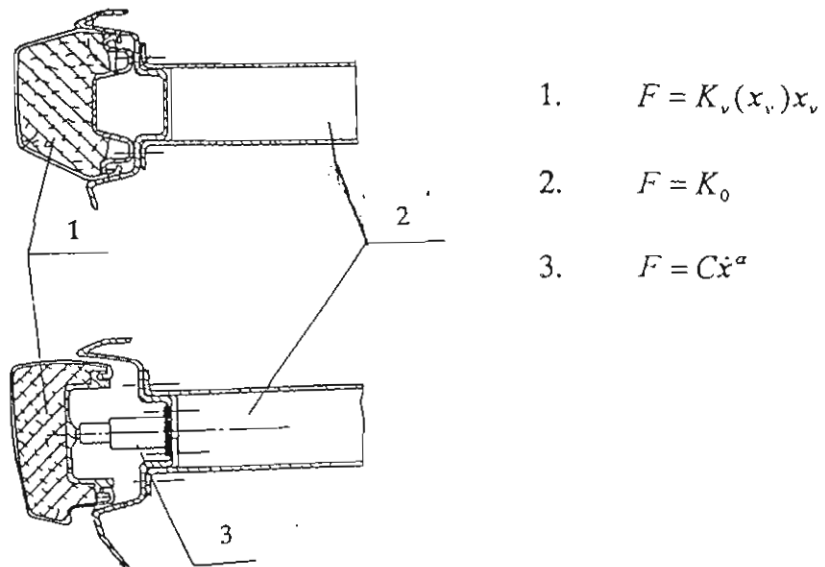


Figure 2: Sketch of the general absorption system
1 - spring; 2 - vehicle frame; 3 - dash-pot

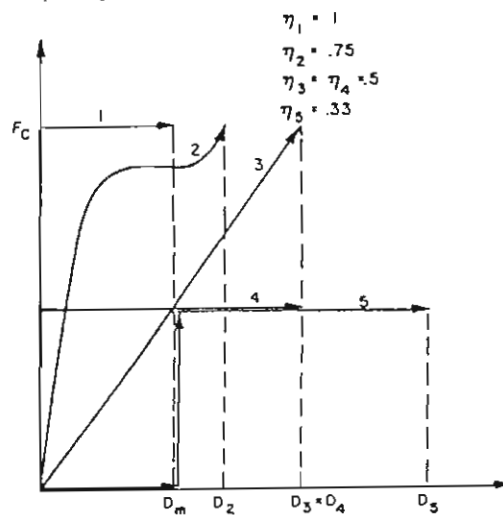


Figure 3: Characteristics at the free moved barrier collision

$\eta = D_m/D$; η - coefficient of the system efficiency; D_m - minimal needed system displacement at the maximal allowable reaction force - F_c in the support, D - system displacement; 1 - Reaction Force; 2 - System Displacement

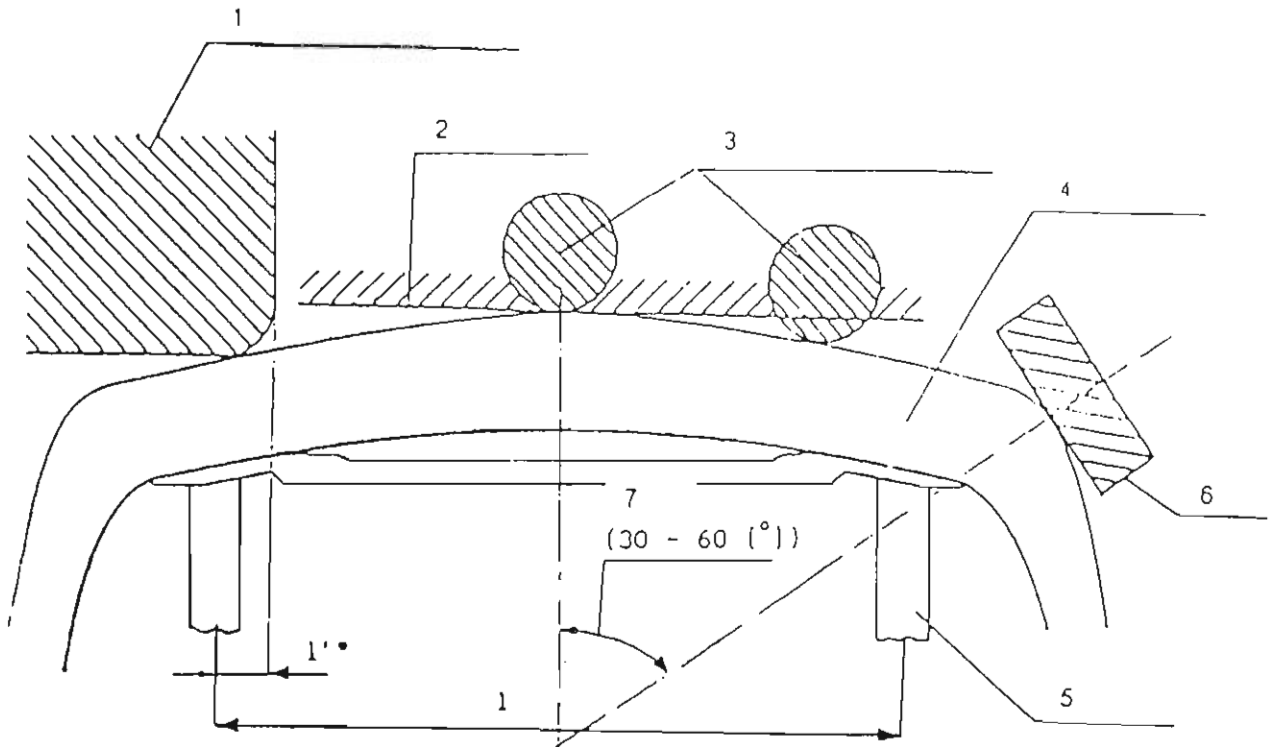


Figure 1: Directions of the vehicle test collisions

1 - pendulum; 2 - wall; 3 - pole; 4 - bumper; 5 - bumper's supports; 6 - wall; 7 - angle; * function of the pendulum centre position; distance between the pendulum centre and the support is from 0.1 to 0.5 distance between the supports l .

Thermoplastic and thermoplastic-composite materials were used as the bumper design materials [8]. Loading speed, loading direction and temperature have great influence on the properties of the materials used for bumpers. The loading speed and temperature have a greater influence on the thermoplastics. Thermoplastics-composites are more sensitive to the load direction according to the fibre orientation.

At the analysis and the optimisation of the evaluated structures, commercial computer program package MACR - K4 [4], [5] based on the finite element method, with a wide application ability in the engineering practice, was used. All evaluated calculations are based on the following assumptions:

1. Pendulum, as well as the complete car (excluding bumper), were modelled as the rigid body.
2. For the modelling of the bumper isoparametric thin shell element with the bi-linear interpolation function including large deformations and Lagrange formulation [3], [4], [5] was used.
3. Because of the bumper symmetry, as well as the symmetry of the initial and boundary conditions only one half of the bumper was modelled.
4. Contact between the pendulum and the bumper has been modelled with the gap-elements (non-linear boundary conditions) [3], [4], [5].

3.0 Self supported bumper - central pendulum test at 4 (km/h)

Requirement of the first design of the self-supporting bumper /11/ is that the bumper has to pass the collision of the pendulum with the prescribed mass into the centre of the bumper at the vehicle velocity of 4 km/h.

Prescribed were: the allowable reaction force in the support, the allowable intrusion of the bumper, weight and the material of the structure as well as the external dimensions and the production technology of the bumper.

The starting point was the unoptimised designer's construction. Intermediate steps are presented in the table on (fig. 4) /11/. For better understanding of the presented data it could be mentioned that the described bumper is consisted from four basic parts:

1. Bumper skin	- wall thickness 3.5 mm
2. U beam	- wall thickness 3.0 mm
3. Internal structure	- wall thickness 2.0 - 3.0 mm
4. Support	- wall thickness 3.0 - 3.5 mm

All structural parts had to be produced from thermoplastics /10/ with the injection moulding process which determinate wall thicknesses and structure design. Bumper skin is welded on the U-beam of the bumper. Main function of the internal structure is to prevent deformation (local buckling) of the bumper cross section or in the other words bumper beam moment of inertia should not be decreased during the test. In the support buckling should be also avoided because maximal intrusion of the bumper is prescribed.

During the optimisation process six different versions of the bumper have been examined (fig. 4) /11/. Main parameter to evaluate if the applied changes in the bumper structure have brought some improvements was the energy absorbed per unit weight of the bumper (J/kg). Intrusion of the system 50 mm for absorbing the 347 J collision energy was allowed, therefore at the end of the table on (fig. 4) energy absorbed is calculated at maximum allowable intrusion and also intrusion needed to absorb prescribed amount of energy is given

After the analysis of the present structure and its improvement we have reached that with 2 % bumper mass increasing results in 28 % efficiency increasing according to the basic version. For the absorption of the prescribed energy amount 93,3 % allowable bumper intrusion was needed.

	original version						modified versions					
	absolute kg	relative [%]	absolute kg	relative [%]	absolute kg	relative [%]	absolute kg	relative [%]	absolute kg	relative [%]	absolute kg	relative [%]
weight	3.21	100	3.35	104.36	3.39	105.61	3.42	106.54	3.27	101.87	96.57	128.09
efficiency	75.32	100	73.38	97.4	86.37	114.67	93.91	124.68				
weight over												
efficiency	1		1.07		0.92		0.85		0.83			0.8
modifications	tubes.						elongated tubes,					
inner structure	tubes.						elongated tubes and ribs,					
skin	tubes.						removed registration plate,					
U beam	stronger support.						stronger support.					
increment	pendulum intrusion [mm]	reaction force [kN]	absorbed energy [J]	total pendulum intrusion [mm]	reaction force [kN]	absorbed energy [J]	pendulum intrusion [mm]	reaction force [kN]	absorbed energy [J]	total pendulum intrusion [mm]	reaction force [kN]	absorbed energy [J]
0	0	0	0	0	0	0	0	0	0	0	0	0
1	2.5	0.31728	0.7932	2.5	0.32016	0.8004	2.5	0.37606	0.94015	2.5	0.4421	1.10525
2	4.2819	0.54078	1.75681	4.2796	0.54546	1.77110	4.2792	0.58274	1.89216	4.2969	0.64336	2.09620
3	6.8186	0.83378	3.92239	6.809	0.86058	3.94785	6.8082	0.9907	4.39764	6.8771	1.02074	4.72991
4	10.443	1.2888	8.59372	10.417	1.298	8.63103	10.404	1.37462	9.34049	10.445	1.54384	10.2381
5	14.996	1.926	17.3628	14.931	1.938	17.3791	13.537	1.88348	15.2414	13.497	2.0922	16.6235
6	18.336	2.628	26.1403	18.243	2.6398	26.1221	17.603	2.7806	26.5473	16.23	2.7382	24.1070
7	21.458	3.5386	37.1878	21.365	3.5772	37.2902	20.654	3.6614	37.7182	18.868	3.523	33.4007
8	24.199	4.3642	49.1501	24.082	4.3708	49.1656	23.39	4.7556	50.7296	22.332	5.0034	50.7325
9	26.981	5.195	63.6023	26.845	5.2062	63.5503	26.097	5.7826	66.3831	25.62	6.1582	70.9806
10	29.777	5.9236	80.1649	29.593	5.9332	79.8548	29.929	7.275	94.2609	30.173	7.8536	106.738
11	32.563	6.5546	98.4260	32.341	6.5634	97.8910	35.319	8.6792	141.041	34.68	9.3346	148.809
12	36.361	7.1806	125.698	36.214	7.653	127.531	42.578	9.7066	211.502	41.386	10.8478	221.554
13	39.802	8.2768	154.178	40.892	8.3844	166.753	50	10.9508	292.778	50	11.3468	319.295
14	43.041	8.2216	180.808	45.222	8.4306	203.257						
15	46.009	8.5764	206.263	49.3	8.9004	239.553						
16	48.876	8.8738	231.704	50	8.9362	245.809						
17	50	8.976	241.793									
modifications	tubes.						stronger support.					
inner structure	tubes.						stronger support.					
skin	tubes.						stronger support.					
U beam	stronger support.						stronger support.					
predicted	total pendulum intrusion [mm]	total energy absorbed [J]	total pendulum intrusion [mm]	total energy absorbed [J]	total pendulum intrusion [mm]	total energy absorbed [J]	total pendulum intrusion [mm]	total energy absorbed [J]	total pendulum intrusion [mm]	total energy absorbed [J]	total pendulum intrusion [mm]	total energy absorbed [J]
calculated	54.04	347	53.594	347	48.61	347	46.325	347	45.61	347	46.65	347
difference [%]	8.1		7.2		-2.8		-7.3		-8.8		-6.7	

Figure 4: The summary of the results of the evaluated bumper versions /11/

4.0 Self supported bumper - central pendulum test at 8 (km/h)

The boundary and the initial conditions of the second also self-supported bumper are in principal the same as for the first one /11/. The difference is only that the pendulum test has to be done at 8 km/h. In addition to design optimisation it was necessary to evaluate the influence of the bumper skin - bumper beam connection. There were two possible connections: spot welding (12 spot welds along the length of the bumper) or continuous weld along the whole length of the bumper. Summary of the evaluated versions is presented on (fig 5) /11/. Main improvements have been done on the internal structure of the bumper (improved version) /11/ and on the transition zone between the bumper support and the bumper beam (final version) /11/. Final version was also evaluated with the differently welded skin. Maximum allowable intrusion was here 60 mm and the energy absorption should be 1443 J.

property	unit	at	version			
			basic	improved	final-w.	final-c.
bumper weight	[kg]		3.63	3.56	3.73	3.73
energy	[J]	40 [mm]	648.87	749.81	777.51	690.69
		60 [mm]			1467.8	1272.1
max. force	[kN]	40 [mm]	23.322	28.066	33.29	29.28
		60 [mm]			34.246	29.224
required energy	[J]	4 [km/h]	360.49	360.49	360.49	360.49
		8 [km/h]	1443	1443	1443	1443
energy, 40 [mm]	[J]	4 [km/h]	180	208	215.68	191.6
		8 [km/h]			53.88	47.86
energy, 60 [mm]	[J]	8 [km/h]			101.72	88.16
max. speed	[km/h]	40 [mm]	5.37	5.77	5.87	5.54
		60 [mm]			8.07	7.51

Figure 5: Summary of the absorption possibilities of the evaluated bumper variants

Calculations have shown, that the given requirements could be satisfied only with the skin which is welded along the whole length to the bumper beam. More details about the evaluated variants could be found in reference /11/.

5.0 Dash-pot + bumper beam system pendulum test at 8 km/h

The third evaluated bumper system dash pot + bumper beam had got in addition to the already described requirements also one additional requirement. Aluminium beam had to be exchanged with a beam from the thermoplastics composites. The new beam should be, because of the market requirements, at least 20 % lighter than the original one with at least the same performances. For the composite variant only 15 % more bumper intrusion has been allowed.

Results of the performed analysis with the selected shapes of the bumper are presented on (fig 6). In general aluminium cross section is closed beam made by extrusion (version A - fig 6). Composite beams could be only made by compression moulding in prescribed material (TPC 2000/TPC 3000 /10/) which means that the cross section should be open /11/ (version B - fig. 6) or closed by gluing two open cross sections /11/ (versions B - closed, F, F+ - fig. 6) All composite cross-section should not be larger than the original one.

version	wall thickness [mm]	material	intrusion of bumper beam [mm]	intrusion of bumper beam [%]	additional intrusion of dashpot [mm]	additional intrusion of dashpot [%]	total intrusion [mm]	total intrusion [%]
A - original	3.00	Al	1.72	100.00	45.59	100.00	45.31	100.00
B - open	6.00	TPC 2000	20.09	1168.14	35.32	81.03	55.41	122.30
B - closed	6.00	TPC 2000	15.51	901.57	36.56	83.89	52.07	114.93
F - open	3.00	TPC 2000	12.12	704.48	35.39	80.07	50.50	111.47
F - open	4.00	TPC 2000	7.92	460.70	40.48	92.88	48.41	106.85
F - open	6.00	TPC 2000	4.73	275.17	42.08	96.54	46.81	103.33
F - closed	3.00	TPC 2000	11.23	652.85	38.83	89.09	50.06	110.49
F - closed	4.00	TPC 2000	7.35	427.15	40.77	93.54	48.12	106.21
F - open	3.00	TPC 3000	10.93	635.17	38.98	89.44	49.91	110.16
F - open	4.00	TPC 3000	7.10	412.85	40.89	93.83	48.00	105.94
F - closed	3.00	TPC 3000	9.98	580.23	39.46	90.52	49.44	109.12
F - closed	4.00	TPC 3000	6.62	385.12	41.13	94.37	47.76	105.41
F+ - open	3.00	TPC 2000	9.22	535.05	39.84	91.40	49.06	108.38
F+ - open	4.00	TPC 2000	6.14	356.74	41.38	94.93	47.51	104.87
F+ - closed	3.00	TPC 2000	8.66	503.26	40.12	92.04	48.77	107.65
F+ - closed	4.00	TPC 2000	5.52	320.81	41.69	95.64	47.20	104.19
F+ - open	3.00	TPC 3000	8.19	475.93	40.35	92.58	48.54	107.14
F+ - open	4.00	TPC 3000	5.41	314.53	41.74	95.77	47.15	104.07
F+ - closed	3.00	TPC 3000	7.66	445.23	40.62	93.19	48.27	106.55
F+ - closed	4.00	TPC 3000	5.16	300.17	41.86	96.05	47.03	103.80

closed - back side of the beam is closed (similar to the original version)

+ - back side of the beam is moved 10 mm backwards (cross section is bigger)

Al: $E_x = E_y = 70 \text{ GPa}$, $S_y = 270 \text{ MPa}$; TPC 2000: $E_x = 30 \text{ GPa}$, $E_y = 10 \text{ GPa}$; TPC 3000: $E_x = 36 \text{ GPa}$, $E_y = 10 \text{ GPa}$

Figure 6: Intrusions of the beam and dash pot for the different bumper variants

With the proper choice of the bumper shape, which had to be adapted to the requirements of the technological process used in manufacturing parts from thermoplastics composites, by increasing the system displacement needed for prescribed energy absorption for 10 % a 25 % mass reduction of the bumper beam has been achieved /11/.

6.0 Conclusions

A bumper beam as one of the essential part of the vehicle passive safety system at low speeds concerning the presented results could be efficiently produced in thermoplastics or in thermoplastics-composites. From the presented results it could be concluded that with the detail observation and determination of the stress/strain field in the bumper and proper distribution of the collision energy in the structure by reconstructioning some bumper parts a bumper efficiency concerning the energy absorption could be significantly increased. By proper selection of the bumper shape and composite material even the metal beams could be efficiently exchanged.

7.0 Literature

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CONTRIBUTION TO THE FINITE ELEMENT METHOD BASED ON THE METHOD OF CHARACTERISTICS IN MODELLING HYDRAULIC NETWORKS

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SUMMARY

The paper deals with the problems related to modelling hydraulic networks. Although the existing models, based on the finite element method and weak formulations of the problems, have certain advantages, they also have certain drawbacks, particularly when dealing with unsteady channel flows. Herein, attention is paid to the method of characteristics. However, since this method implies a complex approach it has not been widely applied in practice. Even when it was successfully applied, certain constraints related to the relative value of the time step compared to the space step, made it expensive. The new approach proposed in this paper is based on the finite element method. The main problem consists in determining the finite element matrix and vector according to the method of characteristics.

1. Introduction

The finite element method and the respective numerical procedures used in modelling unsteady flows in pipes and channels have been recently developed so that they can be further improved, particularly when modelling real network systems. Chronologically speaking, the first numerical models were based on the method of characteristics. The observed drawbacks of this method, such as constraints in the time interval value, have led to the development of the method of finite differences, particularly its implicit schemes.

Both methods display serious disadvantages in flexibility when modelling network systems which, in addition to pipes and channels include other structural elements such as valves, fittings, locks, spillways, chambers, pumps, turbines etc.

Accordingly, the finite element method has obvious advantages. Each structural element is modelled as a finite element; the respective finite elements are assembled into a mesh just as the structural components are connected into a system; i.e. the finite elements matrices form a global nodal matrix when they are superposed. Therein, certain functions of the structures in a system have a nodal character. The methodology for developing such models has been described in lit.[4],[9] together with the presentation of the numerical part.

However, although these methods have great flexibility and efficiency and are used in modelling networks of pipe and channel elements as a unique whole, including all important hydromechanical equipment and structures of the system, see lit. [9], certain drawbacks can be noted, particularly in modelling channel flows.

Actually, the mentioned numerical method has not proved useful in obtaining stable solutions for very short time steps, i.e. it is useful only for the subcritical flow.

The best numerical solutions for unsteady channel flow, which changes either gradually or suddenly, can be obtained by a classical method of characteristics which is valid both for subcritical and supercritical flow. However, the great difficulties encountered in applying this method to networks as well as certain constraints in time intervals make this method unsuitable and expensive so that it has been abandoned; however, this has proved unjustifiable.

This paper presents the possibility of applying the method of characteristics to the finite element method. The procedure will be illustrated on a prismatic channel flow; however, it can be successfully applied also to non-prismatic channels and to all similar wave problems. The presented procedure eliminates the constraints of the time step (high values for the method of characteristics and low values for the presented procedure); thus, it is suitable for all flow regimes in the riverbed. At the same time this approach has maintained all the advantages of the finite element method.

2. Main equations and principles of the procedure for the network system

The unsteady flow in a prismatic river bed is described by the following equations:

a) continuity equation:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial l} = 0 \quad (1)$$

b) dynamic equation:

$$\frac{\partial v}{\partial t} + g \frac{\partial y}{\partial l} + v \frac{\partial v}{\partial l} + g(I_e - I_0) = 0 \quad (2)$$

where:

A - cross-sectional area

Q - discharge

$v = Q/A$ - average channel flow velocity

I_e - energy slope

$$I_e = \frac{n^2 |v| v}{R^{4/3}}$$

I_0 - channel bed slope:

$$I_0 = -\frac{dz}{dl}$$

l - length along the cross-sectional centroid in direction of the flow

z - bed bottom level

t - time

g - gravity acceleration

n - Manning's roughness coefficient

$R = A/O$ - hydraulic radius, O wetted perimeter, see Figure 1.

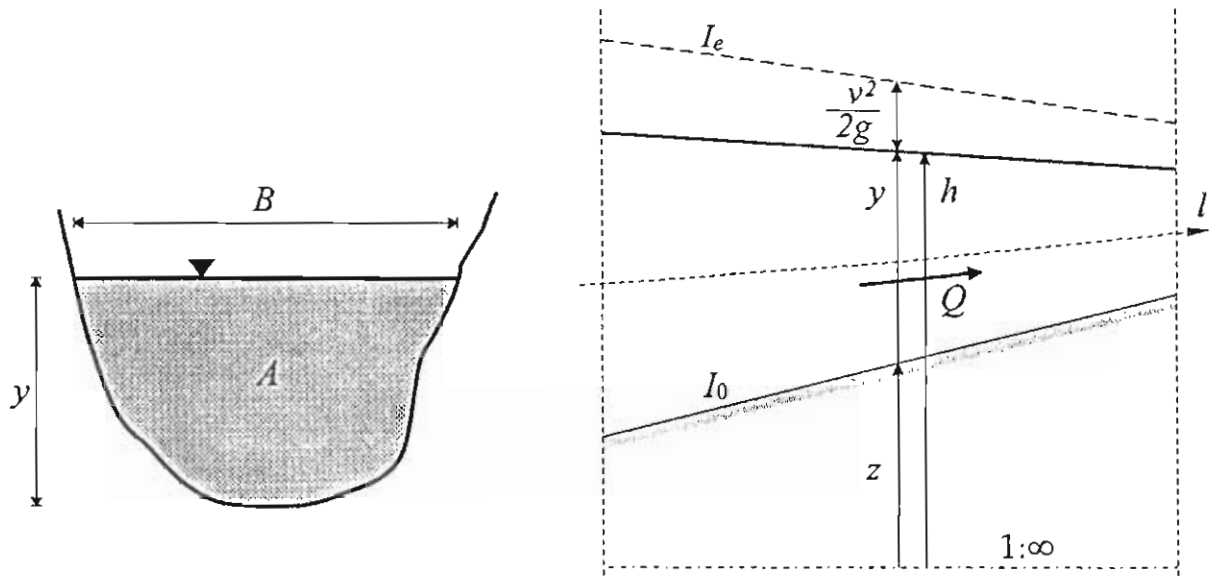


Figure 1 - Basic definitions for channel flow

The main principle in modelling network systems by finite element method consists of integrating the continuity equation (1) and dynamic equation (2) on the finite element. By applying the linear shape functions for variables h and Q , where h expresses piezometric head and the weak formulation of the problem, it is possible to obtain two non-linear equations for each finite element:

$$F_i(h_j, h_j^*, Q_j, Q_j^*) = 0 \quad ; \quad i, j = 1, 2 \quad (3)$$

where:

h_j, Q_j - boundary values of the piezometric head and discharge of the finite element at the beginning of the time interval (i.e. known state t)

h_j^+, Q_j^+ - boundary values of the piezometric head and discharge of the finite element at the end of the time interval (unknown state $t + \Delta t$).

For N finite elements there are $2N$ equations from which it is necessary to compute $2N$ unknown discharges at the end of the time interval, as well as M unknown piezometric heads, where M is the total number of nodes in the finite element mesh. Hence, the piezometric head is considered to be equal for all additional elements converging in a node. In order to solve this system it is necessary to set M nodal equations, i.e. continuity equations in nodes.

Since these equations are non-linear it is necessary to apply the respective algorithm for solving non-linear systems of equations. Newton's modified method has proved efficient.

The following procedure is followed: it is possible to write the Jacobian scheme for the element equations, in the following form:

$$\frac{\partial F_i^{(k)}}{\partial h_j^+} \Delta h_j + \frac{\partial F_i^{(k)}}{\partial Q_j^+} \Delta Q_j = -F_i^{(k)} \quad (4)$$

where:

$\Delta h_j, \Delta Q_j$ - increment of the piezometric head h (nodal value) and discharge Q (element value) in $(k+1)$ -th iterative step, according to expression $h^{(k+1)} = h^{(k)} + \Delta h$, $Q^{(k+1)} = Q^{(k)} + \Delta Q$.

Initial iterative values are the known state from the beginning of the time interval.

The same formal procedure is applied for nodal continuity equations. The algorithm is as follows:

- the iterative increments of the element nodal discharges are expressed from the element equations in the following form:

$$\Delta Q_i = M_{ij} \Delta h_j + q_i \quad (5)$$

where M_{ij} is the matrix and q_i the vector of the finite element which are by a standard procedure of superposition assembled into a global nodal system of equations. The obtained system of equations has a banded matrix form.

- boundary conditions are introduced
- the increments of the piezometric head are solved from the nodal system of equations
- increments of element discharges are determined from the element equations.

After determining the new iterative values of the piezometric heads and discharges, the procedure is repeated until the required accuracy is obtained.

3. Main principles of the method of characteristics

Equations (1) and (2) in the mathematical classification belong to hyperbolic equations, for which it is possible to find simpler differential forms along the characteristic curves, i.e. characteristics. The transformation procedure has a deep physical sense (wave phenomena). More details about the method of characteristics can be found in the classical textbooks on unsteady flow modelling or on equations of mathematical physics, lit. [2],[3],[5],[6],[7], whereas Th. Dracos work, lit. [8] should be noted when considering the problems related to channel flows.

The characteristic forms of equations (1) and (2) cannot be directly written by means of main variables h and Q , but by using variables y and v . Since equation (2) has already been written in that form, the continuity equation (1) has to be transformed into:

$$\frac{\partial y}{\partial t} + v \frac{\partial y}{\partial l} + \frac{A}{B} \frac{\partial v}{\partial l} = 0 \quad (6)$$

which takes into account the fact that prismatic channels are considered, where:

$$A = A(y) \quad ; \quad y = y(l, t) \quad (7)$$

so that the rules of chain derivation can be applied. In equation (6) B is the surface width, see Figure 1.

Equations (6) and (2) are transformed into equations of curves - characteristics:

$$\gamma^{(\pm)}: \quad dx = (v \pm c)dt = w^{(\pm)}dt \quad (8)$$

where c is the velocity of elementary wave on still water:

$$c = \sqrt{g \frac{A}{B}} \quad (9)$$

while $w^{(+)}$ and $w^{(-)}$ are absolute velocities of the positive and negative elementary waves. Thus, characteristics are the paths of the fronts of the positive and negative elementary waves.

The following equations are valid along the characteristic:

$$\Gamma^{(\pm)}: \quad gdy \pm cdv \pm cg(I_* - I_0)dt = 0 \quad (10)$$

The meaning of the obtained equations can be seen in Figure 2, which presents a hodograph for point P in the physical plane (l, t) .

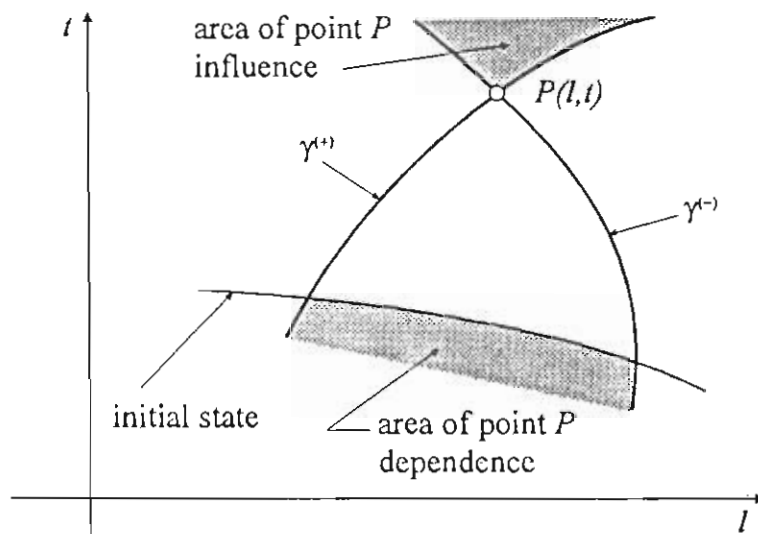


Figure 2. Area of influence and area of dependence for a point from the mesh of characteristics

The position of point $P(l, t)$, i.e. the values of l and t , have been determined by the intersection of two characteristics $\gamma^{(+)}$ and $\gamma^{(-)}$. The values of depth y and velocity v at point P have been determined by the integration of the respective equations $\Gamma^{(+)}$ and $\Gamma^{(-)}$ along the characteristic $\gamma^{(+)}$ and $\gamma^{(-)}$.

It can be easily noted that it would be very difficult to develop a hodograph of the characteristic for the channel network system. The complexity of this procedure has already been revealed in modelling a prismatic channel determined by the irregular position of the cross-sections. Therefore, a fixed grid of time increments and profiles is used instead of developing a hodograph of characteristics, as presented in Figure 3.

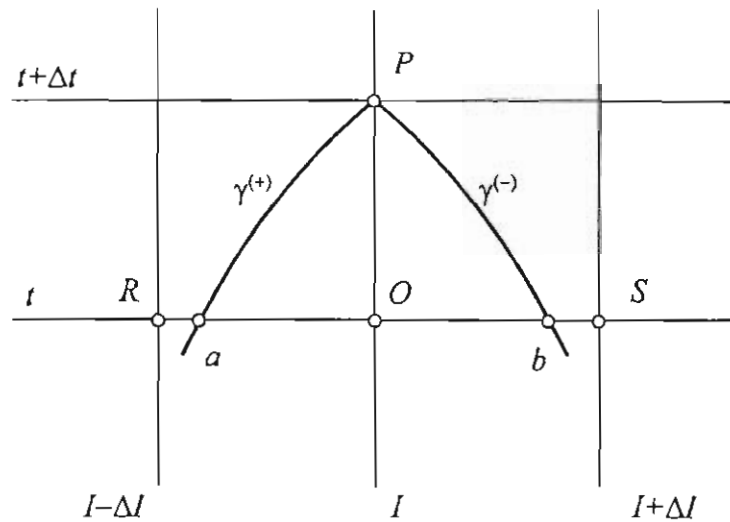


Figure 3. Determination of the state at point P in a fixed grid

The initial state at instant t is known according to the known values at grid points R, O and S . The unknown state at point P is determined by drawing characteristics from points a and b , where values y and v are determined by interpolation. It can be easily seen that there is a constraint for the time increment for the fixed grid, i.e. nodes R and S are the farthest positions of points a and b .

4. Generating finite element matrix and vector by means of characteristics

The main concept of the fixed grid can be directly applied to the generation of the finite element matrices and vectors. Figure 4 presents a typical finite element with the position of the negative characteristic in the upstream node and the position of the positive characteristic in the downstream node. Such a position of the characteristics corresponds to a subcritical flow (solid lines).

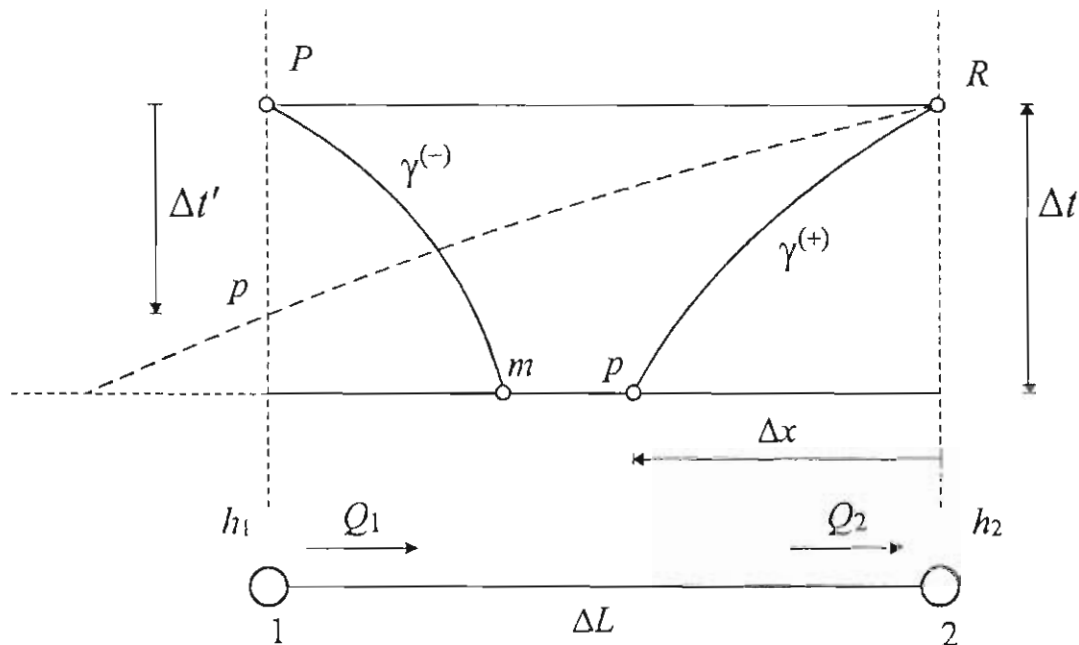


Figure 4. Finite element and characteristics

Firstly, the procedure of integrating the equations of characteristics will be presented. Let two points a and b on characteristic are known, then a general integration rule may be written for the characteristic:

$$\int_a^b dx = \int_a^b w^{(\pm)} dt \quad (11)$$

$$\int_a^b g dy \pm \int_a^b c dv \pm \int_a^b cg(I_e - I_0) dt = 0 \quad (12)$$

where from (11) is obtained:

$$x_b - x_a = \int_a^b w^{(\pm)} dt = \tilde{w}^{(\pm)}(t_b - t_a) \quad (13)$$

and from (12):

$$g(y_b - y_a) = \mp \int_a^b c dv \mp \int_a^b cg(I_e - I_0) dt = \mp \tilde{c}(v_b - v_a) \mp \{cg(I_e - I_0)\} \tilde{t}(t_b - t_a) \quad (14)$$

When computing integrals a theorem on the average integral value is applied, wherein the sign $\sim$ denotes the respective average values of the functions. If points a and b are sufficiently close, the average values of the functions can be taken as an arithmetic average value from the ends of the integration intervals, i.e. generally:

$$\tilde{f} \approx \frac{1}{2}(f_a + f_b) \quad (15)$$

The positions of the characteristics drawn from the local element nodes are determined by the current iterative values. Figure 4 presents the possible positions of the positive characteristic. For short time steps, i.e. for long elements, the characteristic intersects the element at state t , whereas for longer time steps and short elements, the characteristic goes beyond the element (dashed line). In the first case, point p is interpolated at interval ΔL of the finite element, and in the second case it is interpolated on the time interval Δt . The linear interpolations are applied. From approximate geometrical relations for the positive characteristic, node 2, the interpolation parameter can be written as:

$$\lambda_2^{(+)} = \frac{\Delta x}{\Delta L} = \frac{[(v_2 + c_2) + (v_2^+ + c_2^+)]\Delta t}{2\Delta L + [(v_2 + c_2) - (v_1 + c_1)]\Delta t} \quad (16)$$

If $0 < \lambda_2^{(+)} \leq 1$ then point p lies on the element, so that the function values at point p are interpolated according to expression:

$$f_p = \lambda_2^{(+)} f_1 + (1 - \lambda_2^{(+)}) f_2 \quad (17)$$

If $\lambda_2^{(+)} > 1$ then point p lies on the time interval. The interpolation parameter is derived from approximate geometrical relations, in the form:

$$\tau_2^{(+)} = \frac{\Delta t'}{\Delta t} = \frac{1}{\lambda_2^{(+)}} \quad (18)$$

Then the values of the function at point p are interpolated in the following way:

$$f_p = \tau_2^{(+)} f_1 + (1 - \tau_2^{(+)}) f_1^+ \quad (19)$$

The values with indices 1 and 2 refer to nodal values of the function, while $+$ signs refer to the nodal values at the end of the time interval (unknown state $t + \Delta t$); without these signs they refer to the beginning of the time interval (known state t).

Similarly, for the negative characteristic, associated to the upstream node 1, we obtain:

$$\lambda_1^{(-)} = \frac{\Delta x}{\Delta L} = - \frac{[(v_1 - c_1) + (v_1^+ - c_1^+)]\Delta t}{2\Delta L + [(v_1 - c_1) - (v_2 - c_2)]\Delta t} \quad (20)$$

For $0 < \lambda_1^{(-)} \leq 1$ it is valid:

$$f_p = \lambda_1^{(-)} f_2 + (1 - \lambda_1^{(-)}) f_1 \quad (21)$$

For $\lambda_1^{(-)} > 1$ it is valid:

$$\tau_1^{(-)} = \frac{\Delta t'}{\Delta t} = \frac{1}{\lambda_1^{(-)}} \quad (22)$$

and:

$$f_p = \tau_1^{(-)} f_2 + (1 - \tau_1^{(-)}) f_2^+ \quad (23)$$

When $\lambda_1^{(-)} < 0$ there is no negative characteristic in the upstream node for the element (supercritical flow). The influence of the flow regime will be analysed separately in the next section.

Conditional equations of type (3), from which the finite element matrix and vector are generated, of the form (5), are obtained by a consistent application of the presented integration and interpolation rules. Thus, for example, the first conditional equation for the finite element is obtained by setting point a into point m , which is obtained by an interpolation rule for the negative characteristic, and point b corresponds to the unknown state for node 1 (point p):

$$g(y_1^+ - y_p) - \frac{1}{2}(c_1^+ + c_p)(v_1^+ - v_p) - \frac{g}{2}[c_1^+(I_{s1}^+ - I_0) + c_p(I_{sp} - I_0)](t + \Delta t - t_p) = 0 \quad (24)$$

In the previous expression, variables y and v should be expressed by h and Q in order to obtain shape (3) suitable for the finite element method.

A similar procedure is followed with a positive characteristic which finally gives the second conditional equation.

After the formation of equations of type (3), further procedure remains the same as described in point 2.

The described procedure is generally valid for all time steps of the computation, since for consistent boundary conditions the global nodal matrix always remains regular.

5. Application of the procedure to various flow regimes

Figure 5. presents a typical longitudinal channel profile with different flow regimes. Since the finite elements, used for the channel discretizations, are fixed in space, it is necessary to consider various flow regimes at element nodes and their influence upon the generation of finite element matrices and vectors.

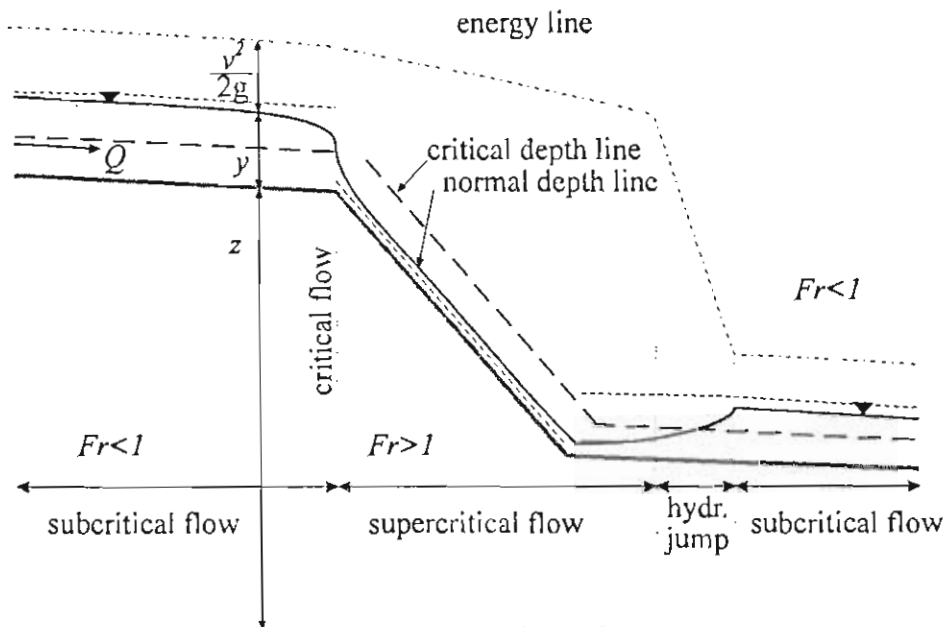


Figure 5. - Presentation of different flow regimes

This paper uses the definition of Froude's number, as commonly used in Europe:

$$Fr = \frac{v^2}{c^2} \quad (25)$$

The relationship between the flow regime and absolute velocities of elementary waves is, as follows:

Subcritical flow $Fr < 1$, $c > v$:

$$\begin{aligned} w_{poz} &= v + c > 0 \\ w_{neg} &= v - c < 0 \end{aligned} \quad (26)$$

Positive and negative waves have opposite directions, the downstream disturbances are transferred upstream, which makes it necessary to determine the downstream boundary conditions.

Critical flow $Fr = 1$, $c = v$:

$$\begin{aligned} w_{poz} &= v + c > 0 \\ w_{neg} &= v - c = 0 \end{aligned} \quad (27)$$

Only a positive wave exists which transfers the upstream disturbances downstream, whereas the downstream disturbances cannot be transferred through the profile with a critical depth. This is the control profile.

Supercritical flow $Fr > 1$, $c < v$:

$$\begin{aligned} w_{poz} &= v + c > 0 \\ w_{neg} &= v - c > 0 \end{aligned} \quad (28)$$

Positive and the negative waves move in the same direction, the downstream disturbances are not transferred upstream which makes it necessary to determine only the upstream boundary condition.

Since different flow regimes are generally possible in the upstream (subscript 1) and downstream (subscript 2) element nodes, the possible positions of the characteristics are presented in Figures 6, 7 and 8.

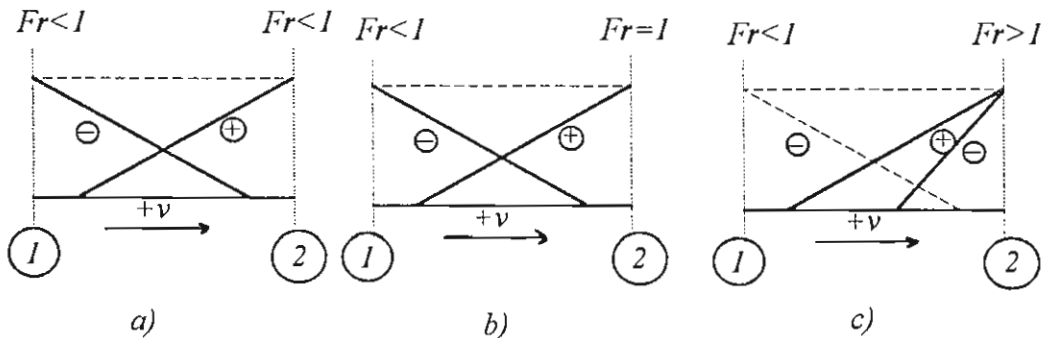


Figure 6. - Subcritical flow upstream, downstream combinations

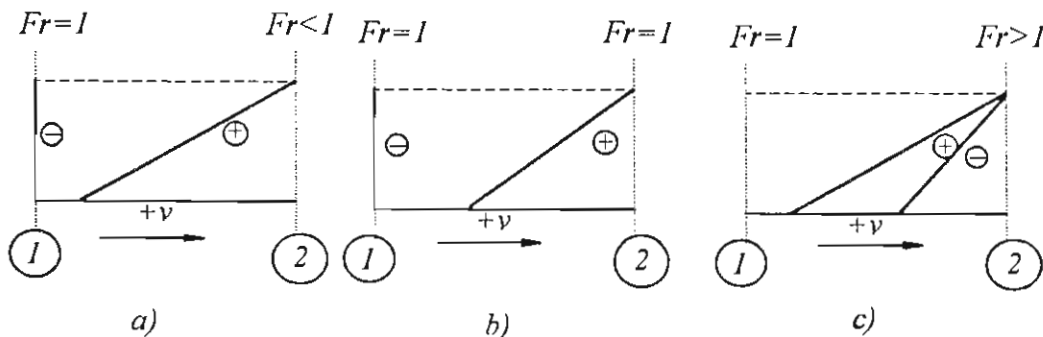


Figure 7. - Upstream critical flow, downstream combinations

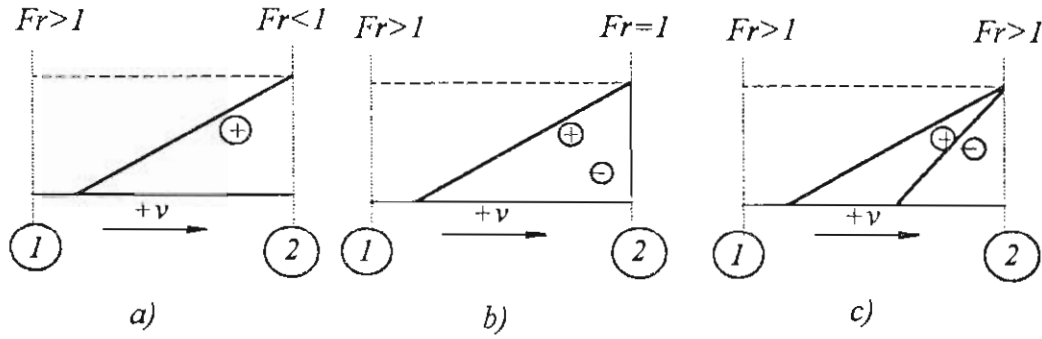


Figure 8. - Upstream supercritical flow, downstream combinations

Criteria for determining the position of the positive characteristic in the downstream node and of the negative characteristic in the upstream node, see (16) and (20), should be expanded by adding a criterion for determining the position of the negative characteristic in the downstream node, as follows:

$$\lambda_2^{(-)} = \frac{\Delta x}{\Delta L} = \frac{[(v_2 - c_2) + (v_2^* - c_2^*)]\Delta t}{2\Delta L + [(v_2 - c_2) - (v_1 - c_1)]\Delta t} \quad (29)$$

For $0 < \lambda_2^{(-)} \leq 1$ and steady regime downstream, it is valid:

$$f_p = \lambda_2^{(-)} f_2 + (1 - \lambda_2^{(-)}) f_1 \quad (30)$$

For $\lambda_2^{(-)} > 1$ it is valid:

$$\tau_2^{(-)} = \frac{\Delta t'}{\Delta t} = \frac{1}{\lambda_2^{(-)}} \quad (31)$$

i.e.:

$$f_p = \tau_2^{(-)} f_2 + (1 - \tau_2^{(-)}) f_1^+ \quad (32)$$

For $\lambda_2^{(-)} < 0$ the negative characteristic does not exist in the downstream node (subcritical flow)

For $\lambda_2^{(-)} = 0$ critical flow occurs in the downstream node.

Evidently, there are numerous combinations, some of which are meaningless and they are consequence of finite discretisation. The actual cases have to be better described, i.e. some practical hints should be given:

- Figure 6:

- realistic case
- realistic case
- case of transition from subcritical flow to supercritical flow on the element

The problem can be solved by neglecting the upstream negative characteristic or by subdividing the element into two or more parts.

- Figure 7:

- hydraulic jump located on the element or beyond it
- a case which is not clear and has not been studied. It can occur at critical flow when the bed bottom slope is equal to the critical depth line slope. The practical solution is suggested by adopting the scheme according to the Figure.
- realistic case, negative null-characteristic from the upstream profile is neglected

- Figure 8

- hydraulic jump located somewhere on the element or outside it
- realistic case, which occurs before the hydraulic jump
- realistic case

Since actual location of the hydraulic jump is not known in advance and if hydraulic jump within numerical procedure occurs on one of the elements, then the solution should be sought by eliminating that element from the analysis, or replacing the actual situation by simple transfer of discharge from upstream to downstream part of the stretch. In other words, the correctly defined upstream and downstream boundary conditions should yield an accurate solution in the remaining part of the network.

6. Conclusion

This paper presents the procedure used in the generation of matrices and vectors of finite elements in order to model network systems, based on the method of characteristics. Thus, it is possible to use all natural advantages of the finite element method and to achieve clarity in modelling various channel flow regimes. The employed procedure makes it possible to eliminate all traditional problems regarding the relative relationship between the value of the time step and element dimensions (Courant's criteria), which at last leads to economical and efficient modelling. The paper also states certain problems which may be encountered in the future and which should be studied in more detail.

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EKSPERIMENTALNO ODREĐIVANJE VISKOZNOSTI OTPORA POTPUNO POTOPLJENIH TIJELA

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SAŽETAK

Slijedeći ideju Betza izveden je izraz primjenjiv za eksperimentalno određivanje ukupnog viskoznog otpora potpuno potopljenog tijela koje se giba konstantnom translatornom brzinom u mirujućem fluidu. Polazne osnove izvedenog izraza su jednačba kontinuiteta i Reynoldsove jednačbe gibanja. Eksperimentalno određivanje ukupnog viskoznog otpora temelji se na usporedbi realnog i fiktivnog bezvrtložnog strujanja fluida. Izvan područja graničnog sloja i vrtložnog traga ova dva strujanja su identična. U radu su dani izrazi kojima je definirana donja i gornja granica ukupnog viskoznog otpora u zavisnosti od veličina koje je moguće eksperimentalno odrediti.

THE EXPERIMENTAL DETERMINATION OF THE VISCOUS RESISTANCE OF SUBMERGED BODIES

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SUMMARY

Following the Betz idea, the expression suitable for the experimental determination of the total viscous resistance of the submerged body, moving at a constant speed in a fluid at rest, is derived, using the Reynolds equations of motion and continuity equation. The experimental method of determining the total viscous resistance is based upon the comparison of the real and fictitious irrotational motion which is identical to the real flow in the domain outside the boundary layer and wake. In the present work the relationships for the lower and upper bounds of the total viscous resistance are derived in terms of the measurable quantities.

1. POSTAVLJANJE PROBLEMA I PRETPOSTAVKE. OPĆE JEDNAČBE

Descartesove ortogonalne osi $\tilde{O}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ čine desni apsolutni koordinatni sustav. Os $\tilde{x}_3$ usmjerena je suprotno od djelovanja sile teže, vertikalno naviše. Pomični sustav $O(x_1, x_2, x_3) \equiv O(x, y, z)$ giba se konstantnom translatornom brzinom $c_j = (c, 0, 0)$, ($j = 1, 2, 3$), u odnosu na apsolutni koordinatni sustav, uz uvjet da je u trenutku $t = 0$, $\tilde{O}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = O(x_1, x_2, x_3) \equiv O(x, y, z)$. Kruto tijelo s dvije ravnine simetrije (ravnina Ox_1x_2 i Ox_1x_3) giba se također konstantnom translatornom brzinom $c_j(c, 0, 0)$, ($j = 1, 2, 3$), u smjeru pozitivne osi $\tilde{x}_1$ u mirujućem neograničenom fluidu, u odnosu na sustav $\tilde{O}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Za bilo koju funkciju $\tilde{f}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t)$ u pojavi vrijedi

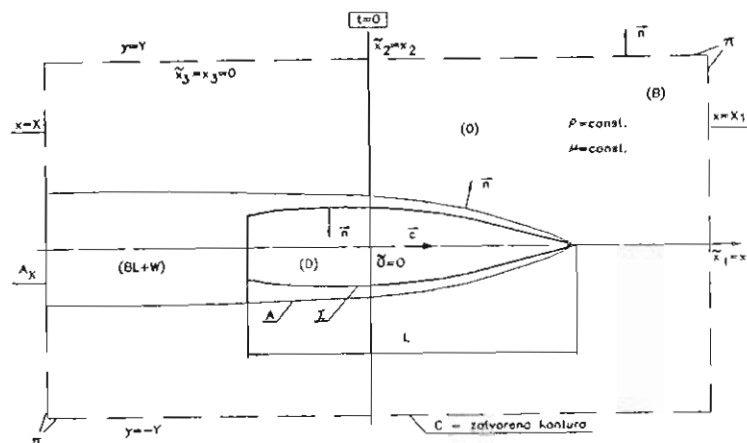
$$\tilde{f}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t) = f(x_1, x_2, x_3) \equiv f(x, y, z), \quad \frac{\partial \tilde{f}}{\partial \tilde{x}_i} = \frac{\partial f}{\partial x_i}, \quad \frac{\partial \tilde{f}}{\partial t} = -c \frac{\partial f}{\partial x_1} \equiv -c \frac{\partial f}{\partial x} \quad (1.1)$$

Pretpostavljamo da su fizikalna svojstva fluida ρ (gustoća), μ (dinamički koeficijent viskoznosti) i ν (kinematička viskoznost) konstantna tijekom realizacije fizikalne pojave. Strujanje fluida može biti viskozno ili neviskozno, ali pretpostavljamo da je nestlačivo.

Polja brzine $v_i = (v_1, v_2, v_3) \equiv (u, v, w)$ i tlaka p , uvjetovana prisustvom tijela, možemo razbiti na slijedeća područja:

- područje graničnog sloja (BL) smješteno u neposrednoj blizini tijela,
- područje vrtložnog traga (W),
- područje izvan (BL + W), označimo ga (Ω), gdje pretpostavljamo da je vektor vrtložnosti $\omega_i = (\omega_1, \omega_2, \omega_3)$, $\omega_i = \varepsilon_{ijk} \partial v_k / \partial x_j$, ($i, j, k = 1, 2, 3$), jednak nuli,
- područje tijela (D) ograničeno zatvorenom površinom (Σ),
- (A) predstavlja graničnu površinu između područja (W + BL) i (Ω).

Uočimo u području (BL + W + Ω + A) cilindričnu površinu Π s izvodnicama paralelnim s $\tilde{x}_3$ osi ograničenu ravninama $\{\tilde{x}_3 \in F_1, \tilde{x}_3 = \text{const.} \ll 0\}$ i $\{\tilde{x}_3 \in F_2, \tilde{x}_3 = \text{const.} \gg 0\}$ i neka je $C(\Pi) \cap (\tilde{x}_3 = 0)$. Analizu vršimo u zatvorenom području (B), ograničenom površinom ($\Sigma + \Pi + F_1 + F_2$), koje zadovoljava uvjet (D) $\subset$ (B). Unutar područja (B) moguća je volumenska razdioba izvora gustoće $\sigma^*[T^{-1}]$. Neka se područje (B) giba zajedno s tijelom istom brzinom $c_j = (c, 0, 0)$ u odnosu na koordinatni sustav $\hat{O}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. Sve relevantne značajke dane su na Sl. 1.



Sl. 1

Polazna osnova su, jednačba kontinuiteta

$$\frac{\partial v_j}{\partial x_j} = \sigma^* \quad (1.2a)$$

i Reynoldsove jednačbe gibanja

$$\rho \left(\frac{\partial \tilde{v}_i}{\partial t} + \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} \right) = \rho f_i + \frac{\partial \tilde{\Phi}_{ij}}{\partial \tilde{x}_j}, \quad (i, j = 1, 2, 3), \quad (1.2b)$$

gdje je

$$\tilde{\Phi}_{ij} = -\tilde{p} \delta_{ij} + \tilde{T}_{ij} + \rho \tilde{t}_{ij} = -\tilde{p} \delta_{ij} + \mu \left(\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{v}_j}{\partial \tilde{x}_i} \right) - \rho \tilde{v}_i \tilde{v}_j, \quad (1.2c)$$

a $[f_i] = [LT^{-2}]$ predstavlja specifičnu masenu silu. Polja brzine i tlaka, kao i ostali članovi jednačbi (1.2a, b) predstavljaju osrednjene veličine u vremenu. $\tilde{v}_i$ je fluktuirajuća komponenta trenutne brzine $\tilde{v}_i$, dok $\tilde{v}_i$ predstavlja osrednju brzinu u vremenu. Veličina

$$\rho \tilde{t}_{ij} = -\rho \tilde{v}_i \tilde{v}_j, \quad (i, j = 1, 2, 3),$$

predstavlja Reynoldsov tenzor naprežanja.

Zbog jednostavnosti pisanja u daljnjem tekstu izostavit ćemo crticu iznad slova zadržavajući dani smisao pojedinih veličina.

Koristeći (1.1), jednačbe (1.2b) prelaze u oblik

$$\rho (v_j - c_j) \frac{\partial v_i}{\partial x_j} = \rho f_i + \frac{\partial \Phi_{ij}}{\partial x_j}, \quad (i, j = 1, 2, 3),$$

ili nakon transformiranja lijevih strana posljednjih jednačbi slijedi

$$\rho \frac{\partial [v_i (v_j - c_j)]}{\partial x_j} - \rho v_i \frac{\partial v_j}{\partial x_j} - \rho f_i - \frac{\partial \Phi_{ij}}{\partial x_j} = 0, \quad (i, j = 1, 2, 3),$$

$$\Phi_{ij} = -p \delta_{ij} + T_{ij} + \rho t_{ij},$$

Koristeći teorem Ostrogradski - Gaussa, slijedi

$$\rho \int_{(\Sigma + \Pi + F_1 + F_2)} v_i (v_j - c_j) n_j dA - \rho \int_{(B)} v_i \frac{\partial v_j}{\partial x_j} d\tau - \rho \int_{(B)} f_i d\tau - \int_{(\Sigma)} \Phi_{ij} n_j dA - \int_{(\Pi + F_1 + F_2)} \Phi_{ij} n_j dA = 0, \quad (i, j = 1, 2, 3) \quad (1.3)$$

Jasno je da vrijede slijedeći odnosi:

- (a) $f_i = (0, 0, -g)$
- (b) $(c_j - v_j) n_j = 0$ na Σ
- (c) $T_{ij} n_j = 0$ na F_1 i F_2 , ne postoje presječne točke $(F_1 \cap \Lambda) \subset (B)$ i $(F_2 \cap \Lambda) \subset (B)$
- (d) $(c_j - v_j) n_j = w$ na F_1 , $(c_j - v_j) n_j = -w$ na F_2 , $|u \cdot w| \cong 0$ na F_1 i F_2 .

Prema definiciji, ukupni otpor tijela je sila kojom fluid djeluje na tijelo i usmjerena je suprotno od smjera gibanja tijela. Po intenzitetu ukupna sila otpora jednaka je

$$R_i = R_p + R_t = \left| - \int_{(\Sigma)} \Phi_{ij} n_j dA \right|, \quad (i, j = 1, 2, 3) \quad (1.4)$$

gdje R_p i R_t predstavljaju komponente ukupnog otpora tijela uslijed djelovanja normalnih i tangencijalnih naprezanja. Koristeći (a) do (d) i (1.4), jednačbe (1.3), za $i = 1$, daju

$$R_1 = - \rho \int_{(B)} u \sigma^* d\tau + \rho \int_{(\Pi)} \left[u v n_2 - \frac{1}{2} (-v^2 + u^2 + w^2) n_1 \right] d\Pi - \int_{(\Pi)} [\rho g \delta H n_1 + (T_{1j} + \rho t_{1j}) n_j] d\Pi, \quad (i, j = 1, 2, 3) \quad (1.5)$$

gdje je

$$\rho g \delta H = \rho g (H_a - H) = \begin{cases} > 0 & \text{u području } M \in (BL + W) \\ = 0 & \text{u području } M \in (\Omega) \end{cases}$$

i

$$\left. \begin{aligned} \rho g H &= p + \frac{1}{2} \rho (v_j - c_j)(v_j - c_j), & (j = 1, 2, 3), \\ \rho g H_a &= p_a + \frac{1}{2} \rho c^2 \end{aligned} \right\} \quad (1.6)$$

Veličine $p/\rho g$ i $p_a/\rho g$ predstavljaju piezometričke visine respektivno, u točki $M \in (BL + W)$ i u području daleko ispred tijela.

Oblik zatvorene konture C je proizvoljan, ali zbog pojednostavljenja dalje analize usvajamo pravokutnu konturu kako slijedi

$$\begin{aligned} x &= X < 0, & y &= \pm Y & (Y > 0) \\ x &= X_1 > 0, \end{aligned} \quad (1.7)$$

Za $X < 0$, konačan, $X_1 = +\infty$, $Y = +\infty$, iz izraza (1.5) dobivamo

$$R_1 = - \rho \int_{(B)} u \sigma^* d\tau + \frac{1}{2} \rho \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} (v^2 + w^2 - u^2) dz + \int_{(A_x)} (\rho g \delta H + T_{11} + \rho t_{11}) dA_x \quad (1.8)$$

gdje je $A_x = (x = X) \cap (BL + W)$.

Za slučaj realnog fluida $\sigma^* = 0$ u (B), ukupni otpor R_1 jednak je ukupnom viskoznom otporu R_v , te iz (1.8) slijedi

$$R_v = \frac{1}{2} \rho \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} (v^2 + w^2 - u^2) dz + \int_{(A_x)} (\rho g \delta H + T_{11} + \rho t_{11}) dA_x \quad (1.9)$$

Prema teoremu Helmholtza polje brzine $\vec{V}$ u području $M \in (BL + W + \Omega + A)$ možemo predstaviti u obliku

$$\vec{V} = \nabla \varphi + \vec{V}_r = \begin{cases} \nabla \varphi + \nabla \chi \vec{A}_1, & M \in (BL + W), \\ \nabla \varphi + \nabla \Phi', & M \in (\Omega), \end{cases} \quad \vec{V}_r = (u_r, v_r, w_r) \quad (1.10a, b)$$

Za $X < 0$, dovoljno velik, za komponentu brzine $\vec{V}_r$ i površinu A_x , unutar područja vrtložnog traga, vrijede slijedeće asimptotske procjene

$$\left| \vec{V}_r \right| = \begin{cases} O(|X|^{-n}), & M \in (BL + W), \\ O(|X|^{-2}), & M \in (\Omega), \end{cases} \quad (1.11a, b, c)$$

$$A_x = O(|X|^n), \quad M \in (BL + W),$$

Vrijednost eksponenta n jednaka je respektivno 1 ili 2/3 zavisno od toga da li je strujanje fluida u vrtložnom tragu laminarno ili turbulentno.

Neka je

$$Q = \lim_{x \rightarrow -\infty} \int_{(A_x)} u_r dA_x \quad (1.12a)$$

Veličina ρQ predstavlja masu fluida koju tijelo "vuče" u jedinici vremena gibajući se konstantnom translatorskom brzinom $\vec{c}$ u mirujućem fluidu. Koristeći (1.11a,c) i (1.12a) slijedi

$$Q = \lim_{x \rightarrow -\infty} \int_{(A_x)} u_r dA_x = 0 \quad (1.12b)$$

Ravninu $x = X < 0$ možemo smjestiti bilo gdje iza tijela bez utjecaja na valjanost izraza (1.9) koji daje vrijednost ukupnog viskoznog otpora tijela. Vodeći računa o (1.11a,b,c) i (1.12b), za $X \rightarrow -\infty$, iz (1.9) slijedi

$$R_v = \lim_{x \rightarrow -\infty} \int_{(A_x)} \rho g \delta H dA_x = \rho c Q \quad (1.13)$$

2. BETZOVI IZVORI ZA SLUČAJ NEOGRANIČENOG FLUIDA

U cilju eksperimentalnog određivanja vrijednosti integrala (1.9) Betz uvodi fiktivni bezvrtložni tok fluida koji zadovoljava slijedeće uvjete:

- (a) Fiktivni bezvrtložni tok fluida definira se unutar područja ispunjenog fluidom, tj. područja s graničnom površinom (Σ) .
 - (b) Izvan područja graničnog sloja i vrtložnog traga, tj. u području (Ω) , fiktivni bezvrtložni tok fluida identičan je s relanim tokom fluida.
 - (c) Potencijal fiktivnog bezvrtložnog toka fluida je neprekidna funkcija s neprekidnim parcijalnim derivacijama prvog i drugog reda za svaku točku $M \in (BL + W + \Omega + A)$.
- Neka Φ_0 predstavlja potencijal fiktivnog bezvrtložnog toka fluida, odnosno, neka je polje brzine definirano u obliku

$$\vec{V}_0 = \nabla \Phi_0 = \nabla(\varphi + \Psi_f), \quad \nabla \Psi_f = (u_f, v_f, w_f), \quad M \in (BL + W + \Omega + A). \quad (2.1)$$

Realni tok fluida definiran je izrazima (1.10a,b). Uspoređujući (1.10b) i (2.1), uz 2/(b), slijedi

$$\vec{V} = \nabla(\varphi + \Phi) \equiv \vec{V}_0 = \nabla(\varphi + \Psi_f), \quad M \in (\Omega + A) \quad (2.2)$$

Potencijal Ψ_f koji zadovoljava 2/(a),(b),(c) očekujemo u obliku

$$\Psi_f(M) = -\frac{1}{4\pi} \left[\int_{(BL+W)} d\tau(M') + \int_{(\Sigma)} d\Sigma(M') \right] \frac{\sigma^*(M')}{MM'} \quad (2.3)$$

tj. u obliku kontinuirane razdiobe σ^* izvora na (Σ) i unutar područja $(BL + W)$. Problem određivanja potencijala $\Psi_f(M)$ može se svesti na određivanje funkcije razdiobe σ^* . Može se pokazati (vidi [1],[2],[3]) da je funkcija razdiobe σ^* rješenje Fredholmove integralne jednadžbe prve vrste sa simetričnom jezgrom.

U daljnjoj analizi pretpostavlja se da postoji rješenje Betzovog problema definiranog uvjetima 2/(a),(b),(c). Eventualna nejedinstvenost rješenja je bez ikakve teorijske ili praktične važnosti.

Ukupni kapacitet Betzovih izvora Q^* određujemo na slijedeći način. Neka je Q_B^* kapacitet Betzovih izvora u području $(B) \cap (\Sigma + BL + W)$. Tada vrijedi

$$Q_B^* = - \int_{(\Sigma)} \vec{n} \cdot \vec{V}_0 d\Sigma + \int_{B \cap (BL+W)} \nabla \cdot \vec{V}_0 d\tau$$

Kako je $\nabla \cdot \vec{V} = 0$ i $\int_{(\Sigma)} \vec{n} \cdot \vec{V} d\Sigma = 0$ slijedi

$$Q_B^* = - \int_{(\Sigma)} \vec{n} \cdot (\vec{V}_0 - \vec{V}) d\Sigma + \int_{B \cap (BL+W)} \nabla \cdot (\vec{V}_0 - \vec{V}) d\tau = - \int_{(\Sigma)} \vec{n} \cdot (\vec{V}_0 - \vec{V}) d\Sigma + \int_{[\Sigma + \Pi \cap (BL+W) + A]} \vec{n} \cdot (\vec{V}_0 - \vec{V}) dA$$

S druge strane $\vec{V}_0 = \vec{V}$ na (A) te iz posljednjeg izraza imamo

$$Q_B^* = \int_{\Pi \cap (BL+W)} \vec{n} \cdot (\vec{V}_0 - \vec{V}) dA \quad (2.4)$$

Neka je $x = X < 0$, $X_1 = +\infty$, $Y = +\infty$. Iz (2.4) dobivamo

$$Q_B^* = Q_X^* = \int_{(A_x)} (u - u_0) dA_x = \int_{(A_x)} (u_r - u_l) dA_x \quad (2.5)$$

Iz (2.5) ukupni kapacitet Betzovih izvora je jednak

$$Q^* = \lim_{X \rightarrow -\infty} Q_B^* \quad (2.6)$$

Za $X \rightarrow -\infty$ brzina u_r je reda veličine $O(|X|^{-2})$, odnosno

$$\lim_{X \rightarrow -\infty} \int_{(A_x)} -u_r dA_x = O(|X|^{-2}) \rightarrow 0$$

Iz (2.6), (2.5), (1.12b) i posljednje jednačbe slijedi

$$Q^* = \lim_{X \rightarrow -\infty} \int_{(A_x)} u_r dA_x = Q = O(1) \quad (2.7)$$

čime je određen ukupni kapacitet Betzovih izvora.

3. EKSPERIMENTALNO ODREĐIVANJE UKUPNOG VISKOZNOSTI OTPORA

Neka $x = X$ predstavlja poprečnu ravninu iza tijela i neka je $(BL+W)_x$ dio područja graničnog sloja i vrtložnog traga ispred ravnine $x = X$. Ukupni otpor R_o tijela koje se nalazi u polju brzine fiktivnog bezvrtložnog strujanja fluida definiranog potencijalom Φ_o , za slučaj da tijelo predstavlja strujnu površinu relativnog fiktivnog toka fluida, tj. da je $\partial\Phi_o/\partial n = \vec{n} \cdot \vec{c}$ na (Σ) , uz $\rho g \delta H = T_{11} = \rho t_{11} = 0$, dan je izrazom (1.8) u obliku

$$R_o = \frac{1}{2} \rho \int_{(x=X)} (v_o^2 + w_o^2 - u_o^2) dy dz - \rho \int_{(BL+W)_x} u_o \sigma^* d\tau \quad (3.1)$$

ili

$$R_o = - \rho \int_{(BL+W)} u_o \sigma^* d\tau \quad (3.2)$$

S druge strane ukupni viskozni otpor za slučaj realnog toka fluida dan je izrazom (1.9). Oduzimanjem jednačbi (3.1)/(3.2) od jednačbe (1.9) te koristeći postavku 2/(b) slijedi

$$R_v = \int \left[\rho g \delta H + T_{11} + \rho t_{11} - \frac{1}{2} \rho (u^2 - u_o^2) + \frac{1}{2} \rho (v^2 - v_o^2) + \frac{1}{2} \rho (w^2 - w_o^2) \right] dA_x - \rho \int_{(x=X)} u_o \sigma^* d\tau \quad (3.3)$$

Izraz (3.3.) je egzakatan. Za $x = X < 0$ dovoljno velik uvodimo slijedeće pretpostavke:

- Član $\frac{1}{2} \rho [(v^2 - v_o^2) + (w^2 - w_o^2)]$ je mali, a zbog unutarnje strukture dijelom se i sam poništava,
- $T_{11} \cong 0$, $\rho t_{11} \cong 0$.

Uvodeći posljednje pretpostavke u (3.3) slijedi

$$R_v = \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u^2 - u_o^2) \right] dA_x - \rho \int_{(x=X)} u_o \sigma^* d\tau \quad (3.4)$$

Izvrši li se analiza:

Za realan tok fluida (1.6), za $X < 0$ dovoljno velik i $u^2 \gg v^2 + w^2$, slijedi

$$\rho g \delta H \cong p_a - p + \rho c u - \frac{1}{2} \rho u^2 \quad \text{za } M \in (BL+W)$$

i

$$0 \cong p_a - p_E + \rho c u_E - \frac{1}{2} \rho u_E^2 \quad \text{za } M \in (A)$$

gdje indeks E označava vrijednosti veličina na površini (A). Iz posljednje dvije jednačbe dobivamo

$$\rho g \delta H \cong (p_E - p) + \rho c (u - u_E) - \frac{1}{2} \rho (u^2 - u_E^2) \quad (3.5)$$

Iz Reynoldsovih jednačbi (1.2b) napisanih u obliku

$$g \nabla \delta H = \vec{\omega} \times (\vec{V} - \vec{c}) + \nu (\nabla \times \vec{\omega}) - \nabla \cdot \vec{t}$$

gdje je $\vec{t} = -\overline{v_i v_j}$, za $X < 0$ dovoljno velik, zanemarujući viskozne efekte, tj. član $\nu (\nabla \times \vec{\omega})$, slijedi

$$\rho g \frac{\partial \delta H}{\partial y} \cong \rho (u - c) \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \rho w \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \rho \left(\frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'^2}}{\partial y} - \frac{\partial \overline{v'w'}}{\partial z} \right)$$

Uvodeći u posljednju jednačbu pretpostavke

$$\left| \frac{\partial v}{\partial x} \right| \ll \left| \frac{\partial u}{\partial y} \right|, \quad w \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \cong 0, \quad \frac{\partial \overline{u'v'}}{\partial x} \cong 0$$

slijedi

$$\rho g \delta H = \rho c(u - u_E) - \frac{1}{2} \rho (u^2 - u_E^2) + \rho \overline{v'^2} - \rho \int_{y(X,z)}^{y_E(X,z)} \frac{\partial v' w'}{\partial z} dy \quad (3.6)$$

Uspoređujući (3.5) i (3.6) dobivamo

$$p - p_E = \rho \left[-\overline{v'^2} + \int_{y(X,z)}^{y_E(X,z)} \frac{\partial v' w'}{\partial z} dy \right] \neq 0 \quad (3.7)$$

Za fiktivan tok fluida, za $X < 0$, vrijedi izraz

$$0 \cong p_a - p_o - \rho c u_o - \frac{1}{2} \rho u_o^2 \quad \text{za } M \in (BL + W)$$

odnosno, u kombinaciji sa (3.5)

$$\rho g \delta H \cong p_o - p + \rho c(u - u_o) - \frac{1}{2} \rho (u^2 - u_o^2) \quad (3.8)$$

Izraz (3.8) pruža dvije mogućnosti:

(I) **Landweberov kriterij:** Pretpostavlja se da vrijedi jednakost

$$u_o(X, y, z) = u_E(X, y_E(X, z), z) \equiv u_E(X, z) \quad (3.9)$$

gdje $y_E = y_E(X, z)$ predstavlja $(x = X) \cap (A)$. Iz (3.8) slijedi:

$$\rho g \delta H = p_o - p + \rho c(u - u_E) - \frac{1}{2} \rho (u^2 - u_E^2), \quad M \in (BL + W),$$

ili, koristeći (3.5), dobiva se:

$$p - p_E = p - p_o, \quad M \in (BL + W), \quad (3.10)$$

Zanemare li se Reynoldsova naprežanja, iz (3.7) i (3.10) slijedi

$$p = p_E \implies p = p_o \quad \text{za } x = X < 0$$

što nije u suglasnosti s eksperimentalnim podacima, ali predstavlja implicitni dio Landweberovog kriterija.

Eksperimentalnim određivanjem veličina $\rho g \delta H$ i p dobiva se polje brzine realnog toka fluida na ravnini $x = X$ a time je i određena i vrijednost prvog člana u izrazu (3.4).

(II) **Betzov kriterij:** Pretpostavlja se da vrijedi jednakost

$$p = p_o \quad \text{za } M \in (BL + W) \quad (3.11)$$

Tada iz (3.8) slijedi

$$\rho g \delta H \cong \rho c(u - u_o) - \frac{1}{2} \rho (u^2 - u_o^2) \quad (3.12)$$

Eksperimentalnim određivanjem veličina $\rho g \delta H$ i p ponovo je poznato polje realnog i fiktivnog toka fluida u ravnini $x = X$ a time je određena i vrijednost prvog člana u izrazu (3.4).

Iz prethodne analize slijedi da su Landweberov i Betzov kriterij praktično jednaki. Daljnju analizu problema vršimo po Landweberovom kriteriju.

Neka je

$$R_{VD} = \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u^2 - u_o^2) \right] dA_x \quad (3.13)$$

Iz (3.4) slijedi

$$R_V = R_{VD} - \rho \int_{(x \leq X)} u_o \sigma^* d\tau \quad (3.14)$$

odnosno

$$R_{VD} < R_V \quad (3.15)$$

Za $X \rightarrow -\infty$, iz (1.13) i (3.14) imamo

$$R_V = \lim_{X \rightarrow -\infty} \left[R_{VD} - \rho \int_{(x \leq X)} u_o \sigma^* d\tau \right] = \rho c Q \quad (3.16)$$

Drugi član zavisnosti (3.4) možemo transformirati kako slijedi

$$\rho \int_{(x \leq X)} u_o \sigma^* d\tau = \rho \bar{u}_o \Big|_{x \leq X} \int_{(x \leq X)} \sigma^* d\tau = \rho \bar{u}_o \Big|_{x \leq X} \left[\int_{(BL+W)} \sigma^* d\tau - \int_{(BL+W)_x} \sigma^* d\tau \right] \quad (3.17a)$$

gdje $\bar{u}_0|_{x \leq X}$ predstavlja srednju brzinu fiktivnog strujanja fluida unutar graničnog sloja i vrtložnog traga za $x \leq X$. Zamjenjujući u posljednjoj jednadžbi izraze (2.5) i (2.7) uz (3.16), dobiva se

$$\rho \int_{(x \leq X)} u_0 \sigma^* d\tau = \frac{\bar{u}_0|_{x \leq X}}{c} R_V - \rho \int_{(A_x)} \bar{u}_0|_{x \leq X} (u - u_0) dA_x \quad (3.17b)$$

što u kombinaciji sa (3.4) daje

$$R_V = \frac{1}{1 - \frac{|\bar{u}_0|_{x \leq X}}{c}} \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u - u_0) (u + u_0 - 2 \bar{u}_0|_{x \leq X}) \right] dA_x \quad (3.18)$$

Neka je

$$R_{VG} = \frac{1}{1 - \frac{|\bar{u}_0|_{x \leq X}}{c}} \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u - u_0)^2 \right] dA_x \quad (3.19)$$

Izraz (3.19) slijedi iz (3.18) ako za vrijednost srednje brzine $\bar{u}_0|_{x \leq X}$ fiktivnog toka fluida usvojimo veličinu $u_0|_{x=X}$. Jasno je da vrijede nejednadžbe

$$R_{VD} < R_V < R_{VG} \quad (3.20)$$

Napominje se da nejednadžba (3.15) slijedi također iz (3.18) ako za veličinu srednje brzine $\bar{u}_0|_{x \leq X}$ fiktivnog toka fluida usvojimo vrijednost nula.

Logično je pretpostaviti da je "stvarna" vrijednost srednje brzine $\bar{u}_0|_{x \leq X}$ bliža vrijednosti $u_0|_{x=X}$, odnosno, da je stvarni otpor tijela po vrijednosti bliži gornjoj granici R_{VG} , tj.

$$R_{VG} - R_V < R_V - R_{VD} \quad (3.21)$$

U izrazima (3.13), (3.18) i (3.19) pojavljuje se kao nepoznata veličina polje brzine $\vec{V}_0$ fiktivnog toka fluida. Koristeći Landweberov kriterij aproksimacije (3.9), izrazi (3.13), (3.18) i (3.19) prelaze u oblik

$$R_{VD} = \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u^2 - u_E^2) \right] dA_x \quad (3.22)$$

$$R_{VG} = \frac{1}{1 - \frac{|\bar{u}_E|_{x=X}}{c}} \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u - u_E)^2 \right] dA_x \quad (3.23)$$

$$R_V = \frac{1}{1 - \frac{|\bar{u}_E|_{x \leq X}}{c}} \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u - u_E) (u + u_E - 2 \bar{u}_E|_{x \leq X}) \right] dA_x \quad (3.24)$$

uz

$$\begin{aligned} R_{VD} &< R_V < R_{VG} \\ R_{VG} - R_V &< R_V - R_{VD} \end{aligned} \quad (3.25a,b)$$

Izrazi (3.22), (3.23) i (3.24) daju donju i gornju granicu te "točnu" vrijednost (u smislu usvojenog Landweberovog kriterija) ukupnog viskoznog otpora tijela u funkciji veličina koje možemo eksperimentalno odrediti. Jasno je, da je eksperimentalno određivanje veličine $\bar{u}_E|_{x \leq X}$ nepouzdanost te koristimo izraz (3.23) kao dovoljno "točan". Znači, u praktičnim primjenama ukupni viskozni otpor tijela nalazimo iz izraza

$$R_V \cong R_{VG} = \frac{1}{1 - \frac{|\bar{u}_E|_{x=X}}{c}} \int_{(A_x)} \left[\rho g \delta H - \frac{1}{2} \rho (u - u_E)^2 \right] dA_x \quad (3.26)$$

Određivanje ukupnog viskoznog otpora R_V definiranog izrazom (3.26) realizira se na slijedeći način:

- Ravnina $x = X < 0$ postavlja se na rastojanje $(L/2) \leq |x| \leq L$ iza tijela, gdje L predstavlja dužinu tijela u pravcu brzine $\vec{c}$. Ovakva pozicija mjerne ravnine osigurava potrebnu točnost mjerenja i umanjuje utjecaj transverzalnih komponenata brzine.
- U ravlini $x = X < 0$ postavlja se horizontalni nosač nepomičan u odnosu na tijelo. Na nosač su postavljene, u smjeru brzine $\vec{c}$, naizmjenično na ekvidistantnim rastojanjima $\Delta y = \text{const.}$ cijevi za mjerenje ukupne visine H i piezometričke visine $p/\rho g$. Otvori za mjerenja piezometričkih visina postavljaju se u ravninu $x = X$. Širina mjerne sekcije l , odnosno, nosača s mjernim cijevima, mora biti takva da je

zadovoljen uvjet $1 > 2|y_{\epsilon_{\max}}(X, z)|$. Vršiti se istovremeno mjerenje ukupne i piezometričke visine na poziciji $(z = \text{const.}) \cap (A_x)$.

4. ZAKLJUČAK

U radu je ostvareno slijedeće:

- Koristeći ideju Betza te uspoređujući realan i fiktivan tok fluida sustavno je izveden egzaktni izraz (vidi (3.3)) koji se koristi kao polazna osnova za eksperimentalno određivanje ukupnog viskoznog otpora potpuno potopljenih tijela.
- Izvršeno je uspoređivanje Betzovog i Landweberovog kriterija aproksimacije fiktivnog toka fluida i pokazano je da su oba kriterija jednaka.
- Uspoređujući Betzov izraz za ukupni viskozni otpor (vidi [7], strana 128, jednadžba (3)) s izrazom (3.18) slijedi da Betzova formula sadrži dvije sistematske pogreške i to:

- ne sadrži koeficijent $\left(1 - \frac{|u_{\epsilon}|}{c}\right)^{-1}$
- podintegralna funkcija ne sadrži član $\rho \bar{u}_0|_{x \leq X}(u - u_0)$

Betzov izraz za ukupni viskozni otpor je identičan donjoj granici ukupnog viskoznog otpora definiranog izrazom (3.13).

- Izraz (3.7) predstavlja promjenu piezometričke visine u jednom poprečnom presjeku vrtložnog traga $x = X < G$ daleko iza tijela. Zanimare li se turbulentni efekti slijedi

$$p = p_{\epsilon}, \quad M(X, y, z) \in (BL + W)$$

što nije u suglasnosti s eksperimentalnim podacima, ali predstavlja implicitni dio i Betzovog i Landweberovog kriterija.

- Koristeći Landweberov kriterij određena je donja i gornja granica te "točna" vrijednost ukupnog viskoznog otpora (izrazi (3.22), (3.23), (3.24)). Za praktične primjene predložen je izraz (3.26).

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APPLICATION OF CORRECTIVE ANALYTICAL DIFFERENCING SCHEME TO SOLVING NAVIER – STOKES EQUATION

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SUMMARY

The corrective analytical differencing (CADI) scheme, based upon the one – dimensional exact solution of the convection – diffusion equation with constant coefficients and the constant source term, is compared to the popular exponential differencing scheme (EDS) of Spalding. Both schemes are applied in solving Navier – Stokes and Euler equations by the finite volume method in body – fitted coordinates. A significant improvement of solution accuracy is obtained by applying the CADI scheme, because of reduced false diffusion.

Key words: Control Volume Method, Body – Fitted Coordinates, Differencing Scheme, Navier – Stokes Equation, False Diffusion

1. INTRODUCTION

A finite volume method in body – fitted (general non – orthogonal) coordinates, which is universal and applicable to complex flow geometries, is adopted in this paper. The goal of every numerical method is to achieve high accuracy on comparatively coarse grids and with reasonable efficiency [1]. It is known that accuracy of a numerical method is affected primarily by the differencing scheme used, while the convective transport modelling presents the major difficulty of differencing schemes. The presence of false diffusion and space oscillations in solutions are the possible consequences of inadequate modelling of this transport which substantially reduce accuracy of numerical results. Until the present day, a great number of differencing schemes has been developed ([2 – 11] and many others). Some of them cannot be applied in non – orthogonal coordinates, others are too complex, and still others are insufficiently accurate in situations of practical interest.

A new corrective analytical differencing (CADI) scheme was proposed in [12]. It is based on the one – dimensional solution of the convection – diffusion equation with the source term taken to be constant between two neighboring nodes (piecewise constant). The CADI scheme reduces to the central – difference scheme when only diffusion transport is present (cell Peclet number $Pe=0$). It becomes similar to the linear upwind scheme when the transport is predominantly convective ($Pe \rightarrow \infty$). In the other cases it corresponds to a differencing scheme based on five point formula – tion in finite – difference method.

In [13], the CADI scheme was applied to solving the general scalar convective – diffusion equation in general non – orthogonal coordinates and compared to Spalding's exponential differencing scheme (EDS). It was concluded that the CADI scheme retained a high degree of accuracy in non – orthogonal coordinates, that it was simple and easily applicable to 3D situations.

In this paper, the CADI differencing scheme is applied for solving the momentum equation and the results have been compared to those obtained by applying the EDS differencing scheme.

2. MATHEMATICAL MODEL AND NUMERICAL METHOD

The adopted mathematical model of laminar, incompressible, steady, plane fluid flow with constant viscosity is given as

$$\frac{\partial(\rho v_j)}{\partial x_j} = 0, \quad (1)$$

$$\frac{\partial}{\partial x_j}(\rho v_j v_i) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}. \quad (2)$$

The finite volume method adopted for integrating the mathematical model is described in [14-15]. The method is based on the SIMPLER algorithm with the staggered grid. The momentum equation is solved for physical, covariant velocity components, as proposed in [16]. A short description of the method follows.

Coordinate transformation

It is assumed that the transformation law between the physical coordinates x_i ($i=1,2$) and the body-fitted coordinates y_j ($j=1,2$) exists in the form $x_i = x_i(y_j)$, and that the Jacobian I is positive and finite.

If $F(x_i)$ represents a scalar field expressed in the physical coordinates and $f(y_j)$ represents the same field in the body-fitted coordinates, the relation

$$\frac{\partial F}{\partial x_j} = \frac{K_{jk}}{I} \frac{\partial f}{\partial y_k} = \frac{1}{I} \frac{\partial}{\partial y_k} [K_{jk} f] \quad (3)$$

is valid, where the curvature parameter K_{jk} is defined as

$$K_{jk} = \begin{bmatrix} K_{11} & K_{21} \\ K_{12} & K_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_2}{\partial y_2} & -\frac{\partial x_1}{\partial y_2} \\ -\frac{\partial x_2}{\partial y_1} & \frac{\partial x_1}{\partial y_1} \end{bmatrix}. \quad (4)$$

Grid geometry and variable layout

Figure 1 shows a part of the computational domain in the physical plane Ox_1x_2 and in the transformed plane Oy_1y_2 . The latter domain is rectangular and the steps Δy_1 and Δy_2 are arbitrarily chosen as $\Delta y_1 = \Delta y_2 = 1$. The grid is completely determined in the physical plane by the control volume vertex coordinates, since the vertices are connected by straight lines. The grid nodes $P, E, \dots$, at which pressure field values are calculated, are the geometrical centres of control volumes. The physical covariant velocity components V_1 and V_2 are calculated at the staggered grid points e and n , respectively. These points are the cell-face midpoints.

The contravariant velocity components are defined as

$$U_i = K_{ji} v_j, \quad (5)$$

while the physical covariant velocities V_1 and V_2 can be written as

$$V_1 = v_i \xi_i, \quad V_2 = v_i \eta_i, \quad (6)$$

where ξ_i and η_i denote unit vectors tangential to $y_2 = \text{const.}$, and $y_1 = \text{const.}$, respectively. The various necessary relations between the components (5), (6) and the Cartesian ones can be derived without difficulty. It is of course always possible to

determine any velocity components from an arbitrary pair of non-collinear velocity components.

The adopted velocity interpolation practice [15] ensures that the continuity equation is satisfied for all control volumes, i.e. for the main control volume surrounding the node P, the staggered control volumes surrounding the nodes e and n, and the vertex control volume around the node ne. It is possible to reconstruct the velocity vector at all nodes, i.e. P, e, n, ne.

Discretization of the continuity equation

Integrating the continuity equation (1) over the main control volume with the central node P (see Fig. 1) yields

$$(\rho U_1)_e - (\rho U_1)_w + (\rho U_2)_n - (\rho U_2)_s = 0. \quad (7)$$

Since the momentum equation is solved for covariant velocity components, the equation (7) also has to be written in these components

$$(\rho A_1 V_1)_e - (\rho A_1 V_1)_w + (\rho A_2 V_2)_n - (\rho A_2 V_2)_s - (\rho B_1 U_2)_e + (\rho B_1 U_2)_w - (\rho B_2 U_1)_n + (\rho B_2 U_1)_s = 0, \quad (8)$$

where the geometrical quantities A_1 , A_2 , B_1 and B_2 can be readily determined from the grid geometry.

Discretization of the momentum equation and the differencing scheme

This equation is solved for covariant components. The inner product of the equation (2) and the unit covariant vector $(\xi_i)_e$ yields

$$\frac{1}{I} \frac{\partial}{\partial y_k} \left(\rho U_k V_i' - \mu K_{jk} K_{ji} \frac{1}{I} \frac{\partial V_i'}{\partial y_k} \right) = \frac{1}{I} K_{ik} \frac{\partial p}{\partial y_k} (\xi_i)_e = S. \quad (9)$$

The relation between the physical covariant velocity component V_i and V_i' is

$$V_i' = V_i + v_i [(\xi_i)_e - \xi_i]. \quad (10)$$

If the grid is curved, V_i' corresponds to the physical covariant velocity component V_i only at the node e.

Integrating the equation (9) over the control volume surrounding the node e (Fig. 1) yields

$$J_E - J_P + J_{ne} - J_{se} = SI, \quad (11)$$

where J_E , J_P , J_{ne} , J_{se} are the fluxes of the "covariant" velocity V' across the corresponding cell-face surfaces. The flux J_E is expressed in the form

$$J_E = \left(\rho U_1 V_1' - \Gamma \frac{K_{j1} K_{j1}}{I} \frac{\partial V_1'}{\partial y_1} - \Gamma \frac{K_{j1} K_{j2}}{I} \frac{\partial V_1'}{\partial y_2} \right)_E = F_{1E} V_{1E}' - D_{1E} \left(\frac{\partial V_1'}{\partial y_1} \right)_E - D_{2E} \left(\frac{\partial V_1'}{\partial y_2} \right)_E \quad (12)$$

The diffusion term D_2 is a consequence of grid non-orthogonality and vanishes in orthogonal grids. The term containing D_2 is modelled explicitly and added to the source term. Modelling of the first two terms in the expression (12) is the task of a differencing scheme.

The one-dimensional form of the equation (9) between the nodes e and ee

$$F_{1E} \frac{\partial V_1'}{\partial y_1} + D_{1E} \frac{\partial^2 V_1'}{\partial y_1^2} = \begin{cases} \hat{S}_e & \text{over } e-E \\ \hat{S}_{ee} & \text{over } E-ee \end{cases} \quad (13)$$

is solved in order to define the cell-face values V_{1E}' and $(\partial V_1' / \partial y_1)_E$. The corrective source terms $\hat{S}_e$ and $\hat{S}_{ee}$, as well as the convection strength F_{1E} and the diffusion term D_{1E} are taken to be constants. After solving the equation (13), the first two

terms in the equation (12) can be written as

$$F_{1E} V_{1E} - D_{1E} \left(\frac{\partial V_1}{\partial y_1} \right)_E = F_{1E} V_{1e} + a_E (V_{1e} - V_{1ee}) + f_{eEE} \hat{S}_e + f_{EEee} \hat{S}_{ee}, \quad (14)$$

where coefficients a_E , f_{eEE} , f_{EEee} are known functions of cell - Reynolds number F_1/D_1 . If $\hat{S}_e$ and $\hat{S}_{ee}$ are set to zero, the equation (14) defines the EDS scheme.

Several differencing schemes are based on the formula (14), or a similar one. In all of them, the corrective source terms $\hat{S}_e$ and $\hat{S}_{ee}$ are determined from the discretized equation (11). On the contrary, in the CADI differencing scheme (by taking into account that the influence of false diffusion is important only in convection dominated flows) the corrective source terms are defined as the discretized left side of the equation (13), with the diffusive transport being neglected. In that case, the corrective source terms $\hat{S}_e$ and $\hat{S}_{ee}$ are defined as

$$\hat{S}_e = F_E \frac{\Delta x_E}{\Delta x_e} (V_{1E} - V_{1P}), \quad \text{and} \quad \hat{S}_{ee} = F_E \frac{\Delta x_E}{\Delta x_{ee}} (V_{1EE} - V_{1E}). \quad (15)$$

In the above equations, the cell - face values of V_1 appear. When the CADI scheme is applied for solving the convection - diffusion equation, the cell - face values are determined by an adequate extrapolation practice. Such an extrapolation practice is unnecessary for solving the momentum equation, since the velocity component V_1 can be easily determined at cell - face surfaces, as mentioned before. The flux J_E is completely defined by the equations (12) to (15). The other fluxes in the equation (11) are modelled in the completely analogue way.

In the source term of the equation (9), only the pressure derivative in y_1 direction remains. This excludes the need of pressure interpolation and makes the numerical method similar to the one in Cartesian coordinates.

In the final discretized equation, the velocity component V_1 is replaced by the local physical covariant velocity component V_1 by using relation (10). The final form of the discretized momentum equation in the direction of $(\xi_1)_e$ is

$$a_e V_{1e} = \sum_{nb \in E, NE, W, SE} a_{nb} V_{1nb} + S_e + A_{1e} (p_P - p_E), \quad (16)$$

where S_e represents all source term components except the pressure term.

The discretization of the equation for the other physical velocity component V_2 is completely analogue.

Pressure equation and pressure correction equation

In the adopted SIMPLER algorithm, the pressure equation and the pressure - correction equation are derived from the momentum equation and the continuity equation.

The equation (16) and the analogue one for the other velocity component can be written in the form

$$V_{1e} = \hat{V}_{1e} + \frac{A_{1e}}{a_e} (p_P - p_E), \quad V_{2n} = \hat{V}_{2n} + \frac{A_{2n}}{a_n} (p_P - p_N), \quad (17)$$

where the pseudovelocities $\hat{V}_{1e}$ and $\hat{V}_{2n}$ do not depend on the pressure. Introducing the equation (17) and the analogue ones in the continuity equation (8) yields the pressure equation.

After solving the momentum equation, the non - convergent velocity field will

generally not satisfy the continuity equation. Therefore, it is necessary to introduce the velocity corrections V_1'' and V_2'' in the imperfect solutions V_1^* and V_2^* , and the pressure correction p'' in the imperfect pressure field p^* . The corrected velocity field can be represented in terms of the pressure corrections as

$$V_1 = V_1^* + \frac{A_{1e}}{a_e} (p_p'' - p_c'') , \quad \text{and} \quad V_2 = V_2^* + \frac{A_{2n}}{a_n} (p_p'' - p_n'') . \quad (18)$$

Introducing the equations (18), and the analogue ones into the continuity equation (8) yields the pressure - correction equation.

The velocity - pressure coupling is accomplished through by using the standard SIMPLER algorithm.

Wall boundary conditions

The velocity at a solid wall is set to zero in laminar flow (no slip condition), while the velocity component perpendicular to such a wall is set to zero in inviscid flow.

3. NUMERICAL RESULTS

It is well known that false diffusion strongly affects solution accuracy when streamlines are not parallel to grid lines and the velocity (or a general physical property) gradient in cross flow direction exists. Two test situations were chosen, for which the exact solutions exist, in order to estimate the accuracy of the CADI differencing scheme.

Inviscid flow around a circular cylinder

A circular cylinder of radius R is submerged into a steady parallel inviscid incompressible fluid flow with constant velocity U perpendicular to the axis of the cylinder. Velocity and pressure fields are symmetric and the pressure force on the cylinder is equal to zero. Every deviation from symmetry in velocity and pressure fields can be explained as a consequence of false diffusion. The flow is governed by the linear Laplace equation, but, in this work, it is solved as a fully non-linear problem and irrotationality of velocity field is not taken into account. The computational domain is defined in polar coordinates (r, θ) with $R < r < 20R$, and $\pi > \theta > 0$. Computational grid is equidistant in θ direction and progressively enlarged in radial direction by a constant factor q , starting from the smallest CV adjacent to the cylinder surface. Calculations are performed with two grids, 26×25 CV (factor $q = 1.2$) and 51×51 (factor $q = 1.1$). The exact boundary conditions are prescribed at the outer boundary ($r = 20R$). Accuracy of the numerical solution is investigated by calculating the relative solution error

$$\epsilon_{\Phi} = \frac{\Phi_{\text{exact}} - \Phi_{\text{numerical}}}{\Phi_{\text{norm}}} \cdot 100\% , \quad (19)$$

where Φ stands for v_1 , v_2 or p , and Φ_{norm} is U , U , or $\rho U^2/2$, respectively. The average error is defined as the sum of absolute values of ϵ_{Φ} at interior nodes divided by the number of interior nodes. Drag coefficient C_F is defined as

$$C_F = \frac{F}{\rho U^2 R} , \quad (20)$$

where F is the total pressure force on the surface of the cylinder (equal to zero in the exact solution). Table 1 shows the results obtained by the CADI and the EDS differencing schemes on both grids.

SCHEME		CADI		EDS	
GRID, CV		COARSE 26X25	FINE 51X51	COARSE 26X25	FINE 51X51
ε_{v_1} %	max	2.22	1.00	7.10	3.66
	min	-0.18	-1.80	-16.2	-6.17
	av	0.157	0.043	3.61	2.00
ε_{v_2} %	max	4.81	2.7	12.9	5.95
	min	-1.08	-0.6	-4.39	-3.08
	av	0.107	0.035	2.21	1.21
ε_p %	max	4.50	2.70	21.8	11.6
	min	-1.30	-1.50	-27.3	-14.9
	av	0.77	0.24	7.92	4.24
C_f		$15 \cdot 10^{-4}$	$4 \cdot 10^{-4}$	$750 \cdot 10^{-4}$	$350 \cdot 10^{-4}$

Table 1. Errors of numerical solution of flow around a circular cylinder

Evidently, the results obtained by the CADI scheme are much more accurate than those of the EDS scheme. It is worth noting that the CADI scheme even gave better results on the coarser grid than the EDS did on the finer. The drag coefficient obtained by the CADI scheme is approximately two orders of magnitude smaller than the one obtained by the EDS. The error distributions of the calculated pressure field for both schemes (both on the coarser grid) are depicted in Figures 2 and 3 in a part of the computational domain. While errors of the CADI solution show a local character (they are distributed near the cylinder surface), errors of the EDS solution are dispersed over the whole domain.

Laminar flow between two parallel infinite plates at rest

The flow situation is sketched in Fig. 4. The velocity profile $U(y)$ of the developed flow is parabolic, and the pressure gradient dp/dx is constant. Streamlines are straight lines, and the pressure change is a consequence of fluid viscosity. If a Cartesian computational grid (in Oxy coordinate frame) had been adopted, the application of both schemes would have given very accurate results, because streamlines and grid lines would have been aligned. Therefore, accuracy of both differencing schemes is tested in a situation with streamlines and grid lines not aligned. The problem is solved in a computational domain defined in O, x, x_2 frame of reference, as shown in Fig. 4. Reynolds number $Re = 2\rho Uh/\mu = 500$ is chosen and the exact boundary conditions (from the analytical solution) are prescribed at all boundaries. The problem is solved on an equally spaced 16×16 CV grid ($\alpha = 25^\circ$), with cell-Reynolds numbers varying from 0.38 to 11. Isobars obtained by application of CADI differencing scheme are depicted in Fig. 5. It is evident that the isobars are approximately equidistant straight lines, and thus in accordance with the exact solution. Maximal discrepancy of calculated pressure with respect to the exact value is 2.92 % of the total pressure difference in the computational domain. Average velocity errors are $(\varepsilon_{v_1})_{av} = 6.6 \cdot 10^{-3} \%$ and $(\varepsilon_{v_2})_{av} = 3.5 \cdot 10^{-3} \%$. Isobars obtained from the EDS numerical solution are depicted in Fig. 6. It is visible that the pressure field is completely wrong (even qualitative picture is wrong, while the maximal relative error in pressure is 141%). This can be explained by the influence of false diffusion, introduced into the numerical solution by the EDS scheme. Average velocity errors are $(\varepsilon_{v_1})_{av} = 1.18 \%$ and $(\varepsilon_{v_2})_{av} = 0.68 \%$.

4. CONCLUSIONS

A comparison of the CADI and the EDS differencing schemes is made by applying both schemes to solving the momentum equation for two benchmark problems with known analytical solutions. In both situations, streamlines were not aligned with grid lines and cell-Reynolds numbers had great values (dominant convection term). These preliminary results show that the CADI scheme is a highly accurate differencing scheme. In the future work, additional tests will have to be done (e.g. calculations with non-uniform, non-smooth and non-orthogonal grids in more complex situations). If results should be as good as those presented here, the CADI differencing scheme might be recommended for general use in Computational Fluid Dynamics.

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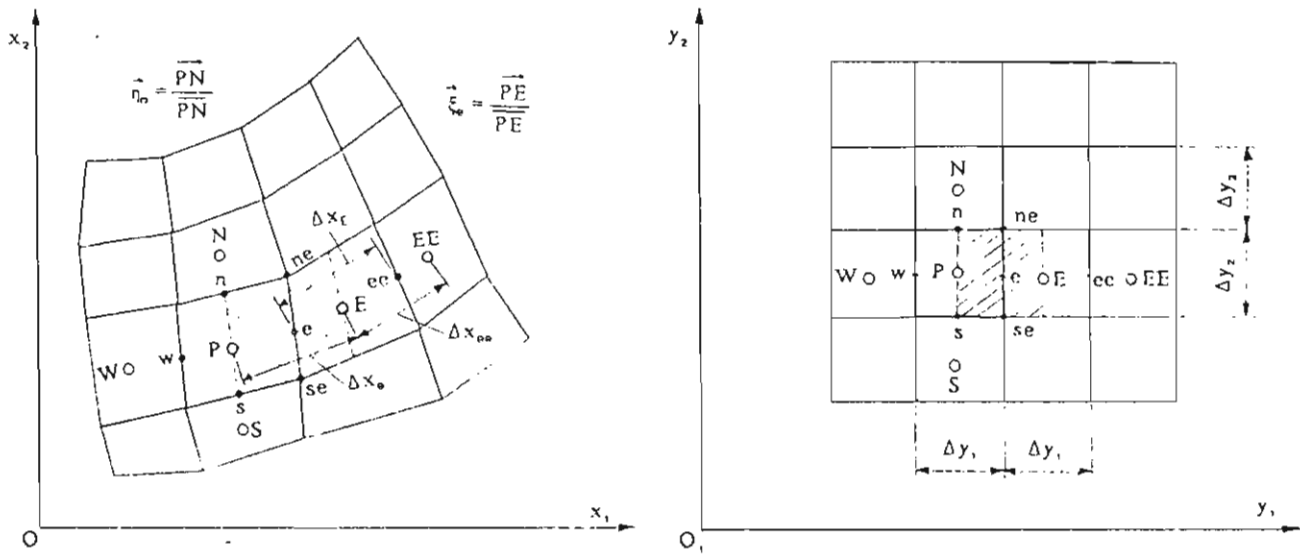


Fig. 1 A part of the computational domain

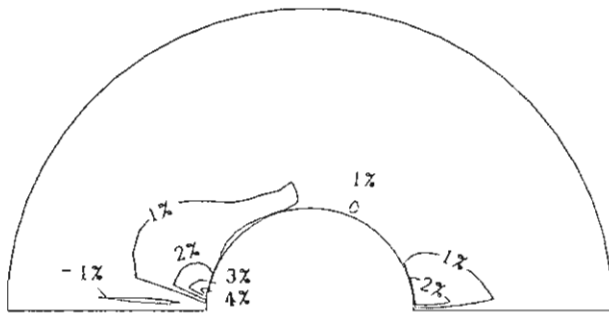


Fig. 2 Pressure error distribution (CADI scheme)

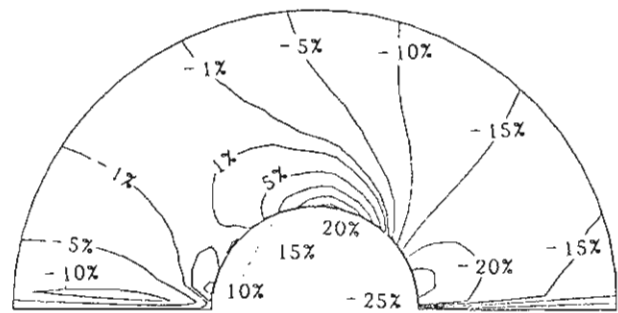


Fig. 3 Pressure error distribution (EDS scheme)

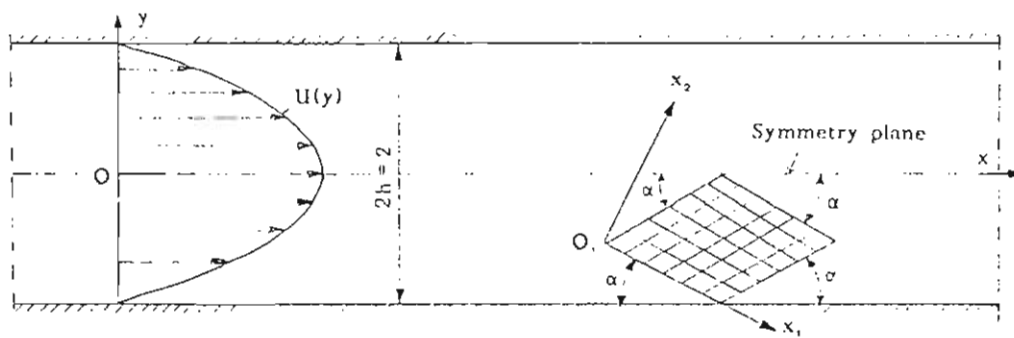


Fig. 4 Computational domain for laminar flow between parallel plates

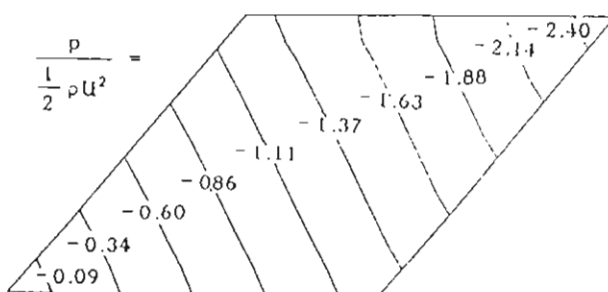


Fig. 5 Isobars (CADI scheme)

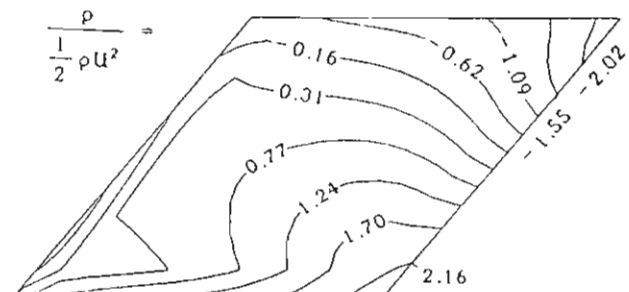


Fig. 6 Isobars (EDS scheme)

Incompressible potential flow in multy-connected domain with periodic boundary

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Abstract. This paper presents decomposition method for the solution of incompressible potential flow in multy-connected domain with periodic boundary. Kutta-Joukowski condition comes as a condition for determination of arbitrary parameters in multy-connected domain. Finite element formulation gives linear systems with the same symetric matrix and the computation costs are equal to the costs of a solution of the linear system for simply-connected domain.

Key words: incompressible potential flow, multy-connected domain, Kutta-Joukowski condition, finite element method

1. Introduction

Model of incompressible potential flow is the simplest one in CFD field, but in some cases can give good results (see [1], [4], [5]).

In literature there are lots of papers using Kutta-Joukowski condition not in the clear way (see[1]).

Using the decomposition method suggested in [2], [3] and [4], this paper presents the method of solution of incompressible potential flow in multy-connected domain with periodic boundary in the clear way. Kutta-Joukowski condition comes as a physically reasonable condition on the blade trailing edge for the flow around a blade. Finite element solution of this problem gives linear systems which have the same matrix, what is the great advantage in reducing the computational costs.

2. Problem formulation

Incompressible potential flow in domain Ω (2D or 3D flow) is governed by equations:

$$\left. \begin{array}{l} \operatorname{div} \bar{v} = 0 \\ \operatorname{rot} \bar{v} = 0 \end{array} \right\} \quad (1)$$

where $\bar{v}$ as velocity field.

One example of multy-connected bounded domain Ω with periodic boundary is shown in Figure 1.

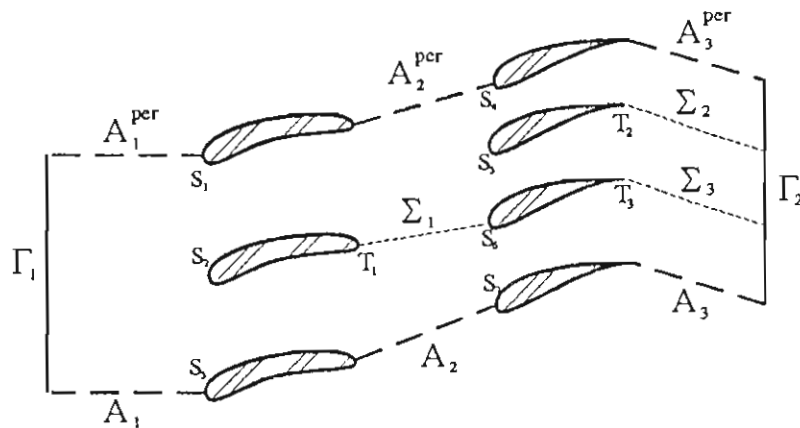


Figure 1. Cascade channel flow

Multy-connected domain can be made in the simply-connected domain with smooth cuts ($\Sigma_1, \Sigma_2, \Sigma_3$ on Figure 1.).

Open domain $\dot{\Omega} = \Omega \setminus \Sigma$, $\Sigma = \bigcup_i \Sigma_i$ is simply connected. For the simply connected domain $\dot{\Omega}$ a solution of (1) can be obtained from following boundary value problem for velocity potential Φ :

$$\begin{aligned} \Delta \Phi &= 0 \quad \text{in } \dot{\Omega}, \\ \frac{\partial \Phi}{\partial n} \Big|_{\Gamma} &= g, \\ \frac{\partial \Phi}{\partial n} \Big|_{A_l} &= - \frac{\partial \Phi}{\partial n} \Big|_{A_l^{per}}, \quad \Phi|_{A_l} = \Phi|_{A_l^{per}} + a_l, \\ \frac{\partial \Phi}{\partial n} \Big|_{\Sigma_k^+} &= - \frac{\partial \Phi}{\partial n} \Big|_{\Sigma_k^-}, \quad \Phi|_{\Sigma_k^+} = \Phi|_{\Sigma_k^-} + b_k, \\ l &= 1, 2, \dots, N_1; \quad k = 1, 2, \dots, N_2 \end{aligned} \tag{2}$$

where g is given function on $\Gamma = \Gamma_1 \cup \Gamma_2$, $A = \bigcup_{l=1}^{N_1} A_l$ and $A^{per} = \bigcup_{l=1}^{N_1} A_l^{per}$ are periodic boundaries, Σ_k^+ one side and Σ_k^- other side of the cut Σ_k . Numbers a_l , $l=1, 2, \dots, N_1$ and b_k , $k=1, 2, \dots, N_2$ are parameters which can be arbitrary chosen.

3. Decomposition of the problem

The boundary value problem (2) can be decomposed in following boundary value problems:

$$\begin{aligned} \Delta \Phi_0 &= 0 \quad \text{in } \dot{\Omega}, \\ \frac{\partial \Phi_0}{\partial n} \Big|_{\Gamma} &= g, \\ \frac{\partial \Phi_0}{\partial n} \Big|_{A_l} &= - \frac{\partial \Phi_0}{\partial n} \Big|_{A_l^{per}}, \quad \Phi_0|_{A_l} = \Phi_0|_{A_l^{per}}, \\ \frac{\partial \Phi_0}{\partial n} \Big|_{\Sigma_k^+} &= - \frac{\partial \Phi_0}{\partial n} \Big|_{\Sigma_k^-}, \quad \Phi_0|_{\Sigma_k^+} = \Phi_0|_{\Sigma_k^-}, \\ l &= 1, 2, \dots, N_1; \quad k = 1, 2, \dots, N_2 \end{aligned} \tag{3}$$

$$\Delta \Phi_k = 0 \quad \text{in } \Omega,$$

$$\left. \frac{\partial \Phi_k}{\partial n} \right|_{\Gamma} = 0,$$

$$\left. \frac{\partial \Phi_k}{\partial n} \right|_{A_l} = - \left. \frac{\partial \Phi_k}{\partial n} \right|_{A_l^{pr}}, \quad \Phi_k|_{A_l} = \Phi_k|_{A_l^{pr}},$$

$$\left. \frac{\partial \Phi_k}{\partial n} \right|_{\Sigma_k^+} = - \left. \frac{\partial \Phi_k}{\partial n} \right|_{\Sigma_k^-}, \quad \partial \Phi_k|_{\Sigma_k^+} = \partial \Phi_k|_{\Sigma_k^-} + \delta_{kk_1},$$

$$l=1,2,\dots,N_1; \quad k=1,2,\dots,N_2; \quad k_1=1,2,\dots,N_1 \\ \delta_{kk_1} = 0 \text{ for } k \neq k_1 \text{ and } \delta_{kk_1} = 1 \text{ for } k = k_1.$$

(4)

$$\Delta \tilde{\Phi}_l = 0 \quad \text{in } \Omega,$$

$$\left. \frac{\partial \tilde{\Phi}_l}{\partial n} \right|_{\Gamma} = 0,$$

$$\left. \frac{\partial \tilde{\Phi}_l}{\partial n} \right|_{A_l} = - \left. \frac{\partial \tilde{\Phi}_l}{\partial n} \right|_{A_l^{pr}}, \quad \tilde{\Phi}_l|_{A_l} = \tilde{\Phi}_l|_{A_l^{pr}} + \delta_{ll_1},$$

$$\left. \frac{\partial \tilde{\Phi}_l}{\partial n} \right|_{\Sigma_{l_1}^+} = - \left. \frac{\partial \tilde{\Phi}_l}{\partial n} \right|_{\Sigma_{l_1}^-}, \quad \partial \tilde{\Phi}_l|_{\Sigma_{l_1}^+} = \partial \tilde{\Phi}_l|_{\Sigma_{l_1}^-},$$

$$l=1,2,\dots,N_1; \quad l_1=1,2,\dots,N_1; \quad k=1,2,\dots,N_2 \\ \delta_{ll_1} = 0 \text{ for } l \neq l_1 \text{ and } \delta_{ll_1} = 1 \text{ for } l = l_1.$$

(5)

Solution Φ of (2) is linear combination of solutions of (3), (4) and (5):

$$\Phi = \Phi_0 + \sum_{l=1}^{N_1} a_l \cdot \tilde{\Phi}_l + \sum_{k=1}^{N_2} b_k \cdot \Phi_k$$

Parameters b_k , $k=1,2,\dots,N_2$ can be obtained from Kutta-Joukowski condition on the blade trailing edges (see [4]):

$$\nabla \Phi \cdot \vec{l}|_{T_k} = 0,$$

where T_k is the trailing edge on the blade and $\vec{l}$ is a unit tangential vector (see Figure 1.).

Parameters $a_l, l=1,2,\dots,N_1$ are arbitrary and usually are determined by setting certain values of tangential velocity components on inflow and outflow boundaries.

4. Variational formulation

For variational formulation of problems (3), (4) and (5) we need Sobolev function spaces:

$$H^1(\Omega) = \left\{ \varphi \in L_2(\Omega), \frac{\partial \varphi}{\partial x_i} \in L_2(\Omega) \right\},$$

$$H_{per}^1(\Omega) = \left\{ \varphi \in H^1(\Omega), \varphi_A = \varphi_{A^{per}} \right\},$$

Multiplying $\Delta \Phi_0 = 0$ with $\Psi \in H_{per}^1(\Omega)$ we get:

$$\int_{\Omega} \nabla \Phi_0 \nabla \Psi - \int_{\Gamma} \frac{\partial \Phi_0}{\partial n} \cdot \Psi - \int_A \frac{\partial \Phi_0}{\partial n} \cdot \Psi - \int_{A^{per}} \frac{\partial \Phi_0}{\partial n} \cdot \Psi - \int_{\Sigma^+} \frac{\partial \Phi_0}{\partial n} \cdot \Psi - \int_{\Sigma^-} \frac{\partial \Phi_0}{\partial n} \cdot \Psi = 0$$

Using boundary conditions in (3), preceeding relation gives following variational formulation: Find $\Phi_0 \in H_{per}^1(\Omega)$ such that

$$\int_{\Omega} \nabla \Phi_0 \nabla \Psi = \int_{\Gamma} g \cdot \Psi, \quad \forall \Psi \in H_{per}^1(\Omega). \quad (6)$$

For the problem (4) we need functions $\beta_k, k=1,2,\dots,N_2$ that satisfy conditions:

$$\beta_k|_{\Sigma_{k_1}^+} = \beta_k|_{\Sigma_{k_1}^-} + \delta_{kk_1}, \quad \beta_k|_{A_i} = \beta_k|_{A_i^{per}},$$

$$\frac{\partial \beta_k}{\partial n} \Big|_{\Sigma_{k_1}^+} = - \frac{\partial \beta_k}{\partial n} \Big|_{\Sigma_{k_1}^-}, \quad \frac{\partial \beta_k}{\partial n} \Big|_{A_i} = - \frac{\partial \beta_k}{\partial n} \Big|_{A_i^{per}},$$

$$l=1,2,\dots,N_1; \quad k_1=1,2,\dots,N_2$$

Substitution $\Phi_k = \varphi_k + \beta_k$ in (4) gives:

$$\Delta \varphi_k = -\Delta \beta_k,$$

$$\frac{\partial \varphi_k}{\partial n} \Big|_{\Gamma} = - \frac{\partial \beta_k}{\partial n} \Big|_{\Gamma},$$

$$\frac{\partial \varphi_k}{\partial n} \Big|_{A_i} = - \frac{\partial \beta_k}{\partial n} \Big|_{A_i^{per}}, \quad \varphi_k|_{A_i} = \varphi_k|_{A_i^{per}}, \quad (7)$$

$$\left. \frac{\partial(\varphi_k + \beta_k)}{\partial n} \right|_{\Sigma_k^+} = - \left. \frac{\partial(\varphi_k + \beta_k)}{\partial n} \right|_{\Sigma_k^-}, \quad \varphi_k|_{\Sigma_k^+} = \varphi_k|_{\Sigma_k^-}.$$

Variational formulation of the problem (7) is: Find $\varphi_k \in H_{per}^1$ such that

$$\int_{\Omega} \nabla \varphi_k \nabla \Psi = - \int_{\Omega} \nabla \beta_k \nabla \Psi, \quad \forall \Psi \in H_{per}^1(\Omega). \quad (8)$$

For the problem (5) we need functions α_l , $l=1,2,\dots,N_l$:

$$\alpha_l|_{A_l} = \alpha_l|_{A_l^{per}} + \delta_{ll_1}, \quad \left. \frac{\partial \alpha_l}{\partial n} \right|_{A_l} = - \left. \frac{\partial \alpha_l}{\partial n} \right|_{A_l^{per}},$$

$$\left. \frac{\partial \alpha_l}{\partial n} \right|_{\Sigma_l} = - \left. \frac{\partial \alpha_l}{\partial n} \right|_{\Sigma_l^-}, \quad \alpha_l|_{\Sigma_l^+} = \alpha_l|_{\Sigma_l^-}.$$

Substitution $\tilde{\Phi}_l = \tilde{\varphi}_l + \alpha_l$ in (5) gives boundary value problem similar to (7) and variational formulation is: Find $\tilde{\varphi}_l \in H_{per}^1(\Omega)$ such that

$$\int_{\Omega} \nabla \tilde{\varphi}_l \nabla \Psi = - \int_{\Omega} \nabla \alpha_l \nabla \Psi, \quad \forall \Psi \in H_{per}^1(\Omega). \quad (9)$$

5. Finite element formulation

In variational formulation (6), (8) and (9), functions α_l , β_k are not harmonic, what is supposed in [2], [3], and this is great advantage in finite element formulation.

We choose functions α_l and β_k with small supports (Figure 2.).

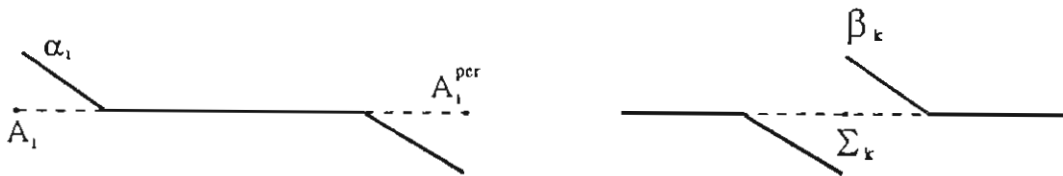


Figure 2. Cross-section of functions α_l and β_k with a plane

In the standard Galerkin finite element method problems (6), (8) and (9) give linear systems:

$$A \cdot \Phi_0 = d_0, \quad (10)$$

$$A \cdot \varphi_k = d_k, \quad (11)$$

$$A \cdot \tilde{\varphi}_l = \tilde{d}_l, \quad (12)$$

$$l=1,2,\dots,N_1; k=1,2,\dots,N_2.$$

This $N_1 + N_2 + 1$ linear systems have the same symmetric matrix A , what is a great advantage in computing.

Cholesky decomposition of matrix A can be used for every linear system (10), (11), (12).

Test example of blade-to-blade cascade channel flow will be given in a next paper.

6. Conclusion

Mathematical model of incompressible potential flow for multi-connected domain with periodic boundary can be formulated in variational form that in finite element formulation gives linear systems with the same matrix. This implies that the cost of computation is almost the same as computation of one linear system.

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Boundary-Domain Integral Formulations of Diffusion-Convective Equations

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Abstract

The paper deals with the numerical solution of transport diffusion-convective equations based on boundary-domain integral representations. Different Green's functions are used to develop an appropriate stable and accurate numerical scheme.

1. Introduction

The governing equations of transport phenomena in fluid flow, e.g. transfer of heat energy, momentum, vorticity, dispersion problems etc., are in general diffusion-convective partial-differential equations. The characteristics of the equations vary strongly from point to point in the flow field due to local Reynolds or Peclet number values physically representing the relationship between diffusion and convection of individual flow quantities. Thus it is not possible to discriminate for a pure elliptic, parabolic and hyperbolic equations since they are of mixed type. This particular character of underlying equations makes the numerical fluid dynamics more difficult as compared to numerical solving of transport phenomena in solids. This is especially true for transport problems characterised by high Reynolds or Peclet number values, when the convection becomes dominant compared with diffusion, or when the hyperbolic character of equation predominates its ellipticity or parabolicity respectively.

2. Governing Equations

Consider a general time dependent nonlinear diffusion-convective equation describing transport of an arbitrary scalar quantity $u(\vec{r}, t)$ in a homogeneous, isotropic and incompressible fluid flow, defined in a solution domain $R = \Omega \times I$ representing the product of space Ω and time interval $I(t_o, t)$

$$\frac{\partial}{\partial x_j} \left(a_c \frac{\partial u}{\partial x_j} \right) - \frac{\partial u}{\partial t} - \frac{\partial v_j u}{\partial x_j} + I_u = 0 \quad \text{in } R, \quad (2.1)$$

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where $\vec{v}(\vec{r}, t)$ is the local solenoidal velocity field. The variable $u(\vec{r}, t)$ can be equated with temperature, concentration, vorticity etc., and will be referred to as a potential. The effective diffusivity $a_e(\vec{r}, u)$ and the source term $I_u(\vec{r}, u)$ are some monotonic space and potential dependent functions. Substituting the expression for effective diffusivity variation in the form of constant a_o and variable part $a_N(\vec{r}, u)$

$$a_e = a_o + a_N(\vec{r}, u) , \quad (2.2)$$

the eq. (2.1) can be partitioned into a linear and nonlinear part in the following manner

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial u}{\partial t} - \frac{\partial v_j u}{\partial x_j} + \frac{\partial}{\partial x_j} \left(a_N \frac{\partial u}{\partial x_j} \right) + I_u = 0 . \quad (2.3)$$

The eqs. (2.1) and (2.3) represent a parabolic initial-boundary values problem, thus some boundary and initial conditions have to be known to complete the mathematical description of the problem, e.g. Dirichlet, Neumann or Cauchy type boundary conditions have to be prescribed on the part of the boundary Γ_1 , Γ_2 and Γ_3 respectively

$$\begin{aligned} u &= \bar{u} & \text{on } \Gamma_1 & \quad t > t_o , \\ \frac{\partial u}{\partial x_j} n_j &= \frac{\partial \bar{u}}{\partial n} & \text{on } \Gamma_2 & \quad t > t_o , \\ \frac{\partial u}{\partial x_j} n_j &= \alpha_u (u - u_f) & \text{on } \Gamma_3 & \quad t > t_o , \end{aligned} \quad (2.4)$$

where α_u is some transfer coefficient between the fluid flow surface defined by unit normal $\vec{n}$ and the surrounding ambient at the potential u_f , while the initial conditions are

$$u = \bar{u}_o \text{ in } \Omega \text{ at } t > t_o . \quad (2.5)$$

In the steady transport problems the eq. (2.3) reduces to the following statement

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial v_j u}{\partial x_j} + \frac{\partial}{\partial x_j} \left(a_N \frac{\partial u}{\partial x_j} \right) + I_u = 0 \text{ in } \Omega \quad (2.6)$$

representing an elliptic boundary values problem subject to boundary conditions given by eq. (2.4).

3. Integral Representation of Steady Diffusion-Convective Equation

3.1 Formulation for Elliptic Fundamental Solution

Consider first an integral representation of a steady transport equation for a variable $u(\vec{r})$ in a known solenoidal velocity field $\vec{v}(\vec{r})$. Eq. (2.6) can be recognised as a nonhomogeneous elliptic Poisson PDE of the form

$$\frac{\partial^2 u}{\partial x_j \partial x_j} + b = 0 \quad \text{in } \Omega , \quad (3.1)$$

where b represents the pseudo body force term. The corresponding boundary-domain integral representation to eq. (3.1) can be formulated by applying a weighted residual technique or simply by Green's theorems for scalar functions, resulting in the following integral statement

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} u^* d\Gamma + \int_{\Omega} b u^* d\Omega, \quad (3.2)$$

where u^* is the elliptic fundamental solution, i.e. the solution of

$$\frac{\partial^2 u^*}{\partial x_j \partial x_j} + \delta(\xi, s) = 0, \quad (3.3)$$

and $\partial u^* / \partial n$ its derivative in direction normal to the boundary, while $c(\xi)$ is the geometrically dependent free term accounting the Cauchy type singularity of the integral on the left hand side of eq. (3.2). The Green's function is known function given by the following expression, i.e. for the plane case one can write

$$u^* = \frac{1}{2\pi} \ln \left(\frac{1}{r} \right), \quad (3.4)$$

and for the space case

$$u^* = \frac{1}{4\pi r}, \quad (3.5)$$

where $\vec{r}(\xi, s)$ is the vector from the source point ξ to field points, i.e. $\vec{r}(\xi, s) = x_i(\xi) - x_i(s)$ for $i = 1, 2$ or $i = 1, 2, 3$, while r is its magnitude $r = |\vec{r}|$. The pseudo body force term b stands for convection, nonlinear diffusion part and source term from eq. (2.6)

$$b = \frac{1}{a_o} \left[\frac{\partial}{\partial x_j} \left(-v_j u + a_n \frac{\partial u}{\partial x_j} \right) + I_u \right] \quad (3.6)$$

rendering equation

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \frac{\partial u}{\partial n} u^* d\Gamma + \frac{1}{a_o} \int_{\Omega} \left[\frac{\partial}{\partial x_j} \left(-v_j u + a_n \frac{\partial u}{\partial x_j} \right) + I_u \right] u^* d\Omega, \quad (3.7)$$

The divergence operator in the domain integral can be shifted to the fundamental solution by applying Gaussian theorem, resulting in the following boundary-domain integral representation

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \frac{1}{a_o} \int_{\Gamma} \left(a_e \frac{\partial u}{\partial n} - u v_n \right) u^* d\Gamma + \frac{1}{a_o} \int_{\Omega} \left[\left(u v_j - a_n \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + I_u u^* \right] d\Omega, \quad (3.8)$$

where $v_n = \vec{v} \cdot \vec{n}$ is the normal velocity component to the boundary. The eq. (3.8) is valid both for space ($j = 1, 2, 3$) as for plane ($j = 1, 2$) flow cases subject to use appropriate space or plane elliptic fundamental solution.

The boundary-domain integral representation eq. (3.8) describes the transport of a scalar function u in an integral form in a physically adequate manner. Due to this fact the numerical scheme resulting from the discretised integral equation is very stable and accurate. It should be noticed, that the linear diffusion part and the total flux of the quantity u across the control surface are given by the boundary integrals. The domain integral is due to the convective effects, nonlinear diffusion part and source term.

3.2 Formulation for Diffusion-Convection Fundamental Solution

Since the fundamental solution exists only for the diffusion-convective equation with constant coefficients, e.g. for constant velocity vector $\vec{v}_o$, the nonhomogeneous velocity field $\vec{v}(\vec{r})$ has to be first decomposed into a constant one $\vec{v}_o$ and a variable or perturbed part $\hat{\vec{v}}_o(\vec{r})$

$$v_j(r_k) = v_{oj} + \hat{v}_j(r_k) , \quad (3.9)$$

enabling to rewrite eq. (2.6) in the following manner

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial v_{oj} u}{\partial x_j} + \frac{\partial}{\partial x_j} \left(a_N \frac{\partial u}{\partial x_j} \right) - \frac{\partial \hat{v}_j u}{\partial x_j} + I_u = 0 \text{ in } \Omega . \quad (3.10)$$

The eq. (3.10) can be seen to be a nonhomogeneous diffusion-convective PDE of the form

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial v_{oj} u}{\partial x_j} + b = 0 , \quad (3.11)$$

where b stands for pseudo body forces. Applying standard procedure, e.g. Green's theorems for scalars, the following boundary-domain integral representation can be formulated

$$c(\xi)u(\xi) + a_o \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \left(a_o \frac{\partial u}{\partial n} - uv_{on} \right) u^* d\Gamma + \int_{\Omega} b u^* d\Omega , \quad (3.12)$$

where $v_{on} = \vec{v}_o \cdot \vec{n}$ and u^* is now the fundamental solution of the diffusion-convective equation, i.e. the solution of equation

$$a_o \frac{\partial^2 u^*}{\partial x_j \partial x_j} + \frac{\partial v_{oj} u^*}{\partial x_j} + \delta(\xi, s) = 0 , \quad (3.13)$$

and given with expressions, e.g. for the plane case

$$u^* = \frac{1}{2\pi a_o} K_o \left(\frac{v_o}{2a_o} r \right) e^{\vec{v}_o \cdot \vec{r} / 2a_o} , \quad (3.14)$$

and for the space case

$$u^* = \frac{1}{4\pi a_o r} e^{(\vec{v}_o \cdot \vec{r} - v_o r) / 2a_o} . \quad (3.15)$$

The pseudo body force term includes the convection only for the perturbed velocity field, nonlinear diffusion and the source term.

$$b = \frac{\partial}{\partial x_j} \left(-\hat{v}_j u + a_N \frac{\partial u}{\partial x_j} \right) + I_u \quad (3.16)$$

rendering equation

$$c(\xi)u(\xi) + a_o \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \left(a_o \frac{\partial u}{\partial n} - uv_{on} \right) u^* d\Gamma + \int_{\Omega} \left[\frac{\partial}{\partial x_j} \left(-\hat{v}_j u + a_N \frac{\partial u}{\partial x_j} \right) + I_u \right] u^* d\Omega , \quad (3.17)$$

and by applying the Gauss theorem to the domain integral, the following final integral statement can be obtained

$$c(\xi)u(\xi) + a_o \int_{\Gamma} u \frac{\partial u^*}{\partial n} d\Gamma = \int_{\Gamma} \left(a_o \frac{\partial u}{\partial n} - uv_n \right) u^* d\Gamma + \int_{\Omega} \left[\left(u \hat{v}_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + I_u u^* \right] d\Omega . \quad (3.18)$$

It may be noted, that in this formulation only the convection due to the perturbed velocity vector $\hat{v}$ is captured by domain integral, making this approach combined with sub-structuring technique very promising for solving transport problems at high Peclet number values.

Substituting $U^* = a_o u^*$ the eq.(3.18) can be rewritten as follows

$$c(\xi)u(\xi) + \int_{\Gamma} u \frac{\partial U^*}{\partial n} d\Gamma = \frac{1}{a_o} \int_{\Gamma} \left(a_e \frac{\partial u}{\partial n} - uv_n \right) U^* d\Gamma + \frac{1}{a_o} \int_{\Omega} \left[\left(u \hat{v}_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial U^*}{\partial x_j} + I_u u^* \right] d\Omega , \quad (3.19)$$

which is formally of the same form as eq. (3.8).

4. Integral Representation of Time Dependent Diffusion-Convective Equation

4.1 Formulation for Parabolic Fundamental Solution

Consider an integral representation of an unsteady transport equation for a variable $u(\vec{r}, t)$ in a known solenoidal velocity field $\vec{v}(\vec{r}, t)$. The eq. (2.3) can be given as a nonhomogeneous parabolic diffusion *PDE* of the form

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial u}{\partial t} + b = 0 \quad \text{in } R , \quad (4.1)$$

where nonhomogeneous term stands for pseudo body forces. The following corresponding boundary-domain integral equation can be obtained by applying weighted residual statement, e.g. written in a time incremental form for the time step $\Delta t = t_F - t_{F-1}$

$$c(\xi)u(\xi, t_F) + a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma = a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} \frac{\partial u}{\partial n} u^* dt d\Gamma + \int_{\Omega} \int_{t_{F-1}}^{t_F} b u^* dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega , \quad (4.2)$$

where u^* is the parabolic diffusion fundamental solution, i.e. the solution of equation

$$a_o \frac{\partial^2 u^*}{\partial x_j \partial x_j} + \frac{\partial u^*}{\partial t} + \delta(\xi, s) \delta(t_F, t) = 0 , \quad (4.3)$$

and given by expression

$$u^*(\xi, s, t_F, t) = \frac{1}{(4\pi a_o \tau)^{d/2}} e^{-\tau^2/4a_o\tau} , \quad (4.4)$$

where (ξ, t_F) and (s, t) are the source and field points respectively, d stands for the dimensionality of the problem and $\tau = t_F - t$.

Equating the pseudo body force term b with convection, nonlinear diffusion part and the source term

$$b = \frac{\partial}{\partial x_j} \left(-v_j u + a_N \frac{\partial u}{\partial x_j} \right) + I_u , \quad (4.5)$$

one can derive the following integral statement

$$c(\xi)u(\xi, t_F) + a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma = a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} \frac{\partial u}{\partial n} u^* dt d\Gamma \\ + \int_{\Omega} \int_{t_{F-1}}^{t_F} \left[\frac{\partial}{\partial x_j} \left(-v_j u + a_N \frac{\partial u}{\partial x_j} \right) + I_u \right] u^* dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega , \quad (4.6)$$

The differential operator in the first domain integral can be shifted to the fundamental solution by using Gaussian theorem, resulting in the following integral representation

$$c(\xi)u(\xi, t_F) + a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma = \int_{\Gamma} \int_{t_{F-1}}^{t_F} \left(a_e \frac{\partial u}{\partial n} - uv_n \right) u^* dt d\Gamma \\ + \int_{\Omega} \int_{t_{F-1}}^{t_F} \left[\left(uv_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + I_u u^* \right] dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega . \quad (4.7)$$

The eq.(4.7) is adequate both for the space ($j = 1, 2, 3$) as for the plane ($j = 1, 2$) flow geometry subject to use of the appropriate space or plane parabolic fundamental solution.

The boundary-domain integral statement eq. (4.7) represents time dependent transport of scalar function u in the integral form on a physically justified way. The diffusion and convection across the solution surface are described by boundary integrals. The first domain integral is due to the convection, nonlinear diffusion and source term, while the last domain integral reflects the initial conditions effects on the development of the scalar field at the next time interval.

Assuming constant variation of all field functions within the individual time increment, the time integrals in eq. (4.7) may be evaluated analytically for space and plane geometry cases, i.e.

$$U^* = a_o \int_{t_{F-1}}^{t_F} u^* dt , \\ \frac{\partial U^*}{\partial n} = a_o \int_{t_{F-1}}^{t_F} \frac{\partial u^*}{\partial n} dt , \\ \frac{\partial U^*}{\partial x_j} = a_o \int_{t_{F-1}}^{t_F} \frac{\partial u^*}{\partial x_j} dt , \quad (4.8)$$

enabling to rewrite the integral statement (4.7) in a form formally identical to steady case one (3.8)

$$c(\xi)u(\xi, t_F) + \int_{\Gamma} u \frac{\partial U^*}{\partial n} d\Gamma = \frac{1}{a_o} \int_{\Gamma} \left(a_e \frac{\partial u}{\partial n} - uv_n \right) U^* d\Gamma \\ + \frac{1}{a_o} \int_{\Omega} \left[\left(uv_j - a_N \frac{\partial u}{\partial x_j} \right) \frac{\partial U^*}{\partial x_j} + I_u U^* \right] d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega . \quad (4.9)$$

4.2 Formulation for Diffusion-Convection Fundamental Solution

Due to the fact that the fundamental solution exists only for the time dependent diffusion-convective equation with constant coefficients, the velocity field has to be first partitioned into the constant $\vec{v}_o$ and the nonhomogeneous part $\hat{v}(\vec{r}, t)$

$$v_j(r_k, t) = v_{oj} + \hat{v}_j(r_k, t) , \quad (4.10)$$

enabling to formulate eq. (2.3) as follows

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial u}{\partial t} - \frac{\partial v_{oj} u}{\partial x_j} + \frac{\partial}{\partial x_j} \left(a_n \frac{\partial u}{\partial x_j} \right) - \frac{\partial \hat{v}_j u}{\partial x_j} + I_u = 0 . \quad (4.11)$$

The eq. (4.11) can be recognised as a nonhomogeneous time dependent diffusion-convective PDE of the form

$$a_o \frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial u}{\partial t} - \frac{\partial v_{oj} u}{\partial x_j} + b = 0 , \quad (4.12)$$

where b represents the nonhomogeneous pseudo body force term, with the following corresponding boundary-domain integral representation, e.g. written in a time incremental manner

$$c(\xi)u(\xi, t_F) + a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma = \int_{\Gamma} \int_{t_{F-1}}^{t_F} \left(a_o \frac{\partial u}{\partial n} - uv_{on} \right) u^* dt d\Gamma + \int_{\Omega} \int_{t_{F-1}}^{t_F} bu^* dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega . \quad (4.13)$$

The u^* is the fundamental solution of time dependent diffusion-convective equation, i.e. the solution of equation

$$a_o \frac{\partial^2 u^*}{\partial x_j \partial x_j} + \frac{\partial v_{jo} u^*}{\partial x_j} + \frac{\partial u^*}{\partial t} + \delta(\xi, s) \delta(t_F, t) = 0 \quad (4.14)$$

and given by expression

$$u^* = \frac{1}{(4\pi a_o \tau)^{d/2}} e^{(v_{oj}\tau_j/2a_o - v_o^2\tau/4a_o - r^2/4a_o\tau)} . \quad (4.15)$$

The pseudo body force term includes the nonhomogeneous convection part, nonlinear diffusion and source term

$$b = \frac{\partial}{\partial x_j} \left(-\hat{v}_j u + a_n \frac{\partial u}{\partial x_j} \right) + I_u \quad (4.16)$$

rendering the following final equation

$$\begin{aligned} c(\xi)u(\xi, t_F) + a_o \int_{\Gamma} \int_{t_{F-1}}^{t_F} u \frac{\partial u^*}{\partial n} dt d\Gamma &= \int_{\Gamma} \int_{t_{F-1}}^{t_F} \left(a_o \frac{\partial u}{\partial n} - uv_n \right) u^* dt d\Gamma \\ &+ \int_{\Omega} \int_{t_{F-1}}^{t_F} \left[\left(u\hat{v}_j - a_n \frac{\partial u}{\partial x_j} \right) \frac{\partial u^*}{\partial x_j} + I_u u^* \right] dt d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega . \end{aligned} \quad (4.17)$$

Again, by assuming constant variation of all field functions within each individual time increment, the time integration can be performed analytically

$$\begin{aligned} U^* &= a_o \int_{t_{F-1}}^{t_F} u^* dt \\ \frac{\partial U^*}{\partial n} &= a_o \int_{t_{F-1}}^{t_F} \frac{\partial u^*}{\partial n} dt \\ \frac{\partial U^*}{\partial x_j} &= a_o \int_{t_{F-1}}^{t_F} \frac{\partial u^*}{\partial x_j} dt \end{aligned} \quad (4.18)$$

enabling to formulate eq. (4.17) in the form identical to eq. (3.19)

$$\begin{aligned} c(\xi)u(\xi, t_F) + \int_{\Gamma} u \frac{\partial U^*}{\partial n} d\Gamma &= \frac{1}{a_o} \int_{\Gamma} \left(a_o \frac{\partial u}{\partial n} - uv_n \right) U^* d\Gamma \\ &+ \frac{1}{a_o} \int_{\Omega} \left[\left(u\hat{v}_j - a_n \frac{\partial u}{\partial x_j} \right) \frac{\partial U^*}{\partial x_j} + I_u U^* \right] d\Omega + \int_{\Omega} u_{F-1} u_{F-1}^* d\Omega . \end{aligned} \quad (4.19)$$

5. Conclusion

Boundary-domain integral representation offers some important features in computational fluid dynamics. By using the fundamental solutions the physics of the transport phenomena is described by boundary integrals, producing a very stable and accurate numerical scheme. Using elliptic or parabolic fundamental solution for example, the diffusion is completely described by boundary elements, while for the convection the integral cells are needed. A more efficient numerical scheme can be formulated by using the fundamental solutions of diffusion-convective equations, whereas domain discretisation is needed only for the nonhomogeneous convection part.

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Finite Element Modeling of Laminar Flow of Incompressible Viscous Fluid with Discretization of Directional Derivative

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ABSTRACT

The finite element method has been developed for laminar flow modeling of incompressible viscous fluid utilizing quadrilateral isoparametric elements. The Galerkin weighted residual method has been chosen with discretization of directional derivative of velocity. Two test cases for lid driven cavity flow, for low and high Reynolds numbers are solved. The results compare well with published ones.

Key Words: Finite elements, Fluid flow modeling, Navier-Stokes equation, Directional derivative discretization, Quadrilateral elements.

1. INTRODUCTION

Modeling of laminar flow of incompressible viscous fluid is of great importance in engineering. Navier-Stokes equation serve as model equation for turbulence modeling too, which is our future goal. Because of nonlinear term in advective acceleration many schemes were proposed, from local linearization [8] to upwinding techniques [8]. We apply discretization of advective derivative of velocity in an iterative solution method. For numerical solution we use Galerkin weighted residual method and finite element approximation with eight node isoparametric elements for velocity components and four node superparametric for pressure to avoid pressure oscillations [9]. Finally, our code is tested on lid driven cavity flow for two Reynolds numbers $Re = 1$ and $Re = 400$.

2. MATHEMATICAL MODEL

The partial differential equation set governing viscous, laminar, steady, two-dimensional flow of incompressible fluid, known as Navier-Stokes equations and equation of continuity, are [1]

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= f_x - \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) & \text{in } \Omega, \\ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= f_y - \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) & \text{in } \Omega, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, & \text{in } \Omega, \end{aligned} \quad (1)$$

where u and v are velocity components, p pressure, f_x, f_y body force component, ν viscosity of the fluid. The boundary conditions treated in this work are

$$\begin{aligned} (u, v) &= (\hat{u}, \hat{v}), & \text{given on } \Gamma_u, \\ \left(\frac{\partial u}{\partial n}, \frac{\partial v}{\partial n} \right) &= \left(\frac{\partial \hat{u}}{\partial n}, \frac{\partial \hat{v}}{\partial n} \right), & \text{given on } \Gamma_n, \end{aligned} \quad (2)$$

where $\Gamma_u \cup \Gamma_n = \partial\Omega$ is the boundary of Ω , $\vec{n}$ outer unit normal to the $\partial\Omega$.

These equations are particularly difficult to integrate for typical application, small ν , because of boundary layers and turbulence. Even the mathematical study of Navier-Stokes equations is not complete.

3. FINITE ELEMENT APPROXIMATION

We introduce a discretization of domain Ω as a union of finite elements in the usual manner. We use quadratic shape function and so called eight node serendipity isoparametric finite element approximation for velocity components and superparametric four node for pressure:

$$\begin{aligned} u &= \sum_{i=1}^8 N_i(\xi, \eta) u_i, & v &= \sum_{i=1}^8 N_i(\xi, \eta) v_i, \\ x &= \sum_{i=1}^8 N_i(\xi, \eta) x_i, & y &= \sum_{i=1}^8 N_i(\xi, \eta) y_i, \\ p &= \sum_{i=1}^4 M_i(\xi, \eta) p_i, \end{aligned}$$

where

$$N_i(\xi, \eta) = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta)(\xi_i \xi + \eta_i \eta - 1)$$

for corner nodes $i = 1, 3, 5, 7$ and

$$N_i(\xi, \eta) = \frac{1}{2}(1 - \xi^2)(1 + \eta_i \eta), \quad \xi_i = 0, \quad i = 2, 6,$$

$$N_i(\xi, \eta) = \frac{1}{2}(1 + \xi_i \xi)(1 - \eta^2), \quad \eta_i = 0, \quad i = 4, 8$$

for midside nodes, and

$$M_i(\xi, \eta) = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta), \quad i = 1, 3, 5, 7$$

for corner nodes. u_i, v_i, p_i are nodal values of primitive variables and x_i, y_i global coordinates of the nodes. The use of lower order finite element approximation elements for pressure is to

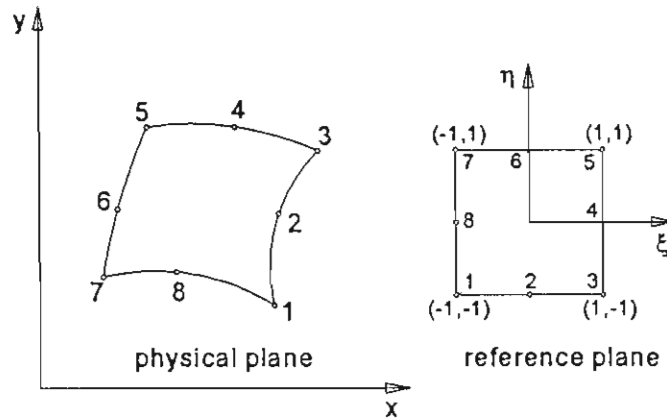


Figure 1: Eight node isoparametric finite element.

avoid the pressure oscillations which occurs for equal order approximation [9].

4. DISCRETIZATION OF DIRECTIONAL DERIVATIVE

Let $\vec{X}(\vec{x}, t; \tau)$ be the orbit (or the trajectory) of the flow field $\vec{w}(x) = (u, v)$ such that

$$\frac{d\vec{X}(\tau)}{d\tau} = \vec{w}[\vec{X}(\tau)], \quad \vec{X}(t) = \vec{x}. \quad (3)$$

Then the directional derivative of velocity becomes [8]

$$(\vec{w} \cdot \nabla) \vec{w} = \frac{\partial}{\partial \tau} \vec{w}[\vec{X}(\vec{x}, t; \tau)] \big|_{\tau=t}. \quad (4)$$

Thus we can write

$$(\vec{w} \cdot \nabla) \vec{w} \simeq \frac{1}{\delta} \{ \vec{w}(\vec{x}) - \vec{w}[\vec{X}_\delta(\vec{x})] \}, \quad (5)$$

where $\vec{X}_\delta(\vec{x})$ is an approximation of $\vec{X}(\vec{x}, t; \tau - \delta)$. We choose Euler scheme for approximate solution of (3)

$$\vec{X}_\delta(\vec{x}) = \vec{x} - \vec{w}(\vec{x})\delta. \quad (6)$$

Euler scheme is of order $O(\delta^2)$. One can use better approximations, of order $O(\delta^3)$ (second order Runge-Kutta for instance), but test calculations show that Euler scheme is sufficiently accurate. The Equation (6) is solved iteratively. First $\vec{w}^0$ is assumed. Knowing $\vec{w}^n$ we calculate $\vec{X}_\delta^{n+1}(\vec{x}) = \vec{x} - \vec{w}^n(\vec{x})\delta$ and $\vec{w}^{n+1}$ as the solution of (12). If the convergence test (18) is satisfied we stop the iteration, otherwise continue.

5. GALERKIN WEIGHTED RESIDUAL APPROACH

Applying Galerkin weighted residual approach to the Navier-Stokes equations and equation of continuity (1) which consists of weighting the residual of the equations, when approximation of the solution is introduced, by test function equal to finite element shape functions and integrating the product over the domain and equating it to zero, we get [1]

$$\sum_{e=1}^{N_e} \int_{A^e} N_i \left[\frac{1}{\delta} \sum_{j=1}^8 (N_j - \tilde{N}_j) u_j + \sum_{l=1}^4 \frac{\partial M_l}{\partial x} p_l - f_x - \nu \sum_{i=1}^8 \left(\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_j}{\partial y^2} \right) u_j \right] dA^e = 0, \quad i = 1, 2, \dots, N_p, \quad (7)$$

$$\sum_{e=1}^{N_e} \int_{A^e} N_i \left[\frac{1}{\delta} \sum_{j=1}^8 (N_j - \tilde{N}_j) v_j + \sum_{l=1}^4 \frac{\partial M_l}{\partial y} p_l - f_y - \nu \sum_{i=1}^8 \left(\frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_j}{\partial y^2} \right) v_j \right] dA^e = 0, \quad i = 1, 2, \dots, N_p, \quad (8)$$

$$\sum_{e=1}^{N_e} \int_{A^e} M_i \left(\sum_{j=1}^8 \frac{\partial N_j}{\partial x} u_j + \sum_{j=1}^8 \frac{\partial N_j}{\partial y} v_j \right) dA^e = 0, \quad i = 1, 2, \dots, N_m, \quad (9)$$

where the outer summation (index e) goes over all N_e elements. N_p is the number of nodes in $\Omega \setminus \Gamma_u$, N_m is number of midside nodes.

The weighting function in the (9) are taken as M_i associated with four node superparametric quadrilateral elements. The shape function $\tilde{N}_j$ in (7,8) denotes the upstream advected shape function N_j :

$$\tilde{N}_j = N_j \circ \tilde{X}_\delta \quad (10)$$

or in detail

$$\tilde{N}_j(x, y) = N_j(x - u(x, y)\delta, y - v(x, y)\delta). \quad (11)$$

The method for calculation of $\tilde{N}_j$ is given later.

The assembled matrix equations after application of divergence theorem to (7,8) take the form

$$A\vec{\phi} = \vec{b} + \vec{c}, \quad (12)$$

where

$$\phi_i = (u_i, p_i, v_i)^T, \quad i = 1, 2, \dots, N_p, \quad (13)$$

$$A_{ij} = \sum_{e=1}^{N_e} \left(\int_{A^e} C dA^e - \int_{\Gamma^e} D ds \right) \quad (14)$$

where C and D are 3×3 matrix with non zero components:

$$\begin{aligned} C_{11} = C_{33} &= \frac{1}{\delta} N_i (N_j - \tilde{N}_j) + \nu \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right), \\ C_{12} &= N_i \frac{\partial M_l}{\partial x}, \quad C_{21} = M_l \frac{\partial N_j}{\partial x}, \quad C_{23} = M_l \frac{\partial N_j}{\partial y}, \quad C_{32} = N_i \frac{\partial M_l}{\partial y}, \\ D_{11} = D_{33} &= \nu N_i \frac{\partial N_j}{\partial n}. \end{aligned}$$

Right hand side vectors $\vec{b}$ and $\vec{c}$ are due to body force densities and given normal derivative of the velocity are

$$b_i = \sum_{e=1}^{N_e} \vec{f} dA, \quad c_i = \sum_{e=1}^{N_e} \vec{q} dA, \quad (15)$$

where $\vec{f}$ and $\vec{q}$ are

$$\vec{f} = \{N_i f_x, 0, N_i f_y\}^T, \quad \vec{q} = \left\{ \nu N_i \frac{\partial \hat{u}_j}{\partial n} \Big|_{\Gamma_2}, 0, \nu N_i \frac{\partial \hat{v}_j}{\partial n} \Big|_{\Gamma_2} \right\}^T.$$

The integration in above formulas is not evaluated in physical plane, because of its difficulties due to the curved boundaries.

Rather the transformation of variables from physical to referential plane is first effected and the approximate integration is evaluated in referential plane using Gauss 3×3 point scheme

$$\int_{A^*} f(x, y) dx dy = \sum_{j=1}^3 \sum_{i=1}^3 a_i a_j F[\xi_i, \eta_i] J_{ij},$$

where a_i are Gauss weightings, J_{ij} is determinant of Jacobian at Gauss point

$$J_{ij} = \det \frac{\partial(x, y)}{\partial(\xi, \eta)} \Big|_{\xi=\xi_i, \eta=\eta_j}$$

and

$$F(\xi_i, \eta_j) = f[x(\xi_i, \eta_j), (\xi_i, \eta_j)].$$

The frontal method, suited for PC computers, is used to solve linear system (12).

6. APPROXIMATE CALCULATION OF SHAPE FUNCTION AT ADVECTED GAUSS POINTS

Rather than exactly calculate values of shape functions at advected points we have chosen its approximate calculation. We only have to calculate nodal function values at advected Gauss points. So we calculate "referential" velocity $\vec{\lambda}$ defined through

$$\begin{aligned} \vec{X}_\delta : \vec{z}_g &\rightarrow \vec{z}_g - \vec{w}\delta, \quad \vec{z} = \sum_{i=1}^8 N_i(\vec{\zeta}) \vec{z}_i, \\ \vec{X}_\delta : \vec{\zeta}_g &\rightarrow \vec{\zeta}_g - \vec{\lambda}\delta, \quad \vec{z} - \vec{w}\delta = \sum_{i=1}^8 N_i(\vec{\zeta} - \vec{\lambda}\delta) \vec{z}_i, \end{aligned} \quad (16)$$

where $\vec{z}_g = (x_g, y_g)$, $\vec{\zeta}_g = (\xi_g, \eta_g)$, $\vec{w} = (u, v)$, $\vec{\lambda} = (\lambda_\xi, \lambda_\eta)$. With expanding nodal function in

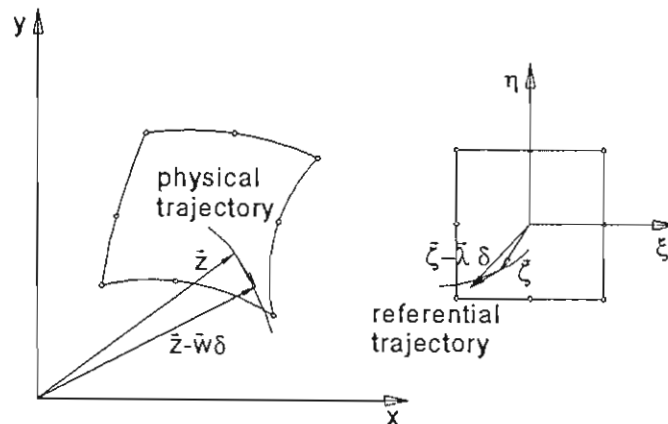


Figure 2: Physical and referential velocity.

Taylor series

$$N_i(\vec{\zeta} - \vec{\lambda}\delta) = N_i(\vec{\zeta}) - \text{grad } N_i \cdot \vec{\lambda}\delta + O(\delta^2),$$

and retaining only first two members we arrive at

$$u = \sum_{i=1}^8 x_i \text{grad } N_i(\vec{\eta}) \cdot \vec{\lambda}, \quad v = \sum_{i=1}^8 y_i \text{grad } N_i(\vec{\eta}) \cdot \vec{\lambda}.$$

Finally, the components of "referential" velocity are:

$$\lambda_x = \frac{a_{x\eta}v - a_{y\eta}u}{a_{y\xi}a_{x\eta} - a_{y\eta}a_{x\xi}}, \quad \lambda_y = \frac{a_{x\xi}v - a_{y\xi}u}{a_{y\eta}a_{x\xi} - a_{y\xi}a_{x\eta}}, \quad (17)$$

where the coefficients are

$$a_{x\xi} = \sum_{i=1}^8 x_i \frac{\partial N_i}{\partial \xi} \Big|_{\xi_0}, \quad a_{x\eta} = \sum_{i=1}^8 x_i \frac{\partial N_i}{\partial \eta} \Big|_{\xi_0},$$

$$a_{y\xi} = \sum_{i=1}^8 y_i \frac{\partial N_i}{\partial \xi} \Big|_{\xi_0}, \quad a_{y\eta} = \sum_{i=1}^8 y_i \frac{\partial N_i}{\partial \eta} \Big|_{\xi_0}.$$

Once advected Gauss points are calculated approximately, shape function value in new points are calculated exactly using (11).

7. NUMERICAL RESULTS

The proposed finite element algorithm based on discretization of directional derivative was tested on lid driven cavity flow, well known test problem. Two examples were tested: low Reynolds number ($Re = 1$), of interest for highly viscous (diffusion dominated) flows, and high Reynolds number ($Re = 400$) typical for advection dominated flows.

The finite element discretization (400 elements) with appropriate boundary conditions is shown in figure 3. Three time steps $\delta = 0.005, 0.01$ and 0.05 were used. The convergence

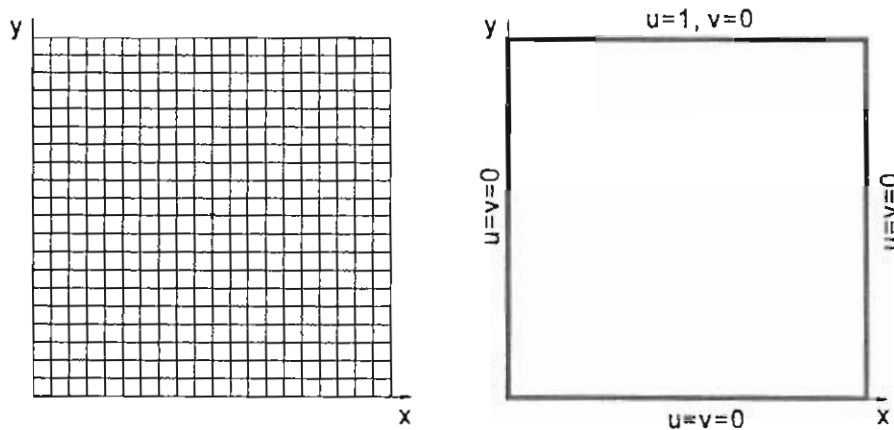


Figure 3: Finite element mesh and boundary conditions.

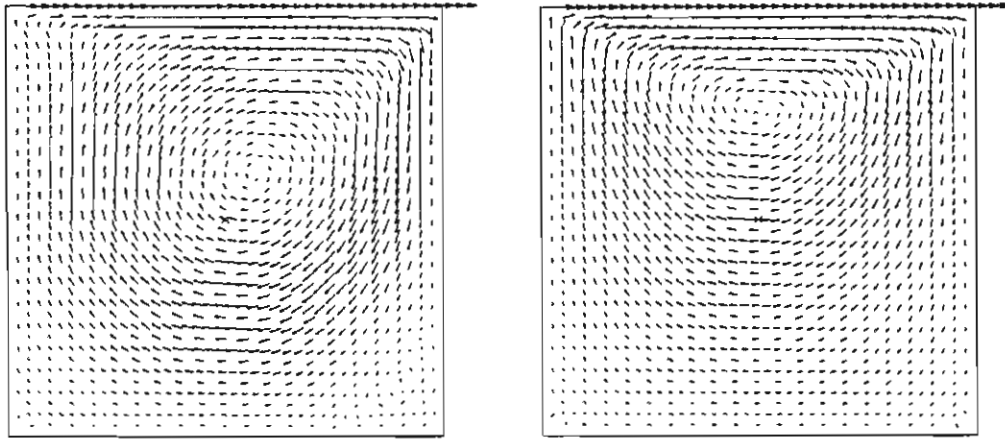
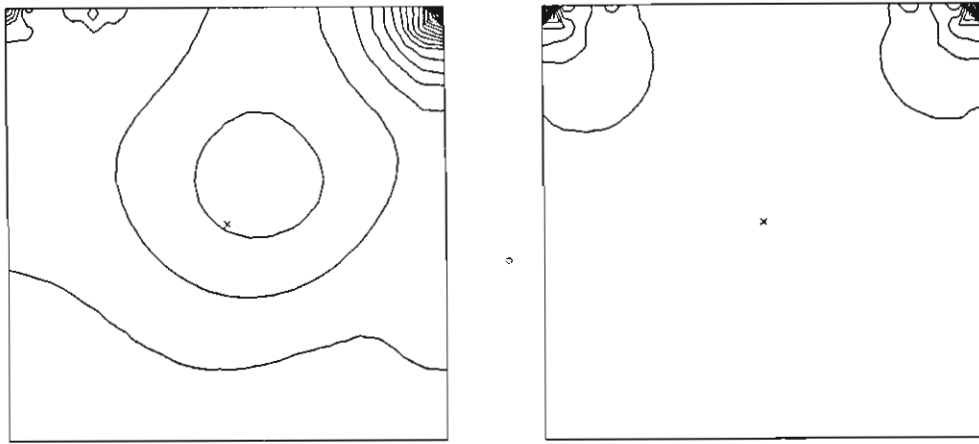
criteria of iteration scheme was chosen to be

$$E = \frac{\|\phi^{n+1} - \phi^n\|}{\|\phi^n\|} < 0.01 \quad (18)$$

in L_∞ norm, with n the number of iteration. The results and number of iterations were not affected significantly by the time step δ .

The velocity field and pressure contours for $Re = 1$ and $Re = 400$ are given in figure 4 and 5 respectively. The corresponding tabulation of vortex center position and velocity components minima and maxima on the cavity centerlines are given in table 1 and 2.

The vertical and horizontal velocity components profiles at centerlines are given in the figures 6 and 7. These results are seen to compare favorably with the literature.

Figure 4: Velocity fields, $Re = 400$ and $Re = 1$.Figure 5: Pressure contours, $Re = 400$ and $Re = 1$.

Source	Vortex center	$u_{min}(0.5, y)$	$v_{min}(x, 0.5)$	$v_{max}(x, 0.5)$
Author	(0.5640, 0.6055)	-0.279	-0.390	0.253
[2]	(0.55, 0.60)	-0.285	-0.390	0.250
[3]	(0.56, 0.61)	-0.285	-	-
[4]	(0.44537, 0.60415)	-	-	-
[5]	(0.44286, 0.60714)	-	-	-
[6]	(0.56, 0.62)	-0.300	-0.450	0.293

Table 1: Comparison of solution against the literature, $Re = 400$.

Source	Vortex center	$u_{min}(0.5, y)$	$v_{min}(x, 0.5)$	$v_{max}(x, 0.5)$
Author	(0.5020, 0.7575)	-0.195	-0.178	0.177
[2]	(0.50, 0.75)	-0.199	-0.177	0.177
[7]	(0.50, 0.75)	-	-	-
[5]	(0.50000, 0.76667)	-	-	-
[6]	(0.50, 0.77)	-0.188	-0.175	0.175

Table 2: Comparison of solution against the literature, $Re = 1$.

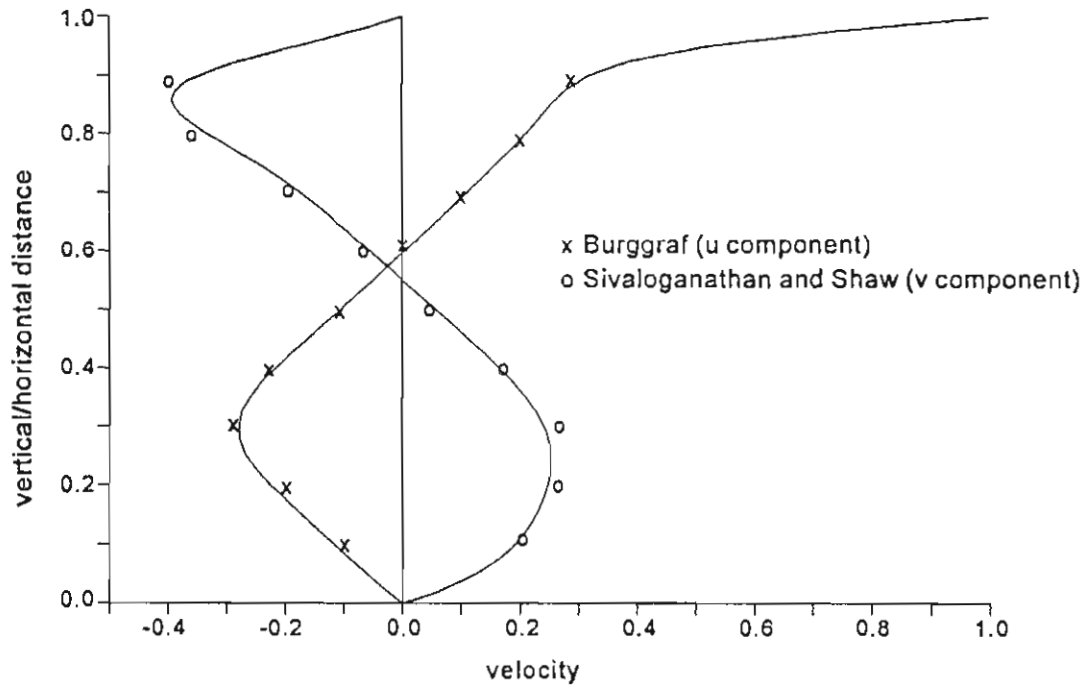


Figure 6: Velocity components profiles, $Re = 400$.

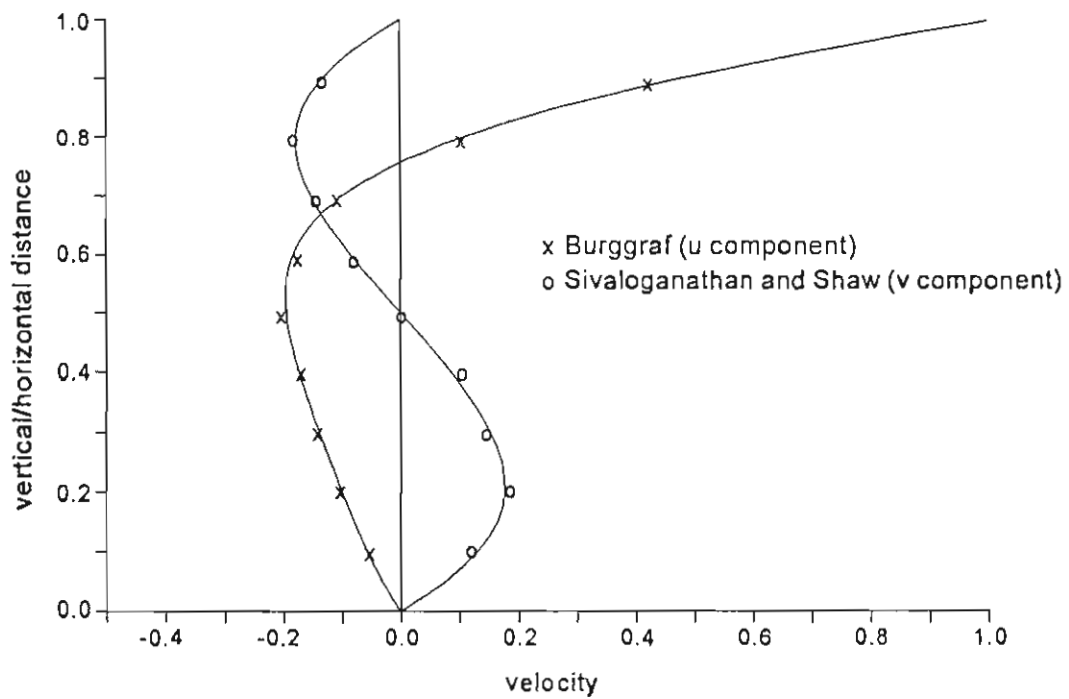


Figure 7: Velocity components profiles, $Re = 1$.

8. CONCLUSIONS

The Galerkin weighted residual finite element technique with discretization of directional derivative (advective acceleration) was developed. The computer code was tested solving the driven cavity flow for two extreme ranges of Reynolds numbers $Re = 1$ and $Re = 400$. The results compare favorably with the literature. This technique will form the basis of a major finite element research program, providing a powerful tool to analyze steady state and transient flows of Newtonian fluids with extension to turbulence modeling. The method showed to be stable, not dependent on time step size.

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The Random Walk Method of Pollutant Transport in Groundwater

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Abstract. The paper discusses the random walk method and the criterion of its appropriateness for the dispersion equation solution. The computer program is completed which simulates by this method the pollutant transport of indicator type in a homogenous porous medium with the developed two-dimensional stationary flow. Analyzing the solutions of one-dimensional and two-dimensional transport problems computed by this program, some advantages of this method compared to the finite element method and the finite difference method have been illustrated.

1. Introduction

Models of pollutant movement through groundwater have been rapidly developed in last twenty years and they present today a qualitative and powerful apparatus for prognosing of possible consequences of uncontrolled pollutant release on the groundwater quality. The finite difference method (FDM) is the first method used in pollutant transport solution. Due to the numerical diffusion problem and because of the numerical oscillations caused by dual character of dispersive equation, Gardner et al. proposed 1964 the method of characteristic (MC) for the solution of dispersion equation. The advantage of this method (MC) i.e. that solutions are free of numerical anomalies is diminished by a very complicated computer program and that is the reason why the method is practically not used at all. One of the first researchers who used finite element method (FEM) in analyzing of the dispersion equation solution, was Gumon. Integral equation formulation in the finite element method by variation principle, which was used in the beginning, was soon replaced by Galerkin's approach to the method of weighted residual, due to the some advantages. 1974 Ahlstrom et al. applied the random walk method (RWM) to transport solutions and Prickett et al. contributed mostly to this method by their article "A Random Walk Solute Transport Model", published 1981. This article discusses the program for the computer by which a big class of transport problems can be simulated. Both the RWM and the MC have considerably less problems with numerical diffusion than the standard methods do.

2. Hydrodynamic Dispersion Equation

If the assumptions on the aquifer and constant flow velocity parallel to x-axis are accepted, the pollutant transport equation of the ideal tracer in one-dimensional case is

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} - D \frac{\partial^2 c}{\partial x^2} = 0 \quad (1)$$

The second member of this equation is called advective and the third one dispersive member. The basic parameter of the equation is dispersion coefficient D . This is a tensor value and the equation of plane case is (Bear, 1977):

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} - \frac{\partial}{\partial x} \left(D_{xx} \frac{\partial c}{\partial x} + D_{xy} \frac{\partial c}{\partial y} \right) - \frac{\partial}{\partial y} \left(D_{yx} \frac{\partial c}{\partial x} + D_{yy} \frac{\partial c}{\partial y} \right) = 0 \quad (2)$$

In the equations c denotes concentration and u_x and u_y denote velocity of water in pore spaces, while

$$\begin{aligned} D_{xx} &= \alpha_L \frac{u_x^2}{u} + \alpha_T \frac{u_z^2}{u} \\ D_{zz} &= \alpha_T \frac{u_x^2}{u} + \alpha_L \frac{u_z^2}{u} \\ D_{xz} &= D_{zx} = (\alpha_L - \alpha_T) \frac{u_z u_x}{u} \end{aligned} \quad (3)$$

Both equations (1) and (2) are linear partial differential equations of second order, which can be observed as a combination of the simple wave equations

$$\frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} = 0 \quad (4)$$

of hyperbolic type and of the diffusive equation

$$\frac{\partial c}{\partial t} - D \frac{\partial^2 c}{\partial x^2} = 0 \quad (5)$$

of parabolic type. However, mathematically viewed, the equation (1) maintains the property of a diffusive equation, which means that the equation of hydrodynamic dispersion belongs to partial difference equations of parabolic type. But when the dispersive member is considerably smaller than the convective one, hyperbolic character of this equation is prevailing.

The equation (5) has the same form as the flow equation and it is simply solved by the FEM and by FDM. The equation (4) describes spreading of waves of the same concentrations along characteristic lines. It can be much harder solved by common numerical methods. In the application of numerical methods the dual character of the dispersive equation can be examined by the grid-Peclet's number, which is in the directions x and y

$$Pe_x = \frac{u_x \Delta x}{D_{xx}} \quad Pe_y = \frac{u_y \Delta y}{D_{yy}} \quad (6)$$

where Δx and Δy are characteristic value of elements in x and y directions. If the grid-Pecklet's number in x respectively y direction is less then one, the parabolic character of dispersive equation is predominant and standard methods can successfully be applied in solving of dispersion equation. Numerical oscillations are increased with the growth of Pe_x and Pe_y so that for the use of the FEM and FDM has to be fulfilled

$$Pe_x < 2 \quad Pe_y < 2 \quad (7)$$

This conditions can be satisfied only by the decrease of the grid elements, in other words by a better discretization of the region. Application of the RWM is not conditioned by the grid-Pecklet's number.

3. The Random Walk Method

The RWM is a stochastic method and the essential characteristic of such methods is the need for a generator of random numbers. Pollutant transport solution by this method is based on the description of dispersion mechanism by pollutant-bearing particles. This means, that tracer solution which comes into contact with groundwater is replaced by certain number of particles, and all particles together give the total mass of solution. Transport simulation is performed by the movement of each particle for a convective and a dispersive step after which the pollution concentrations in certain points is computed. If the same number of pollutant molecules is taken for the number of the particles, the dispersion mechanism can be exactly described.

The main principle of the method will be explained by the simplest one-dimensional case. The analytical solution for the mass ΔM of ideal tracer injected momentary into point $x=0$ is presented by the expression (Kinzelbach, 1986)

$$c(x,t) = \frac{\Delta M}{2wmn_e \sqrt{4\pi\alpha_L ut}} \exp\left(-\frac{(x-ut)^2}{4\alpha_L ut}\right) \quad (8)$$

w is the width and m is thickness of an one-dimensional aquifer. If $\Delta M/n_e mw$ is replaced by $\bar{c}_0$ equation (6) is

$$c(x,t) = \frac{\bar{c}_0}{\sqrt{4\pi\alpha_L ut}} \exp\left(-\frac{(x-ut)^2}{4\alpha_L ut}\right) \quad (9)$$

Comparing this equation with the Gauss' or the normal law of distribution (Fig. 1)

$$n(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (10)$$

it is obviously the question of the same equation were is

$$\sigma = \sqrt{2\alpha_L ut} \quad \mu = ut \quad \bar{c}_0 = 1 \quad n(x) = c(x,t) \quad (11)$$

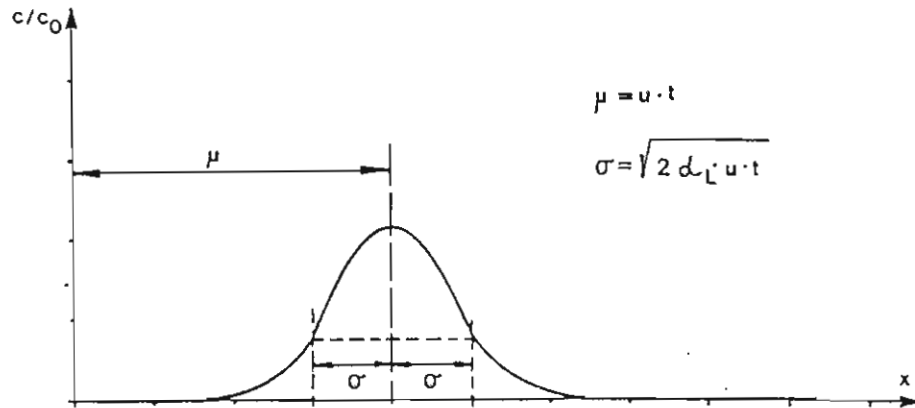


Fig. 1: Solution of dispersion equation for one-dimensional case, which is observed as normal distribution of the particles pathlengths.

Figure (1) illustrates normal distribution around an average $\bar{x} = ut$, with standard deviation $\sigma = \sqrt{2\alpha_L ut}$. The same distribution can be stochastically obtained. N particle is placed in the position $x=0$ in the moment $t=0$. In certain moment particle is moved by value x

$$x = ut + z\sqrt{2\alpha_L ut} \quad (12)$$

where z is normally distributed number within boundaries -6 and $+6$ with average 0 and standard deviations 1 . The number of particles in the interval Δx is directly related to the concentration in the middle of this interval

$$c(x+0.5\Delta x, t) = \frac{\Delta M}{Nmwn_c \Delta x} n(x, x+\Delta x, t) \quad (13)$$

The same procedure is also valid for the permanent source of pollution, but in that case the particles are constantly being added into the source. In the same way the wells and impermeable boundaries can be introduced.

4. The Computer Program

The computer program has been completed which simulates the pollutant transport of tracer type through homogenous porous medium by the RWM. The program has been verified on the one-dimensional problem, the analytical solution of which is known according to Kinzelbach (1986). The input data are as follows:

- velocity in positive direction of the axis x
- longitudinal dispersivity
- transverse dispersivity
- simulation duration
- time step
- value of the element in direction of the axis x
- value of the element in direction of the axis z
- constant concentration at source
- initial concentration at other nodes

$$\begin{aligned}
 u_x &= 1 \text{ m/day} \\
 \alpha_L &= 10 \text{ m} \\
 \alpha_T &= 1 \text{ m} \\
 t &= 200 \text{ day} \\
 \Delta t &= 2.5 \text{ day} \\
 \Delta x &= 2.5 \text{ m} \\
 \Delta z &= 2.5 \text{ m} \\
 c &= 100 \text{ g/m}^3 \\
 c &= 0 \text{ g/m}^3
 \end{aligned}$$

The results obtained by the RWM are compared by analytical solution (Fig. 2). The result quality depends on the number of particles in the system. It is obvious, that by increasing of the particle numbers in the beginning of the simulation the result is improved, but the time needed for computer simulation also increased. For the solution with 200 particles in the beginning of simulation 160 minutes are needed, i.e. ten times more time than for the solution with 20 particles at the beginning of simulation. The estimate is performed on the computer with mathematical co-processor.

Besides, one can notice that the results are not loaded with numerical diffusion so that this method is appropriate for answering the question about the time of pollutants' reaching some critical point, e. g. the well.

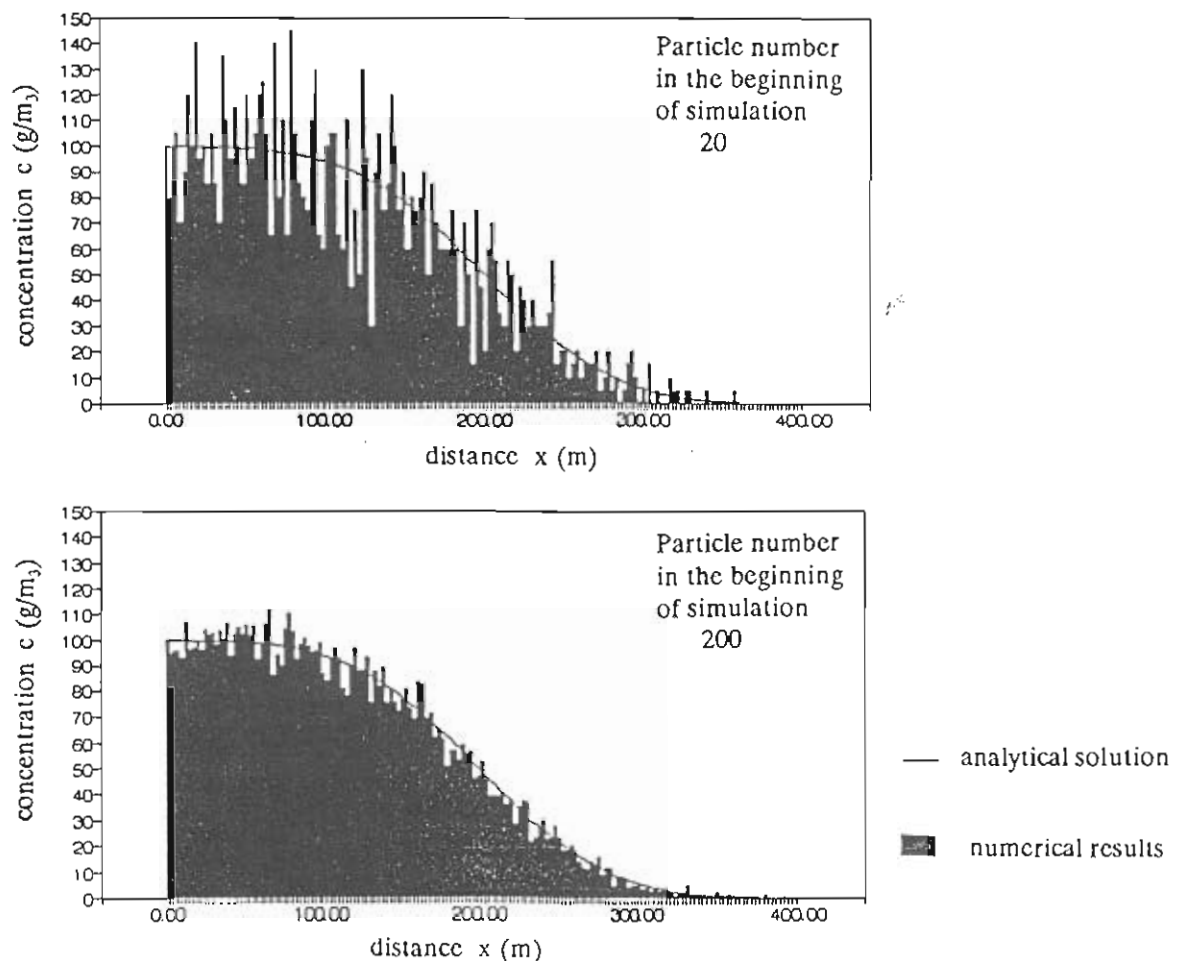


Fig. 2: Comparison of analytical and numerical solution

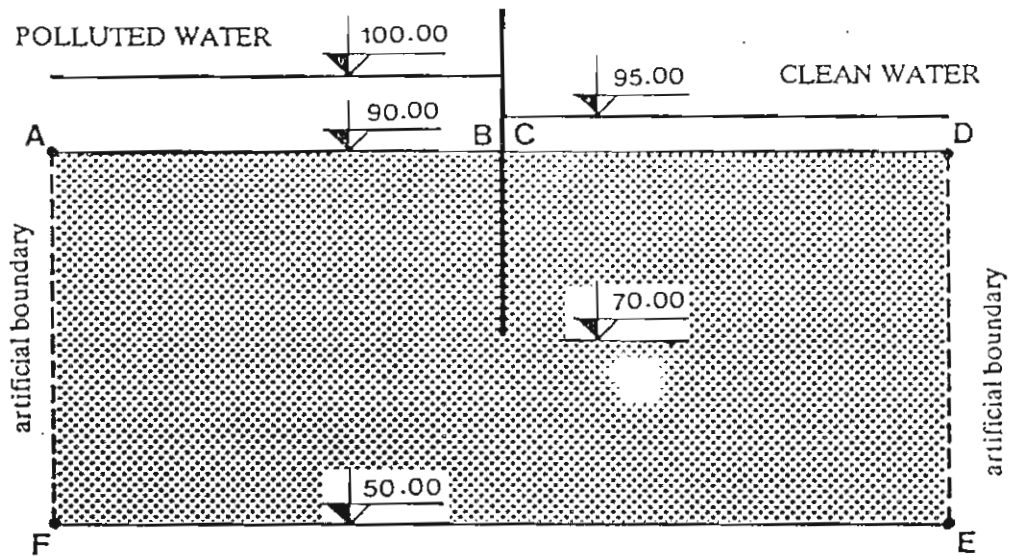


Fig. 3: Schematic illustration of the model used in analyzing of the RWM

The two-dimensional case of transport in stationary groundwater flow has also been solved by the program RAND. Figure (3) schematically illustrates a concrete dam with sheet piles. Chemical solution was added to water in front of the dam, so that this "polluted" water has constant concentration c_0 , while water in porous medium and the second reservoir in the beginning of simulation is clean, so the concentration of solution in this water equals 0.

Other data are:

- longitudinal dispersivity	$\alpha_L = 10 \text{ m}$
- transversal dispersivity	$\alpha_T = 1 \text{ m}$
- effective porosity	$n_e = 0.3$
- size of elements	$\Delta x = \Delta z = 2.5 \text{ m}$
- permeability	$k_x = 2k_z = 10^{-4} \text{ m/s}$
- time step	$\Delta t = 2.5 \text{ day}$
- const.concentration in nodes on the boundary AB	$c = 100 \text{ g/m}^3$
- initial concentration in other nodes	$c = 0 \text{ g/m}^3$
- particle number in the beginning of simulation	$NP = 410$

It can be seen from the solution of the two-dimensional problem (Fig. 4) that pollutant dispersion is bigger in the vicinity of the sheet piles due to bigger velocities in this region. A considerable part of pollutants moves to the aquifer "bottom". The first particle appears in the reservoir after 15 day. Pollutant transport through the boundary is simulated without any problem, which is not the case with the FEM and FDM. In this case the boundary CD is simulated so, that the particle passing this boundary are given the velocity $u_x = u_y = 0.1 \text{ m/day}$, and dispersivities $\alpha_L = 0.5$ and $\alpha_T = 0.005 \text{ m}$.

The presented graphical interpretation of results gives quantitative pollutant distribution on the area. To get a more complete picture of pollutant distribution in form of lines of the same pollutant concentrations, a lot of particles in the system are needed, and even then the lines are "straggling" due to the stochastic character of this method (Fig. 5).

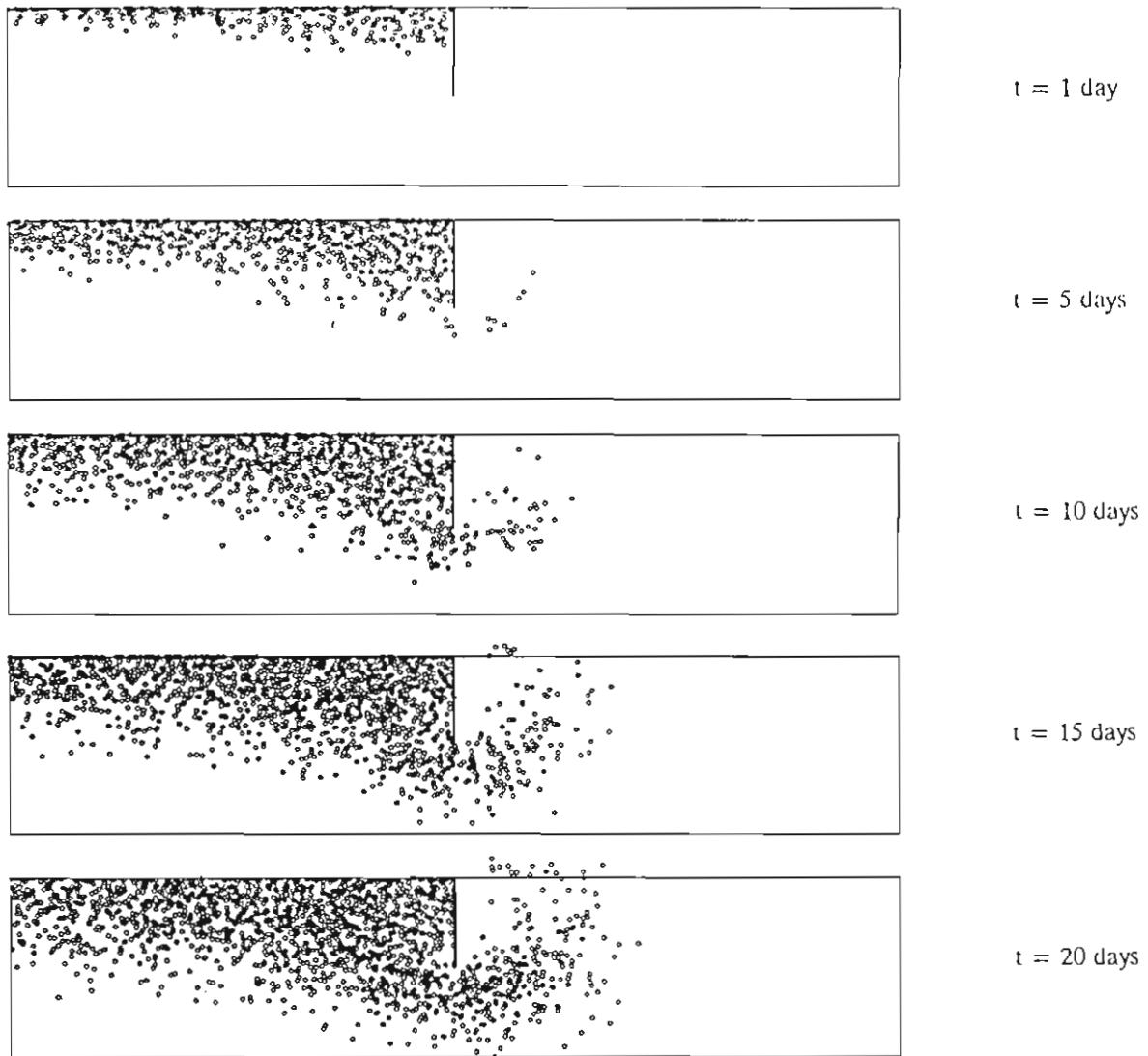


Fig. 4: Graphic presentation of pollutant transport simulation by means of particle themselves

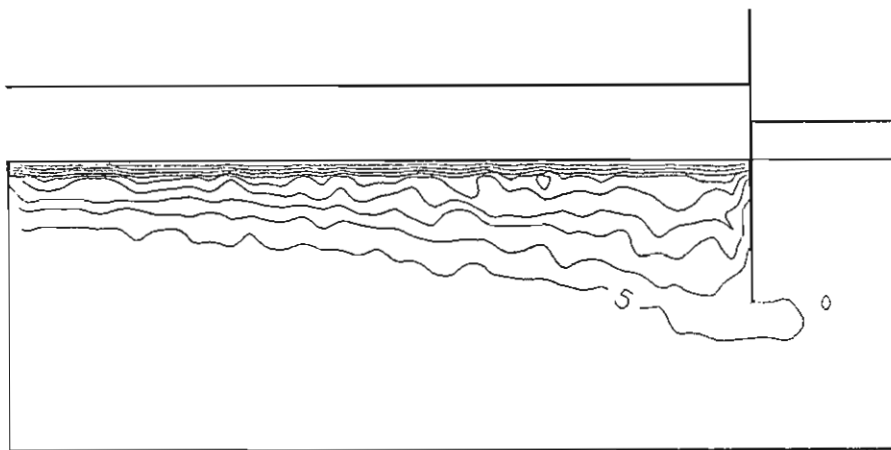


Fig. 5: Graphic presentation of pollution simulation by lines of the same concentration at time $t = 5$ days

5. Conclusion

The RWM is appropriate for transport solution in case when the hyperbolic character of dispersion equations is predominant because the results do not depend on the grid-Pecklet criterion. In the FEM and in the FDM the numerical diffusion cannot be avoided, how small the elements may be, by which the field will be discretized. The results obtained by the RWM are not loaded with numerical diffusion, so that by this method the time of pollutant arrival at some critical point can be exactly computed. Simulation of the pollutant flowing out through the field boundary is simple, compared with other classical numerical methods. RWM is a stochastic method offering rough results when there is a small number of particles in the system. By increasing of the particle number, the simulation time is also considerably increased, which is a basic disadvantage of this method.

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LOAD CARRYING CAPACITY OF HYDROSTATIC SQUEEZE FILM

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ABSTRACT

The paper deals with the problem of load carrying capacity of hydrostatic oil squeeze film, based on " oil-cushion " effect . This effect has been explained in the case of hemispherical bearing squeeze lubrication.

1. INTRODUCTION

The existence of hydrostatic oil squeeze film is very important for the function and longevity of many kinds of bearings . Very often the bearings are loaded by pulsating or impact forces or loads and relativ motions of bearing contact surfaces make impossible the forming of hydrodynamic oil film . In these cases the lubrication is only based on the hydrostatic squeeze film and if conditions are suitable , the oil film is maintained between contact surfaces although the relativ motion is momentary zero . This is known as the *oil-cushion* phenomenon based on the fact that a viscous lubricant cannot be instantaneously squeezed out from the interstice of contact surfaces . This phenomenon is presented in the case of the hemispherical bearing lubrication (Fig. 1. 1).

The analysis is carried out with some hypothesis : the lubricant is Newton's fluid ($\eta = \text{const.}$) and incompressible , the flow is laminar , the oil weight and deformations of contact surfaces are neglected .

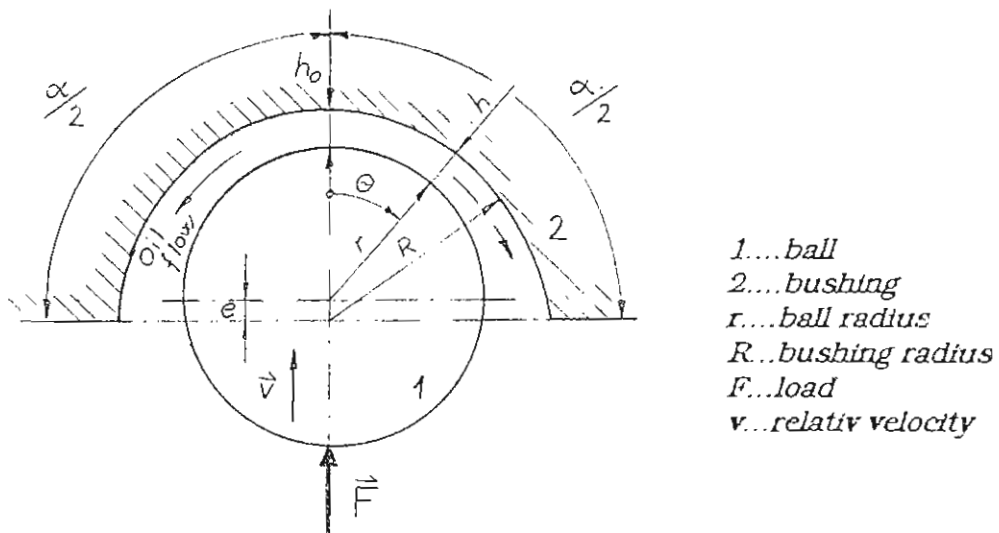


Figure 1.1 Hemispherical bearing

2. LOAD CARRYING CAPACITY OF " OIL-CUSHION "

With regard to the *Hagen-Poiseuille's* law, the flow q through the elemental slot is proportional to the slot width ($2 r \pi \sin \theta$), third power of film thickness (h) and depreciation of pressure into the flow direction ($-dp$), but opposite proportional to the oil viscosity (η) and slot length ($r d\theta$):

$$q = - \frac{2 \cdot r \cdot \pi \cdot \sin \theta \cdot h^3}{12 \cdot \eta \cdot r} \cdot \frac{dp}{d\theta}$$

or

$$q = - \frac{\pi \cdot \sin \theta \cdot h^3}{6 \cdot \eta} \cdot \frac{dp}{d\theta} \quad (2.1)$$

The film thickness (h) is :

$$h = m \cdot r \cdot (1 - \varepsilon \cdot \cos \Theta) \quad , \quad (2.2)$$

where m is the clearance modulus :

$$m = (R - r) / r = \Delta r / r \quad ,$$

and ε is the relativ displacement :

$$\varepsilon = e / \Delta r = e / (m \cdot r) = 0, \dots, 1 \quad .$$

The minimum oil film thickness is :

$$h_0 = m \cdot r \cdot (1 - \varepsilon) \quad . \quad (2.3)$$

From Eqs. (2.2) and (2.1) , becomes :

$$q = - \frac{\pi \cdot \sin \Theta \cdot m^3 \cdot r^3 \cdot (1 - \varepsilon \cdot \cos \Theta)^3}{6 \cdot \eta} \cdot \frac{dp}{d\Theta} \quad (2.4)$$

From the consideration of displacement and the flow continuity , another relation for the squeezed oil q can be written as :

$$q = v \cdot (r \cdot \sin \Theta)^2 \cdot \pi \quad , \quad (2.5)$$

where v is the velocity of ball and bushing relativ approaching . Combining Eqs. (2.4) and (2.5) , the depreciation of pressure is :

$$dp = - \frac{6 \cdot \eta \cdot v}{m^3 \cdot r} \cdot \frac{\sin \Theta}{(1 - \varepsilon \cdot \cos \Theta)^3} \cdot d\Theta \quad , \quad (2.6)$$

and from the boundary condition:

$$\Theta = \Theta_{\max} = \alpha / 2 \quad , \quad p = 0 \quad ,$$

follows :

$$p = \frac{3 \cdot \eta \cdot v}{m^3 \cdot r \cdot \varepsilon} \cdot \left[\frac{1}{(1 - \varepsilon \cdot \cos \Theta)^2} - \frac{1}{(1 - \varepsilon \cdot \cos \alpha/2)^2} \right] \quad (2.7)$$

If $\alpha = 180^\circ$, Eqs. (2.7) becomes :

$$p = \frac{3 \cdot \eta \cdot v}{m^3 \cdot r \cdot \varepsilon} \cdot \left[\frac{1}{(1 - \varepsilon \cdot \cos \Theta)^2} - 1 \right] \quad , \quad (2.8)$$

and the maximum value of pressure (when $\Theta = 0^\circ$) is :

$$p_{\max} = \frac{3 \cdot \eta \cdot v}{m^3 \cdot r \cdot \varepsilon} \cdot \left[\frac{1}{(1 - \varepsilon)^2} - 1 \right] . \quad (2.9)$$

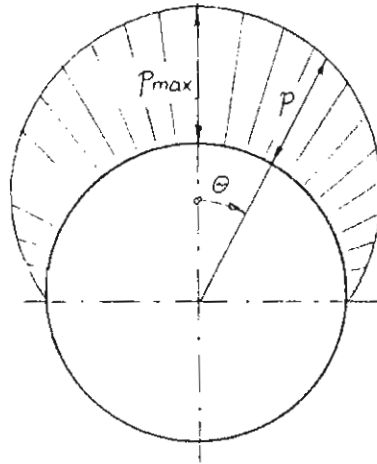


Figure 2.1 Oil film pressure distribution

The load carrying capacity F of the oil film is the result from integration of the elemental force dF (see Fig. 2.2) :

$$dF = p \cdot \cos \theta \cdot dA , \quad (2.10)$$

where $p \cos \theta$ is the vertical component of pressure and dA is the differential area of width $r d\theta$ and length $2 r \pi \sin \theta$:

$$dA = 2 r^2 \pi \sin \theta d\theta , \quad (2.11)$$

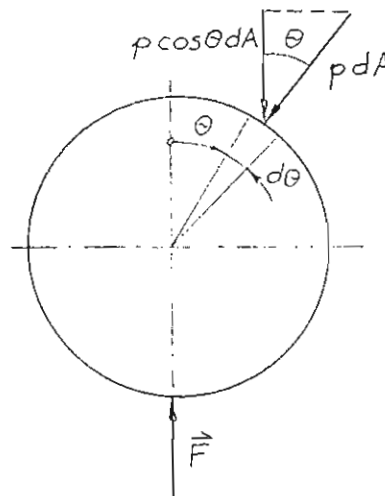


Figure 2.2 Oil film load-carrying capacity

$$F = \int dF = r^2 \cdot \pi \cdot \int_0^{\Theta_{\max} = \alpha/2} p \cdot \sin 2\Theta \cdot d\Theta$$

$$F = \frac{3 \cdot r \cdot \pi \cdot \eta \cdot v}{m^3 \cdot \varepsilon} \cdot \int_0^{\Theta_{\max} = \alpha/2} \left[\frac{1}{(1 - \varepsilon \cdot \cos \Theta)^2} - \frac{1}{\left(1 - \varepsilon \cdot \cos \frac{\alpha}{2}\right)^2} \right] \cdot \sin 2\Theta \cdot d\Theta \quad (2.12)$$

and for $\alpha = 180^\circ$:

$$F = \frac{3 \cdot r \cdot \pi \cdot \eta \cdot v}{m^3 \cdot \varepsilon} \int_0^{\pi/2} \left[\frac{1}{(1 - \varepsilon \cdot \cos \Theta)^2} - 1 \right] \cdot \sin 2\Theta \cdot d\Theta$$

$$F = \frac{3 \cdot r \cdot \pi \cdot \eta \cdot v}{m^3} \left[\frac{1}{\varepsilon^3} \cdot \ln(-1 + \varepsilon)^2 - \frac{2}{\varepsilon^3} - \frac{1}{\varepsilon} + \frac{2}{\varepsilon^3 \cdot (1 - \varepsilon)} \right] \quad (2.13)$$

Eqs. (2.13) can be written as :

$$F = \frac{3 \cdot r \cdot \pi \cdot \eta \cdot v}{m^3} \cdot [K] \quad ,$$

where factor K is :

$$[K] = \left[\frac{1}{\varepsilon^3} \cdot \ln(-1 + \varepsilon)^2 - \frac{2}{\varepsilon^3} - \frac{1}{\varepsilon} + \frac{2}{\varepsilon^3(1 - \varepsilon)} \right] \quad (2.14)$$

On Fig. 2.3 are plotted the computed values of factor K for various values of relative displacement ε :

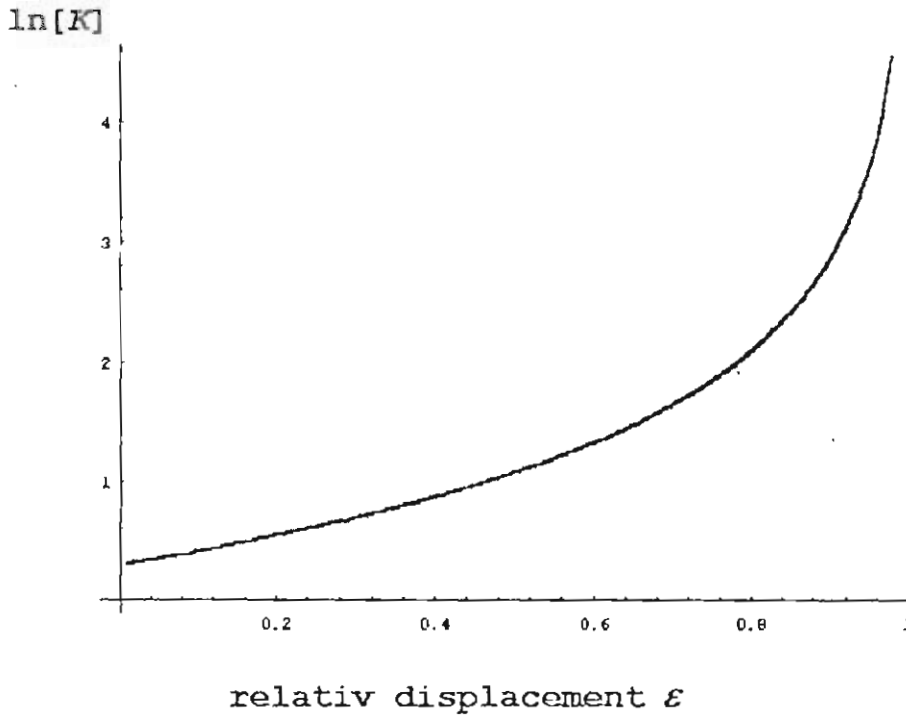


Figure 2.3 Values of factor [K]

The much better feature of oil film is the time Δt required for the ball to move from relativ displacement ε_1 to ε_2 or the clearance at the top of the bearing to change from the thickness h_{01} to h_{02} . The velocity of approaching is the time rate of change of h_0 :

$$v = -\frac{dh_0}{dt} \quad (2.15)$$

and from Eqs. (2.3), follows:

$$v = m \cdot r \cdot \frac{d\varepsilon}{dt} \quad (2.16)$$

Combining Eqs. (2.13) and (2.16) and with the presumption $F = \text{const.}$, the equation for time Δt is:

$$\Delta t = \int_{t_1}^{t_2} dt = \frac{3 \cdot r^2 \cdot \pi \cdot \eta}{m^2 F} \cdot \int_{\varepsilon_1}^{\varepsilon_2} \left[\frac{2}{\varepsilon^3(1-\varepsilon)} - \frac{2}{\varepsilon^3} - \frac{1}{\varepsilon} + \frac{1}{\varepsilon^3} \ln(-1+\varepsilon)^2 \right] \cdot d\varepsilon$$

$$\Delta t = \frac{3 \cdot r^2 \cdot \pi \cdot \eta}{m^2 \cdot F} \cdot \left[-\frac{1}{\varepsilon} - \frac{1}{2} \ln(-1+\varepsilon)^2 - \frac{1}{2\varepsilon^2} \ln(-1+\varepsilon)^2 \right]_{\varepsilon_1}^{\varepsilon_2} \quad (2.17)$$

or:

$$\Delta t = \frac{3 \cdot r^2 \cdot \pi \cdot \eta}{m^2 \cdot F} \cdot [T]_{\varepsilon_1}^{\varepsilon_2} = \frac{3 \cdot r^2 \cdot \pi \cdot \eta}{m^2 \cdot F} \cdot ([T(\varepsilon_2)] - [T(\varepsilon_1)]) \quad (2.18)$$

where factor T is :

$$[T] = \left[-\frac{1}{\varepsilon} - \frac{1}{2} \ln(-1 + \varepsilon)^2 - \frac{1}{2\varepsilon^2} \ln(-1 + \varepsilon)^2 \right] \quad (2.19)$$

Computed values of factor T for various values of relative displacement ε are plotted on Fig. 2.4 .

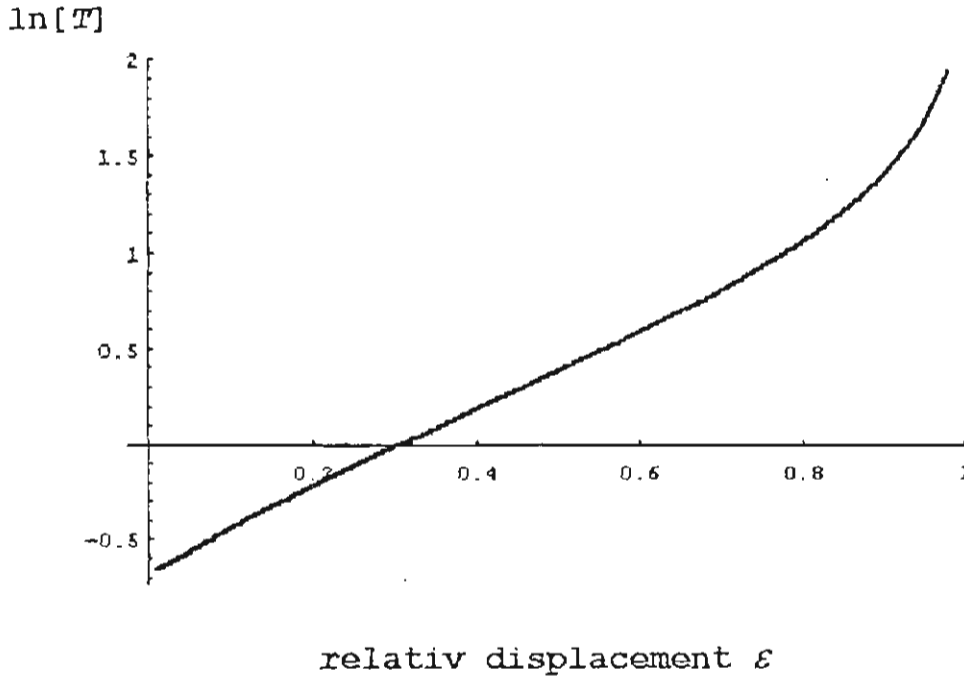


Figure 2.4 Values of factor T

EXAMPLE :

It is necessary to estimate the time Δt required for clearance of hemispherical bearing in hydrostatic transmitter (Fig 2.5) to change from $h_{01}=24 \mu\text{m}$ to $h_{02}=(1/6)h_{01}=4 \mu\text{m}$, if radius of bearing ball $r = 30 \text{ mm}$, clearance modulus $m = 0,1 \%$, and oil viscosity $\eta = 100 \text{ cP} = 0,1 \text{ Ns/m}^2$.

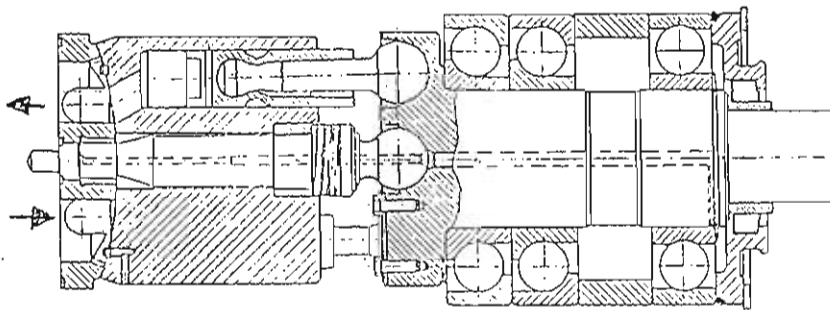


Figure 2.5 Hydrostatic transmitter

The maximum force acting on bearing $F = 30 \text{ kN}$. The duration of this force is one third of power stroke duration and for the transmitter running at $n = 1000 \text{ rpm}$, follows :

$$t_F = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{60}{1000} = 0.01 \text{ sec.}$$

From Eqs.(2.4) the values of relative displacements are :

$$\varepsilon_1 = 1 - \frac{h_{01}}{m \cdot r} = 1 - \frac{0,024}{0,001 \cdot 30} = 0,2$$

$$\varepsilon_2 = 1 - \frac{h_{02}}{m \cdot r} = 1 - \frac{0,004}{0,001 \cdot 30} = 0,86$$

From Fig.2.4 the values of factor T are :

$$\varepsilon_1 = 0,2 \quad [T(\varepsilon_1)] = e^{-0.2} = 0.82$$

$$\varepsilon_2 = 0,86 \quad [T(\varepsilon_2)] = e^{1.3} = 3.67$$

and from Eqs.(2.18), follows :

$$\Delta t = \frac{3 \cdot 0.030^2 \cdot \pi \cdot 0.1}{0.001^2 \cdot 30 \cdot 10^3} \cdot (3.67 - 0.82) \cong 0.081 \text{ sec.}$$

$$\Delta t \cong 81 \text{ msec.}$$

3. CONCLUSIONS

Demonstrated example shows the application of given equations. In this case the time required to produce the oil film of thickness $4 \mu\text{m}$ is about eight times longer then duration of maximum force acting and oil-cushion will prevent ball-bushing metal contact. This artical gives the starting fundations for the quality choosing of the essential parameters just as : lubricant (η), constructional features of bearings (r , m), maximum load (F).

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PRINCIPLES OF CRACK-TIP MIXED MODE CAUSTICS IN MECHANICALLY ANISOTROPIC MATERIALS

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SUMMARY

This paper focuses on the theoretical development of the method of caustics and its application to mechanically anisotropic materials. The presented analysis of the shape of the optical effects for mixed I and II mode crack-opening shows the influence of mechanical characteristics to the shape of caustic curves. The solutions cannot be given in the general form and the numerical simulations of the experiments are necessary for each situation. The experimentation is performed in order to determine the stress intensity factors K_I and K_{II} by evaluation of caustics, automatically recorded by image scanner and digitized in the adapted form for data processing.

1. INTRODUCTION

The experimental mechanics is constituted by numerous number of optical methods employed for determination and evaluation stresses and strains of deformed body. The optical method of caustics is one among these techniques that is very sensitive to stress gradients and changes in surface deformation. The idea of forming caustic curve is first founded for transparent materials in 1964 by Mannog [1, 2], developed for non-transparent materials by Theocaris and his school [3] and improved for dynamic cases of loading by Kalthoff and his school [4]. Relevant information about the method can be found in review type papers [5, 6].

The method of caustics is originally applied to mechanically isotropic materials for stress-intensity factors determination, for the cracks in constructions made of mechanically isotropic materials converged by static-dynamic loading. The process of stress-intensity factors determination is reduced to the epicycloidal single or double dark spot measuring, while the shape of the optical effect is always of the same form [7, 8]. This contribution explains the employment of reflection method of caustics to the non-transparent mechanically anisotropic fibre reinforced materials.

2. THE OPTICAL PRINCIPLES

The basic optical principle of the caustic method will be illustrated on the specimen with the crack, loaded and illuminated by the concentrated field of collected light beams in the zone surrounding the crack-tip. The change of the optical path length of the reflected light is a consequence of nonuniform surface deformation, resulting as the optical effect on the virtual plane positioned at the distance z_0 behind the specimen (fig. 1a). The point $P(x,y)$ will be mapped onto the point $P'(x',y')$ on the screen (fig. 1b) by the basic equation

$$\vec{r}' = m \vec{r} + z_0 h \nabla \varepsilon_z, \quad (1)$$

where m is a scale factor (parallel right beams $m=1$, convergent light $m<1$, divergent light $m>1$) and h is a thickness of the specimen.

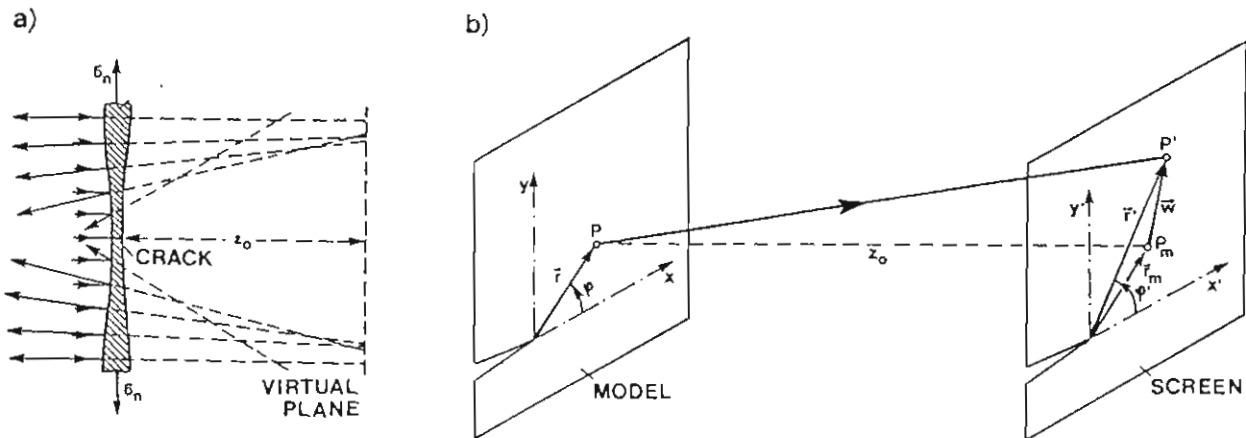


Figure 1. The optical principles of light beams deflections

The light will be concentrated on the edge of a dark spot, where the Jacobian determinant of the mapping equation is equal to zero:

$$J = \frac{\partial x'}{\partial r} \frac{\partial y'}{\partial \varphi} - \frac{\partial x'}{\partial \varphi} \frac{\partial y'}{\partial r} = 0. \quad (2)$$

The points that represent the position of the light beams on the specimen are responsible for generation of the caustic curve on the virtual plane. The size and shape of caustic curve is a container of informations of a penetrated stress raiser, material elastic characteristics and type of applied loading.

3. CRACK IN MECHANICALLY ANISOTROPIC PLATE

One can consider the mixed (I and II) crack opening mode for the body made of mechanically anisotropic material subjected to the external force system. The deformation of the surface in the direction of light beams attack is derived from two-dimensional theory of elasticity in terms of complex stress function [9]:

$$\begin{aligned} \epsilon_z = \frac{K_I}{\sqrt{2\pi r}} \left\{ \operatorname{Re} \left\{ \frac{1}{\mu_1 - \mu_2} \left[\frac{\mu_1}{\sqrt{\cos \varphi + \mu_2 \sin \varphi}} (\mu_2^2 a_{13} + a_{23} - \mu_2 a_{36}) - \right. \right. \right. \\ \left. \left. - \frac{\mu_2}{\sqrt{\cos \varphi + \mu_1 \sin \varphi}} (\mu_1^2 a_{13} + a_{23} - \mu_1 a_{36}) \right] \right\} + \\ + \mu \operatorname{Re} \left\{ \frac{1}{\mu_1 - \mu_2} \left[\frac{1}{\sqrt{\cos \varphi + \mu_2 \sin \varphi}} (\mu_2^2 a_{13} + a_{23} - \mu_1 a_{36}) - \right. \right. \\ \left. \left. - \frac{1}{\sqrt{\cos \varphi + \mu_1 \sin \varphi}} (\mu_1^2 a_{13} + a_{23} - \mu_1 a_{36}) \right] \right\} \right\}, \end{aligned} \quad (3)$$

where μ_i are the solutions of the characteristic equation [10], a_{ij} are the compliance matrix components, μ is a ratio between the stress-intensity factors $\mu = K_{II}/K_I$.

By introducing functions [11]

$$L_I = \operatorname{Re} \{L_I(\varphi, \mu, \mu_i)\}, \quad L_{II} = \operatorname{Re} \{L_{II}(\varphi, \mu, \mu_i)\}, \quad (4)$$

and by using equation (1), the components of light beams position vector will be:

$$x' = mr \cos \varphi + z_0 h \frac{K_I}{\sqrt{2\pi}} r^{-3/2} \left[\left(-\frac{1}{2} L_I \cos \varphi - \frac{\partial L_I}{\partial \varphi} \sin \varphi \right) + \mu \left(-\frac{1}{2} L_{II} \cos \varphi - \frac{\partial L_{II}}{\partial \varphi} \sin \varphi \right) \right], \quad (5)$$

$$y' = mr \sin \varphi + z_0 h \frac{K_I}{\sqrt{2\pi}} r^{-3/2} \left[\left(-\frac{1}{2} L_I \sin \varphi + \frac{\partial L_I}{\partial \varphi} \cos \varphi \right) + \mu \left(-\frac{1}{2} L_{II} \sin \varphi + \frac{\partial L_{II}}{\partial \varphi} \cos \varphi \right) \right].$$

Zeroing of the Jacobian determinant (eq. 2) leads to the function $r_0 = r_0(\varphi, \mu, \mu_i)$ that represents the shape of the initial curve positioned on the specimen surface. The light beams are partially reflected directly from this initial curve and transferred into the singular caustic curve.

Function r_0 is a constant for isotropic materials, e.g. for all kinds of materials the initial curve is a circle. On the other hand, the solutions for r_0 depend on mechanical characteristics of mechanically anisotropic material and the crack position with respect to principal axes of orthotropy. The idea for this solution is presented in 1990 by Jecić [12, 13] and developed

for different types of materials by analyzing their mechanical properties [14, 15, 16]. This solution includes the exact result for the initial caustic curves:

$$r_0 = c \left[-\frac{M_1}{2} \pm \sqrt{\left(\frac{M_1}{2}\right)^2 - M_2} \right]^{2/5}, \quad (6)$$

with

$$\begin{aligned} M_1 &= M_{1I} + \mu M_{1II}, \\ M_2 &= M_{2I} + \mu \left(\frac{1}{2} L_I L_{II} - \frac{1}{2} \frac{\partial^2 L_I}{\partial \varphi^2} L_{II} + 3 \frac{\partial L_I}{\partial \varphi} \frac{\partial L_{II}}{\partial \varphi} - \frac{1}{2} L_I \frac{\partial^2 L_{II}}{\partial \varphi^2} \right), \end{aligned} \quad (7)$$

and

$$\begin{aligned} M_{1I} &= \frac{1}{4} L_I + \frac{\partial^2 L_I}{\partial \varphi^2}, & M_{1II} &= \frac{1}{4} L_{II} + \frac{\partial^2 L_{II}}{\partial \varphi^2}, \\ M_{2I} &= \frac{1}{4} L_I^2 - \frac{1}{2} L_I \frac{\partial^2 L_I}{\partial \varphi^2} + \frac{3}{2} \left(\frac{\partial L_I}{\partial \varphi} \right)^2, & M_{2II} &= \frac{1}{4} L_{II}^2 - \frac{1}{2} L_{II} \frac{\partial^2 L_{II}}{\partial \varphi^2} + \frac{3}{2} \left(\frac{\partial L_{II}}{\partial \varphi} \right)^2. \end{aligned} \quad (8)$$

That is a difference in approach and methods of solving the problem concerning Rossmanith's solution [17].

The coordinates of the singular caustic curve can be expressed in the exact form by inserting the function r_0 into the eq. (5), where the optical constant of caustics c is defined as a factor:

$$c = \left(z_0 h \frac{K_I}{m \sqrt{2 \pi}} \right)^{2/5}. \quad (9)$$

4. SIMULATIONS OF THE OPTICAL EFFECT

The shape of the optical effect extremely depends on mechanical characteristics of material for each material separately, so the effect must be simulated for each situation. The simulations must be carried out by varying the variables r and φ in the eq. (5) for the specific material. The quality of the obtained picture will be as much as good as the number of the light beams-field initial points can be increased. The simulations of singular caustic curves follow directly by the described procedure.

Simulations (figures 2 and 3) in this paper are given for the unit value of the constant c (eq. 9). Material constants are $E_L = 149$ GPa, $E_T = 30$ GPa, $G_{LT} = 15$ GPa, $\nu_{LT} = 0.33$, representative for the unidirectional graphite-fibre reinforced composite.

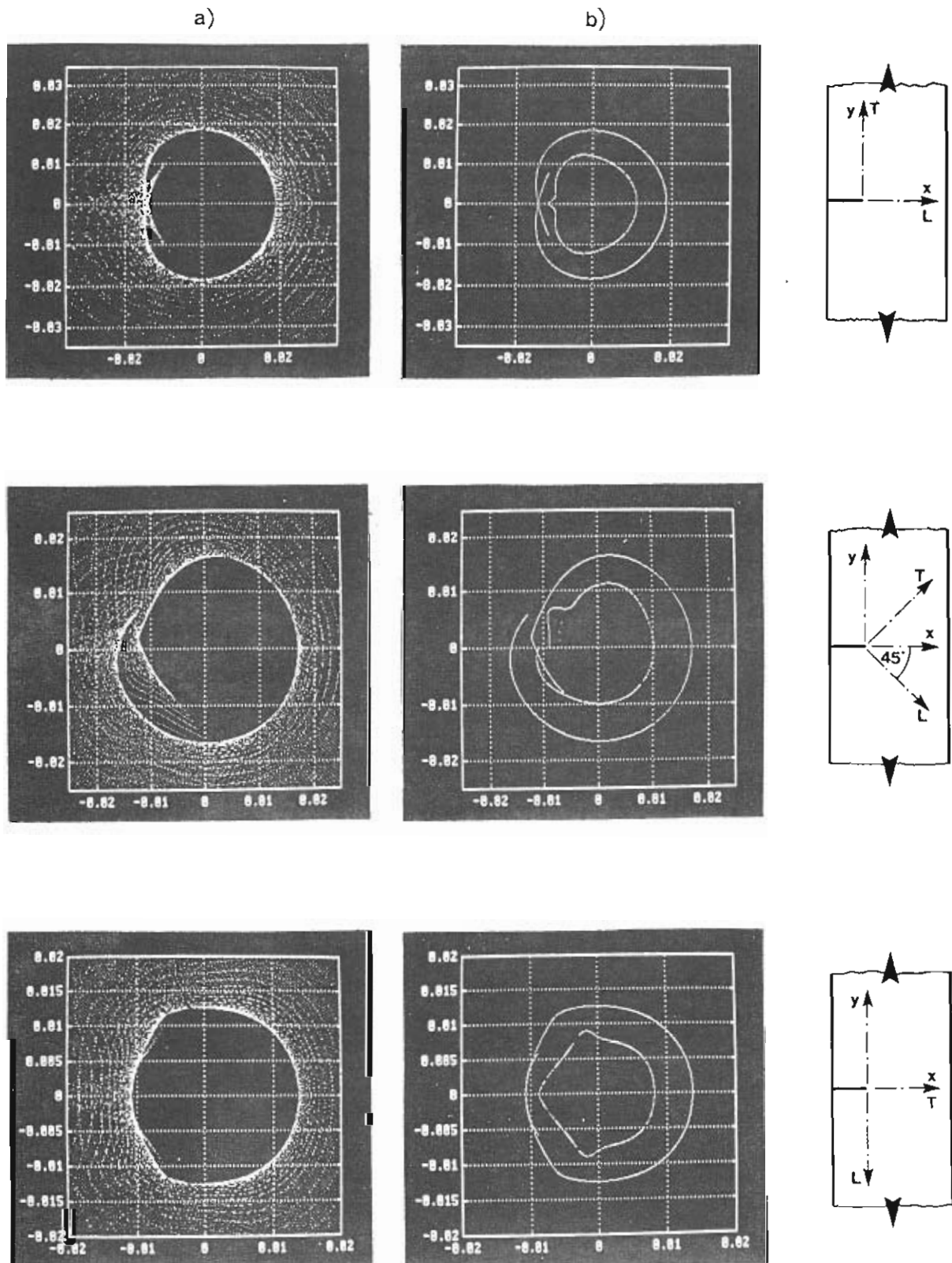


Figure 2. Simulations for the light beams positions (a) and initial and caustic curve (b) for mode I crack opening

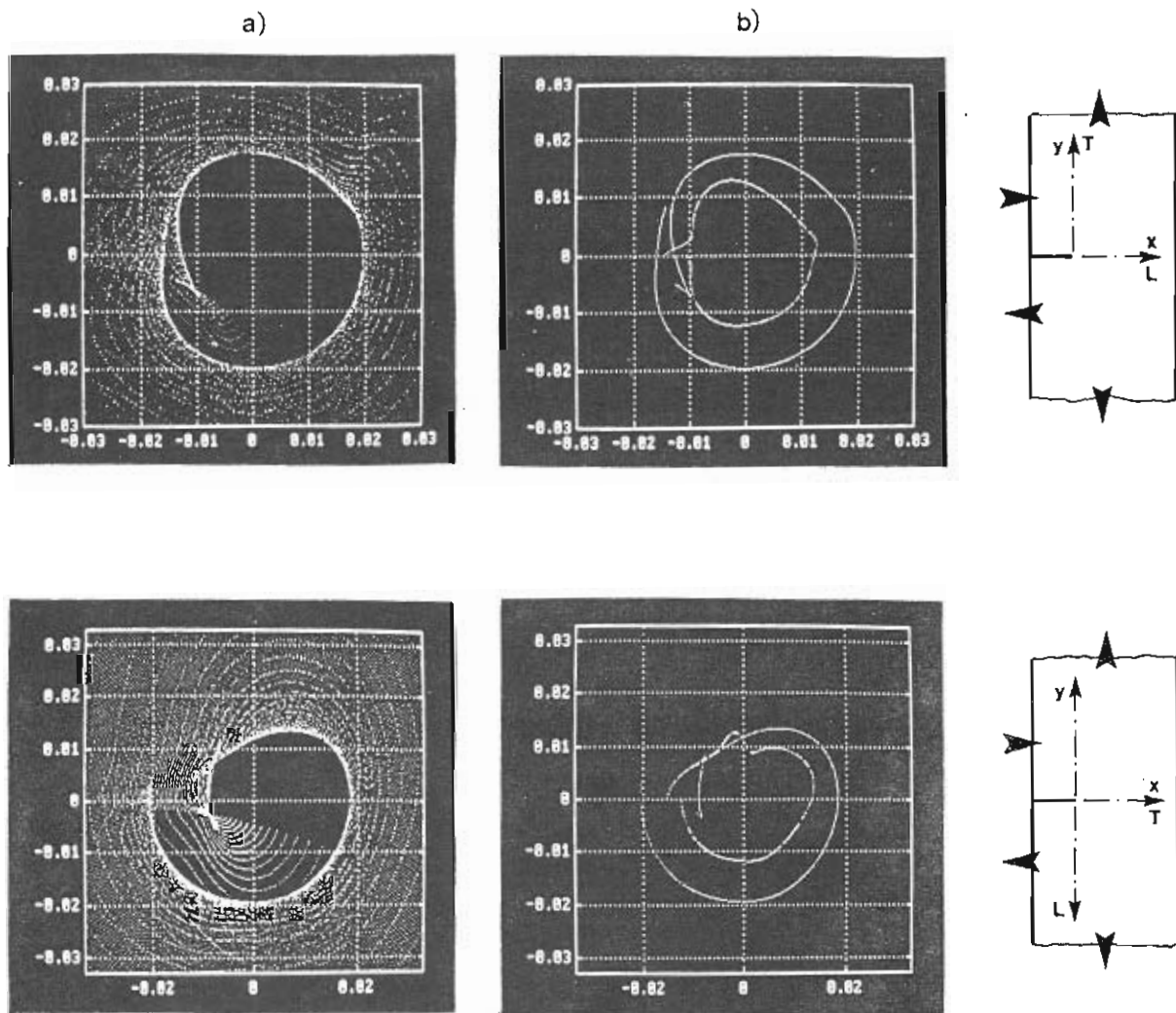


Figure 3. Simulations for the light beams positions (a) and initial and caustic curve (b) for I and II mixed mode crack opening ($\mu = K_{II}/K_I = 1$)

5. K_I AND K_{II} DETERMINATION TECHNIQUE

The determination of stress intensity factors K_I and K_{II} in mixed-mode stress field is of great importance for estimating of the life-time of mechanical part containing a crack. The required equipment for analyzing experimental optical effect consists an image scanner (e.g. a CCD-camera with A/D converter), where the digitized and stored caustic pattern can be displayed on the monitor in the form adapted for interactive selection of certain points on the screen (fig. 4).

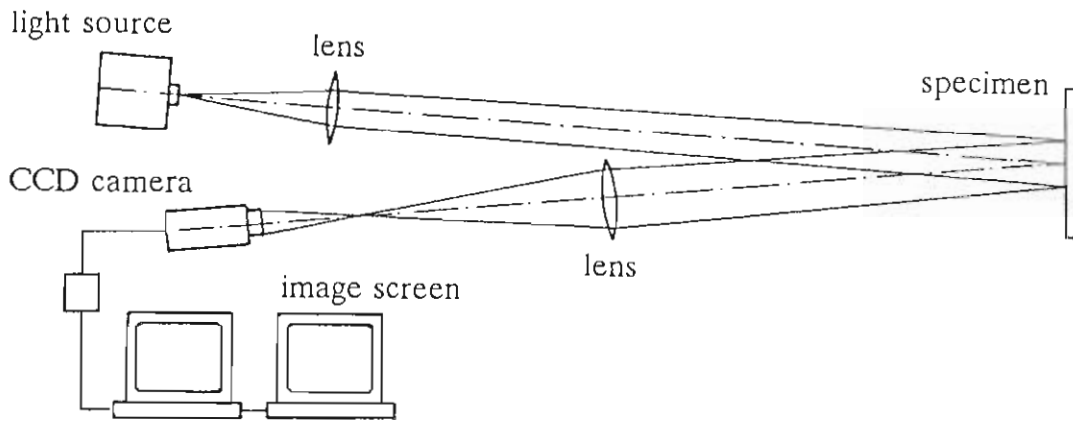


Figure 4. *Experimental setup*

In experimental data analysis it is usually enough to compare one (for pure mode I or II) or two (for mixed-mode) characteristic diameters of the experimental effect (fig. 5a) with the theoretical results numerically obtained for the unit value of constant c . More complicated technique is the multi-point over-deterministic data reduction method [18]. The distances between conjugated experimental data points and the best-fit theoretical caustic is minimized on the basis of a least squares technique. In some cases the distortions of caustics present a serious problem because some parts of experimentally recorded caustics must be discarded (foggy blurred zone, fig. 5b) for data selection. The large number of data points make this method more reliable comparing to the diameters measuring technique.

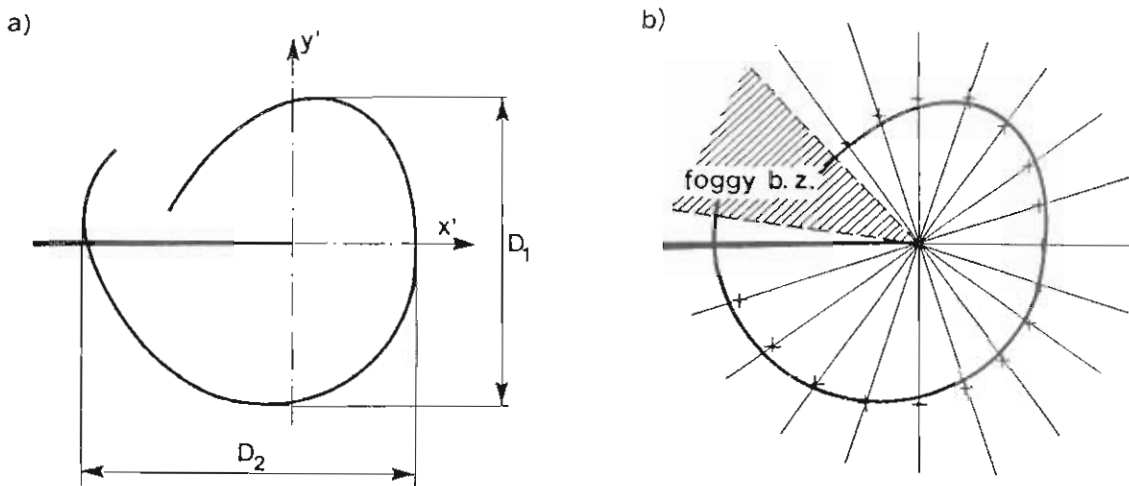


Figure 5. *Measuring of the diameters on the experimental curve (a) and fitting the points with the theoretical solution (b)*

6. CONCLUSION REMARK

The optical method of caustics is one of the most progressive experimental techniques in fracture mechanics in last few years. Its extension to a field of mechanically anisotropic materials is a logical consequence of the application of these materials in modern industry. This contribution shows briefly the principles of theoretical approach, that is confirmed experimentally. The very first results can predict its practical use in presented way in near future.

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APPLICATION OF THERMOELASTIC EFFECT IN EXPERIMENTAL STRESS ANALYSIS

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Summary

Starting from the thermoelasticity presumptions an equation has been set. It connects the sum of principal stresses on the body surface with the exit signal of thermographic equipment (the main equation of thermographic stress analysis). On the basis of a specific expression for parameter K , thermoelastic effect for steel and aluminium as well as photoelastic materials Araldit B and Makrolon has been analysed in order to establish the appropriateness of these materials for the experimental determination for stresses, along with the possibility of introducing hybrid method in which stress pattern analysis by thermal emission would be applied simultaneously with the photoelasticity analysis with the aim to solve the problem of principal stress separation.

1. Introduction

Photoelasticity is the experimental method of determining stress and strain, based on the fact that many transparent materials in the state of stress become optically birefringent. As a result of the photoelastic research, we get the difference between the main stresses on a model surface.

Thermoelastic stress analysis with the infra-red radiometric method is a relatively new technique of the experimental stress analysis of structures under cyclic loading conditions. This technique based on the thermoelastic effect, was described as early as in 1853 by Lord Kelvin. Under adiabatic and isentropic conditions, the temperature changes of homogeneous isotropic linear elastic solid are directly related to the change in material volume, i.e. to change in the sum of principal stresses.

The separation of main stresses in the both methods is a demanding task. Therefore, this particular phenomenon that in photoelasticity we get the difference, while in thermography we get the sum of the main stresses in the analyzed point of a model as a result, leads to a hybrid method for solving the problem of the main stresses separation. This hybrid method would use the results of the both, photoelastic and thermographic, methods.

Employing such a hybrid method requires the defining of the criteria of individual material applicability in the experimental analysis of stress in view of the demanded precision of stress determination and the resolution of thermographic equipment.

2. The Energy of Isotropic Body under Elastic Deformation

According to the first law of thermodynamics which represents the principle of energy conservation, the sum of the increase of the internal energy U and the kinetic energy E_k is equal to the sum of the increase of the external forces work W and the heat energy of the solid Q , or rather

$$dU + dE_k = dW + dQ. \quad (1)$$

With energy changes through time, it follows that

$$\dot{U} + \dot{E}_k = \dot{W} + \dot{Q}, \quad (2)$$

where the points above the symbols represent differentiation by time.

By inserting the adequate amounts into (2), and with the use of Ostrogradski-Gauss's theorem we get the following:

$$\int_V [\dot{u} - (\sigma_{ji,j} + f_i - \rho \dot{v}_i) v_i - \sigma_{ji} v_{i,j} - \dot{q} + \psi_{i,i}] dV = 0, \quad (3)$$

where u stands for: internal heat of the unit volume,

ρ	mass density of the body,
v_i	speed vector in index form,
f_i	volume forces
$\dot{q}$	internal heat rate per unit volume and
ψ_i	heat flow.

This expression must be applicable to each part of the volume of the solid, and therefore also for an infinitely small part. This means that everything that stands in the square brackets can equal zero. Furthermore, the expression in the round brackets also equals zero, because it is the equilibrium equation known from the theory of elasticity. Therefore we get the final expression:

$$\dot{u} - \sigma_{ji} v_{i,j} - \dot{q} + \psi_{i,i} = 0. \quad (4)$$

The development of the expression (4) [1] results in the generalization of Hook's law in the field of thermoelasticity known as Duhamel-Neumann's law,

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda \varepsilon_{kk} - \gamma \Delta T) \delta_{ij}, \quad (5)$$

and expression for entropy (6) is

$$s = \gamma \varepsilon_{kk} + c_s \frac{\Delta T}{T_0}. \quad (6)$$

In these expressions μ and λ are Lamé's elastic constants, δ_{ij} Kronecker's delta, $c_s = cp$ specific heat of the unit volume, while constant γ in which α is the coefficient of linear thermal expansion is:

$$\gamma = \alpha (2\mu + 3\lambda). \quad (7)$$

3. The Equation for Thermoelastic Stress Analysis

The principle on which the SPATE equipment functions is based upon the measuring of thermal radiation, which is a result of the dynamic load put upon the elastic solid. This equipment is made in such a way that its exit signal Σ is in proportion to the change in thermal emission from the surface of the solid $\dot{E}_z$.

Apart from the detector response factor R , that depends upon the characteristics of the thermographic equipment, this proportionality is defined by the following equation

$$\Sigma = R \Delta \dot{E}_z. \quad (8)$$

According to Stefan-Boltzmann's law the energy of temperature radiation emitted in a time unit from the surface of the gray body $[W/m^2]$ equals

$$\dot{E}_z = \epsilon B T^4, \quad (9)$$

where $B = 5,67 \cdot 10^{-8} W/(m^2 K^4)$ is Stefan-Boltzmann constant, and ϵ the surface emittance. If the changes of temperature are small we can assume that surface emittance does not change.

By differentiating the expression (9) we can see that radiant flux changes

$$\Delta \dot{E}_z = 4 \epsilon B T^3 \Delta T. \quad (10)$$

The first stress invariant according to (5) equals

$$\sigma_{kk} = (2\mu + 3\lambda) \epsilon_{kk} - 3\gamma \Delta T, \quad (11)$$

and on account of the equation the first strain invariant is

$$\epsilon_{kk} = \frac{\sigma_{kk} + 3\gamma \Delta T}{2\mu + 3\lambda}. \quad (12)$$

By the cycling loading of the solid with external forces, an isentropic change takes place, so in this case in the expression (6) $s=0$. On this condition and after applying (12), for the temperature change ΔT we have

$$\Delta T = -K \sigma_{kk} = -K (\sigma_1 + \sigma_2 + \sigma_3), \quad (13)$$

and the parameter K equals

$$K = \frac{\gamma T_0}{3 T_0 \gamma^2 + (2\mu + 3\lambda) c_t} = \frac{\alpha T_0}{\frac{3 T_0 \alpha^2 E}{1 - 2\nu} + c_t}, \quad (14)$$

where E and ν are Young's modulus and Poisson's ratio.

For a large number of materials the first denominator on the right side of the expression (14) is significantly smaller than the second, so after its omission we can assume that the thermoelastic constant C is

$$C = \frac{\alpha T_0}{c_t} = \frac{\alpha T_0}{c_p}. \quad (15)$$

In this case the temperature change equals

$$\Delta T = -C \sigma_{kk} = -C (\sigma_1 + \sigma_2 + \sigma_3). \quad (16)$$

It can be distinguished from (13) and (16) that the tensile stress provokes a decrease, and the compressive stress provokes an increase of temperature.

On the free surface of the solid, one of the principal stresses equals to zero (for example $\sigma_3=0$), so with the signal Σ after the insertion of (13) into (10) and into (8) as well, the obtained interdependence obtained is

$$\Sigma = -4 R \epsilon B T^3 K (\sigma_1 + \sigma_2). \quad (17)$$

Due to the fact that $T^3 = (T_0 + \Delta T)^3 \approx T_0^3$, the normal stress sum on the surface, seen as dependent upon the signal Σ is defined by the following expression

$$\sigma_1 + \sigma_2 = A \Sigma. \quad (18)$$

The expression (18) represents the main equation of thermographic stress analysis in which A is the calibration factor that takes into consideration all constants on the right side of the equation (17), so that

$$A = -\frac{1}{4 R \epsilon B K T_0^3}. \quad (19)$$

A reliable experimental or analytical determination of some parameters of the factor A is very complicated. Because of that, in the quantitative use of thermographic stress analysis, the calibration of the factor A is usually undertaken. The size of the factor A determined in such a way gives a constant specific for a certain experiment. If we use some other equipment or change physical parameters that have some influence upon the parameter A , a new calibration is required.

4. Thermography in Experimental Stress Analysis

Stress pattern analysis by thermal emission is a new experimental technique, the theoretic foundation of which is known and solved. However, the technique is still in the phase of development. There are two main advantages of thermographic stress analysis opposed to the majority of classical methods:

- measurement is made without physical contact; therefore the disturbance of the actual state is avoided,
- the thermogram is a permanent registration of the temperature state (and therefore the sum of normal stress as well) and its distribution of the whole visual field of device.

The basic problems that are arising in a practical carrying out of thermographic analysis are the determination of the value of the calibration factor, specific for the type and manner of experiment performance, and the problem of separation of principal stress, because the obtained results of normal stress sum cannot be correspondingly used in engineering practice. Also, so far the experimental research of a plain stress condition has been possible only on the basis of the thermogram that shows the infra-red radiation of surface, so we can not judge about stress in the case of three dimensional stress conditions. This represents the major limitation for this method application in practice.

5. Choice of Model Material

The occurrence of energy radiation concentration on a shorter wavelength with the rise of temperature allows the practical measurement of body temperature by measuring the intensity of radiation. Also, the basic expression of thermographic stress analysis allows us to connect measured temperature field with the first stress invariant.

The required precision of stress determination is connected with the equipment resolution and the choice of model material.

Even though the parameter K in the practical execution of thermographic analysis, is not analytically determined, it enables us to determine the thermoelastic effect for certain materials. If we suppose that the variables that are a part of the parameter K do not change their amounts with the change of temperature, which is again a result of the thermoelastic effect, it is evident that the increase of temperature will be proportional to the value of the parameter K , and that in choosing the type of material for the model, advantage should be given to the material the parameter of which possesses a greater value (the dependency of temperature rise on the amount of cyclic load is shown in the diagram 1.).

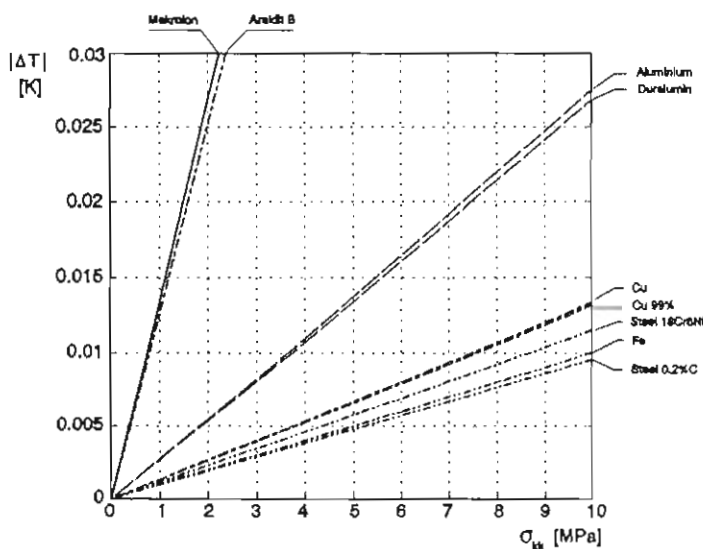


Diagram 1.

On the bases of the factor K expression, the diagrams 2-7 are determined. They show the dependency of the intensity of thermoelastic effect upon the properties of materials as well as the change of certain thermoelastic properties of materials that are included in the description of parameter K (the shaded areas show actual values of specific features for each material). From these diagrams it is obvious that thermoelastic effect is the most prominent in Makrolon and also that Makrolon is the most sensitive of all four compared materials towards the change of some properties. Diagram of dependency $K=K(\nu)$ is interesting, because only for $\nu > 0.4$ we can register the change of thermoelastic effect.

Thermoelastic effect completely disappears with $\nu=0.5$. This is understandable, because $\nu=0.5$ describes an incompressible material. It is obvious that thermoelastic effect, with brought in the hypothesis of small deformations, does not depend upon angular deformations which appear in it but only upon linear, so that the cyclic load that provokes pure shearing stress will not induce change in body temperature.

Diagrams showing the dependency of K upon the thermoelastic properties of solids leads us to the complexity of an analytical determination of this parameter, because of a wide range of material properties mentioned in the literature. This requires that the exact determination of certain thermoelastic properties for each model precedes the carrying out of the experiment. Therefore, the determination of the factor of calibration A which itself contains the parameter K is necessary in the performance of each experiment.

It is already mentioned that with large number of materials, that are used in mechanical engineering, instead of the parameter K we can use the thermoelastic constant C in order to display the intensity of thermoelastic effect. As the diagrams for aluminium 8-13 show, it is possible to substitute C for K except in the case of $\nu > 0.4$. For this case a simplification leads to the error that can not be neglected. This error is the result of the fact that the expression for the thermoelastic constant C does not depend upon ν .

Besides the intensity of thermoelastic effect described by the parameter K , for a more complete evaluation of materials applicable in thermography, we need to know one more factor. In spite of the fact that they possess a very high parameter K , certain materials are not suitable for thermographic stress analysis, because, due to the low strength, they are applicable only for lower loads. The figure that shows us the maximum possible temperature change on a cyclically loaded model is the parameter K multiplied by the limit of proportionality R_p . The multiplication of K by R_p is the second thermoelastic sensitivity indicator that takes into account the limitation of permitted loading. The material is also as much better as this multiplication is greater, supposing that with a cyclic loading appliance we can produce the amount of force with which we can load a model to the proportional limit.

In the table 1. the thermo-mechanic properties of certain materials applicable in thermographic are presented.

Table 1.

MATERIAL	α [1/K]	E [Pa]	ν	ρ [kg/m ³]	c [J/kgK]	K [K/°Pa]	E/ f_σ [1/cm]	σ_p/f_σ [1/cm]	σ_p [Pa]	K· R_p [K]
MAKROLON	65·10 ⁻⁶	2.44·10 ⁹	0.37	1200	1172	13.215·10 ⁻⁹	3430	54.3	38·10 ⁶	0.5022
ARALDIT B	65·10 ⁻⁶	3.53·10 ⁹	0.37	1250	1172	12.567·10 ⁻⁹	3200	38	40·10 ⁶	0.5027
ALUMINIUM	23.7·10 ⁻⁶	70·10 ⁹	0.34	2700	896	2.7489·10 ⁻⁹	-	-	75·10 ⁶	0.2062
STEEL 0.2% C	12.0·10 ⁻⁶	210·10 ⁹	0.26	7850	461	0.9569·10 ⁻⁹	-	-	210·10 ⁶	0.2009

Apart from the most prominent thermoelastic effect, Makrolon is characterized by very good photoelastic properties. It shows high optical sensitivity, linearity and slight optical and mechanical creep. It is not liable to edge effect. On a model of the same thickness and under the same load, Makrolon shows approximately 50% greater number of isochromatics than Araldit B, otherwise commonly used in photelasticity. This combination of properties would make Makrolon suitable for a hybrid method which would simultaneously make a thermographic and photoelastic stress analysis. The main expressions that connect stresses and characteristic signal are:

$$\text{-in stress pattern analysis by thermal emission} \quad \sigma_1 + \sigma_2 = A\Sigma$$

$$\text{-in plane photoelasticity} \quad \sigma_1 - \sigma_2 = \frac{Nf_\sigma}{h}$$

On the right side of the both expressions there is the value that describes a received signal on the observed spot of the model, multiplied by a constant containing parameters characteristic of the experiment itself as well as of used devices. On the left side in the first case there is a sum, and in the second there is a difference of principle stresses on the analyzed spot of the model. It is the analogy between the given expressions that opens the possibility of solving the problem of main stresses separation in the mentioned methods by using the results of the both methods in the case of the equivalent load of the same model.

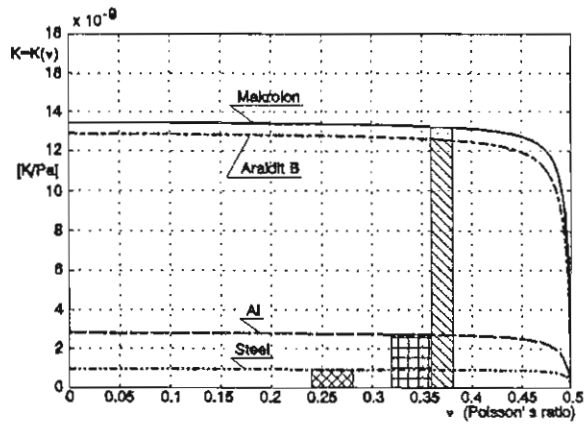


Diagram 2.

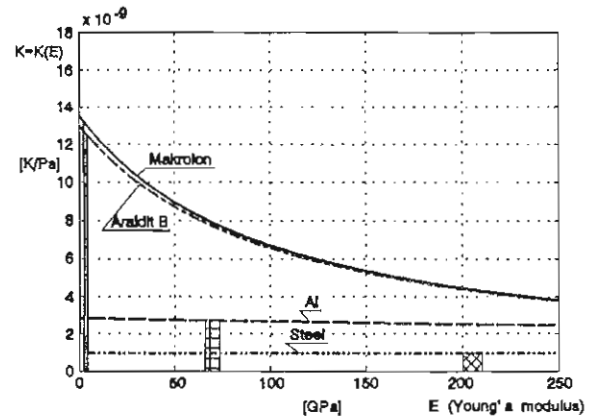


Diagram 3.

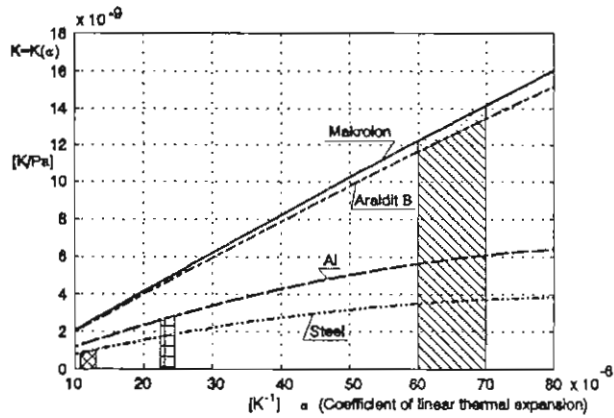


Diagram 4.

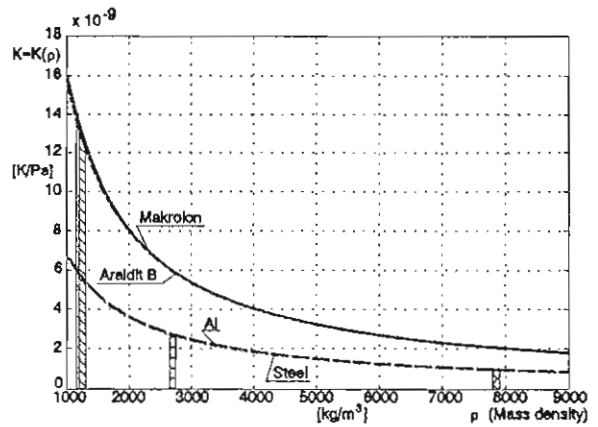


Diagram 5.

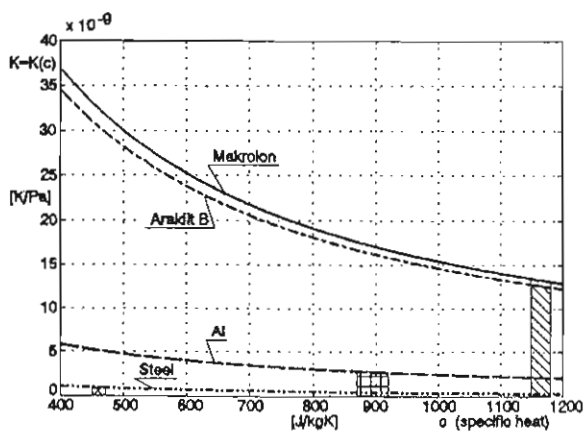


Diagram 6.

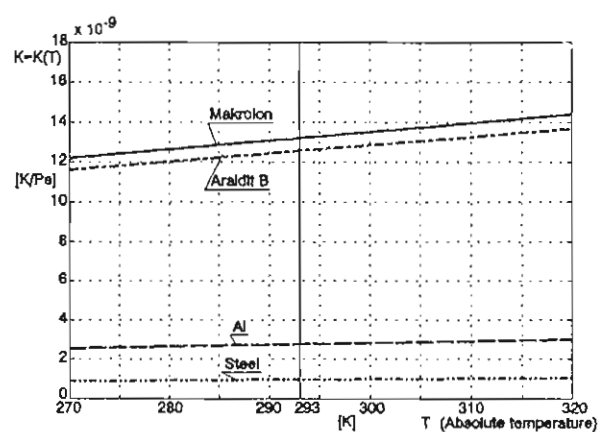


Diagram 7.

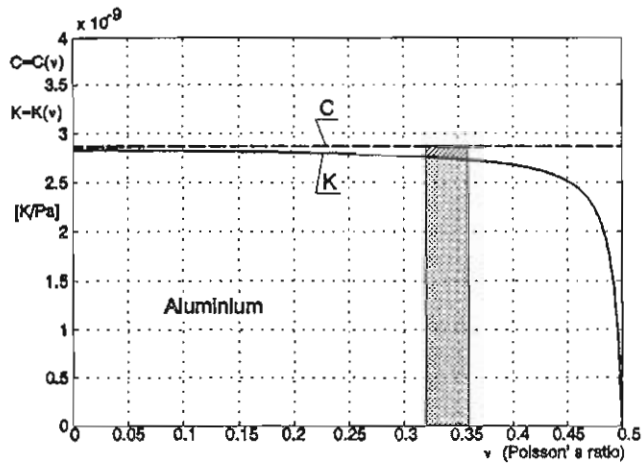


Diagram 8.

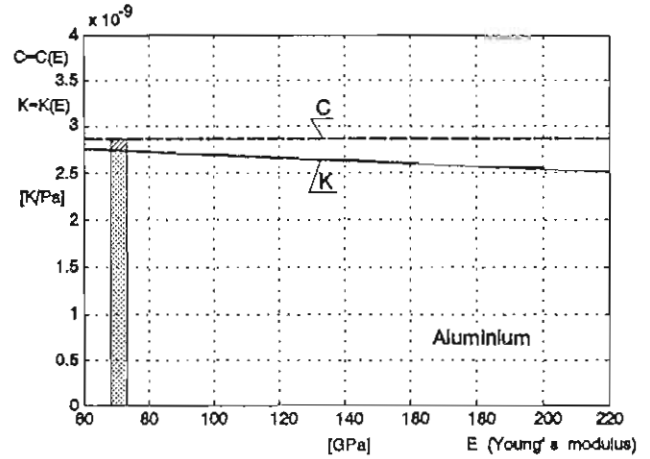


Diagram 9.

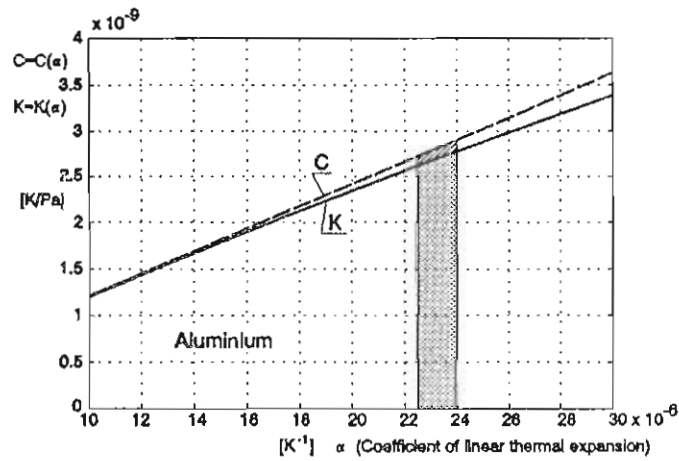


Diagram 10.

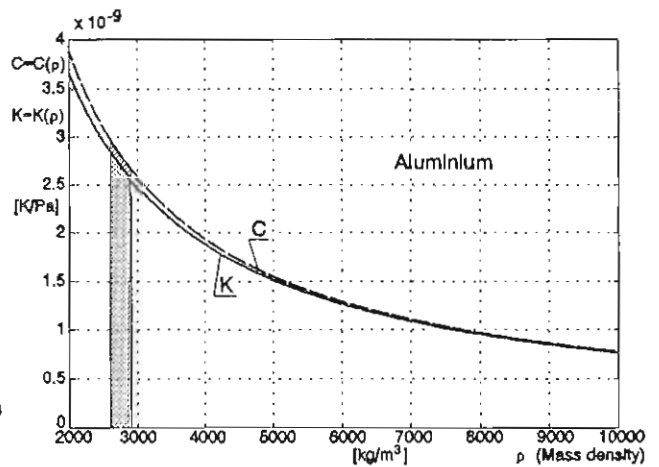


Diagram 11.

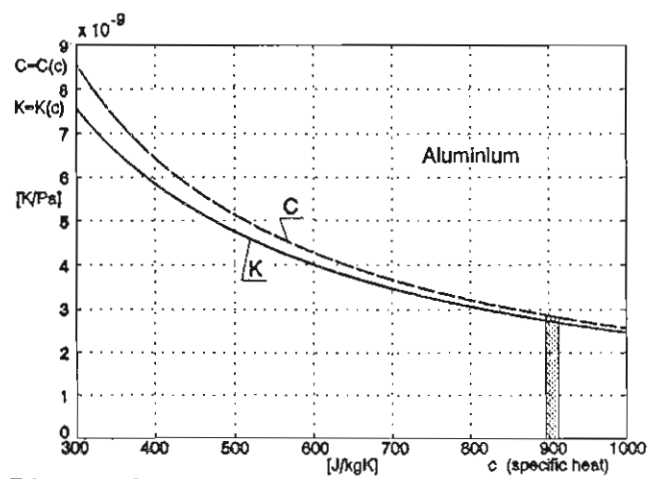


Diagram 12.

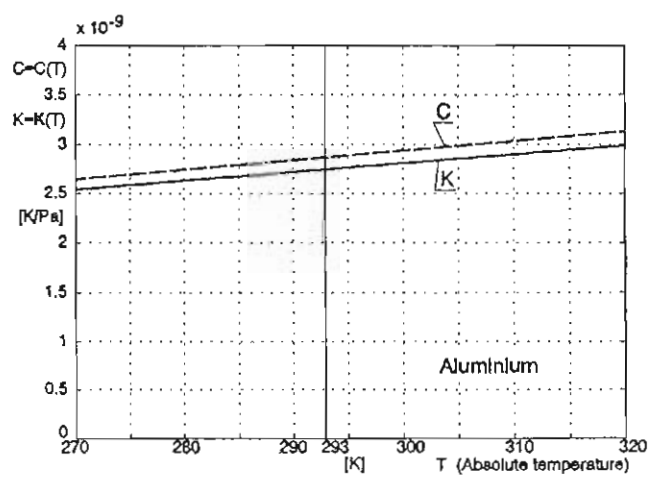


Diagram 13.

6. Conclusion

The main equation of thermographic stress analysis connects temperature distribution on the surface of a cyclic loaded body with the first invariant of stress tensor. It is the analogy of this equation with the basic statement of plane photoelasticity in which the main stresses difference appears, that opens a possibility of the main stresses separation by the simultaneous application of thermography and photoelasticity on an equivalently loaded model.

Model material used in such a hybrid method should be transparent and it should manifest as great thermo and photoelastic effect as possible. On the basis of the thermoelastic parameter K which is linearly proportional to the temperature change in thermoelastic effect, as well as on the phenomenon that Makrolon shows $\approx 50\%$ higher number of isochromatics than otherwise commonly used Araldit B, Makrolon distinguishes itself as the most appropriate model material in such a hybrid method. However, the facts that the maximum temperature change of Araldit B and Makrolon during the thermographic analysis are almost the same, and that the price of Makrolon is about 40% higher than the price of Araldit B, are the main disadvantages of this material.

On the other hand, the stress analysis of steel materials requires the equipment of the highest temperature resolution. This will surely represent a problem in the analysis of real constructions mainly made of steel.

On account of the given insight into the application of thermoelastics in photoelasticity further improvements of thermographic equipment and devices which will connect the results provided by photoelasticity and thermography are expected. Also, the criteria of the evaluation of the materials applicable in this hybrid method are to be established.

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MECHANICAL CHARACTERIZATION OF HIGH PRECISION MANIPULATORS BASED ON FLEXIBLE ELEMENTS

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ABSTRACT: The study of theoretical and experimental aspects of the mechanical design of high precision manipulators based on spring-strips is presented. The meaningfulness of the approximations introduced by the main analytical methods of characterization of these manipulators is established experimentally using laser interferometry. This measurement technique is particularly suitable when accurate measurements must be performed since it is a non-contact technique, it has a nanometric resolution and it permits to perform measurements characterized by small intervals of uncertainty.

1. INTRODUCTION

Precision engineering has recently gained increasing importance [1]. This is mainly related to the strong demand of miniaturization coming from fields such as integrated circuit technology, hard disk drive production, innovative optics, microsensors, microactuators, etc. [2] Since the technological processes involved in these fields make it necessary to perform high precision movements, a growing attention has been dedicated to the mechanical design of new kinds of manipulators characterized by accuracies in the nanometric scale [3].

In designing high precision devices two approaches are possible: either traditional sliding and rolling elements or mechanical devices based on flexible elements can be used. In the first case high accuracies can be obtained only when complex adaptive control procedures are employed [4], [5]. In fact, the presence of Coulomb friction and stiction causes non-zero steady state error or the mechanical systems' response to go into a limit cycle [6]. Mechanisms based on flexible elements must be therefore preferred in several applications since they are free from backlash, friction and wear [7], [8]. These devices were used for years in scientific instrumentation [9], [10] as well as in aeronautical [11], [12] and measuring equipment [13], [14]. However, in high precision applications the analytical modelling of the mechanical behaviour of manipulators based on flexible elements has to comprise the evaluation of several non-linear effects whose significance can be established only following an accurate experimental approach. This was made possible only recently with the availability of laser interferometric measurement techniques.

The aim of this work is to summarize the theoretical and experimental aspects that must be considered in order to optimize manipulators' design depending on the required level of accuracy. The main analytical methods for the characterization of manipulators based on leaf springs will be therefore shortly presented. The principle of operation of Michelson-type optical heterodyne interferometry will then be described. As an example of application of the described methods, the case of a parallel spring translator and that of a flexural pivot will be illustrated.

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2. THEORETICAL CONSIDERATIONS

In the characterization of high precision manipulators it is particularly important to evaluate the influence of second-order "parasitic" deflections on the primary degree of freedom. In order to perform this evaluation a non-linear approach is usually adopted.

In the case of manipulators based on leaf springs the "parasitic" deflections are mainly induced by geometric non-linearities. In fact, following the mounting procedures given in [15] and using thin and narrow spring-strips [13], the magnitude of non-linearities caused by mounting inaccuracies, by the compliance of the constraints and by the anticlastic curvature can be generally made smaller than the interval of uncertainty of the measures [16]. The study of these manipulators can be therefore reduced to the problem of geometric non-linearities of thin cantilever beams that are loaded with torques and with forces of various directions (Fig. 1) [14], [17].

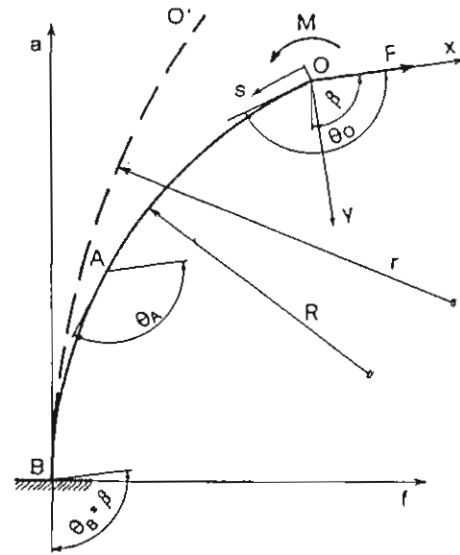


Fig. 1: Cantilever beam loaded at the free end

The analytical solution of the problem of large deflections of a straight cantilever beam loaded at the free end with merely an axial force is known in literature as the Elastica method (EM) [18]. Recently it was proven that this procedure can be extended also to the general case of loading represented on Fig. 1 [19].

However, the Elastica approach is rather complex and time consuming since in this case an iterative procedure including the calculation of elliptic integrals must be performed [19]. For this reason several authors ([17], [18], [20], [21]) use the usual approximated expression for the curvature of the beam to calculate the "parasitic" deflections of manipulators based on spring-strips. In this "approximated curvature" method (ACM) the influence of the axial load on the flexural behaviour of the beam is taken into account but, due to the linearization of the expression for the curvature, the method introduces errors that increase as the axial load approaches the critical (buckling) load [18] limiting in this way the field of its applicability.

Manipulators based on leaf springs can be also modelled as rigid pinned frames with legs of length equal to that of the spring-strips [22]. This "geometric" model (GM) permits a first estimation of the behaviour of the analysed mechanisms but, despite its great simplicity, introduces a significant error even in the case of small deflections and limited axial loads. In fact, in this model the actual deflection of the beam is not taken into account and the influence of the axial load is not considered [16].

3. LASER INTERFEROMETRY FOR DISPLACEMENT MEASUREMENTS

Up until recently the experimental techniques employed for the evaluation of the mechanical behaviour of manipulators based on spring-strips were rather vague and permitted to achieve only limited degrees of accuracy [23], [24], [25], [26]. In the last two decades, however, the availability of laser sources characterized by spectral purity and

temporal and spatial coherence has permitted to make use of an optical approach to displacement measurements. This approach is usually based on triangulation techniques, time of flight techniques and interferometry.

One of the major advantages of laser interferometry with respect to the other techniques is that the accuracy of the measures is independent on the measurement range. The only limitation of this technique is the sensitivity of the interferometric signal to fluctuations of the laser source and to the quality of the target. To overcome this problem heterodyne techniques can be used [27].

At present Michelson-type optical heterodyne interferometer is the most widely used approach to microdisplacement measurements [3]. This is a Doppler laser system whose resolution is increased following an optical scheme similar to the Michelson interferometer in order to reveal the variation of frequency with the heterodyne technique, i.e. from the interference of two laser beams of slightly different known frequencies (Fig. 2). The two beams are emitted from the same laser head, but they have opposite linear perpendicular polarizations so that, with proper optical elements, it is possible to split them. If the beams are directed along two different optical paths and then made to interfere, the measure of the variation of the difference of beams' frequencies gives a signal proportional to the displacement that is being measured. This measurement technique permits to achieve resolutions of down to 2.5 nm [3].

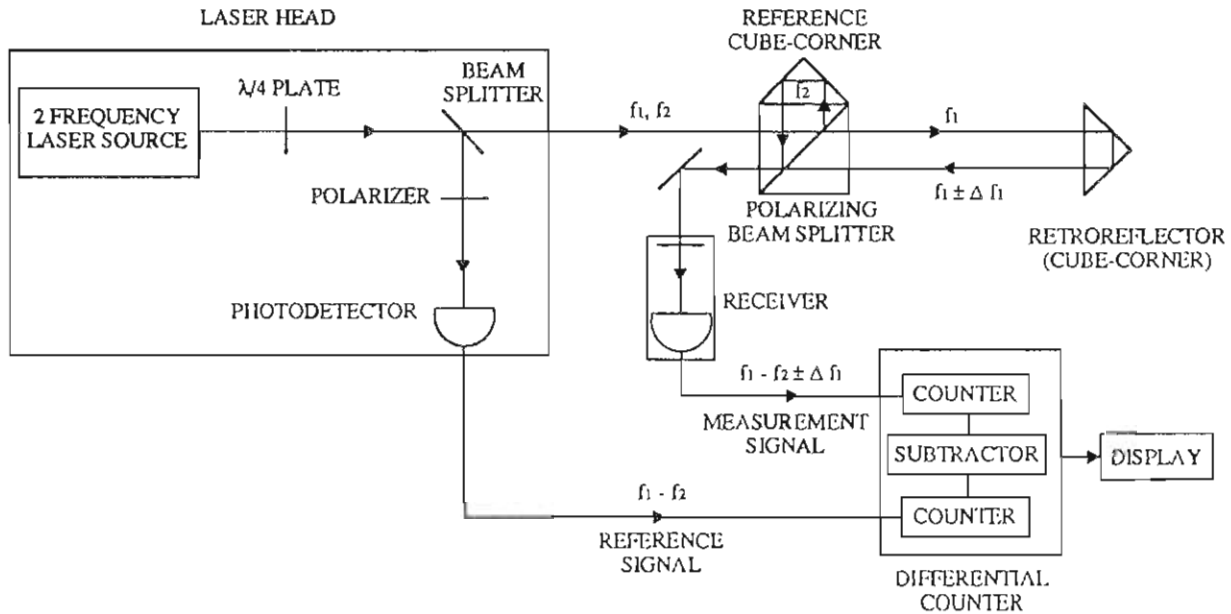


Fig. 2: Michelson-type optical heterodyne interferometry

In the study of high precision manipulators it is particularly important to evaluate the most significant sources of uncertainty of experimental data. In Michelson-type optical heterodyne interferometry the interval of uncertainty of the measurements is determined by several systematic and accidental errors whose effect was thoroughly studied in [3], [28] and [29].

The misalignment of the laser beam path to the axis of motion of the manipulator is very often the main source of systematic errors. This error can be considerably diminished following an iterative approach based on the evaluation of the laser beam slope angle that minimizes the error function which can be evaluated experimentally.

Accidental errors include atmospheric compensation errors and deadpath errors. These errors are determined by the change of the refractive index of air and can be minimized carrying out the experiments in short periods of time

(the change of the environmental characteristics is small) or monitoring the changes of temperature, pressure and humidity and compensating for them.

The Michelson-type optical heterodyne measurement technique can be used to develop different optical layouts and thus different kinds of measurements can be performed (linear, angular, straightness, single or multiple axes, etc.). Two examples of the application of this technique in the case of manipulators based on spring-strips will be illustrated in the following paragraphs.

4. PARALLEL SPRING TRANSLATORS

Parallel spring translators (Fig. 3) are mechanical devices in which the employment of spring-strips allows to obtain very accurate translations. However, the travel f , that constitutes the main degree of freedom, is accompanied by a vertical "parasitic" displacement a . It can be shown that in this case the problem is analytically equivalent to the study of large deflections of cantilever beams of length $L/2$ loaded at the free end with a variably directed force [14].

In order to evaluate experimentally the "parasitic" deflection a the approach described in [22] employing Michelson-type interferometry can be followed.

While the travel f can be measured easily with a linear interferometer, the corresponding vertical displacements a must be measured with a straightness measurement optics.

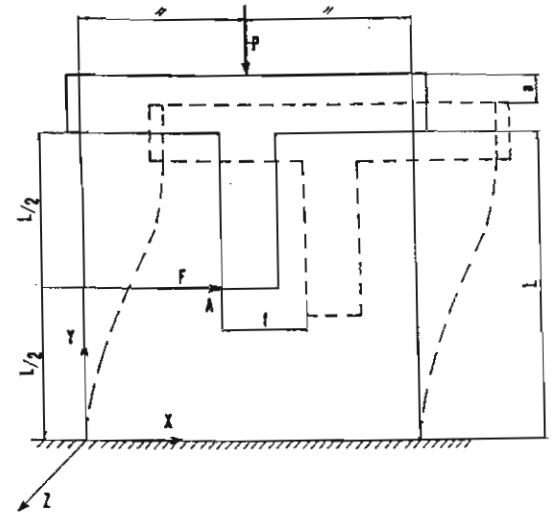


Fig. 3: Parallel spring translator

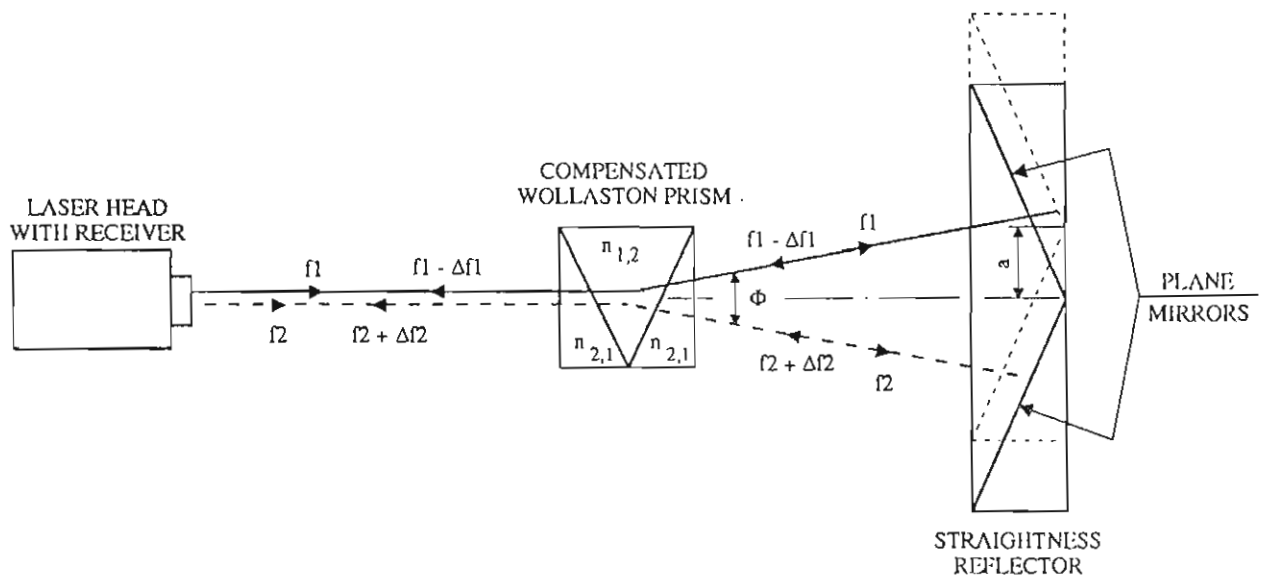


Fig. 4: Straightness measurement principle

The measurement principle based on an interferometer with a Wollaston prism is shown on Fig. 4. The prism that is mounted on the translator has a different refractive index for each of the two polarized components of the laser beam so that it deviates these two components by two equal and opposite angles towards a fixed retroreflector. The

reflector contains two plane mirrors which reflect the beam components back along their respective paths. The difference in optical paths of the two beam components will be proportional to the displacements perpendicular to the beam and the value of a will be consequently obtained simply by multiplying the interferometric measurement by $1/[2\sin(\Phi/2)]$.

The greatest error in the measurement of the "parasitic" displacement a was the systematic error caused by the misalignment of the laser beam to the axis of travel of the parallel spring translator. To minimize this error an iterative procedure employing an autocollimator has been used. The laser head was positioned on an adjustable mount and a plane mirror was mounted on it. In accordance with the symmetry of the problem, the inclination of the laser head has then been monitored with the autocollimator and regulated until the vertical displacements of the translator were found to be equal in positions $\pm f$.

The residual error of misalignment is the major source of the interval of uncertainty represented by the error bars on Fig. 5. From the given graph it is evident that a level of uncertainty lower than $\pm 10^{-4}$ had to be achieved in order to emphasise the influence of the non-linear term in the expression for the curvature. In fact, while GM introduces a systematic error of the order of 20%, ACM gives results that fit well with those obtained with EM and with the measurements in the case of small travels, but introduces significant errors in the case of long travels [16].

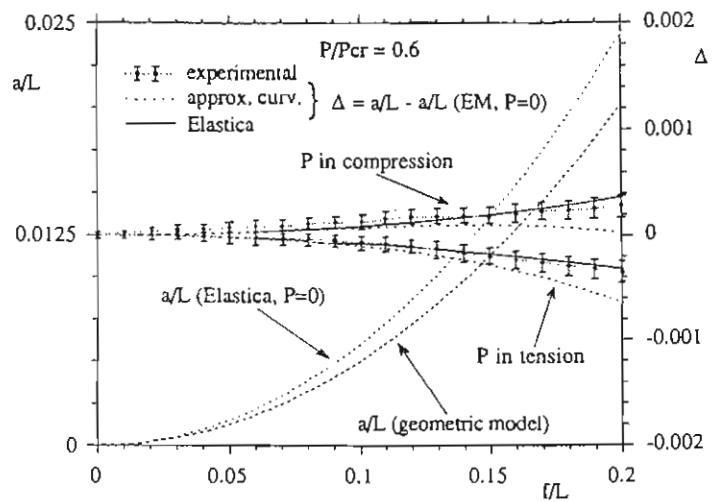


Fig. 5: a/L versus f/L

5. FLEXURAL PIVOTS

Flexural pivots (Fig. 6) are devices that are characterized by a high stiffness along translation coordinates x and y , and an elevated compliance along the rotational coordinate ϑ . A customary constructive solution is obtained using two spring-strips of equal dimensions that cross at an angle 2α .

In this case the main "parasitic" deflection is constituted by the translation $\overline{OO'}$ by which the geometric centre of the pivot O , defined as the point of intersection of the tangents to the elastic line of the spring-strips at their movable joints A_1 and A_2 , moves to position O' . This phenomenon can be characterized analysing the variation of the amplitude d and the phase angle φ of $\overline{OO'}$ as function of rotation ϑ . In this case the problem is reduced to the study of the elastic line of a spring-strip fixed at one end and loaded at the other with torques and with forces of various directions [17].

The interferometric measurement technique that can be used in this case is described in [30] and shown schematically on Fig. 7. While in the case of parallel spring translators the measure of the travel f is less sensitive to errors than that of the "parasitic" motion a [31], in the case of flexural pivots the displacement of a single point must be followed in a two-dimensional space and both coordinates of the displacement have to be established with the same

degree of accuracy. In view of that, the property of the "cube-corner" that it always reflects laser beams parallel to the incoming beams was used. The "cube-corner" allows to measure only the component of the displacement of its optical center in the direction of the beam independently from its rotation. Using two laser beams makes it thus possible to obtain two equal-accuracy components of this displacement and therefore to follow the position of a point in a two-dimensional space. In accordance with this principle, two laser beams that intersect at a precisely calibrated angle γ and a "cube-corner" whose optical centre K coincides with the geometric centre O of the pivot were used to measure very accurately the travel of O in O' and thus to obtain the dependence of d and φ on the rotation angle ϑ [30].

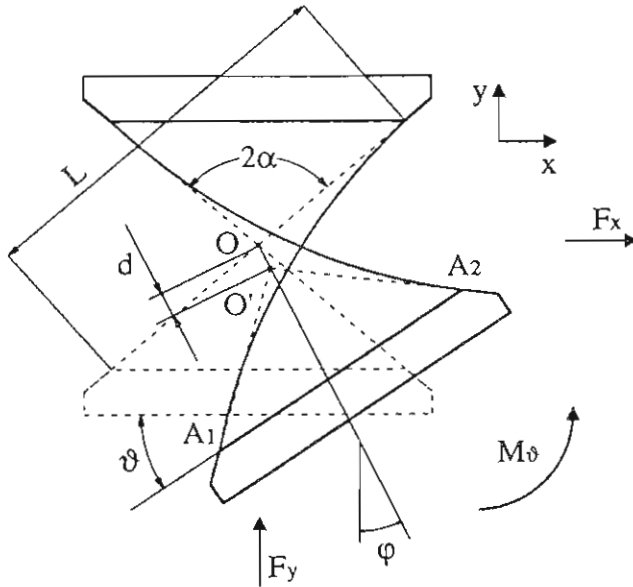


Fig. 6: Flexural pivot

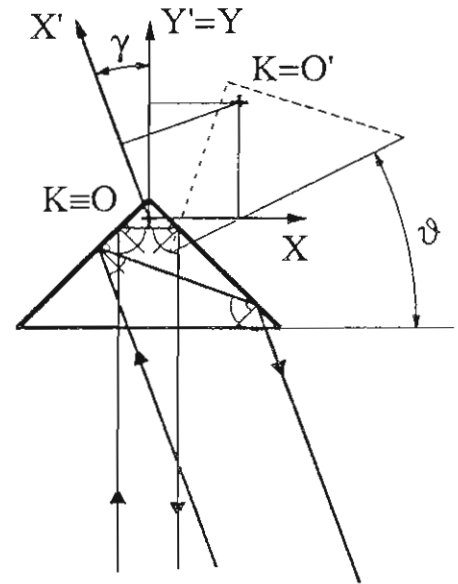
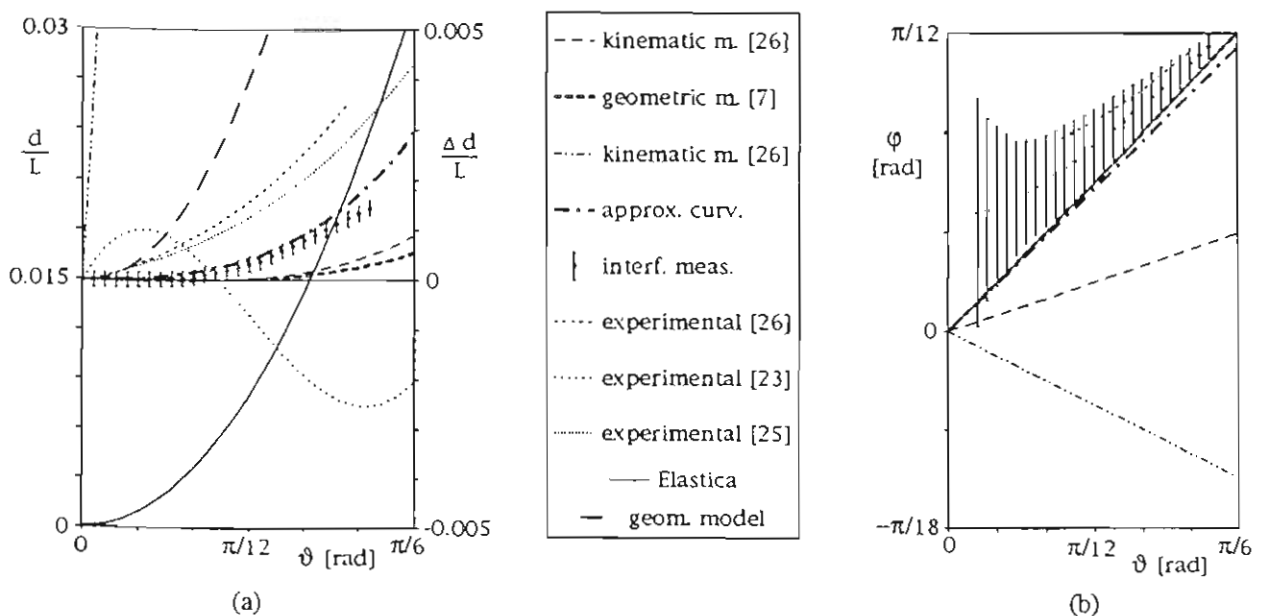


Fig. 7: Principle of measurement

On Fig. 8 are given the results obtained with the various analytical and experimental methods. The experimental results given in [23], [25] and [26] are characterized by an unacceptable dispersion caused by the low accuracy of the employed techniques. Only the interferometric measurements permit to emphasise the limitations of the analytical methods proposed in literature.

Fig. 8: Dependence of the amplitude (a) and the phase angle (b) of $\overline{OO'}$ as function of ϑ ($F_y = -12.5$ N)

The results obtained with GM are fairly good when the phase angle ϕ is considered (the data coincide with that obtained with EM) but introduce increasing errors in the calculation of the amplitude d/L of the "parasitic" deflection $\overline{OO'}$. In fact, the influence of the axial loads and that of the deflection of the beam is significant even for small deflections. The approximated expressions given in literature for the calculation of $\overline{OO'}$ and obtained using geometric [7] or kinematic [26] graphical-analytical considerations do not permit to obtain a satisfactory solution either.

In the case of flexural pivots EM is applicable easily only when the mechanism is loaded merely with a torque M_0 because in this case the symmetry of the problem can be used to determine the reactions at the spring-strips' fixed ends [17].

In the general case in which lateral loads F_x and F_y are present, a close-form solution based on ACM can be developed expanding the considerations given in [12] for the calculation of the stiffness of flexural pivots [30]. This approach permits to obtain the closest results to the actual behaviour of the pivot.

6. CONCLUSIONS

Laser interferometry permits to evaluate easily the significance of the various non-linear effects on the mechanical behaviour of high precision manipulators. In this way the design of such devices can be optimized choosing the most suitable analytical model depending on the required degree of accuracy.

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DETERMINATION OF INTRAABDOMINAL PRESSURE USING HARMONICAL METHOD OF HUMAN FIGURE DESIGN

Prof. dr Osman MUFTIĆ

SUMMARY

Many kinds of ergonomical or biomechanical analysis usually start with the formation of the biomechanical model of the human body.

That way a simple method is developed of the human body figure drawing and determining of the value of the abdominal cross section. Definitions of the human body segments, as well as dimensions of the cross section of abdomen, then the centres of gravity are determined by the using the harmonic circle and the canon of the eight head length.

The paper deals with the simple method for estimating of abdominal resultant force for males as well for females depending on their age and individual distribution of the antropomeasures.

Recorded forces are very good tool for future study of modelling the biomechanical models in work analysis.

Results stipulated by this method are compared with the usual standard measures, giving very good correspondence.

S A Ž E T A K

Brojne ergonomske i biomehničke analize obično započinju s oblikovanjem odgovarajućeg biomehničkog modela subjekta koji se proučava.

To je istodobno bilo razlogom da se razvije jednostavna metoda za crtanje ljudske figure kao i za određivanje dimenzija poprečnog presjeka trbušne šupljine. Definicije su segmenata ljudskog tijela kao i odgovarajuće segmentalne mase utvrđene putem harmonijske kružnice i kanona osam visina glave.

U radu je razvijena jednostavna metoda za utvrđivanje veličine rezultirajuće intraabdominalne sile za muškarce i za žene a u zavisnosti od njihove dobi i individualne razdiobe njihovih antropomjera

Utvrđene su sile veoma dobar alat za daljnje studije oblikovanja tzv. biomehničkih modela u analizi ljudskog rada.

Rezultati što su utvrđeni ovom metodom uspoređeni su s rezultatima koji se određuju standardnim putem pri čemu su pokazali vrlo dobru podudarnost.

1. INTRODUCTION

Precise analysis of the human figure partially in motion in space is a very complex task. The reasons for this complexity are numerous and particularly important in the formation of biomechanical models. This fact incited us to the elaboration of this work.

In the existing foreign and domestic literature available, there are lots of data on parameters of the human body and data from works of Grandjean (1980), Bernstein (1974), Kaleps and collaborators (1984), Donskij and Zatsiorskij (1979) and in our country Božičević J. (1986), Keros (1977), Rudan (1978) and Muftić (1984, 1988) are particularly distinguished. In the mentioned sources, values of so called static anthropomeasures are predominantly pointed out. Linear dimensions of body parts, namely lengths of the body and its parts, are in them as well. Besides, data on the extent of some typical motions of body parts also appear in this literature and are taken as kinematic anthropomeasures.

Mentioned results are usually presented on charts accompanied with drawings provided with signs to which individual figures relate. The charts are provided either with mean or limiting values presenting the highest and the lowest values or are distributed into percentile groups (usually three or five groups). Disregarding the clarity, the use of these results is relatively unsuccessful when one wants to include the figure of the acting subject into the drawn-in work place to serve for ergonomical analysis.

The above difficulty has motivated this work with clear goal to establish a method of drawing those characteristic sizes of human body which could determine its special configuration together with the characteristic point of its parts which are necessary for the intraabdominal pressure or force determination.

Several investigations have shown that the stress imposed on the trunk in various handling tasks induce a pressure rise in the abdominal cavity (Morris, Davis, Stubbs, Troup, Muftić).

This pressure acting upwards on the diaphragm, downwards on the pelvic floor, frontally on the stomach muscles and on the back side on the spine and lumbar muscles. Also this pressure relieving some part of the stress applied to the spine. According to Morris this pneumatic mechanism could sustain about 30% of calculate compressive force

In our opinion the projective area of diaphragm and the projective area of the pelvic floor are of the same value, which should depend on statistical anthropometrical distribution as for males as well for females

In a such case the value of resultant intraabdominal force instead of pressure could be sometimes more practical and even precise in the biomechanical analysis.

In that direction in the present work we tried to determine relations between average cross-section of the abdomen in the function of human height.

2. METHOD AND MATERIALS

To reach the given goal we have availed ourselves to the literature proposal (6) that the analysis of anthropomeasures is done by help of eight head length canon adds to the special geometrical construction so called harmonical circle. Using this idea we draw the net on Figure 1., that makes the limits of normal man's contour.

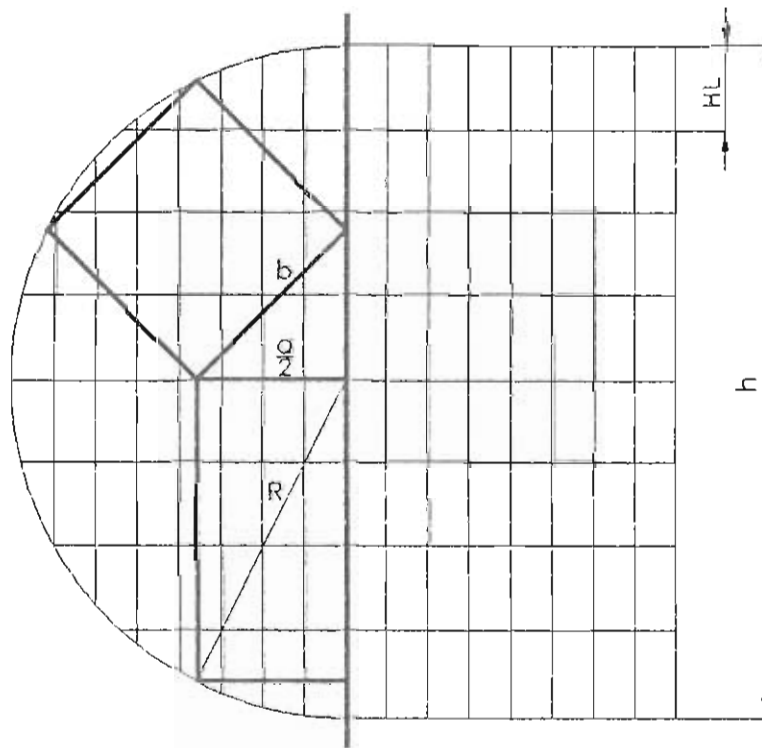


Fig. 1. The connection between the harmonical circle and the net of the eight head length canon. The harmonical circle is the geometrical construction based on shown triangle and the circle (R) passing through the ortexes of the square (b) and square (a). The condition of the connection is done by relation $2R = h$, where h is height of a standing subject.

Using the limits done by the geometrical net we drew into the Fig. 1, now we determine the characteristical points A, B, C, D, ...M in Fig. 2, that represent either positions of joints or contour edges. Coordinates of the points are determined as functions of the total height of the man and its eighths, i.e. the lengths of head (HL). The following group of data was available which were determined from the relation of the drawn man according to the photography to whom the earlier mentioned net from Figure 1. was added.

The general results of the mentioned harmonical analysis are given as the function of the standing height $h = 2R$ of the regarded subject, as follows:

Table 1. The lengths of human body segments as the function of human heights.

length of arm = $(25/84)h$;	length of upper arm = $(5/32)h$,
length of lower arm = $h/8$,	length of hand = $(97/64)h$,
length of leg = $(17/32)h$;	length of upper leg = $(9/32)h$;
length of lower leg = $(7/32)h$;	length of foot = $(1/32)h$.

With the such relations the next step is to draw the corresponding links between the mentioned characteristic points A, B, C, ...etc. This has determined a kind of an auxiliary skeleton presented in Fig. 2.

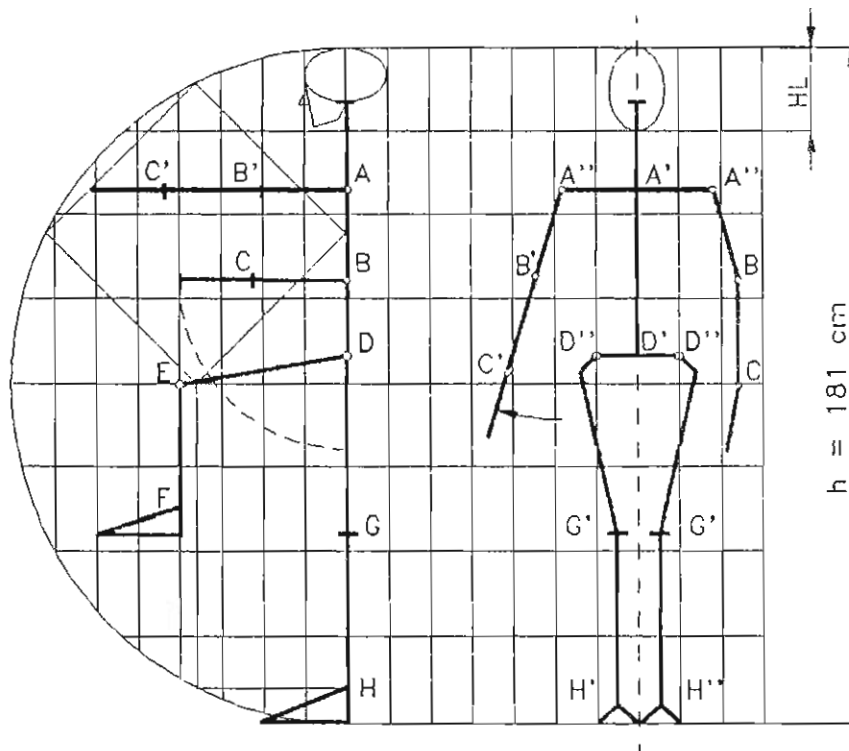


Fig. 2. The planar skelton model is done by definitions of the characteristic points A,B, ..., H. All defined lengths between the points represent characteristic linear anthropomeasures according to table 1.

Upon the determined "skeleton", as the next step, one can draw-in the countour of the subject, but usually it is not necessary. However, the following should be kept in mind:

a) all characteristical joints, as e.g. hips and shoulders should be in positions to make possible movement of the extremities;

b) sizes of muscular structures and fat as well as the skin should fit the auxilliary skeleton so that the legs can be put together and the arms lowered along the body; from that dimensional situation one can suppose, that approximative dimensions of the trunk cross section could be done as the area made by rectangular form of

$$h/8 \times h/12 \text{ (or } h/13 \text{ or } h/14)$$

According to the results of Mairiaux intraabdominal pressure IAP (in kPa) was related to the lumbar moment M_L (in Nm) at the level of lumbar spine L4/L5 by regressive equation:

$$IAP = 0,079 M_L - 1,127 \quad (\text{kPa})$$

with the coefficient of correlation $r = 0,75$. On a such way IAP measurements may be used as an index of spinal stress in real life tasks. We doubted to upper formula, because at moment $M_L=0$, IAP should be even negative, or moment must have some value to make IAP always positive. It was reason that we repeated the measuring of IAP for known lumbar level L4/L5 where we performed 32 males and 30 females in a series of 96 or respectively 90 pullings the rope task. The lumbar moment was determined through a biomechanical model deriving data from the subject anthropometry and photographically recorded postures. Using the regression formula of Donskij and Zatscijorskij we determined the elements of the biomechanical model. (Lit. 6).

3. RESULTS

The realibility of the above method of the human countour was checked as follows:

For a selected height and mass of the tasted persons which corresponds to the highest value of the height of 181 cm and mass of 86 kg, the measuring of the characteristical lengths was performed (e.g. of the upper leg, foot etc.).

The obtained values of anthropomeasures, quoted in Table 2, can easilly be compared with measured Kroemer's subject of mean values of 172 cm height were the corresponding variations range of antropomeasures are shown in the mentioned Table 2.

Table2. Anthropometrical Kroemer's values (in cm) comparing with obtained results from the Fig. 2.

Item No.	Body part	After Kroemer	Variation zone	Obtained results
1.	Standing height	172	160-184	181
2.	Eyes height, standing	161	150-172	169,5
3.	Shoulders height, standing	142	131-153	150
4.	Head top above seat	90	84-96	97,5
5.	Eyes height above seat	79	73-85	86,5
6.	Shoulders height above seat	59	54-64	64
7.	Elbow height, standing	106	18-114	109
8.	Elbow height, above seat	24	20-28	27
9.	Stretched arm, reach	82	75-87	83
10.	Length of lower arm and hand	47	45-51	49
11.	Armes span	175	159-191	180
12.	Thigh thickness	59	54-64	63
13.	Knee height above feet	55	51-59	53
14.	Height below knee from feet	45	42-48	42
15.	Shoulders breadth	45	41-49	47
16.	Thight breadth	35	31-39	35
17.	Lower leg thickness	10	12-17	13
18.	Distance from back to below knee	50	46-54	51,5

In the diagram of the Figure 3. we have three directions, that represent respective regression formulae which contain relations between IAP and lumbar moments. For our purpose we used the middle direction that represent the mean values. From the real abdominal cross-section we found that the values of the magnitude of the cross sections of the abdomen are the units defined by the length of the head, which is one eights of the standing body height h . Such a construction of the idealized geometrical form of the abdominal cross-section defined by means of the harmonical circle and so values too are shown in the work of Jurčević T. and coll. (). From that work we have the functional relation between area of cross section as the function of the body height represented as the statistical function of percentiles as:

$$A = 104,72 \cdot 10^{-4} \cdot h^2,$$

From this formula Jurčević T. et coll. (4) calculated relevant values for five percentiles groups for males and females. Here we decided to calculate on a different basis (the individual heights of the examinee) the interrelations between a lowest and upper values of the heights given in Figure 3. and calculated cross sections areas, as follows:

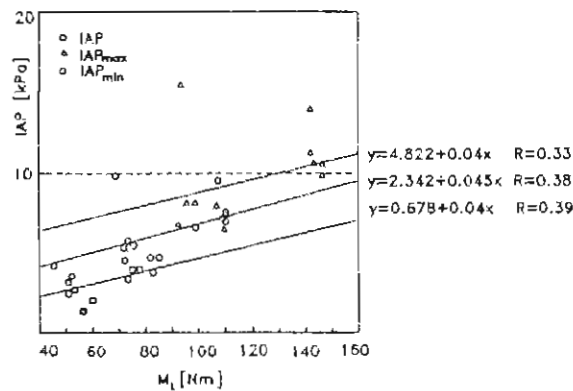


Figure 3. Relations between IAP and lumbar moment M_L

According to determined area of the abdominal cross section we calculated the three different groups of cross sections for three different heights of subjects.

Table 3. Calculated values of the abdominal cross sections area magnitude as the function of the individual heights of examinee

	for $h = 160$ cm	for $h = 172$ cm	for $h = 184$ cm
Area in m^2	0,0227	0,0307	0,0352
Area in m^2	0,02470	0,0284	0,0325
Area in m^2	0,02660	0,0263	0,03

4. D I S S C U S I O N

The established method of drawing the man's contours by the application of harmonical circle and canon of the eight head lengths represents an easy and simple geometrical method with the use of which it is possible to determine the linear anthropomeasures of the man of any height within normal limits from the knowledge of his height from head to heel. Thus it is possible, in two planes, to create biomechanical models of determined person by using either Dempster's data (6) or Donskij's and Zatscijorskij's (7) data, and draw in the centres of segmental masses or segmental weight forces. Thus the biomechanical model is supplemented with the definition of the plan of external forces.

Comparing our values of the calculated anthropomeasures with those of Kroemer's one can notice that our values are in most cases close to the upper limit of Kroemer's variations (see Table 2.). This factor is evidently satisfactory since the height of our average tested person is nearer to the upper limit of Kroemer's results.

Besides the above, it should also be pointed out at the connection of our results with the results of other authors particular of those who are concerned with the dynamic anthropometry, having in mind that the lengths of the extremities, according to the our method, are measured from the centres of joints and not from external characteristical anthropological points. This has particular importance in the calculation of dynamic moments of inertia when assuming the extremities to be cylindrical bodies of a determined length.

The realibility of the above method of the human countour determination was checked as follows:

For a selected height and mass of the tasted persons which corresponds to the highest value of the height of 181 cm and mass of 86 kg, the measuring of the characteristical lengths was performed (e.g. of the upper leg, foot etc.).

The obtained values of anthropomeasures, quoted in Table 1, can easilly be compared with measured Kroemer's subject of mean values of 172 cm height were the corresponding variations range of anthropomeasures are shown in the Table 2.

Knowing the fundamental anthropomeasures it is very easy to apply created model in the human work analysis. For example, from the photos of the chosen work which we analise, we can very easy calculate the reduced moment to the L4/L5 level of the spine. As the result of the mechanical balance to this moment, intraabdominal pressure comes to the necessary value. If we divide the value of the lumbar moment with the supposed arm we define so called lumbar force wich is necessary for the equilibrium. Dividing of this force with the respective area of the abdominal cross section we determine IAP.

Finally, it should be mentioned that method needs further development searching for even more accurate solutions which will take into account varieties in total masses of the tested persons.

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PRIMJENA FUNKCIJA KOMPLEKSNE PROMJENLJIVE U RAVNINSKIM PROBLEMIMA TERMOELASTIČNOSTI

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Sažetak

U radu je prikazano da se metodama funkcije kompleksne promjenljive ravninski problem termoelastičnosti može svesti na ravninski problem izotermičke teorije elastičnosti odnosno na određivanje dviju analitičkih funkcija kompleksne promjenljive. Kompleksni potencijali se tada mogu odrediti iz rubnih uvjeta pomoću metode najmanjih kvadrata

THE APPLICATION OF A COMPLEX VARIABLE FUNCTIONS TO THE PLANE PROBLEMS OF THERMOELASTICITY

Summary

In this paper is shown that the plane problem of the thermoelasticity can be reduced to the plane problem of the isothermic theory of elasticity by means of the complex variable methods. In this case the problem is reduced to determination of two holomorphic functions of a complex variable. The complex potentials then can be determined from the boundary conditions by means of the least squares method.

1. UVOD

Rješenje problema termoelastičnosti se sastoji u određivanju polja pomaka u_i , deformacija ε_{ij} , naprezanja σ_{ij} i temperature T u promatranom tijelu izloženom mehaničkim i toplinskim utjecajima. Potpuni sustav jednačbi potreban za rješenje tog problema sadrži pored osnovnih jednačbi izotermičke teorije elastičnosti (diferencijalne jednačbe ravnoteže, jednačbe neprekidnosti deformacija i fizikalne jednačbe) i jednačbu dotoka topline koja u kvazistatičkoj formulaciji glasi:

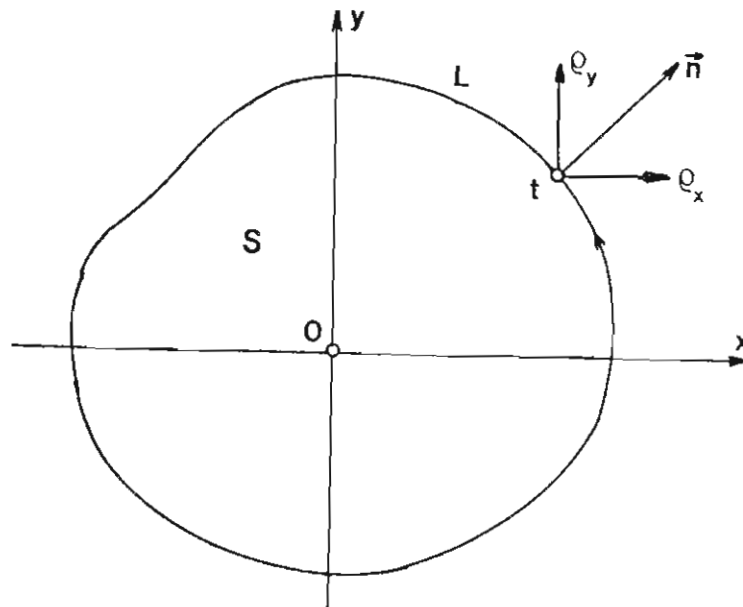
$$\lambda_q \nabla^2 \Theta - c \frac{\partial \Theta}{\partial t} + w = 0 \quad (1)$$

gdje je:

λ_q - koeficijent toplinske vodljivosti materijala, c - specifični toplinski kapacitet materijala, w - specifični kapacitet izvora topline.

$\Theta = T - T_0$ označava prirast temperature u usporedbi s početnim stanjem T_0 , za koje se pretpostavlja da je $|\frac{\Theta}{T_0}| \ll 1$, (teorija malih deformacija).

Kod rješavanja kvazistatičkih problema termoelastičnosti prvo treba riješiti jednačbu dotoka topline (1) uz zadane početne i rubne uvjete. Zatim se za poznato temperaturno polje $\Theta(x, y, z, t)$, gdje vrijeme t igra ulogu parametra, iz osnovnih jednačbi izotermičke teorije elastičnosti odredi polje naprezanja i pomaka. Rješavanje te zadaće neposrednim integriranjem sustava diferencijalnih jednačbi općenito je vrlo složeno. Međutim, pri rješavanju ravninskih problema termoelastičnosti može se uspješno primijeniti teorija funkcija kompleksne promjenljive slično kao i u izotermičkoj teoriji elastičnosti.



Slika 1. Ravninski problem termoelastičnosti

Fig.1. The plane problem of the thermoelasticity

Pretpostavit ćemo da nam je poznato ravninsko temperaturno polje

$$\Theta(x, y, t) = T - T_0. \quad (2)$$

Komponente tenzora deformacija ϵ_{ij} za ravninsko stanje naprezanja određene su izrazima:

$$\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha_T \Theta \\ \epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha_T \Theta \\ \epsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\ \epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha_T \Theta. \end{aligned} \quad (3)$$

Pri malim promjenama temperature, pretpostavlja se da se modul elastičnosti E , Poissonov koeficijent ν i koeficijent linearnog toplinskog rastezanja α_T odnose na izotermičke deformacije koje odgovaraju određenoj temperaturi $T = T_0$.

Uz pretpostavku da su volumenske sile $X = Y = 0$, komponente naprezanja se mogu izraziti funkcijom naprezanja $U(x, y)$:

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}. \quad (4)$$

Funkcija naprezanja $U(x, y)$ zadovoljava jednadžbu:

$$\nabla^2 \nabla^2 U + E\alpha_T \nabla^2 \Theta = 0 \quad (5)$$

ili:

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} + E\alpha_T \left(\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} \right) = 0 \quad (6)$$

i zadane rubne uvjete.

Ako su na konturi L zadane komponente ρ_x i ρ_y vektori punog naprezanja (sl.1), rubni se uvjeti za funkciju $U(x, y)$ mogu prikazati u obliku:

$$\frac{\partial U}{\partial x} = -\int_0^s \rho_y ds + C_1, \quad \frac{\partial U}{\partial y} = \int_0^s \rho_x ds + C_2. \quad (7)$$

Za jednostruko suvislo područje može se uzeti da su integracijske konstante $C_1 = C_2 = 0$.

Opće rješenje jednadžbe (5) glasi:

$$U = U^* + U_p \quad (8)$$

gdje je U^* opće rješenje biharmonijske jednadžbe

$$\nabla^2 \nabla^2 U^* = 0, \quad (9)$$

a U_p je partikularno rješenje jednačbe

$$\nabla^2 U_p + E\alpha_T \Theta = 0. \quad (10)$$

2. KOMPLEKSNI OBLIK OPĆEG RJEŠENJA RAVNINSKOG PROBLEMA TERMOELASTIČNOSTI

U izotermičkoj teoriji elastičnosti ($\Theta = 0$) funkcija naprežanja $U(x, y)$ zadovoljava biharmonijsku jednačbu

$$\nabla^2 \nabla^2 U = 0. \quad (11)$$

Biharmonijska funkcija $U(x, y)$ može se u kompleksnom obliku izraziti s dvije analitičke funkcije $\varphi(z)$ i $\chi(z)$, kompleksne promjenljive $z = x + iy$:

$$2U(x, y) = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}. \quad (12)$$

Iz izraza (12) dobiva se linearna kombinacija parcijalnih derivacija funkcije naprežanja $U(x, y)$ u kompleksnom obliku:

$$f(x, y) = \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = \varphi'(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} \quad (13)$$

gdje je:

$$\psi(z) = \frac{d\chi}{dz}. \quad (14)$$

Kompleksni oblik za komponente naprežanja glasi:

$$\begin{aligned} \sigma_x + \sigma_y &= \nabla^2 U = 2[\varphi'(z) + \overline{\varphi'(z)}] \\ \sigma_y - \sigma_x + 2i\sigma_{xy} &= \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 2i\frac{\partial^2}{\partial x \partial y}\right)U = 2[\bar{z}\varphi''(z) + \psi'(z)]. \end{aligned} \quad (15)$$

Linearna kompleksna kombinacija komponenata pomaka ima oblik:

$$\frac{E}{1+\nu}(u + iv) = \frac{3-\nu}{1+\nu}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \quad (16)$$

Rubni uvjeti (7) se mogu pomoću izraza (13) izraziti u kompleksnom obliku:

$$f(t) = \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = \varphi'(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = i \int_0^s (\rho_x + i\rho_y) ds + const. \quad (17)$$

gdje $t = x + iy$ označava točku na konturi L , a x, y su koordinate promatrane točke na konturi.

Kod ravninskih problema termoelastičnosti funkcija naprežanja $U(x, y)$ zadovoljava jednačbu (5). Uvrstimo li u izraze (15) funkciju naprežanja $U = U^* + U_p$, i uzimajući u obzir da je $U^*(x, y)$ biharmonijska funkcija koja ima sva ranije navedena svojstva biharmonijske funkcije naprežanja u izotermičkoj teoriji elastičnosti, dobit ćemo kompleksni oblik za komponente naprežanja u ravninskoj termoelastičnosti:

$$\sigma_x + \sigma_y = \nabla^2(U^* + U_p) = 2[\varphi^{*'}(z) + \overline{\varphi^{*'}(z)}] - E\alpha_T \Theta$$

$$\begin{aligned}\sigma_y - \sigma_x + 2i\sigma_{xy} &= \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 2i\frac{\partial^2}{\partial x\partial y}\right)(U^* + U_p) = \\ &= 2[\overline{z}\varphi^{*''}(z) + \psi^{*'}(z)] + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 2i\frac{\partial^2}{\partial x\partial y}\right)U_p.\end{aligned}\quad (18)$$

gdje je:

$$2U^* = \overline{z}\varphi^* + z\overline{\varphi^*(z)} + \chi^*(z) + \overline{\chi^*(z)} \quad (19)$$

$$\psi^*(z) = \frac{d\chi^*}{dz}. \quad (20)$$

Izraz (3) može se prikazati u slijedećem obliku:

$$\begin{aligned}E\frac{\partial u}{\partial x} &= \left(\frac{\partial^2}{\partial y^2} - \nu\frac{\partial^2}{\partial x^2}\right)(U^* + U_p) + E\alpha_T \Theta \\ E\frac{\partial v}{\partial y} &= \left(\frac{\partial^2}{\partial x^2} - \nu\frac{\partial^2}{\partial y^2}\right)(U^* + U_p) + E\alpha_T \Theta.\end{aligned}\quad (21)$$

Iz izraza (21) i (10) dobiva se kompleksni oblik za komponente pomaka u ravninskim problemima termoelastičnosti, koji glasi:

$$\frac{E}{1+\nu}(u + iv) = \frac{3-\nu}{1+\nu}\varphi^*(z) - \overline{z\varphi^{*'}(z)} - \overline{\psi^*(z)} - \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)U_p. \quad (22)$$

U navedenim izrazima sve veličine vezane na biharmonijsku funkciju $U^*(x, y)$ označene su gornjim indeksom *.

Ako su zadani pomaci točaka na konturi L :

$$u = g_1(t) = g_1(s), \quad v = g_2(t) = g_1(s)$$

rubni uvjeti glase:

$$\frac{3-\nu}{1+\nu}\varphi^*(t) - t\overline{\varphi^{*'}(t)} - \overline{\psi^*(t)} = \frac{E}{1+\nu}(g_1 + ig_2) + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)U_p. \quad (23)$$

Izraz (13) se pomoću izraza (8) može svesti na oblik:

$$\begin{aligned}f(x, y) &= \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \frac{\partial U^*}{\partial x} + i\frac{\partial U^*}{\partial y} + \frac{\partial U_p}{\partial x} + i\frac{\partial U_p}{\partial y} = \\ &= \varphi^*(z) + \overline{z\varphi^{*'}(z)} + \overline{\psi^*(z)} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)U_p.\end{aligned}\quad (24)$$

Ako su na konturi L zadane komponente ρ_x i ρ_y vektora punog naprezanja, tada se na temelju izraza (7) i (24) rubni uvjeti mogu izraziti u obliku:

$$\varphi^*(t) + t\overline{\varphi^{*'}(t)} + \overline{\psi^*(t)} = i \int_0^s (\rho_x + i\rho_y)ds - \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)U_p + const. \quad (25)$$

Ako umjesto ravninskog stanja naprezanja promatramo ravninsko stanje deformacija tada u mjesto E , ν i α_T u navedenima izrazima dolazi E_1 , ν_1 , α_{T1} :

$$E_1 = \frac{E}{1 - \nu^2}, \quad \nu_1 = \frac{\nu}{1 - \nu}, \quad \alpha_{T1} = \alpha_T(1 + \nu). \quad (26)$$

Iz izraza (18), (22), (23) i (25) je uočljivo da ako je poznato rješenje U_p jednadžbe (10), ravninski problem termoelastičnosti se svodi na ravninski problem izotermičke teorije elastičnosti, koji se može riješiti metodama funkcije kompleksne promjenljive. U slučaju stacionarnog temperaturnog polja bez izvora, rješenje U_p jednadžbe (10) može se izraziti u kompleksnom obliku.

3. ODREĐIVANJE KOMPLEKSNIH POTENCIJALA METODOM NAJMANJIH KVADRATA

Razmatrat ćemo ravninski problem sa zadanim pomacima na zatvorenoj konturi L (sl.1). Pretpostavljamo da nam je poznata funkcija U_p . Rubne uvjete (23) napisat ćemo u obliku:

$$\kappa \varphi^*(t) - t \overline{\varphi^{*'}(t)} - \overline{\psi^*(t)} = G(t) \quad (27)$$

gdje je za ravninsko stanje naprezanja

$$\kappa = \frac{3 - \nu}{1 + \nu}. \quad (28)$$

a za ravninsko stanje deformacija

$$\kappa = 3 - \nu$$

$$G(t) = \frac{E}{1 + \nu}(g_1 + i g_2) + \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) U_p. \quad (29)$$

Budući su funkcije $\varphi^*(z)$ i $\psi^*(z)$ analitičke u promatranom području S , mogu se unutar zatvorenog područja S prikazati u obliku reda potencija:

$$\varphi^*(z) = \sum_{k=1}^{\infty} \alpha_k z^k, \quad \psi^*(z) = \sum_{k=0}^{\infty} \beta_k z^k. \quad (30)$$

Izostavljanje konstantnog člana u prvom redu potencija (30) odgovara dopunskom uvjetu $\varphi^*(0) = 0$. Ako u redu potencija (30) zadržimo n prvih članova dobit ćemo polinome:

$$\varphi_n^*(z) = \sum_{k=1}^n \alpha_k z^k, \quad \psi_n^*(z) = \sum_{k=0}^n \beta_k z^k \quad (31)$$

koji prema Uolševom teoremu za bilo kakav zadani broj $\varepsilon > 0$ ispunjava nejednakost:

$$|\varphi^*(z) - \varphi_n^*(z)| < \varepsilon, \quad |\psi^*(z) - \psi_n^*(z)| < \varepsilon,$$

ako sa n idemo dovoljno daleko. Kad $n \rightarrow \infty$ onda $\varepsilon \rightarrow 0$, pa $\varphi_n^*(z) \rightarrow \varphi^*(z)$ i $\psi_n^* \rightarrow \psi(z)$ odnosno polinomi (31) prelaze u redove (30).

Približne vrijednosti funkcija $\varphi^*(z)$ i $\psi^*(z)$ tražimo u obliku polinom (31). Rubne uvjete (27) svodimo na slijedeći oblik:

$$\kappa\varphi_n^*(t) - \overline{t\varphi_n^*(t)} - \overline{\psi_n^*(t)} = G_n(t). \quad (32)$$

Koeficijente α_k i β_k polinoma (31) odredit ćemo iz uvjeta da integral kvadrata razlike tražene funkcije $G_n(t)$ i zadane funkcije $G(t)$ duž zatvorene konture L ima najmanju vrijednost:

$$\int_L |G_n(t) - G(t)|^2 ds = \min. \quad (33)$$

Definiramo li skalarni produkt dviju funkcija $\varphi_m^*(t)$ i $\varphi_n^*(t)$:

$$(\varphi_m^*, \varphi_n^*) = \int_L \varphi_m^*(t) \overline{\varphi_n^*(t)} ds, \quad ds = |dt| \quad (34)$$

onda je norma funkcije $\varphi_m^*(t)$ na konturi L definirana izrazom:

$$\|\varphi_m^*(t)\|^2 = (\varphi_m^*, \varphi_m^*) = \int_L \varphi_m^*(t) \overline{\varphi_m^*(t)} |dt| = \int_L |\varphi_m(t)|^2 ds. \quad (35)$$

Izraz (33) možemo sad prikazati u slijedećem obliku:

$$\int_L |G_n(t) - G(t)|^2 ds = \|G_n(t) - G(t)\|^2 = \min. \quad (36)$$

Ako u izraz (31) uvedemo nove oznake:

$$\alpha_k = a_{2k-1} + ia_{2k}, \quad \beta_k = a_{2k+2n+1} + ia_{2k+2n+2},$$

$$\varphi_{(2k-1)}^*(z) = z^k, \quad 1 \leq k \leq n, \quad \varphi_{(2k)}^*(z) = iz^k, \quad 1 \leq k \leq n, \quad (37)$$

$$\varphi_{(m)}^*(z) = 0, \quad m > 2n, \quad \psi_{(m)}^*(z) = 0, \quad 1 \leq m \leq 2n,$$

$$\psi_{(2k+2n+1)}^*(z) = z^k, \quad 0 \leq k \leq n, \quad \psi_{(2k+2n+2)}^*(z) = iz^k, \quad 0 \leq k \leq n,$$

onda polinome (31) možemo prikazati u obliku:

$$\varphi_n^*(z) = \sum_{k=1}^{4n+2} a_k \varphi_{(k)}(z), \quad \psi_n^*(z) = \sum_{k=1}^{4n+2} a_k \psi_{(k)}(z). \quad (38)$$

U izrazima (38) a_k su realni koeficijenti. Ako u izraz (32) uvrstimo izraz (38), dobit ćemo:

$$G_n(t) = \sum_{k=1}^{4n+2} a_k G_{(k)}(t) \quad (39)$$

gdje je

$$G_{(k)}(t) = \kappa\varphi_{(k)}^*(t) - \overline{t\varphi_{(k)}^*(t)} - \overline{\psi_{(k)}^*(t)}. \quad (40)$$

Uvjet (33) možemo napisati u slijedećem obliku:

$$\left\| \sum_{k=1}^{4n+2} a_k G_{(k)}(t) - G(t) \right\|^2 = \min. \quad (41)$$

Budući da su koeficijenti a_k realni dobivamo:

$$\sum_{j,k=1}^{4n+2} a_j a_k (G_{(j)}, G_{(k)}) - 2 \sum_{k=1}^{4n+2} a_k \operatorname{Re}(G, G_{(k)}) + (G, G) = \min. \quad (42)$$

Da bi izraz (42) primio minimalnu vrijednost, njegove derivacije po a_j i a_k moraju biti jednake nuli. Tako dolazimo do sustava linearnih algebarskih jednačbi s nepoznicama a_k :

$$\sum_{k=1}^{4n+2} a_k \operatorname{Re}(G_{(k)}, G_{(j)}) = \operatorname{Re}(G, G_{(j)}); \quad j = 1, 2, \dots, 4n + 2. \quad (43)$$

Kad iz sustava jednačbi (43) odredimo konstante a_k , funkcije $\varphi_n^*(z)$ i $\psi_n^*(z)$ su dane izrazom (31) koji daje približne vrijednosti funkcije $\varphi^*(z)$ i $\psi^*(z)$ u zatvorenom području S . U nekoj točki područja S stanje naprezanja je određeno izrazom (18), a stanje pomaka izrazom (22).

Da bismo ocijenili konvergenciju približnog rješenja točnom, rješenje problema tražimo za neki povolji odabrani n_0 . Pri rješavanju sustava (43) za n_0 može se na elektroničkom računalu istodobno dobiti rješenje za $n_1 < n_0$, jer je matrica sustava za n_1 podmatrica sustava za n_0 . Na osnovi analize tih rješenja može se ocijeniti konvergenciju čitavog postupka i vidjeti dali je potrebno ići s $n > n_0$ tj. na proširenje postojećeg sustava. U proširenom sustavu potrebno je odrediti samo nove koeficijente matrice, koji se pojavljuju zbog proširenja matrice sustava. Pritom stara matrica postaje podmatrica nove matrice.

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THE ANALYSIS OF THE CRACK TIP OPENING DISPLACEMENT (CTOD) AND THE MAGNITUDE OF THE PLASTIC ZONE AROUND A CRACK TIP BASED ON DUGDALE'S STRIP YIELD MODEL AND THE ESTIMATION OF ITS RELIABILITY

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Abstract - The article deals with the analytical determination of stresses and displacements and some other fracture mechanics parameters, such as: the crack tip opening displacement (CTOD) and the magnitude of the plastic zone around a crack tip, in thin infinite cracked plates. The plates are loaded biaxially at the infinity, while the crack surface is free of loading. It is supposed that the magnitude of applied loading will be caused the appearance of the plastic deformations in the neighbourhood of the crack tip. In this paper are used the analytical methods based on the theory of the functions of a complex variable. For describing of the plastification process in an infinite plate with an embedded crack the Dugdale strip yield model originally based on the Tresca yield criterion has been utilized. The influence of the remote normal traction which is parallel to the crack surface, on those parameters, has also been investigated. In connection with this purpose, it has been necessary to modify the original Dugdale strip yield model and to apply the von Mises yield criterion.

1. INTRODUCTION

From the point of view of the Linear Elastic Fracture Mechanics (LEFM), and on the assumption of linear elastic response and a brittle material behaviour such as: iron, glass, ceramics, rock, stone etc, the analysis of a single fracture parameter - stress intensity factor K - is sufficient for the assessment of the fracture process. Up to the present such opinion was widely accepted in the case of elastic state of strains in the material and the analysis of a fracture parameter K forms a basis of the Linear Elastic Fracture Mechanics (LEFM) approach. On the other hand, assuming a non-linear elastic material behaviour, respectively, in the case of elastic-plastic or elastic-viscoplastic state of strains in the material, the analysis of a stress intensity factor K will be not reliable for the estimation of the fracture process.

Since the magnitude of the stress intensity factor K according to LEFM depends only upon the magnitude of the in-plane loading applied in a direction perpendicular to the crack plane but not upon the in-plane loading applied in a direction parallel to the crack plane, it

might lead to a conclusion that this latter in-plane loading has no influence on the material behaviour in the fracture process. Such opinion was usually and generally accepted to date what is seen from the next citation [1]: "... for bodies containing plane cracks, loads that are applied in a direction that is parallel to the plane of the crack, in addition to the tensile load acting in the direction perpendicular to it, can have no influence upon the fracture behaviour of the body". However, the results of numerous experimental and theoretical investigations such as, e. g. Eftis', Liebowitz' and their co-authors, demonstrate that the loading acting parallel to the crack plane also exerts influence upon the shape and the magnitude of plastic zones around the crack tip and upon the magnitude of the crack tip opening displacements (CTOD), even though the stress intensity factor K is kept constant during the experiment [1], [2], [3] and others. This is a fact of vital importance and ought to be taken into account, especially when the fracture parameter CTOD is utilized as a fracture criterion.

If it is necessary to assess whether a construction element built of ductile material is fracture-proof, the reliability estimation will be again based on the analysis of a single fracture parameter. The most frequently used fracture parameters are either the J-integral or the crack tip opening displacement CTOD. This will be a subject-matter of the Non-Linear Fracture Mechanics investigation or of the Elastic-Plastic Fracture Mechanics (EPFM) approach.

2. FUNDAMENTAL EQUATIONS OF THE PLANE THEORY OF ELASTICITY BASED ON THE COMPLEX VARIABLE THEORY

In order to determine the stress field, respectively the displacement field in thin, infinite, cracked plate (Griffith's crack), the numerous analytical methods are employed. Among the numerous and powerful analytical methods, these based on the Muskhelishvili's theory of the functions of the complex variable still play an important role in exact solving of some classes of physical problems and the explicit determination of the solution in a closed form. A great popularity in applying of these methods has been achieved with the attractiveness of the theory of the complex variable itself, subsequently with an elegance in the strategy of inventing of the solution and finally, concise and compact formulation of the solution in a whole domain of a definition of the complex variable $z=x+iy$. A short review of the fundamental equations of the plane theory of elasticity based on the complex variable theory, according to [6], will be shown in a sequence.

By introducing the two analytical functions of the complex variable $\varphi(z)$ and $\psi(z)$, where is $\psi(z)=\chi'(z)$, a some biharmonic function $F(x, y)$ - Airy's stress function, can be represented in the following way [6]

$$F(x,y) = \operatorname{Re} \left[\bar{z} \varphi(z) + \int \psi(z) dz \right]. \quad (1)$$

A stresses, strains and the displacements can also be represented by means of the two analytical functions of the complex variable $\varphi(z)$ and $\psi(z)$. Kolosov [7] was the first to succeed

in deriving general expressions for the stresses and displacements in the complex expressing for the general plane problems of the theory of elasticity

$$\begin{aligned}\sigma_{xx} + \sigma_{yy} &= 2[\varphi'(z) + \overline{\varphi'(z)}] = 4 \operatorname{Re} [\varphi'(z)] \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2[\bar{z}\varphi''(z) + \psi'(z)] \\ 2\mu(u + iv) &= C_0\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}.\end{aligned}\quad (2)$$

where μ denotes the shear modulus and C_0 is a constant which in the case of plane state of strain takes the value $C_0 = (3\mu + \lambda)/(\mu + \lambda) = 3 - 4\nu$, respectively in the case of plane state of stress it amounts to $C_0 = (3 - \nu)/(1 + \nu)$.

In the paper, the thin, infinite plates ($z \in D$, $D: |z| \geq 0$) with an embedded straight crack of length $2a$ lying on the x -axis ($z \in L$, $L: |\operatorname{Re} z| \leq a$, $\operatorname{Im} z = 0$) will be analysed. An in-plane loading is assumed and it is symmetrically distributed with respect to the x - and y -axes. In general, the boundary conditions at infinity are defined in the form

$$\sigma_{xx} = k\sigma_\infty, \quad \sigma_{yy} = \sigma_\infty, \quad \sigma_{xy} = 0, \quad \text{as } |z| \rightarrow \infty. \quad (3)$$

If on the crack surface acts up the uniformly distributed internal pressure $p_0(z)$, which opens the crack, the additional boundary conditions on the crack surface must be satisfied, as follows

$$\sigma_{xx} \neq 0, \quad \sigma_{yy} = -p_0(z), \quad \sigma_{xy} = 0, \quad \text{as } z \in L, \quad p_0(-z) = p_0(z). \quad (4)$$

The applied loading is symmetrically distributed with respect to the x - and y -axes, the generated stress field in the plate will be characterized by the following relations

$$\begin{aligned}\sigma_{xx}(z) &= \sigma_{xx}(\bar{z}) = \sigma_{xx}(-\bar{z}) \\ \sigma_{yy}(z) &= \sigma_{yy}(\bar{z}) = \sigma_{yy}(-\bar{z}) \quad \text{for } z \in D. \\ -\sigma_{xy}(z) &= \sigma_{xy}(\bar{z}) = \sigma_{xy}(-\bar{z})\end{aligned}\quad (5)$$

The analysed stress field will also have the property that the shear stress is equal a zero along the both co-ordinate x - and y -axes, i. e.

$$\sigma_{xy}(z) = 0, \quad \text{for } \operatorname{Im} z = 0 \quad \text{and} \quad \operatorname{Re} z = 0. \quad (6)$$

Considering the fact that the shear stress vanishes along the x -axis, i. e. for $\operatorname{Im} z = 0$, an additional condition is obtained from the second Kolosov's formula (2) which the analytical functions of the complex variable $\varphi(z)$ and $\psi(z)$ have to satisfy

$$\operatorname{Im} [\bar{z}\varphi''(z) + \psi'(z)]_{(\operatorname{Im} z = 0)} = 0. \quad (7)$$

At this point we shall introduce a new complex function $\Omega(z)$ in the form

$$\Omega(z) = z\varphi''(z) + \psi'(z). \quad (8)$$

This complex function will be equal of the complex function in expression (7) under condition that $\operatorname{Im} z = 0$. The function $\Omega(z)$ must be an analytical function in the whole domain of a

definition of the complex variable z . In fact, due to its imaginary part which equals zero for $\text{Im } z=0$ it follows that the function $\Omega(z)$ is constant all over the domain [8]. Consequently, the following equation must be satisfied

$$\Omega(z) = A, \quad z \in D, \quad (9)$$

where A is a real constant. Now, an additional condition (7) can express in the following form

$$\begin{aligned} x \operatorname{Re} [\varphi''(z)] - y \operatorname{Im} [\varphi''(z)] + \operatorname{Re} [\psi'(z)] &= A, \\ x \operatorname{Im} [\varphi''(z)] + y \operatorname{Re} [\varphi''(z)] + \operatorname{Im} [\psi'(z)] &= 0. \end{aligned} \quad (10)$$

$\operatorname{Re} [\psi'(z)]$ and $\operatorname{Im} [\psi'(z)]$ are eliminated from these additional conditions (10) and put in the equations (2). The general expressions for the stresses and displacements, expressed only by means of the one analytical function of the complex variable $\varphi(z)$ and their derivatives, are obtained from the system of equations (2), as follows

$$\begin{aligned} \sigma_{xx} &= 2 \operatorname{Re} [\varphi'(z)] - 2y \operatorname{Im} [\varphi''(z)] - A \\ \sigma_{yy} &= 2 \operatorname{Re} [\varphi'(z)] + 2y \operatorname{Im} [\varphi''(z)] + A \\ \sigma_{xy} &= -2y \operatorname{Re} [\varphi''(z)] \\ 2\mu u &= (C_0 - 1) \operatorname{Re} [\varphi(z)] - 2y \operatorname{Im} [\varphi'(z)] - Ax \\ 2\mu v &= (C_0 + 1) \operatorname{Im} [\varphi(z)] - 2y \operatorname{Re} [\varphi'(z)] + Ay. \end{aligned} \quad (11)$$

Equations (11) were first derived by Sih in 1966. His intention has been to correct the traditional Westergaard equations where the constant A was either overlooked or omitted arbitrarily arguing that it has no influence on the stress intensity factor K_I . By allowing that the real constant A , in expressions for the stresses and displacements (11), takes the values different from zero ($A \neq 0$), in that case it is possible to determine the stress field and the displacement field in thin, infinite, cracked plate at the arbitrary nonuniform biaxial in-plane loading with $k \neq 1$ in equation (3). From the system of equations (11) also yields, the difference of the normal stresses along the x -axis ($\text{Im } z=0$) will be constant. Its magnitude is determined by the boundary conditions at infinity

$$\sigma_{yy}(z) - \sigma_{xx}(z) = 2A = (1 - k) \sigma_{\infty}, \quad \text{for } \text{Im } z = 0. \quad (12)$$

3. MAGNITUDE OF THE PLASTIC ZONE AROUND THE CRACK TIP AND THE CRACK TIP OPENING DISPLACEMENT (CTOD)

If it is wanted to analyse the stress field and the displacement field at the arbitrary nonuniform biaxial in-plane loading ($k \neq 1$) of an infinite cracked plate, with the boundary conditions defined by the equations (3), the analytical function $\varphi(z)$ takes the following form

$$\begin{aligned} \varphi(z) = \frac{1}{\pi} \left\{ \sigma_0 z \operatorname{arctg} \left(\frac{z}{a} \sqrt{\frac{b^2 - a^2}{z^2 - b^2}} \right) + \sigma_0 \frac{a}{2} \left[\operatorname{arctg} \left(\frac{az - b^2}{\sqrt{b^2 - a^2} \sqrt{z^2 - b^2}} \right) - \right. \right. \\ \left. \left. - \operatorname{arctg} \left(\frac{az + b^2}{\sqrt{b^2 - a^2} \sqrt{z^2 - b^2}} \right) \right] \right\} - \frac{(1-k)\sigma_\infty}{4} z - \frac{\sigma_0}{2} a. \end{aligned} \quad (13)$$

The magnitude of the plastic zone around the crack tip and the crack tip opening displacement (CTOD) have been determined using the Dugdale strip yield model [5]. The original Dugdale model is supposed on the part of the imaginary sharp, elastic crack, of the length $2r_p$, the constant cohesive stresses, their magnitude being equal to the yield stress $\sigma_{yy} = \sigma_0$. However, it will be strictly valid only in a one special case of biaxial in-plane loading of the infinite cracked plate. Namely, only in the case when the in-plane loading in a direction parallel to the crack surface $k\sigma_\infty$ is equal to the in-plane loading in a direction perpendicular to the crack surface σ_∞ , i. e. when is $k=1$ and according to (12), $A=0$. Under such conditions of the loading, it will be $\sigma_{xx}(x,0) \equiv \sigma_{yy}(x,0)$ at all points along the crack line so that the material becomes yielded whenever the stress $\sigma_{yy}(x,0)$ reaches the uniaxial yield stress σ_0 . It is, in fact, the Tresca yield criterion. In all other cases of biaxial in-plane loading of the infinite cracked plate, i. e. when $k \neq 1$, the relation between normal stresses along the x-axis; $\sigma_{xx}(x,0)$ and $\sigma_{yy}(x,0)$, is defined by the relation (12). Accordingly, the magnitude of the constant cohesive stresses $\sigma_{yy}(x,0)$ should generally be determined so that the yield conditions within the strip band around the crack tip are satisfied.

The von Mises yield criterion, in combination with a condition (12), gives the quadratic equation of the form

$$\sigma_{yy}^2(x,0) - (1-k)\sigma_\infty\sigma_{yy}(x,0) + \left[(1-k)^2\sigma_\infty^2 - \sigma_0^2\right] = 0. \quad (14)$$

By solving of this equation it is immediately obtained that

$$\sigma_{yy}(x,0) = \sigma_0 \left[\sqrt{1 - \frac{3}{4}(1-k)^2 \left(\frac{\sigma_\infty}{\sigma_0} \right)^2} + \frac{1-k}{2} \frac{\sigma_\infty}{\sigma_0} \right]. \quad (15)$$

If in the case of the original Dugdale model the constant cohesive stress is equal to $\sigma_y = \sigma_0$, in the case of the modified Dugdale model it will be equal to

$$\sigma_y = \frac{\sigma_0}{\sqrt{1 - \frac{3}{4}(1-k)^2 \left(\frac{\sigma_\infty}{\sigma_0} \right)^2} + \frac{1-k}{2} \frac{\sigma_\infty}{\sigma_0}}. \quad (16)$$

For example, for $k=2$ and $\sigma_\infty/\sigma_0=1/2$ it is obtained: $\sigma_{yy}(x,0)=0,65\sigma_0$, $\sigma_{xx}(x,0)=1,15\sigma_0$ and $\sigma_y=1,5385\sigma_0$.

From the expressions (11) it is seen that the real constant A has no influence on the magnitude of the displacements v of the points lying on the x-direction. At the same time, it

will influence on the magnitude of the displacements of these points in a x-direction. Knowing the analytical, complex function $\phi(z)$, according to (13), the displacements of the points lying on the crack line, in a direction of the x-axis, can be determined by arbitrary biaxial loading of the cracked plate as follows

$$u(x, 0) = \frac{C_0 + 1}{8\mu} (k - 1) \sigma_\infty x + \frac{C_0 - 1}{4\mu} \sigma_0 \begin{cases} (x + a) & \text{for } -b < x \leq -a \\ 0 & \text{for } -a < x < a \\ (x - a) & \text{for } a \leq x < b \end{cases} \quad (17)$$

The influence of the loading parallel to the crack surface takes into account through the load biaxiality factor k . From the equations (17), it is seen that the crack tip will be "fixed" in the direction of the x-axis only in the case of equal biaxial loading with $k=1$. In the case of uniaxial loading where $k=0$, in a direction of the y-axis, the crack tip ($x=a$) will be moved along the crack line at a distance $-(C_0+1)(\sigma_\infty a)/(8\mu)$, towards the crack center due to the Poisson ratio effect. That phenomenon cannot be predicted if the real constant A has been arbitrarily omitted.

The original Dugdale strip yield model is implicitly assumed the use of the Tresca yield criterion. In the case of the biaxial in-plane loading of the cracked plate (for $k=1$), the condition $\sigma_x(x,0) \equiv \sigma_y(x,0)$ is satisfied at all points along the crack line so that the material becomes yielded whenever the stress $\sigma_y(x,0)$ reaches the uniaxial yield stress σ_0 . On the other hand, along the crack line, $\sigma_y(x,0) \equiv 0$, $\sigma_x(x,0)$ and $\sigma_y(x,0)$ are consequently the two principal stresses. The third principal stress is zero for the plane stress, $\sigma_z(x,0) \equiv 0$.

The modified Dugdale strip yield model retains the assumption about the elastic-perfectly plastic material model like in the original Dugdale model, but employs the von Mises yield criterion instead of the Tresca yield criterion such that the influence of biaxial stress states on the material yield condition may be appropriately taken into account. The magnitude of the plastic zone around the crack tip, based on the modified Dugdale strip yield model, is given as follows

$$\frac{r_p}{a} = \sec \left\{ \frac{\pi}{2} \frac{\sigma_\infty}{\sigma_0} \left[\sqrt{1 - \frac{3}{4} (1 - k)^2 \left(\frac{\sigma_\infty}{\sigma_0} \right)^2} + \frac{1 - k}{2} \frac{\sigma_\infty}{\sigma_0} \right] \right\} - 1, \quad (18)$$

while the crack tip opening displacement (CTOD) is equal

$$\delta_t = \frac{8}{\pi} \frac{\sigma_0}{E} \frac{a \ln \left\{ \sec \left[\frac{\pi}{2} \frac{\sigma_\infty}{\sigma_0} \left(\sqrt{1 - \frac{3}{4} (1 - k)^2 \left(\frac{\sigma_\infty}{\sigma_0} \right)^2} + \frac{1 - k}{2} \frac{\sigma_\infty}{\sigma_0} \right) \right] \right\}}{\sqrt{1 - \frac{3}{4} (1 - k)^2 \left(\frac{\sigma_\infty}{\sigma_0} \right)^2} + \frac{1 - k}{2} \frac{\sigma_\infty}{\sigma_0}}. \quad (19)$$

Based on the equations (18) and (19), the magnitudes of the parameters r_p and δ_t are calculated for the different values of the load biaxiality factor k , and in the dimensionless forms

$\frac{r_p}{a}$ and $\left(\frac{\pi E}{8\sigma_0 a}\right)\delta_t$ are graphically represented in Fig. 1 and Fig. 2 for both negative and positive k values, in dependence upon the applied loading σ/σ_0 , which is the load acting in a direction perpendicular to the crack surface.

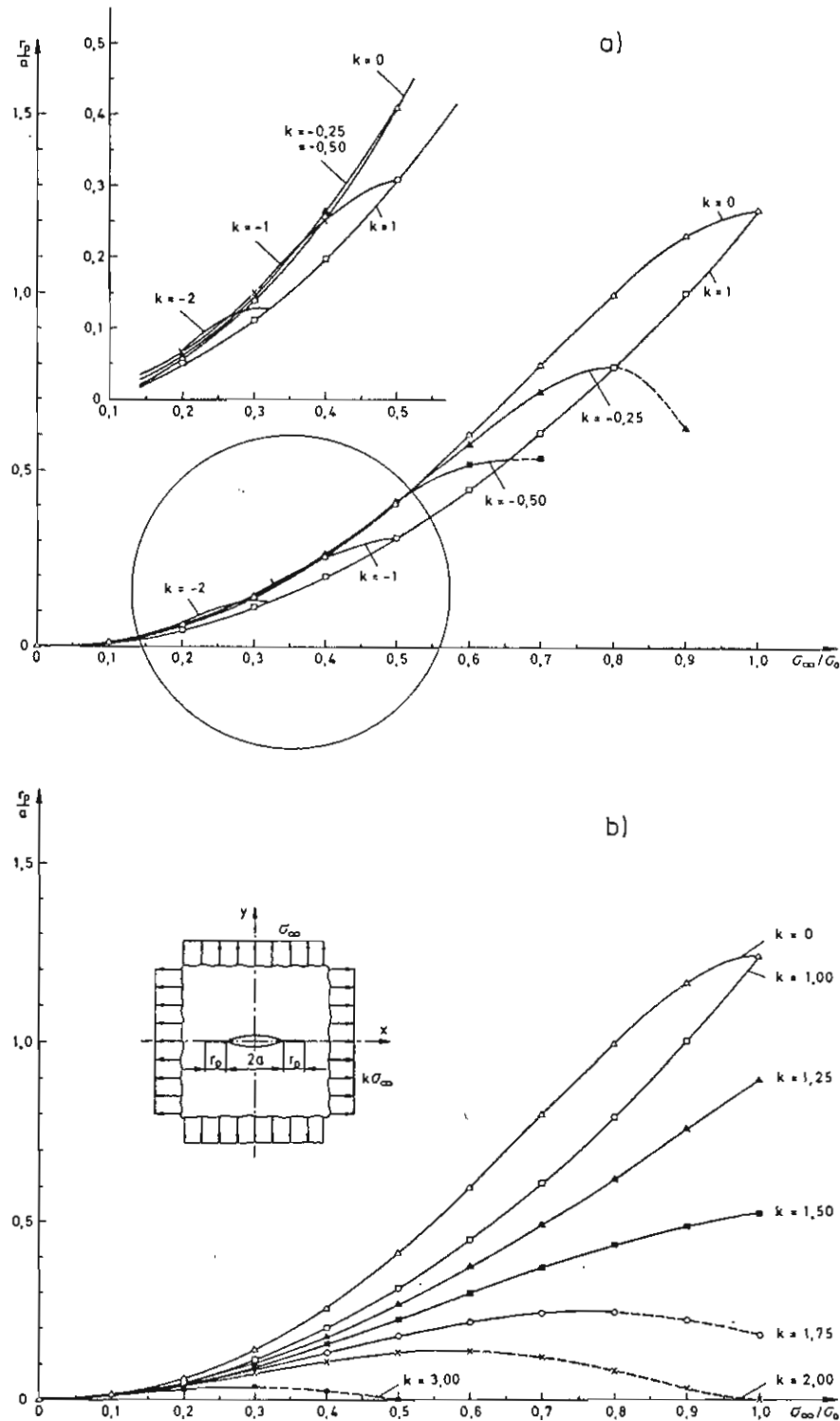


Figure 1. The dependence of the magnitude of the plastic zone around the crack tip r_p on the loading of the plate in a direction perpendicular to the crack surface as well as on the loading parallel to the crack surface (factor k); a) for $k \leq 1.00$; b) for $k \geq 1.00$.

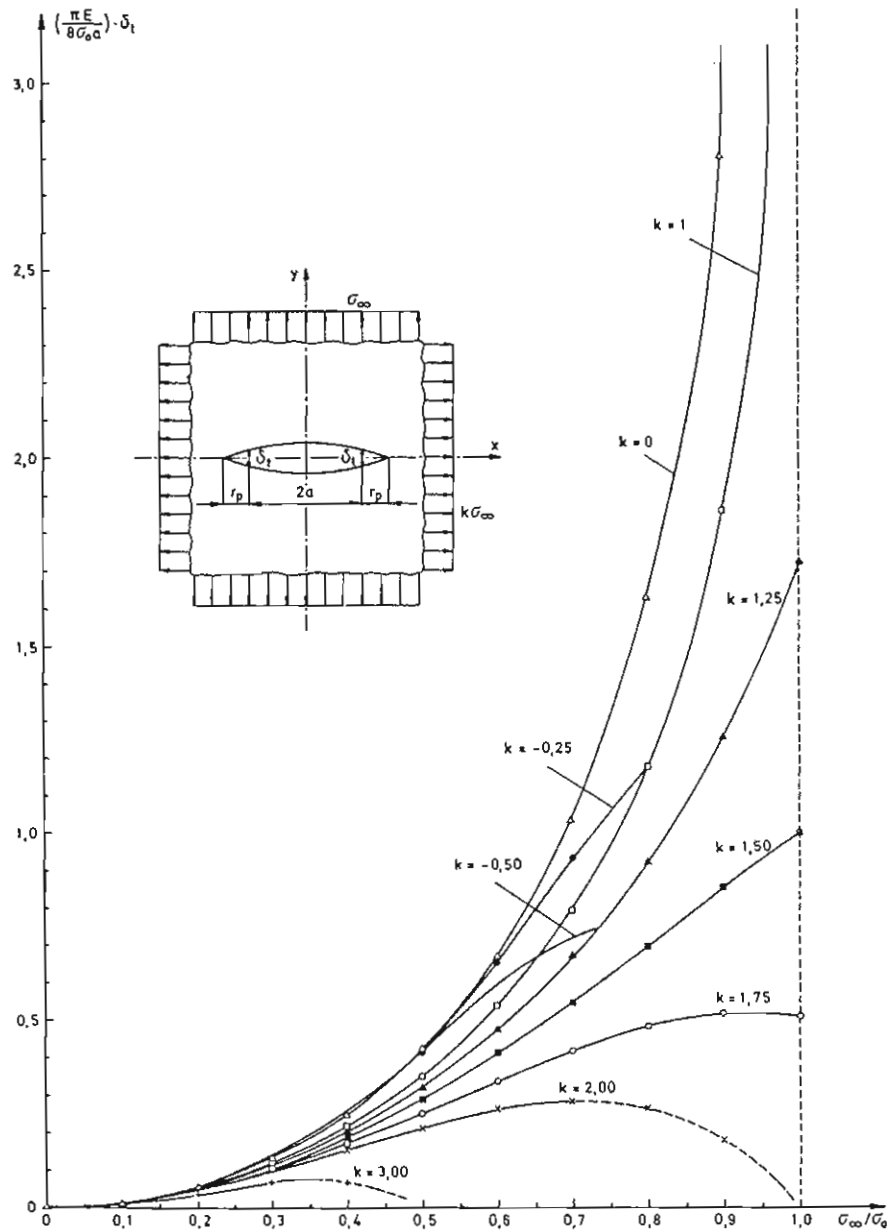


Figure 2. The dependence of the crack tip opening displacement δ_t on the loading of the plate in a direction perpendicular to the crack surface as well as on the loading parallel to the crack surface (factor k).

4. CONCLUSION

In this paper an analytical approach in solving some elastic-plastic-mechanical problems in the thin, infinite, cracked plates has set forth. The theory of functions of a complex variable has been utilized. The theoretical bases for the analysis of plane problems by means of the theory of functions of a complex variable with special reference to the Westergaard method have been represented. The expressions for the stresses and the displacements in a complex expressing (11) have been derived, where at these fields are defined only by means of an analytical function of a complex variable $\phi(z)$. The real constant A has also been included into expressions (11), making it possible for the influence of biaxial in-plane loading of the plate upon the stresses and the displacements to be taken into account.

For describing the plastification process in the thin, infinite, cracked plates an original Dugdale strip yield model based on the Tresca yield criterion has been employed. In order to take into account the influence of the loading applied in a direction parallel to the crack plane the conventional Dugdale strip yield model for crack tip plasticity has been modified in this paper, with the help of the von Mises yield criterion. The expression (17) gives the displacements in the direction of the x-axis, of the points lying on the crack line, by arbitrary biaxial in-plane loading of the cracked plate. It is seen that the load acting in a direction parallel to the crack plane will also influence on the magnitude of the displacements of these points. The analytical expressions (18) and (19) and the diagrams on Fig. 1 and Fig. 2 are showing the influence of the loads parallel to the crack surface on the magnitude of the plastic zone r_p around the crack tip and on the crack tip opening displacement (CTOD).

Finally, it can be concluded that the Dugdale strip yield model, both in its original (conventional) and modified version, is appropriate and reliable for determination and analysis of some fracture mechanics parameters such as: the crack tip opening displacement (CTOD), the J-integral and the magnitude of the plastic zone r_p around the crack tip, and particularly for the analysis of the displacements field and the stress field in the thin, infinite, cracked plates. The accuracy and the reliability has been verified by means of the finite elements method (FEM).

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THE INVESTIGATION OF THE METAL FORMING PROCESSES USING NONLINEAR PROGRAMMING TECHNIQUE

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Abstract

The problem of compression of a long compact or perforated spring by a flat punch is investigated. The solution is based on the slip-line theory. To get numerical results are used matrix operators and nonlinear programming technique. The distribution of stresses and displacements in the strip is obtained.

1. INTRODUCTION

The slip-line method is largely used in solving the problems of metal forming processes. But the effectiveness of this method is reduced by the necessity of making use of trial and error procedure in constructing slip-line field. The matrix analysis of the slip-line fields developed by Ewing, Johnson and Dewhurst make possible to avoid this procedure and to reduce the problem to that of finding the initial slip-lines from which the complete field can be constructed from the matrix equation. In this paper the nonlinear programming technique is used to solve this equation. As illustration the procedure is applied of the problem of the action of the flat rigid smooth punch at the rigid-plastic strip, reposing on a smooth foundation and that of the action of two such punches, located oppositely on strip which quadratic hole. The calculations were performed by the help of the program Complex, which included the main program, the set of subroutines to build matrix operators and nonlinear optimization program.

2. MATHEMATICAL DESCRIPTION AND RESULTS

The quasistatistical plane plastic flow of metal may be described by the following set of equations (1):

$$\frac{\partial \sigma}{\partial x} - 2K \left(\cos 2\alpha \frac{\partial \alpha}{\partial x} + \sin 2\alpha \frac{\partial \alpha}{\partial y} \right) = 0 \quad (1)$$

$$\frac{\partial \sigma}{\partial y} - 2K \left(\sin 2\alpha \frac{\partial \alpha}{\partial x} + \cos 2\alpha \frac{\partial \alpha}{\partial y} \right) = 0 \quad (2)$$

$$\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} + \operatorname{tg} 2\alpha \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) = 0 \quad (3)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \quad (4)$$

where $\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2}$ the halfsum of the normal stresses at x, y plane, the angle between x-axis and a normal to the maximum shear stresses are counted in a clockwise sense of a positive direction of the x-axis;

V_x, V_y - the projections of velocity vector to the axes;

K - yield shear stress.

The closed set of partial equations (1-4) permits to specify the functions σ, α, V_x, V_y which determine stresses and velocities of a plastic region of a body. To do it practically one can make use of a slip-line method. Let ξ, η to be the axes which touch the maximum stresses areas at every point. The directions of these axes are such, that ξ -axis makes the angle equal $\pi/4$ with the positive direction of x-axis, counted in clockwise sense of the positive direction of the σ -stress, and η -axis - the angle equal $\pi/4$ in the opposite direction. In these axes the plastic deformations of material are described by the equations [1]:

$$\sigma - 2K\alpha = c \quad (\eta = \text{const}) \quad (5)$$

$$\sigma + 2K\alpha = c \quad (\xi = \text{const})$$

$$dV\xi - V\eta d\alpha = 0 \quad (\eta = \text{const})$$

$$dV\eta + V\xi d\alpha = 0 \quad (\xi = \text{const}) \quad (6)$$

Relations (5), (6) are equivalent to the equations (1) - (4) and give the solution of the problem. At the same time the slip-line field is not known beforehand, depends upon the boundary conditions and may be determined as a result of solution or at the processes of it. Such is the effect of nonlinearity of the initial set of equations. The usual way of solving such problems is based on trial and error procedure. This way becomes not effective in a number of technical problems. It is much more convenient to use the approach, based on the matrix operators technique, permitting to reduce the problem to the inverting nonlinear matrix

equation [2]. Boundary and other conditions may be regarded as objective functions and restrictions and the inversion of matrix equation may be performed with the methods of nonlinear programming [3], [4]. The radius of curvature of a slip-line is defined by

$$\frac{1}{\rho} = \pm \left| \frac{\partial \varphi}{\partial S} \right| \quad (7)$$

where φ is the inclination of the local tangent to that at the base point and dS is the differential of arc length. The plus or minus sign is taken according to the φ increase in counter-clockwise or clockwise sense along the slip-line [5].

The radii of curvature of the two given slip-line may be expanded as a power series in the angular coordinate

$$R_0(\xi) = \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}, \quad S_0(\eta) = \sum_{n=0}^{\infty} b_n \frac{\eta^n}{n!} \quad (8)$$

The matrix representation of these expansions are

$$R_0(\xi) = [\xi_0 \ \xi_1 \ \dots \ \xi_n] \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad S_0(\eta) = [\eta_0 \ \eta_1 \ \dots \ \eta_n] \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix} \quad (9)$$

where

$$\xi_n = \frac{\xi^n}{n!}, \quad \eta_n = \frac{\eta^n}{n!} \quad (10)$$

For the arbitrary slip-lines through the point p the corresponding representations are

$$\bar{Z}_3 = P_\psi^* \bar{Z}_1 + Q_\psi^* \bar{Z}_2, \quad \bar{Z}_4 = P_\theta^* \bar{Z}_2 + Q_\theta^* \bar{Z}_1 \quad (11)$$

where

$$P_\psi^* = \begin{bmatrix} \varphi_0 & 0 & 0 & \dots \\ \varphi_1 & \varphi_0 & 0 & \dots \\ \varphi_2 & \varphi_1 & \varphi_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad Q_\psi^* = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \dots \\ \varphi_2 & \varphi_3 & \varphi_4 & \dots \\ \varphi_3 & \varphi_4 & \varphi_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (12)$$

matrix operators, which generate the field on the convex side of a given slip-line φ , θ , ψ - angular span [2].

Using the superposition principle of Hill one can obtain other matrix operators necessary for constructing slip-lines field, such as the reversion matrix operator R , which reverses the intrinsic direction of a given slip-line with angular span and the shift operator S , which translates the origin of a slip-line through an angular distance in the intrinsic direction

$$R_\varphi = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \dots \\ 0 & -\varphi_0 & -\varphi_1 & \dots \\ 0 & 0 & \varphi_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad S_\varphi = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \dots \\ 0 & \varphi_0 & \varphi_1 & \dots \\ 0 & 0 & \varphi_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (13)$$

the operators of constructing a field in a centered fan

$$P_{\theta\psi} = R_{\theta} P_{\psi}^*, \quad Q_{\theta\psi} = R_{\psi} Q_{\theta}^* \quad (14)$$

and the operator of a smooth boundary

$$T_{\varphi} = -(P_{\varphi\varphi} + Q_{\varphi\varphi}) \quad (15)$$

The Cartesian co-ordinates x, y of any point of a particular slip-line is convenient to determine indirectly by means of moveable co-ordinates $\bar{x}, \bar{y}$, which are introduced by relations

$$x = \bar{x} \cos \varphi \pm \bar{y} \sin \varphi, \quad y = \pm \bar{x} \sin \varphi + \bar{y} \cos \varphi \quad (16)$$

In moveable co-ordinates exists a simple relation between the coefficients in power series expansions of these and of radii of curvature. Thus if $R(\xi) = \sum_{n=0}^{\infty} r_n \frac{\varphi^n}{n!}$, then

$$\bar{x}(\varphi) = \pm \sum_{n=0}^{\infty} t_n \frac{\varphi^n}{n!}, \quad \bar{y}(\varphi) = \sum_{n=0}^{\infty} t_n \frac{\varphi^{n+1}}{(n+1)!} \quad (17)$$

$$t_{n+1} = t_{n-1} = |r_n|, \quad t_0 = 0, \quad t_1 = |r_0|$$

Force components, acting on the concave side of the slip-line, are

$$\frac{F_x}{K} = \left(\frac{F_x^-}{K} \right) \cos \varphi - \left(\frac{F_y^-}{K} \right) \sin \varphi, \quad (18)$$

$$\frac{F_y}{K} = \left(\frac{F_y^-}{K} \right) \cos \varphi - \left(\frac{F_x^-}{K} \right) \sin \varphi,$$

where

$$\begin{aligned} \frac{F_x^-}{K} &= -\bar{x} + \frac{P_0}{K} \bar{y} - 2 \sum_{n=0}^{\infty} C_{n-1} \frac{\varphi^n}{n!}, \\ \frac{F_y^-}{K} &= \bar{y} + \frac{P_0}{K} \bar{x} - 2 \sum_{n=0}^{\infty} C_n \frac{\varphi^n}{n!}, \end{aligned} \quad (19)$$

$$C_{n+1} = C_{n-1} = n r_{n-1}, \quad C_0 = -C_{-1} = 0.$$

Series representation for the moment M does not lead to any simple recurrence relations and recourse must be made to a numerical integrations. The corresponding integral in terms of moveable co-ordinates for the slip-line is

$$\frac{M}{K} = \int_0^{\varphi} \left[\left(\frac{P_0}{K} - 2\chi \right) \bar{x}(\chi) + \bar{y}(\chi) \right] R(\chi) d\chi \quad (20)$$

where χ is the current value of the angle.

The practice of solving problems of plastic flow by means of above described relations consists in a preliminary fixing of the slip-lines netting, forming the nonlinear matrix equation, inverting of it using nonlinear programming and obtaining so more precise radii of curvature of the initial slip-lines netting. The process is repeated until the radii of curvature remain unchangeable within prescribed limits. As an example and proof of the validity of the method

was investigated the problem of the pressure of the flat smooth punch at the width $2a$ the rigid-plastic strip, of the thickness b , reposing on a smooth foundation. The punch is assumed to come down at a speed $V = -V_0$, along the contacting part of the x-axis acts the uniform pressure $-p$ and do here $\sigma_y = -P$, $\tau_{xy} = 0$, $V_n = V_0$. Along the y-axis in view of symmetry for shear stresses, and displacements V along x-axis, are valid relation $\tau_{xy} = 0$, $u = 0$. The slip-line field and godograf are given in Figures 1 a and 1 b. The slip-lines AC and EC can be presented through basic vectors

$$\bar{Z}_{AB} = \bar{Z}_1, \quad \bar{Z}_{EC} = \bar{Z}_2 \quad (21)$$

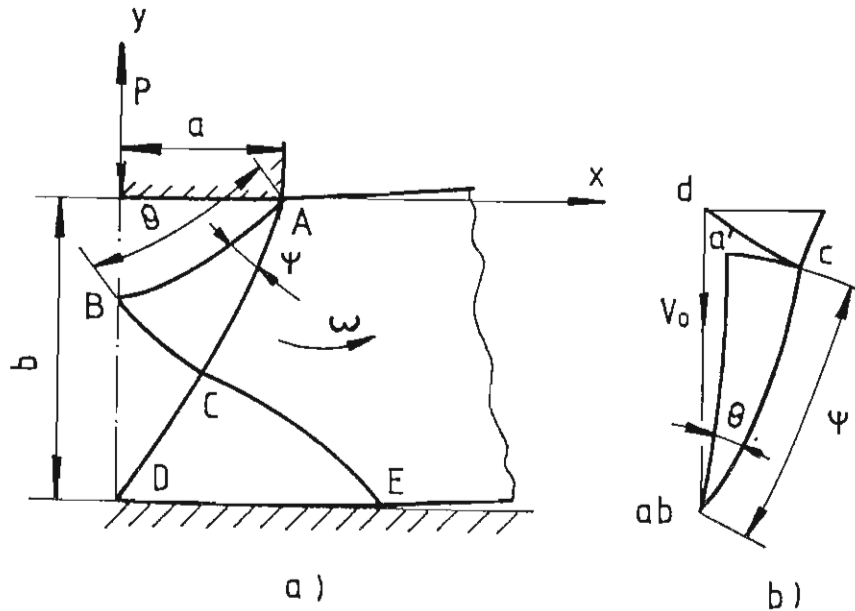


Fig. 1 The pressure of a smooth punch at the rigid-plastic strip:
a - slip-lines net, b- godograf.

The corresponding lines ac and ec of godograf diagram are the:

$$\bar{Z}_{ac} = \bar{Z}_1 \omega, \quad \bar{Z}_{ec} = \bar{Z}_2 \omega \quad (22)$$

For the physical plane

$$\bar{Z}_{AB} = P_\psi^* \bar{Z}_1 \quad (23)$$

$$\bar{Z}_{BC} = Q_\theta^* P_\psi^* \bar{Z}_1 \quad (24)$$

$$\bar{Z}_{DC} = T_\psi \bar{Z}_2 \quad (25)$$

$$\bar{Z}_{BC} = T_\psi^2 \bar{Z}_2 \quad (26)$$

For the godograf diagram

$$\bar{Z}_{bc} = T^{-2} \bar{Z}_2 \omega \quad (27)$$

$$\bar{Z}_{bc} = P_0 Q_\psi^{*-1} \bar{Z}_1 \omega \quad (28)$$

The equations (27) and (28) give one relation between Z and Z_1 . One more relation give equations (24) and (26). After simplifications these relations give the final matrix equation

$$(Q_{\psi}^* P_{-\theta} T_{\psi}^A Q_{\theta}^* P_{-\psi}) \bar{Z}_1 = \bar{Z}_1$$

The numerical results are obtained with help of the program "Flexiplex" [6], which realizes the above mentioned technique of the slipping tolerance criterion.

The results are presented in terms of nondimensional variables with a-characteristic length and $2K$ -characteristic stress $\sigma_x = n \cdot \bar{p}$, where $\bar{p} = p/2K$, $n=1,2,\dots$

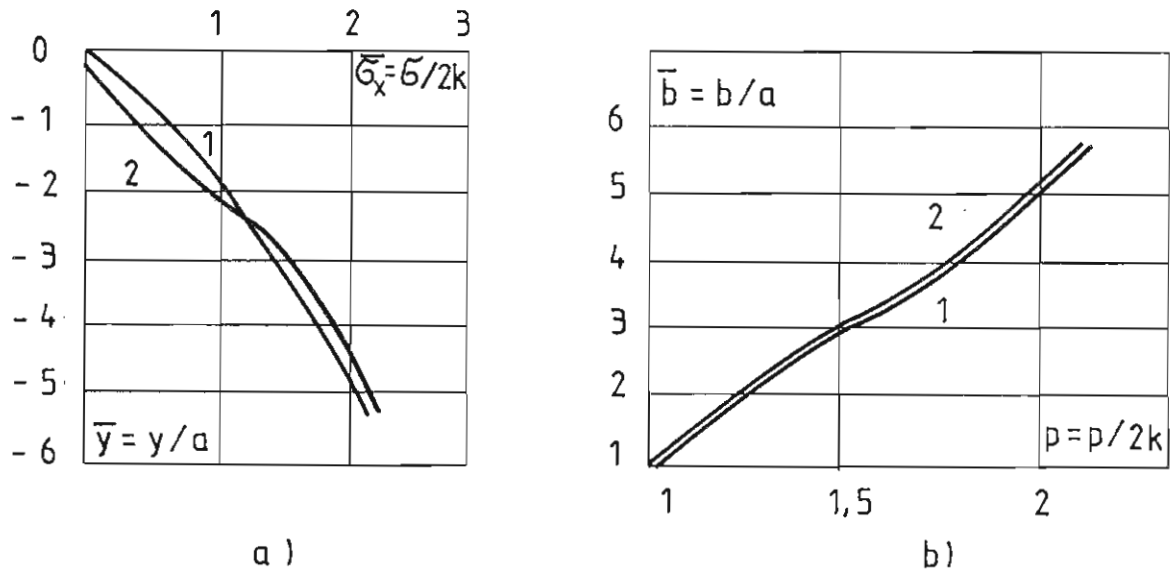


Fig. 2. Numerical results: a - the distribution of σ_x component along the negative half-axis y at the pressure $p=1$ (1 - authors' results, 2 - results from the reference [1]); b - the undimensional pressure $\bar{p}$ versus undimensional width $\bar{b}$ (1 - authors' results, 2 - results from the reference [1])

In Fig. 2 (curve a) is shown the relation between the nondimensional pressure and nodimensional thickness of the strip. In the same figure (curve b) is produced the result received by Sokolowsky [1].

In Fig. 3. are presented the results for the strip with a rectangular hole (the dependence of the nondimensional normal stress σ_x upon the co-ordinate). The curve (a) corresponds to the case of the absence of a hole, the curve (b) to the quadrat hole with the dimensions 4×4 mm, and the curve (c) - to the hole 10×10 mm, the total thickness of the strip being 200 mm. The graphs show the noticeable influence of the holes on the magnitude of the normal stresses along y -axis.

The comparison of the numerical results with the data, obtained by other authors, testify the effectiveness of the proposed approach and the possibility of using it in solving different problems of the metal forming. Besides of this, after choosing the available restrictions and objective function, the method can be used for controlling the parameters of the metal forming processes.

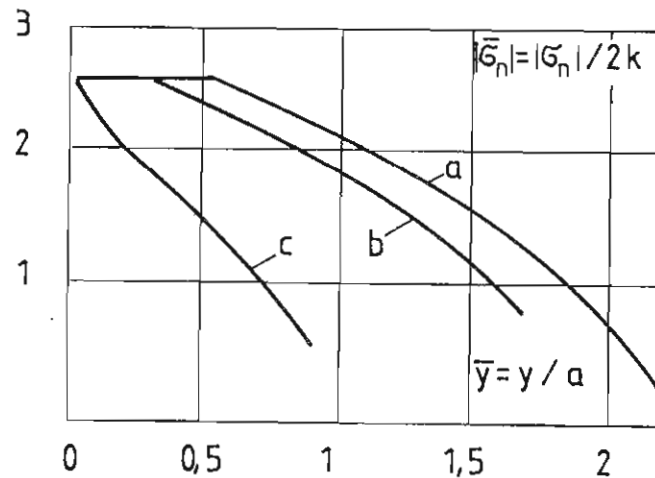


Fig. 3. The distribution of the nondimensional normal stress $|\bar{\sigma}_n| = |\sigma_n|/2K$ upon the co-ordinate axis $\bar{y}=y/a$ in the strip of the thickness 100 mm
 a) the strip without a hole
 b) the strip with a hole 4 x 4 mm
 c) the strip with a hole 10 x 10 mm

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Radially Symmetric Solutions of Some Elliptic Equations

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Abstract. This paper is concerned with the qualitative analysis of solutions of some elliptic equations of the form $\Delta u + f(u, |x|) = 0$ for $|x| \geq a > 0$, $x \in \mathbb{R}^n$, $n \geq 2$. Under the corresponding sufficient conditions on the function f , the existence, behavior and approximation of certain solutions are established.

1. INTRODUCTION

Consider the semilinear elliptic equation

$$\Delta u + f(u, |x|) = 0 \quad (1)$$

in the annulus $\Omega = \{x \in \mathbb{R}^n \mid |x| > a\}$, $n \geq 2$, with boundary conditions

$$u(a) = B \text{ and } u'(|x|) = 0 \text{ at } |x| = \infty \text{ or} \quad (2)$$

$$u(|x|) = C \text{ and } u'(|x|) = 0 \text{ at } |x| = \infty, \quad (3)$$

where Δ is the n -dimensional Laplacian, $f: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a C^1 nonlinear function, $a \in \mathbb{R}^+ = (0, \infty)$, $B, C \in \mathbb{R}$.

The equations of the form (1) arise in many areas of applied mathematics including many physical fields, fluid mechanics and population genetics and have been studied extensively in recent years.

Radial solutions of such equations have been considered by various authors by variational techniques or by shooting methods (see e.g. [1-6, 9]). Here we use the qualitative analysis theory and the topological retraction method ([7-8]). The results obtained in this paper extend and supplement the known results and it gives the new proceeding for studying of radial solutions of the equations of the form (1).

As usual, because we are seeking for radial solutions $u(|x|)$ of (1), we consider the following ordinary differential equation ($r = |x|$)

$$u''(r) + \frac{n-1}{r} u'(r) + f(u(r), r) = 0 \quad (4)$$

with boundary conditions (2) or (3) $\left(' = \frac{d}{dr} \right)$. Equations of the form (4) appear in various problems in mechanic.

In this paper we establish the existence, behavior and approximation:

(i) a one-parameter family of radial solutions $u(r)$ of the problem (1), (2) such that

$$u'(r) = O(r^{1-n}) \text{ as } r \rightarrow \infty; \quad (5)$$

(ii) at least one radial solution $u(r)$ of the problem (1), (3) with the asymptotic behavior

$$u(r) = C + O(r^{2-n}) \quad \text{as } r \rightarrow \infty. \quad (6)$$

We shall study the equation (4) by means of the equivalent system

$$u' = v, \quad v' = -\frac{n-1}{r}v - f(u, r), \quad r' = 1, \quad (7)$$

which satisfies the conditions for existence and uniqueness of solutions of Cauchy problem in the set $\Sigma = \mathbb{R}^2 \times \mathbb{R}^+$.

In Section 2 we give the main results of this paper, and in Section 3 we give some particular results. We apply some of these results in Section 4 to the well-known Matukuma equation, as an illustration of possibilities of application of the obtained results in Sections 2 and 3. In Section 5 we give the remark on approximation of solutions whose existence are established.

2. MAIN RESULTS

Let

$$\varphi(r) = \begin{cases} Ar^{2-n} + C & \text{for } n \geq 3 \\ C & \text{for } n = 2 \end{cases}, \quad A, C \in \mathbb{R}.$$

We consider the behavior of the integral curves $(u(r), r)$, $r \geq a$, of (4) in the corresponding neighborhoods of the curve $(\varphi(r), r)$, $r \geq a$.

Theorem 1. *If the function f satisfies the condition*

$$|f(u, r)| < \alpha\beta(\beta - n + 2)r^{-\beta-2} \quad \text{on} \quad (8)$$

$$\omega_1 = \{(u, r) \in \mathbb{R}^2 \mid |u - \varphi(r)| \leq \alpha r^{-\beta}, r \geq a\},$$

where $\alpha, \beta \in \mathbb{R}^+$ and $\beta > n - 2$, then the problem (1), (3) has at least one radial solution $u(r)$ satisfying the conditions

$$|u(r) - \varphi(r)| < \alpha r^{-\beta} \quad \text{and} \quad |u'(r) - \varphi'(r)| < \alpha\beta r^{-\beta-1} \quad \forall r \geq a. \quad (9)$$

Proof. Since we are only interested in radial solutions of (1), we consider the equivalent equation (4). We shall study the equation (4) by means of the equivalent system (7). Let us consider the behavior of integral curves $(u(r), v(r), r)$ of the system (7) with respect to the sets Σ and

$$\sigma_1 = \{(u, v, r) \in \Sigma \mid |u - \varphi(r)| < \alpha r^{-\beta}, |v - \varphi'(r)| < \alpha\beta r^{-\beta-1}, r > a\}.$$

The boundary surfaces of σ_1 are

$$U_i = \{(u, v, r) \in \text{Cl}\sigma_1 \mid \Phi_i \equiv (-1)^i [u - \varphi_i(r)] = 0\}, \quad (10)$$

$$V_i = \{(u, v, r) \in \text{Cl}\sigma_1 \mid \Psi_i \equiv (-1)^i [v - \varphi'_i(r)] = 0\}, \quad i = 1, 2, \quad (11)$$

where $\varphi_1(r) = \varphi(r) - \alpha r^{-\beta}$, $\varphi_2(r) = \varphi(r) + \alpha r^{-\beta}$,
 $\psi_1(r) = \varphi'(r) - \alpha\beta r^{-\beta-1}$, $\psi_2(r) = \varphi'(r) + \alpha\beta r^{-\beta-1}$.

Let us denote by X the vector field defined by (7). By means of the sign of the scalar products

$$\Pi u_i = (\nabla \Phi_i, X) \text{ on } U_i, \quad \Pi v_i = (\nabla \Psi_i, X) \text{ on } V_i, \quad i = 1, 2$$

we shall establish the behavior of integral curves of (7) with respect to the sets Σ and σ_1 . The vector $\nabla \Phi_i$ and $\nabla \Psi_i$ are vectors of external normal on surfaces U_i and V_i respectively.

Now, in view of (8), we have

$$\begin{aligned} \Pi u_1 &= -v + \varphi'_1(r) > -\psi_2(r) + \varphi'_1(r) = 0 && \text{on } U_1 \setminus L, \\ \Pi u_1 &= -\psi_2(r) + \varphi'_1(r) = 0 && \text{on } L = U_1 \cap V_2, \\ \Pi u_2 &= v - \varphi'_2(r) > \psi_1(r) - \varphi'_2(r) = 0 && \text{on } U_2 \setminus K, \\ \Pi u_2 &= \psi_1(r) - \varphi'_2(r) = 0 && \text{on } K = U_2 \cap V_1, \\ \Pi v_1 &= \frac{n-1}{r} v + f(u, r) + \psi'_1(r) \geq \frac{n-1}{r} \psi_1(r) + \psi'_1(r) + f(u, r) \\ &= \alpha\beta(\beta - n + 2)r^{-\beta-2} + f(u, r) > 0 && \text{on } V_1, \\ \Pi v_2 &= -\frac{n-1}{r} v - f(u, r) - \psi'_2(r) \geq -\frac{n-1}{r} \psi_2(r) - \psi'_2(r) - f(u, r) \\ &= \alpha\beta(\beta - n + 2)r^{-\beta-2} - f(u, r) > 0 && \text{on } V_2. \end{aligned}$$

Moreover, $u'(L) = v(L) = \psi_2(r) = \varphi'_1(r)$, $u''(L) = v'(L) = -\frac{n-1}{r} \psi_2(r) - f(\varphi_1, r) > \varphi''_1(r)$,
 $u'(K) = v(K) = \psi_1(r) = \varphi'_2(r)$, $u''(K) = v'(K) = -\frac{n-1}{r} \psi_1(r) - f(\varphi_2, r) < \varphi''_2(r)$.

Thus, the curves L and K are the sets of points of exit, but also strict exit because $\Pi v_1(K) > 0$ and $\Pi v_2(L) > 0$. Consequently, the set $S_1 = \partial\sigma_1 = U_1 \cup U_2 \cup V_1 \cup V_2$ is a set of points of strict exit of integral curves of (7) with respect to the sets σ_1 and Σ . Hence, according to the retraction method ([7]), the system (7) has at least one solution $(u(r), v(r))$ which satisfies the conditions

$$\varphi_1(r) < u(r) < \varphi_2(r) \text{ and } \psi_1(r) < v(r) < \psi_2(r) \quad \forall r > a. \quad (12)$$

For $r=a$ the conditions (12) also hold, because the system (7), $\forall r > 0$, satisfies the conditions for existence of solutions.

Therefore, the equation (4), as well as (1), has at least one solution $u(r)$ satisfying the conditions (9), (3) and (6).

Theorem 2. *If*

$$|f(u, r)| < \gamma \delta a^\delta (\delta - n + 2) r^{-\delta-2} \quad \text{on} \quad (13)$$

$$\omega_2 = \left\{ (u, r) \in \mathbb{R}^2 \mid |u - \varphi(r)| \leq \gamma \left[1 - \left(\frac{a}{r} \right)^\delta \right], r \geq a \right\},$$

where $\gamma, \delta \in \mathbb{R}^+$ and $\delta > n-2$, then the problem (1), (2), where $B = \varphi(a)$, has a one-parameter family of radial solutions $u(r)$ which satisfy the conditions

$$|u(r) - \varphi(r)| < \gamma \left[1 - \left(\frac{a}{r} \right)^\delta \right] \quad \text{and} \quad |u'(r) - \varphi'(r)| < \gamma \delta a^\delta r^{-\delta-1} \quad \forall r > a. \quad (14)$$

Proof. Let us consider the behavior of integral curves $(u(r), v(r), r)$ of (7) with respect to the sets Σ and

$$\sigma_2 = \left\{ (u, v, r) \in \Sigma \mid |u - \varphi(r)| < \gamma \left[1 - \left(\frac{a}{r} \right)^\delta \right], |v - \varphi'(r)| < \gamma \delta a^\delta r^{-\delta-1}, r > a \right\}.$$

The boundary surfaces of σ_2 are U_i and V_i defined by (10) and (11) respectively, where

$$\begin{aligned} \varphi_1(r) &= \varphi(r) - \gamma \left[1 - \left(\frac{a}{r} \right)^\delta \right], & \varphi_2(r) &= \varphi(r) + \gamma \left[1 - \left(\frac{a}{r} \right)^\delta \right], \\ \psi_1(r) &= \varphi'(r) - \gamma \delta a^\delta r^{-\delta-1}, & \psi_2(r) &= \varphi'(r) + \gamma \delta a^\delta r^{-\delta-1}. \end{aligned}$$

Now, in view of (13), we have

$$\begin{aligned} \Pi u_1 &= -v + \varphi'_1(r) < -\psi_1(r) + \varphi'_1(r) = 0 & \text{on } U_1 \setminus M, \\ \Pi u_1 &= -\psi_1(r) + \varphi'_1(r) = 0 & \text{on } M = U_1 \cap V_1, \\ \Pi u_2 &= v - \varphi'_2(r) < \psi_2(r) - \varphi'_2(r) = 0 & \text{on } U_2 \setminus N, \\ \Pi u_2 &= \psi_2(r) - \varphi'_2(r) = 0 & \text{on } N = U_2 \cap V_2, \\ \Pi v_1 &= \frac{n-1}{r} v + f(u, r) + \psi'_1(r) \geq \frac{n-1}{r} \psi_1(r) + \psi'_1(r) + f(u, r) \\ &= \gamma \delta a^\delta (\delta - n + 2) r^{-\delta-2} + f(u, r) > 0 & \text{on } V_1, \\ \Pi v_2 &= -\frac{n-1}{r} v - f(u, r) - \psi'_2(r) \geq -\frac{n-1}{r} \psi_2(r) - \psi'_2(r) - f(u, r) \\ &= \gamma \delta a^\delta (\delta - n + 2) r^{-\delta-2} - f(u, r) > 0 & \text{on } V_2. \end{aligned}$$

Moreover, $u'(M) = v(M) = \psi_1(r) = \varphi'_1(r)$, $u''(M) = v'(M) = -\frac{n-1}{r} \psi_1(r) - f(\varphi_1, r) < \varphi''_1(r)$,
 $u'(N) = v(N) = \psi_2(r) = \varphi'_2(r)$, $u''(N) = v'(N) = -\frac{n-1}{r} \psi_2(r) - f(\varphi_2, r) > \varphi''_2(r)$.

Thus, the curves M and N are the sets of points of entrance, but they are not the sets of points of strict entrance, because $\Pi v_1(M) > 0$ and $\Pi v_2(N) > 0$. Consequently, the set $U = (U_1 \cup U_2) \setminus (V_1 \cup V_2)$ is a set of points of strict entrance and the set $S_2 = (V_1 \cup V_2) \setminus (U_1 \cup U_2)$ is a set of points of strict exit of integral curves of (7) with respect to the sets σ_2 and Σ . Furthermore, we can note that all points of exit are points of strict exit.

Hence, according to the retraction method, the system (7) has a one parameter family of solutions $(u(r), v(r))$ satisfying the corresponding conditions (12), as well as $u(a) = \varphi(a)$ and $\varphi'_1(a) < v(a) < \varphi'_2(a)$, because the system

(7) (for $r > 0$) satisfies conditions for existence of solutions. Thus, the equation (4), as well as (1), has a one-parameter family of solutions $u(r)$ satisfying the conditions (14), (2) (with $B = \varphi(a)$) and (5).

Theorem 3. Assume that the function f satisfies the condition

$$\begin{aligned} |f(u, r)| &< \mu \left(\eta - \frac{n-1}{r} - \frac{1}{2} \right) e^{-\eta r} \quad \text{on} \\ \omega_3 &= \left\{ (u, r) \in \mathbb{R}^2 \mid |u - \varphi(r)| \leq \mu e^{-\eta r}, r \geq a \right\}, \end{aligned} \quad (15)$$

where $\mu, \eta \in \mathbb{R}^+$ and $2a\eta > a + 2(n-1)$. Then the problem (1), (3) has at least one radial solution $u(r)$ with the property

$$[u(r) - \varphi(r)]^2 + [u'(r) - \varphi'(r)]^2 < \mu^2 e^{-2\eta r} \quad \forall r \geq a. \quad (16)$$

Proof. Here we consider the behavior of integral curves of (7) with respect to the sets Σ and

$$\sigma_3 = \left\{ (u, v, r) \in \Sigma \mid [u(r) - \varphi(r)]^2 + [u'(r) - \varphi'(r)]^2 < \mu^2 e^{-2\eta r}, r > a \right\}.$$

The boundary surface of σ_3 is

$$W = \left\{ (u, v, r) \in \text{Cl}\sigma_3 \mid H \equiv [u - \varphi(r)]^2 + [v - \varphi'(r)]^2 - \mu^2 e^{-2\eta r} = 0, r > a \right\}.$$

The scalar product of the vector ∇H of external normal on surface W and the vector field X defined by (7) is

$$\begin{aligned} \Pi w &= \left(\frac{\nabla H}{2}, X \right) = [u - \varphi(r)]v - [v - \varphi'(r)] \left[\frac{n-1}{r} v + f(u, r) \right] \\ &\quad - [u - \varphi(r)]\varphi'(r) - [v - \varphi'(r)]\varphi''(r) + \mu^2 \eta e^{-2\eta r} \\ &= [u - \varphi(r)][v - \varphi'(r)] - [v - \varphi'(r)] \left[\frac{n-1}{r} v + f(u, r) + \varphi''(r) \right] + \mu^2 \eta e^{-2\eta r} \\ &\geq -\frac{\mu^2}{2} e^{-2\eta r} - \mu e^{-\eta r} \left| \frac{n-1}{r} [v - \varphi'(r)] + f(u, r) + \varphi''(r) + \frac{n-1}{r} \varphi'(r) \right| + \mu^2 \eta e^{-2\eta r} \\ &\geq -\frac{\mu^2}{2} e^{-2\eta r} - \mu e^{-\eta r} |f(u, r)| - \frac{n-1}{r} \mu^2 e^{-2\eta r} + \mu^2 \eta e^{-2\eta r} > 0 \quad \text{on } W. \end{aligned}$$

Hence, the set $S_3 = \partial\sigma_3 = W$ is a set of points of strict exit of integral curves of (7). Thus, according to the retraction method, the system (7) has at least one solution $(u(r), v(r))$ which satisfies the condition

$$[u(r) - \varphi(r)]^2 + [v(r) - \varphi'(r)]^2 < \mu^2 e^{-2\eta r} \quad \forall r \geq a.$$

Therefore, the equation (4), as well as (1), has at least one solution $u(r)$ satisfying the conditions (16), (3) and (6).

Theorem 4. Suppose

$$\begin{aligned} |f(u, r)| &< \theta \left(\lambda - \frac{n-1}{r} - 1 \right) e^{-\lambda r} \quad \text{on} \\ \omega_4 &= \left\{ (u, r) \in \mathbb{R}^2 \mid |u - \varphi(r)| \leq \theta e^{-\lambda r}, r \geq a \right\}, \end{aligned} \quad (17)$$

where $\theta, \lambda \in \mathbf{R}^+$, $a\lambda > a+n-1$. Then the problem (1), (3) has at least one radial solution $u(r)$ which satisfies the condition

$$|u(r) - \varphi(r)| + |u'(r) - \varphi'(r)| < \theta e^{-\lambda r} \quad \forall r \geq a. \quad (18)$$

Proof. Here, in the neighborhood of the curve $(\varphi(r), r)$, $r \geq a$ we use the set

$$\sigma_4 = \{(u, v, r) \in \Sigma \mid |u - \varphi(r)| + |v - \varphi'(r)| < \theta e^{-\lambda r}, r > a\}.$$

The boundary surfaces of σ_4 are

$$P_1 = \{(u, v, r) \in \text{Cl}\sigma_4 \mid G_1 \equiv (-1)^i [u - v - \varphi(r) + \varphi'(r)] - \theta e^{-\lambda r} = 0, r > a\},$$

$$Q_i = \{(u, v, r) \in \text{Cl}\sigma_4 \mid H_i \equiv (-1)^i [u + v - \varphi(r) - \varphi'(r)] - \theta e^{-\lambda r} = 0, r > a\}, \quad i=1, 2.$$

Now, in view of (17), for the scalar products

$$\Pi p_i = (\nabla G_i, X) \text{ on } P_i, \quad \Pi q_i = (\nabla H_i, X) \text{ on } Q_i, \quad i=1, 2$$

we have

$$\begin{aligned} \Pi p_1 &= (-1)^i \left[\left(\frac{n-1}{r} + 1 \right) v + f(u, r) + \varphi''(r) - \varphi'(r) \right] + \theta \lambda e^{-\lambda r} \\ &= (-1)^i \left\{ \left(\frac{n-1}{r} + 1 \right) [v - \varphi'(r)] + f(u, r) + \varphi''(r) + \frac{n-1}{r} \varphi'(r) \right\} + \theta \lambda e^{-\lambda r} \\ &\geq - \left(\frac{n-1}{r} + 1 \right) \theta e^{-\lambda r} - |f(u, r)| + \theta \lambda e^{-\lambda r} > 0 \quad \text{on } P_1, \\ \Pi q_1 &= -(-1)^i \left[\left(\frac{n-1}{r} - 1 \right) v + f(u, r) + \varphi''(r) + \varphi'(r) \right] + \theta \lambda e^{-\lambda r} \\ &= -(-1)^i \left\{ \left(\frac{n-1}{r} - 1 \right) [v - \varphi'(r)] + f(u, r) + \varphi''(r) + \frac{n-1}{r} \varphi'(r) \right\} + \theta \lambda e^{-\lambda r} \\ &\geq - \left| \frac{n-1}{r} - 1 \right| \theta e^{-\lambda r} - |f(u, r)| + \theta \lambda e^{-\lambda r} > 0 \quad \text{on } Q_1. \end{aligned}$$

Consequently, the set $S_4 = \partial\sigma_4 = P_1 \cup P_2 \cup Q_1 \cup Q_2$ is a set of points of strict exit of integral curves of (7) with respect to the sets σ_4 and Σ . Thus, according to the retraction method, the system (7) has at least one solution $(u(r), v(r))$ which satisfies the condition $|u - \varphi(r)| + |v - \varphi'(r)| < \theta e^{-\lambda r} \quad \forall r \geq a$.

Hence, the equation (4), as well as (1), has at least one solution $u(r)$ satisfying the conditions (18), (3) and (6).

3. SOME PARTICULAR RESULTS

Let $Q(r) = \alpha\beta(\beta - n + 2)r^{-\beta-2}$ and $0 \leq \varepsilon < 1$. Using the Theorem 1 we can give the following corollaries.

Corollary 1. *If the function f satisfies the condition*

$$\varepsilon Q(r) < f(u, r) < Q(r) \quad \text{on}$$

$$\omega'_1 = \{(u, r) \in \mathbb{R}^2 \mid \varphi(r) - \alpha r^{-\beta} \leq u \leq \varphi(r) - \varepsilon \alpha r^{-\beta}, r \geq a\},$$

then the problem (1), (3) has at least one radial solution $u(r)$ satisfying conditions

$$\varphi(r) - \alpha r^{-\beta} < u(r) < \varphi(r) - \varepsilon \alpha r^{-\beta} \quad \text{and} \quad \varphi'(r) + \varepsilon \alpha \beta r^{-\beta-1} < u'(r) < \varphi'(r) + \alpha \beta r^{-\beta-1} \quad \forall r \geq a.$$

Corollary 2. *If*

$$-Q(r) < f(u, r) < -\varepsilon Q(r) \quad \text{on}$$

$$\omega''_1 = \{(u, r) \in \mathbb{R}^2 \mid \varphi(r) + \varepsilon \alpha r^{-\beta} \leq u \leq \varphi(r) + \alpha r^{-\beta}, r \geq a\},$$

then the problem (1), (3) has at least one radial solution $u(r)$ satisfying conditions

$$\varphi(r) + \varepsilon \alpha r^{-\beta} < u(r) < \varphi(r) + \alpha r^{-\beta} \quad \text{and} \quad \varphi'(r) - \alpha \beta r^{-\beta-1} < u'(r) < \varphi'(r) - \varepsilon \alpha \beta r^{-\beta-1} \quad \forall r \geq a.$$

$$\text{Let } t(r) = \gamma \left[1 - \left(\frac{a}{r} \right)^\delta \right], \quad T(r) = \gamma \delta a^\delta (\delta - n + 2) r^{-\delta-2}, \quad \delta > n - 2 \quad \text{and} \quad 0 \leq \varepsilon < 1.$$

Using the Theorem 2 we can give the following corollaries.

Corollary 3. *Suppose that*

$$\varepsilon T(r) < f(u, r) < T(r) \quad \text{on}$$

$$\omega'_2 = \{(u, r) \in \mathbb{R}^2 \mid \varphi(r) + \varepsilon t(r) \leq u \leq \varphi(r) + t(r), r \geq a\}.$$

Then the problem (1), (2) ($B = \varphi(a)$) possesses a one-parameter family of radial solutions $u(r)$ with the properties

$$\varphi(r) + \varepsilon t(r) < u(r) < \varphi(r) + t(r) \quad \text{and} \quad \varphi'(r) + \varepsilon t'(r) < u'(r) < \varphi'(r) + t'(r) \quad \forall r > a.$$

Corollary 4. *Suppose that*

$$-T(r) < f(u, r) < -\varepsilon T(r) \quad \text{on}$$

$$\omega''_2 = \{(u, r) \in \mathbb{R}^2 \mid \varphi(r) - t(r) \leq u \leq \varphi(r) - \varepsilon t(r), r \geq a\}.$$

Then the problem (1), (2) ($B = \varphi(a)$) possesses a one-parameter family of radial solutions $u(r)$ with the properties

$$\varphi(r) - t(r) < u(r) < \varphi(r) - \varepsilon t(r) \quad \text{and} \quad \varphi'(r) - t'(r) < u'(r) < \varphi'(r) - \varepsilon t'(r) \quad \forall r > a.$$

4. EXAMPLE

In this Section we consider the Matukuma equation,

$$\Delta u + \frac{u^p}{1 + |x|^2} = 0, \quad p > 1, \tag{19}$$

which have been investigated extensively by various authors (see e.g. [1-4, 6]). Using the Theorem 1 and the Corollary 1, with $\varphi(r) = Ar^{2-n}$, $n \geq 3$, $A > 0$, we can give the following corollaries 5 and 6 respectively.

Corollary 5. For

$$n \geq 3, \quad A \geq \alpha > 0, \quad n-2 < \beta \leq (n-2)p \quad \text{and} \quad (A+\alpha)^p \leq \alpha \beta (\beta-n+2),$$

the problem (19), (3) (with $C=0$) has at least one positive radial solution $u(r)$ such that

$$\left| u(r) - Ar^{2-n} \right| < \alpha r^{-\beta} \quad \text{and} \quad \left| u'(r) - (2-n)Ar^{1-n} \right| < \alpha \beta r^{-\beta-1} \quad \forall r \geq a \geq 1.$$

Corollary 6. For

$$n \geq 3, \quad A > \alpha > 0, \quad 2\epsilon \alpha p(p-1)(n-2)^2 \leq (A-\alpha)^p \quad \text{and} \quad A^p \leq \alpha p(p-1)(n-2)^2$$

the problem (19), (3) (with $C=0$) has at least one positive radial solution $u(r)$ such that

$$Ar^{2-n} - \alpha r^{(2-n)p} < u(r) < Ar^{2-n} - \epsilon \alpha r^{(2-n)p} \quad \text{and} \\ (2-n)Ar^{1-n} + \epsilon \alpha (n-2)p r^{(2-n)p-1} < u'(r) < (2-n)Ar^{1-n} + \alpha (n-2)p r^{(2-n)p-1} \quad \forall r \geq a \geq 1.$$

5. REMARK ON APPROXIMATION OF SOLUTIONS

We can note that the obtained results also contain the answer to the approximation of solutions $u(r)$ whose existence is established. The errors of the approximation for solution $u(r)$ and the first derivative $u'(r)$, in the case of the Theorem 1, are defined by the functions $q(r) = \alpha r^{-\beta}$ and $-q'(r) = \alpha \beta r^{-\beta-1}$ respectively, and in the case of the Theorems 3 and 4 with the functions of the form $\rho(r) = \vartheta e^{-\zeta r}$ and $-\rho'(r) = \vartheta \zeta e^{-\zeta r}$ ($\vartheta, \zeta > 0$). These functions are sufficiently small $\forall r \geq a$ when parameters α and ϑ are sufficiently small, and tend to zero as $r \rightarrow \infty$, and the inequalities (9), (16) and (18) present sufficiently good approximations for established solutions. In the case of Corollaries 1 and 2 we have better estimates of the accuracy of approximation solutions.

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On a Hypercomplex Analysis

by

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Abstract: In this paper a hypercomplex analysis in more general algebras over hypercomplex numbers is studied and some results of the classical analysis are generalized.

It is natural to look for generalizations of the theory of functions over the complex numbers to higher dimensions. The development of physics at the end of the last century proposed new questions in mathematics. Above all that, it was necessary to find algebraic possibilities in order to advantageously carry out calculations with vector functions over 3-dimensional domains. Two discoveries in the middle of the last century were decisive for the algebraic assumption for such applications, i.e. for the development of a hypercomplex analysis. In 1843, the Irishman W.R. HAMILTON invented the quaternions as a generalization of the complex numbers by loss of commutativity. In a paper of 1844, the German mathematician H. GASSMANN introduced the algebraic ideas of the inner and outer product of multivectors.

It is well known that complex methods may be well used for the treatment of boundary value problems of partial differential equations in the plane. Moreover, it is very important to transfer results of the classical function theory to function theories over domains in $\mathbb{R}^s$ ($\mathbb{R}$ denotes the set of real numbers). A comprehensive description of Clifford Analysis¹ has been done by the research group of R. DELANGHE. The application to solving linear and nonlinear boundary value problems of partial differential equations of mathematical physics by the help of hypercomplex function theories can be found in [1].

The present paper is meant as a brief introduction to an analysis which is based on a non-associative algebra against the Clifford Algebra, where the associativity of multiplication holds. Both algebras are generated by the same elements $e_0, \dots, e_{s-1}$, $s \geq 1$, satisfying the same relations.

Let $x = (x_0, \dots, x_{s-1})$, $y = (y_0, \dots, y_{s-1})$ be ordered s -tuples of the s -dimensional real vector space $\mathbb{R}^s$, $s \geq 1$. $x = y$ iff $x_j = y_j$ for all j . Furthermore we define $x + y = (x_0 + y_0, \dots, x_{s-1} + y_{s-1})$ and $tx = (tx_0, \dots, tx_{s-1})$, $t \in \mathbb{R}$. This implicates that we can write every element x in the form $x_0 e_0 + x_1 e_1 + \dots + x_{s-1} e_{s-1}$ with the orthonormal bases $e_0 = (1, 0, \dots, 0)$, $e_1 = (0, 1, 0, \dots, 0)$, ..., $e_{s-1} = (0, \dots, 0, 1)$.

¹ which is a special hypercomplex analysis

$\langle x, y \rangle$ denotes a bilinear product $\langle \cdot, \cdot \rangle: R^s \times R^s \rightarrow R^s$ with the relations

$$\begin{aligned} \langle e_j, e_j \rangle + 1 &= 0, \quad 1 \leq j \leq s-1, & \langle e_0, e_j \rangle &= e_j, \quad 0 \leq j \leq s-1, \\ \langle e_j, e_k \rangle + \langle e_k, e_j \rangle &= 0, \quad j \neq k, \quad 1 \leq j, k \leq s-1. \end{aligned}$$

We demand that each product $\langle e_j, e_k \rangle$ again is an element of R^s and so it can be written in the form $\langle e_j, e_k \rangle = p_0 e_0 + \dots + p_{s-1} e_{s-1}$ with determined real numbers p_j ². We restrict our considerations to such hypercomplex numbers. In this way we have defined an antisymmetric hypercomplex algebra $H = (R^s, +, \langle \cdot, \cdot \rangle)$.

If $s = 1$ resp. $s = 2$ we obtain the real resp. the complex numbers. With $s = 4$ we preserve the quaternions if $\langle e_1, e_2 \rangle = e_3$, $\langle e_2, e_3 \rangle = e_1$ and $\langle e_3, e_1 \rangle = e_2$. By requiring 21 special products $\langle e_j, e_k \rangle$, $j \neq k$, with $e_1, \dots, e_7$ we can receive Cayley's octaves which form a non-commutative and non-associative 8-dimensional system of hypercomplex numbers.

Introducing the abbreviation $\tilde{x} = \sum_{j=1}^{s-1} \text{Im}_j(x) e_j$ we obtain $x = x_0 e_0 + \tilde{x} \in H$. We call $\text{Re}(x) = x_0$ the real part and $\text{Im}_j(x) = x_j$, $j = 1, \dots, s-1$, the imaginary parts of x . By $\bar{x} = x_0 e_0 - \tilde{x}$ we define the conjugate hypercomplex number of $x = x_0 e_0 + \tilde{x}$. Finally an Euclidian norm of x resp. $\tilde{x}$ is given by

$$r = \|x\| = \langle x, \bar{x} \rangle^{1/2} = \langle \bar{x}, x \rangle^{1/2} = (x_0^2 + x_1^2 + \dots + x_{s-1}^2)^{1/2}$$

resp.

$$w = \|\tilde{x}\| = (x_1^2 + \dots + x_{s-1}^2)^{1/2}.$$

In the following we consider only hypercomplex systems H where the condition

$$(1) \quad \|\langle x, y \rangle\| = \|x\| \|y\| \quad \text{for all} \quad x, y \in H$$

holds. It can be shown that the quaternions and octaves satisfy this relation.

The inverse of the hypercomplex number x ($x \neq 0$) is obtained as $x^{-1} = x \|x\|^{-2}$. A straightforward computation leads to the identities

$$\overline{x^{-1}} = (\bar{x})^{-1}, \quad \|x^{-1}\| = \|x\|^{-1}, \quad \langle x, \bar{y} \rangle = (x, y) e_0 - \sum_{0 \leq j < k \leq s-1} (x_j y_k - x_k y_j) \langle e_j, e_k \rangle,$$

$((x, y))$ is the scalar product of x and y where x, y are arbitrary elements of H . Also inequalities as

$$|\|x\| - \|y\|| \leq \|x + y\| \leq \|x\| + \|y\| \quad \text{and} \quad \langle x, \bar{y} \rangle + \langle y, \bar{x} \rangle = 2(x, y) \leq 2\|x\|^2 \|y\|^2$$

² Generally this condition is not valid in Clifford Algebras. Their elements can be written in the form $a_0 e_0 + \sum a_v E_v$ where $a_0, a_v \in R$ and v ranges over all multi-indices $v = (v_1, \dots, v_j)$ with $1 \leq v_1 < v_2 < \dots < v_j \leq s-1$ and $E_v = \langle e_{v_1}, e_{v_2}, \dots, e_{v_j} \rangle$.

can be verified. By using the norm we can resolve the equation $\langle p, x \rangle = q$ resp. $\langle x, p \rangle = q$ for any given $p, q \in H$ if the law of associativity holds. We get

$$x_L = \langle \bar{p}, q \rangle \|p\|^{-2} \quad \text{resp.} \quad x_R = \langle q, \bar{p} \rangle \|p\|^{-2}.$$

To enable the computation with polynomials and series we introduce powers with positive exponents. If n is any positive integer and $x \in H$ we define $x^n = \langle x, x^{n-1} \rangle$, $x^0 = 1$ inductively. On account of $x = x_0 e_0 + \tilde{x}$ and by the use of the relations³

$$(2) \quad \tilde{x}^{2k} = (-1)^k w^{2k}, \quad \tilde{x}^{2k+1} = (-1)^k w^{2k} \tilde{x}, \quad k \in N_0,$$

holds

Proposition 1:

If $\sum_{k=0}^{-1} := 0$ and $\left[\frac{n}{2} \right]$ denotes the integer part of $\frac{n}{2}$, then

$$\operatorname{Re}(x^n) = \sum_{0 \leq k \leq [n/2]} (-1)^k \binom{n}{2k} x_0^{n-2k} w^{2k},$$

$$\operatorname{Im}_j(x^n) = x_j \sum_{0 \leq k \leq [(n-1)/2]} (-1)^k \binom{n}{2k+1} x_0^{n-2k-1} w^{2k}, \quad j = 1, \dots, s-1,$$

for every hypercomplex number x and $n \in N_0$.

The introduction of the angle $\varphi = \arctan(w/x_0)$ allows another decomposition of the n -th power x^n . We get (see [2])

$$(3) \quad \operatorname{Re}(x^n) = r^n \cos(n\varphi), \quad \operatorname{Im}_j(x^n) = r^n \frac{x_j}{w} \sin(n\varphi), \quad w \neq 0, \quad j = 1, \dots, s-1.$$

As for complex values we define the exponential function and the trigonometric functions on H by the Taylor series

$$e^x = \sum_{n \geq 0} \frac{x^n}{n!}, \quad \sin x = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos x = \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} x^{2n},$$

which converge for all finite $x \in H$.

³ $N_0 = \{0, 1, 2, 3, \dots\}$

Proposition 2:

If $x \in H$, then

- (i) $(e^x)^n = e^{nx}$, $n \in N_0$, (ii) $\sin^2 x + \cos^2 x = 1$,
 (iii) $\sin(2x) = 2 \sin x \cos x$, (iv) $\cos(2x) = \cos^2 x - \sin^2 x$.

In principal, all identities with one variable can be transfered to H . On the contrary relations with two or more different variables only have validity under restrictions, because the bilinear product is not associative in general.

If $n = 1$ in (3), it is easy to see that a hypercomplex number x may be written as

$$(4) \quad x = x_0 e_0 + \tilde{x} = r \left(\cos \varphi + \frac{\sin \varphi}{w} \tilde{x} \right).$$

Proposition 3:

Let $x = r \left(\cos \varphi + \frac{\sin \varphi}{w} \tilde{x} \right)$, $y = \hat{r} \left(\cos \hat{\varphi} + \frac{\sin \hat{\varphi}}{\hat{w}} \tilde{y} \right)$ and $s \geq 3$. Then

- (i) $e^{x+y} = e^x e^y$ and (ii) $x y^{\pm 1} = r \hat{r}^{\pm 1} \left(\cos(\varphi \pm \hat{\varphi}) + \frac{\sin(\varphi \pm \hat{\varphi})}{w} \tilde{x} \right)$ hold iff $\tilde{x} = \tilde{y}$.

Corollary: (De Moivres Theorem)⁴

$$x^n = r^n \left(\cos(n\varphi) + \frac{\sin(n\varphi)}{w} \tilde{x} \right), \quad n \in \mathbb{Z} \quad (\text{see}[3]).$$

In the following we intend to find a connection between the hypercomplex exponential and trigonometric functions. I.e., we require a function $e(\varphi)$ with the norm 1 and for which $e(\varphi) = \cos \varphi + \frac{\sin \varphi}{w} \tilde{x}$ holds (see(4)). We obtain

Proposition 4: (Euler's Formular)

$$\exp \left(\frac{\varphi}{w} \tilde{x} \right) = \cos \varphi + \frac{\sin \varphi}{w} \tilde{x}, \quad \varphi = \arctan \left(\frac{w}{x_0} \right), \quad w = \|\tilde{x}\|.$$

Proof:

Substituting $\frac{\varphi}{w} \tilde{x}$ in the hypercomplex exponential series yields the desired formula.

⁴ $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Especially $\exp\left(\frac{\pi}{w}\tilde{x}\right) = -1$. Subtracting or adding

$$\exp\left(\frac{\varphi}{w}\tilde{x}\right) = \cos\varphi + \frac{\sin\varphi}{w}\tilde{x} \quad \text{and} \quad \exp\left(-\frac{\varphi}{w}\tilde{x}\right) = \cos\varphi - \frac{\sin\varphi}{w}\tilde{x}$$

and replacing the real φ by the hypercomplex x leads to

Proposition 5:
$$\sin x = -\frac{\tilde{x}}{2w}\left(e^{\frac{\tilde{x}}{w}x} - e^{-\frac{\tilde{x}}{w}x}\right), \quad \cos x = \frac{1}{2}\left(e^{\frac{\tilde{x}}{w}x} + e^{-\frac{\tilde{x}}{w}x}\right).$$

To verify these identities which also may be used as definitions for $\sin x$ and $\cos x$ we use the power series.

Closely related to the hypercomplex trigonometric functions $\sin x$ and $\cos x$ are the hypercomplex hyperbolic functions $\sinh x$ and $\cosh x$ defined by

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{1}{2}(e^x + e^{-x}).$$

From the definitions of $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ by the exponential functions we have

Proposition 6:

$$(5) \quad \sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \sinh(-x) = -\sinh x, \quad \cosh(-x) = \cosh x,$$

$$(6) \quad \sin x = -\frac{\tilde{x}}{w}\sinh\left(\frac{\tilde{x}}{w}x\right), \quad \cos x = \cosh\left(\frac{\tilde{x}}{w}x\right), \quad \sinh x = w\tilde{x}^{-1}\sin\left(\frac{\tilde{x}}{w}x\right), \quad \cosh x = \cos\left(\frac{\tilde{x}}{w}x\right),$$

$$(7) \quad \cosh^2 x - \sinh^2 x = 1, \quad \sinh(2x) = 2 \sinh x \cosh x,$$

$$(8) \quad \sin x = \sin x_0 \cosh w + \frac{\tilde{x}}{w} \cos x_0 \sinh w, \quad \cos x = \cos x_0 \cosh w - \frac{\tilde{x}}{w} \sin x_0 \sinh w,$$

$$\sinh x = \sinh x_0 \cosh w + \frac{\tilde{x}}{w} \cosh x_0 \sinh w, \quad \cosh x = \cosh x_0 \cosh w + \frac{\tilde{x}}{w} \sinh x_0 \sinh w,$$

$$(9) \quad \|\sin x\|^2 = \sin^2 x_0 + \sinh^2 w = \cosh^2 w - \cos^2 x_0,$$

$$\|\cos x\|^2 = \cos^2 x_0 + \sinh^2 w = \cosh^2 w - \sin^2 x_0,$$

$$\|\sinh x\|^2 = \sinh^2 x_0 + \sin^2 w = \cosh^2 x_0 - \cos^2 w,$$

$$\|\cosh x\|^2 = \cosh^2 x_0 + \cos^2 w = \sinh^2 x_0 - \sin^2 w.$$

- (10) *The Addition Theorems for the trigonometric and hyperbolic functions only hold for each pair of hypercomplex numbers $x = x_0 + \tilde{x}$ and $y = y_0 + \tilde{y}$ iff $\tilde{x} = \tilde{y}$.*

Note that the formulas (8) which imply the real and imaginary parts of our functions also show that the zeros of $\sin x$ resp. $\cos x$ only occur at the real points $x = n\pi$ resp. $x = (2n+1)\pi/2$. The only zeros of $\sinh x$ resp. $\cosh x$ are purely imaginary for the values $x = \tilde{x}$ with the norm $\|\tilde{x}\| = n\pi$ resp. $\|\tilde{x}\| = (2n+1)\pi/2$, where n is any integer.

Also, we deduce from (8) that $\sin x$ and $\cos x$ are periodic, with period 2π . Similarly, it can be verified easily that $(2\pi/w)\tilde{x}$ is the fundamental period of the hyperbolic functions. Furthermore, the relations (8) conclude the identities (9) which imply the following inequalities:

Corollary:

$$\left. \begin{array}{l} |\sin x_0| \\ |\sinh w| \end{array} \right\} \leq \|\sin x\| \leq \cosh w, \quad \left. \begin{array}{l} |\cos x_0| \\ |\sinh w| \end{array} \right\} \leq \|\cos x\| \leq \cosh w,$$

$$\left. \begin{array}{l} |\sinh x_0| \\ |\sin w| \end{array} \right\} \leq \|\sinh x\| \leq \cosh x_0, \quad \left. \begin{array}{l} |\sinh x_0| \\ |\cos w| \end{array} \right\} \leq \|\cosh x\| \leq \cosh x_0.$$

Other inequalities can be found in [3].

Further hypercomplex trigonometric and hyperbolic functions as well as inverse functions can be defined in the usual sense, as for complex values.

The definition of the hypercomplex logarithm for $x \neq 0$ and $x \in H$ is made in [3] by

$$z_n = \log x = \log r + \frac{\tilde{x}}{w}(\varphi + 2n\pi), \quad n \in \mathbb{Z}.$$

It is an infinitely many-valued function and z_0 is called the principal hypercomplex logarithm if $0 \leq \varphi < 2\pi$. A Taylor series expansion of the form

$$\log(1+x) = \log\|1+x\| + \frac{\tilde{x}}{w} \arg(1+x) = \sum_{n \geq 1} (-1)^{n-1} \frac{x^n}{n}, \quad \|x\| < 1,$$

with the derivation

$$\frac{d}{dx}(1+x) = (1+x)^{-1} = \sum_{n \geq 0} (-1)^n x^n, \quad \|x\| < 1,$$

is also given.

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CONTRIBUTION TO THE MECHANICS OF SANDWICH BARS

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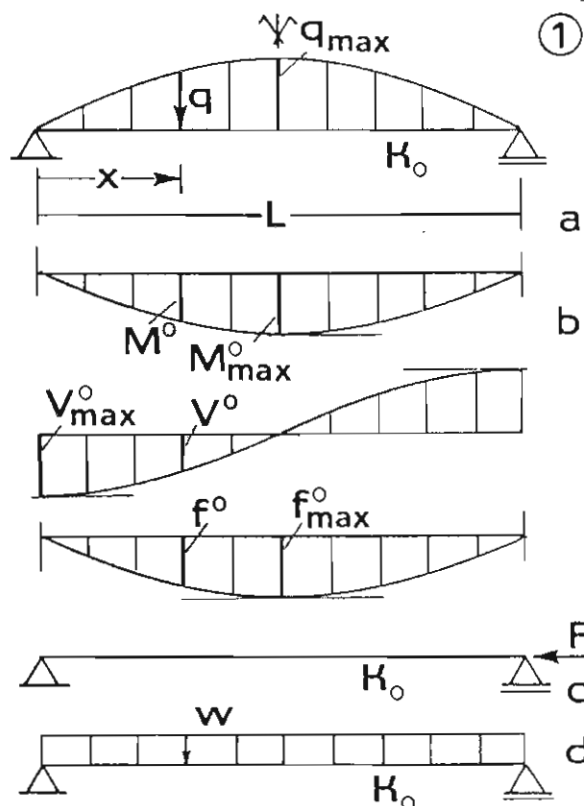
ABSTRACT

On the basis of the author's theory of coupled shearwalls a simple exact analysis method is developed for the one-span sandwich bar simply supported at both ends. Dealt with are the effect of a sine lateral load which can also approximate the effect of an uniform load. Then, the fundamental bifurcation value of end axial loads and the fundamental circular frequency of the bar with an uniform mass distribution are derived. Finally there is an investigation of the effect of end axial loads on the laterally loaded bar, i.e. the 2nd-order theory.

1. INTRODUCTION. REFERENCE BAR

On the basis of the author's theory of coupled shearwalls [1, 2] a simple exact analysis method is developed for the one-span sandwich bar simply supported at both ends and subjected to a lateral and/or longitudinal load. For example, such bars acting as transfer girders enable to create large open areas in large buildings. The bar's response actions and eigenvalues will be referred to those of the corresponding simple flexural bar.

Let the reference bar be subjected to a sine lateral load (Fig. 1a). The notation is as follows: L ..span, K_0 ..bending stiffness, x ..abscissa oriented from the bar's left end rightwards, $q_{\max}$..maximum load's intensity. The maximum shear ($x=0$), the maximum bending moment ($x=L/2$) and the maximum deflection ($x=L/2$) are easily determined applying known relationships between the load and the response actions,



$$\begin{aligned} V_{\max}^0 &= \frac{q_{\max} L}{\pi}, \quad M_{\max}^0 = \frac{V_{\max}^0 L}{\pi}, \\ f_{\max}^0 &= \frac{M_{\max}^0 L^2}{\pi^2 K_0}. \end{aligned} \quad (1)$$

At an arbitrary x the load intensity and the response actions amount to

$$\begin{aligned} q &= c q_{\max}, \quad V^0 = c_V V_{\max}^0, \\ M^0 &= c M_{\max}^0, \quad f^0 = c f_{\max}^0, \end{aligned} \quad (2)$$

where the dimensionless coefficients

$$c = \sin \frac{\pi x}{L}, \quad c_V = \cos \frac{\pi x}{L} \quad (3)$$

describe the actions' variation along the span (Fig. 1b).

The effect of a lateral load Q uniformly distributed along L

can be approximated by that of the sine load; then $q_{\max} = (4/\pi)(Q/L)$.

The fundamental bifurcation value of axial loads applied at the bar's ends (Fig. 1c) and the fundamental circular frequency of the bar whose mass intensity w/g is uniform along x and its rotatory inertia negligible (Fig. 1d) amount

$$F_{\text{cr}}^0 = \frac{\pi^2 K_0}{L^2}, \quad p_0 = \left(\frac{\pi}{L}\right)^2 \sqrt{K_0 \frac{g}{w}}; \quad (4)$$

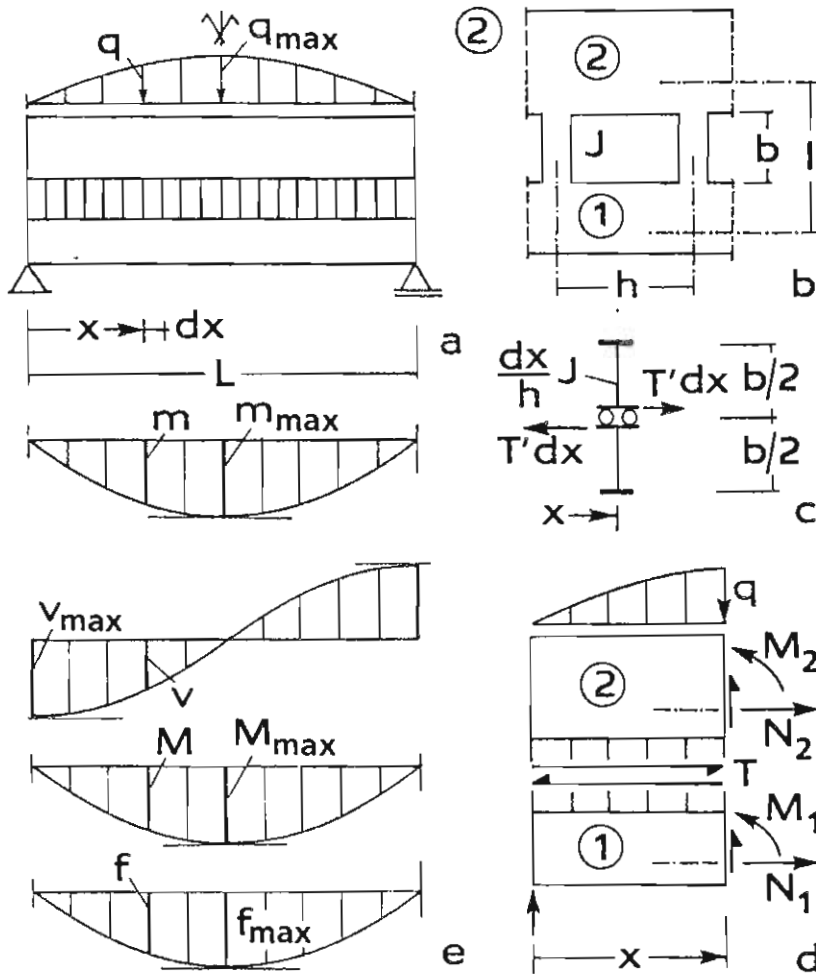
both the buckling and the vibration mode are described by the sine curve c .

If the laterally loaded bar is simultaneously subjected to axial loads F , the response actions V^0 , M^0 and f^0 due to the lateral load are to be multiplied by the amplification factor a_0 , and the bar's reduced or 2nd-order bending stiffness and the corresponding circular frequency become K_0^{II} and p_0^{II} , respectively,

$$a_0 = \frac{1}{1 - F/F_{\text{cr}}^0}, \quad K_0^{\text{II}} = \frac{K_0}{a_0}, \quad p_0^{\text{II}} = \left(\frac{\pi}{L}\right)^2 \sqrt{K_0^{\text{II}} \frac{g}{w}}. \quad (5)$$

2. MECHANICAL SCHEME

A typical sandwich bar consists of the bottom chord 1, the top chord 2 and the connecting medium (Fig. 2a). The connection is con-



stituted by inclined bars forming a truss, for example, or, very often, by stubs, i.e. short columns (Fig. 2b). As the number of columns is usually large, they are simulated by a continuum or laminas in an infinitesimal distance dx .

The bar's cross sectional properties are the axial stiffnesses D and the bending eigenstiffnesses K of its chords,

$$D_1 = EA_1,$$

$$D_2 = EA_2,$$

$$K_1 = EI_1, \quad K_2 = EI_2,$$

(6)

the total bending eigenstiffness K_o , the total bending stiffness K_m and the axial-flexural stiffness K_d of the chord system and the shear stiffness K_s of the connection,

$$K_o = K_1 + K_2, \quad K_d = \frac{D_1 D_2 l^2}{D_1 + D_2}, \quad K_m = K_o + K_d, \quad K_s = \frac{12 EJ l^2}{hb^3}. \quad (7)$$

The meaning of the symbols is: E ..modulus of elasticity; A, I, l .. cross-sectional areas, eigenmoments of inertia and axial distance of the chords; J, h, b ..reduced eigenmoments of inertia, axial distance and clear length of the columns. - The bar's dimensionless stiffness parameters are

$$A = \frac{x^2 K_o K_d}{L^2 K_m K_s}, \quad B = \frac{K_d}{K_m}. \quad (8)$$

The bar is statically indeterminate to the 1st degree. A suitable base system is obtained by introducing zero-shear fields at the mid-length of the laminas (Fig. 2c). The free-body diagram (Fig. 2d)

of a bar's part shows the external and internal forces. Let m be the base system's bending moment at the arbitrary cross section x due the shear force T acting along the connection's mid-height and $M = M_1 + M_2$ the chords' total bending moment. - The bar's free bending moment M^0 due to the lateral load is shared by the connection and the chords,

$$M + m = M^0. \quad (9)$$

Let $m = m(x)$ be the redundant action. By definition

$$m = \int_0^x m' dx, \quad T = \int_0^x T' dx, \quad (10)$$

where primes denote derivatives with respect to x and m' and T' are the intensity of the connection bending moment or connection moment per unit length and the intensity of the shear force T , shear force per unit length or shear flow, respectively. - After m has been determined, M follows from Eq. (9) and is shared by the chords, because their deflection lines are equal, proportionally to their bending eigenstiffnesses,

$$M = M^0 - m, \quad M_1 = \frac{K_1}{K_0} M, \quad M_2 = \frac{K_2}{K_0} M. \quad (11)$$

3. FORMULATION AND SOLUTION OF THE BOUNDARY-VALUE PROBLEM

To determine the redundant action let apply the principle of minimum complementary energy. The bar's complementary energy per unit length consists of the contributions of the chords' total bending moment, the connection moment and the connection shear,

$$U = \frac{(M^0 - m)^2}{2K_0} + \frac{m^2}{2K_d} + \frac{m'^2}{2K_s}. \quad (12)$$

The Euler differential equation

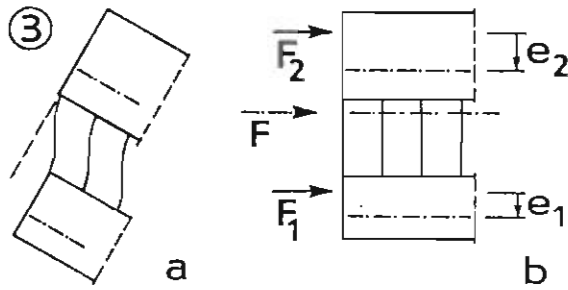
$$\frac{\partial U}{\partial m} - \frac{d}{dx} \frac{\partial U}{\partial m'} = 0 \quad (13)$$

associated with U yields the linear 2nd-order differential equation of the redundant action,

$$-AL^2 m'' + \pi^2 m = B\pi^2 M^0. \quad (14)$$

The boundary conditions are that the connection's outer lamellae freely deform (Fig. 3a) and, thus, m vanishes at both bar's ends.

The solution of the considered boundary-value problem, i.e. the equation of m and, herewith, the equations of $m' = v$, M and f (Fig.



2e) follow to be

$$\begin{aligned} m &= c m_{\max}, \quad v = c v_{\max}, \\ M &= c M_{\max}, \quad f = c f_{\max}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} m_{\max} &= r_m M_{\max}^0, \quad v_{\max} = r_m v_{\max}^0, \\ M_{\max} &= r_M M_{\max}^0, \quad f_{\max} = r_M f_{\max}^0. \end{aligned} \quad (16)$$

The mid-span deflection can, alternatively, be expressed through the bar's effective bending stiffness K_e corresponding to a sine lateral load,

$$f_{\max} = \frac{M_{\max} L^2}{\pi^2 K_e}, \quad K_e = \frac{K_0}{r_M}. \quad (17)$$

The dimensionless connection-bending-moment and chords' total-bending-moment coefficients are

$$r_m = \frac{B}{1+A}, \quad r_M = 1 - r_m = \frac{1+A-B}{1+A}. \quad (18)$$

For design purposes it is important to know the coefficients' and K_e 's regions given by the limit values of K_S and B ,

	$K_S = 0$	$K_S = \infty$	$B = 1$
r_m	0	B	$1/(1+A)$
r_M	1	$1-B$	$A/(1+A)$
K_e	K_0	$K_0/(1-B)$	$(1+A)/A \cdot K_0$

(19)

The other internal forces follow to be

$$\begin{aligned} T &= c T_{\max}, \quad T' = c v_{\max}, \\ M_1 &= c M_{1,\max}, \quad M_2 = c M_{2,\max}, \quad N_1 = -N_2 = c T_{\max}, \end{aligned} \quad (20)$$

where

$$\begin{aligned} T_{\max} &= m_{\max}/l, \quad T'_{\max} = v_{\max}/l, \\ M_{1,\max} &= \frac{K_1}{K_0} M_{\max}, \quad M_{2,\max} = \frac{K_2}{K_0} M_{\max}, \quad N_{1,\max} = -N_{2,\max} = T_{\max}. \end{aligned} \quad (21)$$

4. BAR'S EIGENVALUES. 2ND-ORDER EFFECT

When investigating the lateral stability of the bar subjected to longitudinal forces at its ends it is assumed that 1) the forces are applied along the bar's gravity axis, i.e. F is shared by the

chords proportionally to their axial stiffnesses and 2) $F_1 e_1 + F_2 e_2 = 0$, so that there are no initial deflections (Fig. 3b). When investigating the bar's free lateral vibrations the assumption is made that its mass-intensity, w/g , is constant along x . Herewith both the fundamental buckling mode and the fundamental vibration mode are sine curves, c. The bifurcation value of F and the bar's circular frequency are

$$F_{cr} = \frac{\pi^2 K_e}{L^2}, \quad p = \left(\frac{\pi}{L}\right)^2 \sqrt{K_e \frac{g}{w}}. \quad (22)$$

The effect of axial loads F on the laterally loaded bar can be taken into account multiplying the 1st-order response actions m , v , M and f by the amplification factor a and the bar's 2nd-order bending stiffness corresponding to a sine lateral load becomes K_e^{II} ,

$$a = \frac{1}{1 - F/F_{cr}}, \quad K_e^{II} = \frac{K_e}{a} = \frac{K_0}{a r_M} = K_e - \left(\frac{L}{\pi}\right)^2 F. \quad (23)$$

Herein $(L/\pi)^2 F$ is the diminution of the stiffness due to the effect of the axial loading. The 2nd-order mid-span deflection and the reduced circular frequency can be formulated as

$$f_{max}^{II} = \frac{q_{max} L^4}{\pi^4 K_e^{II}}, \quad p^{II} = \left(\frac{\pi}{L}\right)^2 \sqrt{K_e^{II} \frac{g}{w}}. \quad (24)$$

5. SANDWICH BARS WITH LIMIT VALUES OF THE CONNECTION'S SHEAR STIFFNESS

In the engineering design process it is useful to know the extreme values of the bar's response actions and eigenvalues corresponding to the limit values of the connection's shear stiffness K_s .

For the limit case $K_s = \infty$ (Fig. 4a), identified by the sub- or superscript m , the general formulae derived above or the direct application of the mechanic's of materials methods yield

$$m^m = B M^0 \quad (25)$$

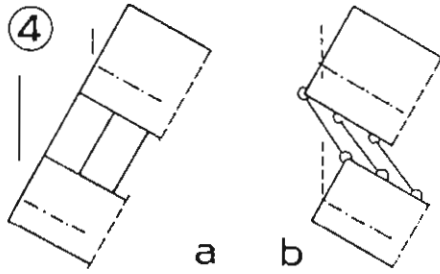
and herewith

$$T^m = c T_{max}^m, \quad T^{\cdot m} = c_V T_{max}^{\cdot m}, \quad M^m = c M_{max}^m, \quad f^m = c f_{max}^m, \quad (26)$$

where

$$T_{max}^m = \frac{B}{1} M_{max}^0, \quad T_{max}^{\cdot m} = \frac{B}{1} V_{max}^0, \quad M_{max}^m = (1 - B) M_{max}^0, \quad f_{max}^m = \frac{M_{max}^0 L^2}{\pi^2 K_m} \quad (27)$$

and



$$F_{cr}^m = \frac{\pi^2 K_m}{L^2}, \quad p_m = \left(\frac{\pi}{L}\right)^2 \sqrt{K_m \frac{g}{w}},$$

$$a_m = \frac{1}{1 - F/F_{cr}^m}. \quad (28)$$

If, moreover, $B = 1$, i.e. the chords are axially rigid, there is $M^m = 0$ and $K_m = \infty$; the free bending moment M^0 is entirely transformed into a pair of axial forces in the chords and the beam is undeformable.

For the limit case $K_s = 0$ (Fig. 4b) the formulas of Section 1 are to be applied directly.

6. APPLICATION OF THE DECOMPOSITION METHOD

The laterally loaded bar's mid-span deflection can also be formulated as

$$f_{\max} = \frac{A}{1+A} f_{\max}^0 + \frac{1}{1+A} f_{\max}^m, \quad (29)$$

i.e. superposing the results $f_{\max}^0$ and $f_{\max}^m$ for the corresponding bars with $K_s = 0$ and $K_s = \infty$ loaded, each, by the total load. More precisely, $f_{\max}$ is the weighted arithmetic mean of $f_{\max}^0$ and $f_{\max}^m$ associated with the weighting factors A and 1 , respectively.

Herewith the author's decomposition method for the determination of the eigenvalues of complex structural systems [3] yields

$$F_{cr} = \frac{1}{\frac{A}{1+A} \frac{1}{F_{cr}^0} + \frac{1}{1+A} \frac{1}{F_{cr}^m}}, \quad p = \sqrt{\frac{1}{\frac{A}{1+A} \frac{1}{p_0^2} + \frac{1}{1+A} \frac{1}{p_m^2}}}; \quad (30)$$

it can easily be shown that these results agree with those obtained earlier [Eqs. (22)]. - The amplification factor which introduces the 2nd-order effect becomes

$$a = \frac{1}{\frac{A}{1+A} \frac{1}{a_0} + \frac{1}{1+A} \frac{1}{a_m}}. \quad (31)$$

7. EXACTNESS PROOF OF THE 2ND-ORDER AND EIGENVALUES

The exactness of the previously obtained results can be proved starting from the differential equation of the laterally loaded composite bar [4],

$$f'''' = \frac{\pi^2}{AL^2} f'' = -\frac{M^0}{K_0} + \frac{\pi^2}{AK_m L^2} M^0; \quad (32)$$

primes again denote derivatives with respect to x and $M^{0''}$ is the negative intensity, $-q$, of the load.

To investigate the bar's behavior when subjected simultaneously to a sine lateral load and axial loads at its ends, $M^0 + Ff$ must be substituted for M^0 and $-q + Ff''$ for $M^{0''}$ in Eq. (32). Omitting for the sake of simplicity the 2nd-order superscript II it becomes

$$f^{(4)} - \left(\frac{\pi^2}{AL^2} - \frac{F}{K_0} \right) f'' - \frac{\pi^2 F}{AK_m L^2} f = \frac{q}{K_0} + \frac{\pi^2}{AK_m L^2} M^0. \quad (33)$$

The boundary conditions of this 4th-order linear differential equation are that both f and f'' vanish at both bar's ends. - The solution of the boundary-value problem yields again the results for $f_{\max}^{II}$ and K_e^{II} obtained earlier [Eqs. (23) and (24)].

To investigate the bar's free lateral vibrations the inertia-load intensity $(w/g)\ddot{f}$, where dot superscripts indicate derivatives with respect to the time coordinate, must be substituted for $M^{0''}$ in Eq. (32). Differentiating twice with respect to x and setting $f = Y \sin(\pi x/L)$ it becomes

$$Y^{(6)} - \frac{\pi^2}{AL^2} Y^{(4)} - \frac{wp^2}{gK_0} Y'' + \frac{\pi^2 wp^2}{AL^2 gK_m} Y = 0, \quad (34)$$

the 6th-order linear differential vibration-mode equation. With the boundary conditions that Y , Y'' and $Y^{(4)}$ vanish at both bar's ends its solution gives again the result for the fundamental circular frequency p obtained earlier [Eq. (22)]. The effect of axial end loads can be taken into account by substituting K_e^{II} for K_e .

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METODA REDUKCIJE DIMENZIJE PRI PRORAČUNU CILINDRIČNOG TIJELA

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Sažetak

Prikazana je metoda redukcije dimenzija (MRD) primijenjena na proračun elastičnog tijela. Promatra se vertikalno osnosimetrično kružno cilindrično tijelo. Tijelo je elastično homogeno i izotropno i nalazi se pod djelovanjem sile $\vec{F}$. Gornja i donja osnovka tijela je učvršćena a plašt je slobodan. Numeričko rješavanje problema dano je MRD i metodom konačnih elemenata (MKE). Izračunat je pomak $\vec{u} = (u, v, w)$ točaka tijela. Proračun je izveden na računalu AT 386. MRD znatno smanjuje opseg proračuna.

DIMENSIONAL REDUCTION METHOD IN A CYLINDRICAL BODY ANALYSIS

Summary

The paper presents the dimension reduction method (DRM) applied to the analysis of an elastic body. Consider a vertical axisymmetrical circular cylindrical body. The body is elastic homogeneous, isotropic and the force $\vec{F}$ is acting on the body. The upper and lower bases of the body are fixed and the lateral surface of the body is free. The DRM and finite element method (FEM) are applied to the numerical solving of the problem. The displacement $\vec{u} = (u, v, w)$ of points of the body are computed. The computation was carried out on AT 386 computer. DRM significantly reduces the extent of the computation.

1. Uvod

Za rješavanje problema mehanike kontinuuma koristi se niz numeričkih metoda. Posljednjih deset godina M. Vogelius i I. Babuška razvili su metodu redukcije dimenzija (MRD). Primjenom MRD na cilindrično tijelo trodimenzionalni problem sveo se na jednodimenzionalni koji se može lakše riješiti s manje računanja. Prva i druga aproksimacija rješenja daje dobre rezultate, pa nije potrebno promatrati treću ili neku višu aproksimaciju.

2. Formulacija problema

Vertikalno osnosimetrično kružno cilindrično tijelo nalazi se pod djelovanjem sile $\vec{F}$. Cilindrično tijelo je elastično, homogeno i izotropno. Jednadžbe ravnoteže elastičnog tijela imaju oblik

$$A\vec{u} = \vec{F} \quad (1)$$

gdje je $A\vec{u} = (\lambda + \mu) \text{grad div } \vec{u} + \mu \Delta \vec{u}$.

Jednadžba (1) može se napisati u obliku

$$\begin{aligned} L_1 u + L_2 v + L_3 w + X &= 0 \\ L_4 u + L_5 v + L_6 w + Y &= 0 \\ L_7 u + L_8 v + L_9 w + Z &= 0 \end{aligned} \quad (2)$$

u području V , gdje su

$$\begin{aligned} L_1 &= (\lambda + 2\mu) \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} + \mu \frac{\partial^2}{\partial z^2}, L_2 = (\lambda + \mu) \frac{\partial^2}{\partial x \partial y}, L_3 = (\lambda + \mu) \frac{\partial^2}{\partial z \partial x}, \\ L_4 &= (\lambda + \mu) \frac{\partial^2}{\partial y \partial x}, L_5 = \mu \frac{\partial^2}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2}{\partial y^2} + \mu \frac{\partial^2}{\partial z^2}, L_6 = (\lambda + \mu) \frac{\partial^2}{\partial y \partial z}, \\ L_7 &= (\lambda + \mu) \frac{\partial^2}{\partial z \partial x}, L_8 = (\lambda + \mu) \frac{\partial^2}{\partial z \partial y}, L_9 = \mu \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} + (\lambda + 2\mu) \frac{\partial^2}{\partial z^2}, \end{aligned}$$

Gornja i donja osnovka cilindričnog tijela je učvršćena pa rubni uvjeti glase

$$u = v = w = 0 \quad \text{na } S_1 \text{ i } S_2 \quad (3)$$

plašt je slobodan

$$\vec{t}_{/s} = 0 \quad (4)$$

gdje je $\vec{t} = (t_1, t_2, t_3)$, $t_i = \sum_{k=1}^3 \sigma_{ik} \nu_k$ $i = 1, 2, 3$, ν_k su kosinusi smjera vektora $\vec{n}$ vanjske normale na S , σ_{ik} su komponente tenzora naprezanja. $\vec{F} = (X, Y, Z)$ vektor volumne sile, λ i μ su Lameovi koeficijenti, $\vec{u} = (u, v, w)$ vektor pomaka. Operator A je operator teorije elastičnosti.

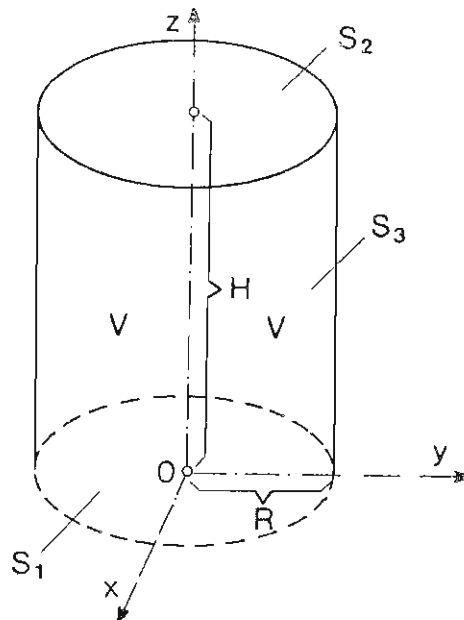
$$V = \{(x, y, z) : x^2 + y^2 < R^2, \quad z \in (0, H)\}$$

$$S_1 = \{(x, y, z) : x^2 + y^2 \leq R^2, \quad z = 0\}$$

$$S_2 = \{(x, y, z) : x^2 + y^2 \leq R^2, \quad z = H\}$$

$$S_3 = \{(x, y, z) : x^2 + y^2 = R^2, \quad z \in (0, H)\}$$

V je elastično tijelo, a S je njegova granica $S = S_1 \cup S_2 \cup S_3$, H je visina tijela, a R je polumjer gornje i donje osnovke (slika 1).



Slika 1. Fig.1.

3. Numeričko rješavanje problema MRD

Približno rješenje problema (2), (3) i (4) traži se u obliku

$$\begin{aligned} u_n &= \sum_{i=1}^n f_i(z) g_i(x, y), \\ v_n &= \sum_{i=1}^n f_{i+n}(z) g_{i+n}(x, y) \\ w_n &= \sum_{i=1}^n f_{i+2n}(z) g_{i+2n}(x, y) \end{aligned} \quad (5)$$

gdje su $g_i(x, y)$ ($i = 1, 2, \dots, 3n$) odabrane linearno nezavisne funkcije. Njihova izgradnja ovisi o operatoru A problema i rubnim uvjetima. Funkcija $f_i(z) \in C^2[0, H]$ ($i = 1, 2, \dots, 3n$) zadovoljavaju rubne uvjete problema i njih treba odrediti. Primjenom Galerkinove metode za funkcije $f_i(z)$ ($i = 1, 2, \dots, 3n$) dobiju se jednačbe

$$\begin{aligned} \int \int_{S_1} (L_1 u_n + L_2 v_n + L_3 w_n + X) g_k(x, y) dx dy &= 0, \quad k = 1, \dots, n \\ \int \int_{S_1} (L_4 u_n + L_5 v_n + L_6 w_n + Y) g_{k+n}(x, y) dx dy &= 0, \quad k = 1, \dots, n \\ \int \int_{S_1} (L_7 u_n + L_8 v_n + L_9 w_n + Z) g_{k+2n}(x, y) dx dy &= 0, \quad k = 1, \dots, n \end{aligned} \quad (6)$$

i rubni uvjeti

$$f_k(0) = 0, \quad f_k(H) = 0, \quad k = 0, 1, 2, \dots \quad (7)$$

Funkcije $g_i(x, y)$ ($i = 1, \dots, 3n$) za problem (2), (3) i (4) dobiju se iz sustava funkcija

$$\omega(x, y)x^k y^l \quad k, l = 0, 1, 2, \dots \quad (8)$$

gdje je

$$\omega(x, y) = (R_1^2 - x^2 - y^2), \quad (9)$$

$R_1 = 2R$ i tada se dobije najbolja točnost u izračunavanju pomaka točaka tijela. Sustav (6) predstavlja sustav običnih diferencijalnih jednadžbi drugoga reda s rubnim uvjetima (7).

4. Primjer

Promatra se kružno cilindrično tijelo od betona. Tijelo se nalazi u polju sile teže. Zadane su fizikalne i geometrijske veličine.

Fizikalne veličine: akceleracija sile teže $g = 9.81 \text{ m/s}^2$, gustoća betona $\rho = 2500 \text{ kg/m}^3$, Youngov modul $E = 3 \cdot 10^{10} \text{ N/m}^2$, Poissonov koeficijent $\nu = 0.2$. Geometrijske veličine: polunjer osnovke $R = 0.5 \text{ m}$, visina tijela $H = 3 \text{ m}$, duljina konačnog elementa $H_1 = 0.3 \text{ m}$, $R_1 = 1 \text{ m}$. Prva aprosimacija rješenja u smislu MRD ima oblik

$$\begin{aligned} u_1 &= f_1(z)g_1(x, y), & v_1 &= f_2(z)g_2(x, y) \\ w_1 &= f_3(z)g_3(x, y) \end{aligned} \quad (10)$$

za $k = l = 0$ iz sustava funkcija (8) dobije se

$$g_1(x, y) = g_2(x, y) = g_3(x, y) = R_1^2 - x^2 - y^2 \quad (11)$$

Na temelju (6), (7), (10) i (11) dobije se sustav običnih diferencijalnih jednadžbi za nepoznate funkcije $f_1(z)$, $f_2(z)$ i $f_3(z)$

$$\begin{aligned} 9635.32f_1''(z) - 80208.33f_1(z) &= 0 \\ 9635.32f_2''(z) - 80208.33f_2(z) &= 0 \\ 25694.18f_3''(z) - 43749.99f_3(z) &= 0.021459 \end{aligned} \quad (12)$$

i rubni uvjeti

$$f_k(0) = 0, \quad f_k(H) = 0, \quad k = 1, 2, 3 \quad (13)$$

Rješenja sustava (12) i (13) imaju oblik

$$f_1(z) = 0, \quad f_2(z) = 0, \quad f_3(z) = C_{13}e^{rz} + C_{23}e^{-rz} + FP_3 \quad (14)$$

gdje je FP_3 partikularno rješenje treće diferencijalne jednadžbe iz sustava (12).

$$FP_3 = -0.4905 \cdot 10^{-6}, \quad r = 1.28261, \quad C_{13} = 0.10249 \cdot 10^{-7}, \quad C_{23} = 0.48026 \cdot 10^{-6}.$$

Pomaci u_1 , v_1 , w_1 u zadanim točkama kružnog cilindričnog tijela izračunati po formulama (10) dani su u tablici 1.

Druga aproksimacija rješenja u smislu MRD ima oblik

$$\begin{aligned} u_2 &= f_1(z)g_1(x, y) + f_2(z)g_2(x, y) \\ v_2 &= f_3(z)g_3(x, y) + f_4(z)g_4(x, y) \\ w_2 &= f_5(z)g_5(x, y) + f_6(z)g_6(x, y) \end{aligned} \quad (15)$$

Tablica 1. Pomaci kružnog cilindričnog tijela
Table 1. The displacement of the circular cylindrical body.

x	y	z	u_1	v_1	w_1
0.	0.	0.3	0.	0.	$-0.148589 \cdot 10^{-6}$
0.	0.	1.5	0.	0.	$-0.350233 \cdot 10^{-6}$
0.2	0.2	1.5	0.	0.	$-0.322213 \cdot 10^{-6}$
0.5	0.	1.5	0.	0.	$-0.262674 \cdot 10^{-6}$
0.	0.	2.7	0.	0.	$-0.148589 \cdot 10^{-6}$

Funkcije $g_i(x, y)$ ($i = 1, \dots, 6$) dobiju se iz sustava funkcija (8) i glase

$$\begin{aligned} g_1(x, y) &= g_3(x, y) = g_5(x, y) = R_1^2 - x^2 - y^2 \\ g_2(x, y) &= g_4(x, y) = g_6(x, y) = (R_1^2 - x^2 - y^2)(x + y) \end{aligned} \quad (16)$$

Na temelju (6), (7), (15) i (16) dobije se sustav običnih diferencijalnih jednadžbi za funkcije $f_i(z)$ ($i = 1, \dots, 6$)

$$\begin{aligned} 9635.32 f_1''(z) + 13888.89 f_6'(z) - 80208.33 f_1(z) &= 0 \\ 1090.49 f_2''(z) - 2170.14 f_5'(z) - 19097.23 f_2(z) - 4340.27 f_4(z) &= 0 \\ 9635.32 f_3''(z) + 13888.89 f_6'(z) - 80208.33 f_3(z) &= 0 \\ 1090.49 f_4''(z) - 2170.14 f_5'(z) - 4340.27 f_2(z) - 19097.23 f_4(z) &= 0 \\ 25694.19 f_5''(z) + 13888.89 f_2'(z) + 13888.89 f_4'(z) - 43749.99 f_5(z) &= 0.0214594 \\ 2907.98 f_6''(z) - 2170.14 f_1'(z) - 2170.14 f_3'(z) - 10416.66 f_6(z) &= 0 \end{aligned} \quad (17)$$

i rubni uvjeti

$$f_k(0) = 0, \quad f_k(H) = 0, \quad k = 1, \dots, 6 \quad (18)$$

Sustav (17) s rubnim uvjetima (18) riješen je metodom konačnih elemenata. Na segmentu $[0, 3]$ promatra se 10 konačnih elemenata a duljina jednog konačnog elementa $H_1 = 0.3m$. Oblik približnog rješenja problema (17) i (18) po MKE je

$$\begin{aligned} f_{1,9} &= \sum_{k=1}^9 C_k \varphi_k(z), \quad f_{2,9} = \sum_{k=1}^9 D_k \varphi_k(z), \quad f_{3,9} = \sum_{k=1}^9 E_k \varphi_k(z) \\ f_{4,9} &= \sum_{k=1}^9 F_k \varphi_k(z), \quad f_{5,9} = \sum_{k=1}^9 G_k \varphi_k(z), \quad f_{6,9} = \sum_{k=1}^9 H_k \varphi_k(z) \end{aligned} \quad (19)$$

gdje su $\varphi_k(z)$ ($k = 1, 2, \dots, 9$) bazične funkcije, linearne iznad svakog konačnog elementa, dakle polinomi prvog reda.

U smislu MKE za nepoznanice C_k, D_k, E_k, F_k, G_k i H_k ($k = 1, 2, \dots, 9$) dobije se linearni sustav

$$Bx = b \quad (20)$$

gdje je B matrica krutosti, x matrica stupac čiji elementi su $C_k, D_k, E_k, F_k, G_k, H_k$ ($k = 1, 2, \dots, 9$), a b matrica stupac čiji su elementi slobodni članovi linearnog sustava. Rješenje sustava (20) dano je u tablici 2.

Pomaci u_2, v_2, w_2 u zadanim točkama kružnog cilindričnog tijela izračunati po formulama (15) dani su u tablici 3. u slučaju druge aproksimacije rješenja. Za sve proračune izrađeni su programi na jeziku FORTRAN 77 verzija 5.0 i realizirani na računalu AT-386. Za izračunavanje dvostrukih i jednostrukih integrala i za ostale matematičke operacije korišten je paket programa MATHEMATICA.

Tablica 2. Vrijednosti za $C_k, D_k, E_k, F_k, G_k, H_k$

Table 2. The values of $C_k, D_k, E_k, F_k, G_k, H_k$

Čvor k	C_k	D_k	E_k	F_k	G_k	H_k
1	0.	$0.28219 \cdot 10^{-7}$	0.	$0.28219 \cdot 10^{-7}$	$-0.15253 \cdot 10^{-6}$	0.
2	0.	$0.24239 \cdot 10^{-7}$	0.	$0.24239 \cdot 10^{-7}$	$-0.25545 \cdot 10^{-6}$	0.
3	0.	$0.15517 \cdot 10^{-7}$	0.	$0.15517 \cdot 10^{-7}$	$-0.32037 \cdot 10^{-6}$	0.
4	0.	$0.73569 \cdot 10^{-8}$	0.	$0.73569 \cdot 10^{-8}$	$-0.35522 \cdot 10^{-6}$	0.
5	0.	0.	0.	0.	$-0.36622 \cdot 10^{-6}$	0.
6	0.	$-0.73569 \cdot 10^{-8}$	0.	$-0.73569 \cdot 10^{-8}$	$-0.35522 \cdot 10^{-6}$	0.
7	0.	$-0.15517 \cdot 10^{-7}$	0.	$-0.15517 \cdot 10^{-7}$	$-0.32037 \cdot 10^{-6}$	0.
8	0.	$-0.24239 \cdot 10^{-7}$	0.	$-0.24239 \cdot 10^{-7}$	$-0.25545 \cdot 10^{-6}$	0.
9	0.	$-0.28219 \cdot 10^{-7}$	0.	$-0.28219 \cdot 10^{-7}$	$-0.15253 \cdot 10^{-6}$	0.

Tablica 3. Pomaci kružnog cilindričnog tijela

Table 3. The displacement of the circular cylindrical body

x	y	z	u_2	v_2	w_2
0.	0.	0.3	0.	0.	$-0.15253 \cdot 10^{-6}$
0.	0.	1.5	0.	0.	$-0.36622 \cdot 10^{-6}$
0.2	0.2	1.5	0.	0.	$-0.33692 \cdot 10^{-6}$
0.5	0.	1.5	0.	0.	$-0.27466 \cdot 10^{-6}$
0.	0.	2.7	0.	0.	$-0.15253 \cdot 10^{-6}$

5. Zaključak

Može se dati algoritam MRD primjenjen na problem (2), (3), (4).

1. Izgradi se približno rješenje problema (2), (3), (4) u obliku (5). Funkcije $g_i(x, y)$ ($i = 1, 2, \dots, 3n$) odaberu se iz sustava (8).
2. Na temelju (5) i (8) izgradi se sustav običnih diferencijalnih jednadžbi drugog reda (6) za funkcije $f_i(z)$ ($i = 1, 2, \dots, 3n$) s rubnim uvjetima (7).
3. Rješi se sustav (6) s rubnim uvjetima (7). Rješenja su funkcije $f_i(z)$ ($i = 1, 2, \dots, 3n$).
4. Nadene funkcije $f_i(z)$ ($i = 1, 2, \dots, 3n$) uvrste se u formule (5) i tada se mogu izračunati pomaci točaka kružnog cilindričnog tijela.

Za numeričko rješavanje problema (2), (3), (4) metodom redukcije dimenzije izgrađen je sustav koordinatnih funkcija (8) te za $R_1 = 2R$ dobije se najbolja točnost kod izračunavanja pomaka točaka tijela. Iz tablica 1 i 3 vidi se da vrijednosti pomaka točaka cilindričnog tijela u slučaju prve i druge aproksimacije rješenja malo se razlikuju. Proračuni su pokazali da prva i druga aproksimacija rješenja za cilindrično tijelo daju dobru točnost, osim toga proračuni su po opsegu manji od proračuna numeričkih metoda koje rješavaju problem trodimenzionalno.

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NAPREZANJA PRI RASTEZANJU ŠTAPOVA OTVORENOG TANKOSTJENOG PRESJEKA S UTJECAJEM SMICANJA

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Sažetak

Razmatran je štap otvorenog tankostjenog presjeka opterećen na rastezanje s utjecajem smicanja. Pretpostavlja se da poprečni presjek ima dvije osi simetrije. Opterećenje je uzdužno i simetrično raspoređeno po konturi poprečnog presjeka. Izrazi za naprezanja dani su u analitičkom obliku. Točnost izraza provjerena je na jednostavnim primjerima, za koje postoje egzaktna rješenja u teoriji elastičnosti.

STRESSES IN THIN-WALLED BEAMS OF OPEN CROSS-SECTION UNDER TENSION WITH INFLUENCE OF SHEAR

Abstract

The thin-walled beam of open cross-section under tension with influence of shear is considered. It is assumed that the cross-section has double axes of symmetry. The loading is longitudinal and symmetrically distributed over the cross-section contour. Expressions for stresses are given in an analytic form. The validity of the expressions were verified on some simple examples, for which the exact solutions are solved in the theory of elasticity.

1. UVOD. Klasična teorija štapova otvorenog tankostjenog presjeka (Vlasov) počiva na dvjema osnovnim pretpostavkama: (1) da se oblik poprečnog presjeka ne mijenja pri deformiranju; (2) da su kutne deformacije u središnjoj površini jednake nuli [1,2]. Pri proizvoljno raspoređenom uzdužnom opterećenju po konturi presjeka štap se rasteže, savija i uvija. U posebnom slučaju, kad poprečni presjek ima dvije osi simetrije i kad je opterećenje po konturi presjeka raspoređeno simetrično, štap se samo rasteže. Poprečni presjeci ostaju ravni i okomiti na uzdužnu os štapa.

U slučaju rastezanja štapova otvorenog tankostjenog presjeka prva pretpostavka zadovoljena je prema tome samim ograničenjima - simetričnošću poprečnog presjeka i opterećenja. Ako se isključi druga pretpostavka, poprečni presjeci će se iskrivljavati u smjeru uzdužne osi štapa, uslijed kutnih deformacija. Normalna naprezanja neće biti jednoliko raspoređena po konturi poprečnog presjeka. Štap se rasteže s utjecajem smicanja.

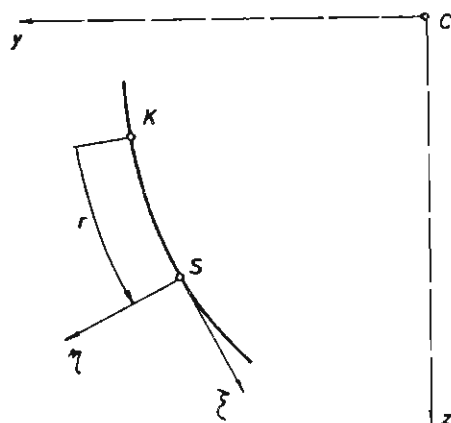
U teoriji elastičnosti poznata su rješenja za ravninsko stanje naprezanja u tankim pločama ("stijenama") opterećenim smičnim silama [3,4]. Ta se rješenja mogu smatrati točnim rješenjem za štap jednostavnog tankostjenog poprečnog presjeka, pravokutnog oblika. U [5] dan je izraz za normalno naprezanje za takav štap, a rezultati su uspoređeni s točnim rješenjem [4], za opterećenje duž uzdužne osi štapa.

2. DEFORMACIJE I NAPREZANJA. Razmatranje će se ograničiti na prizmatične štapove otvorenog tankostjenog presjeka s dvije osi simetrije. Opterećenje je uzdužno i simetrično raspoređeno po konturi poprečnog presjeka. Deformacije u smjeru tangente na konturu poprečnog presjeka mnogo su manje od deformacija u smjeru uzdužne osi štapa.

Uzdužni pomak proizvoljne točke S središnje linije poprečnog presjeka može se u tom slučaju prikazati u obliku (slika 1)

$$u_S(x, r) = u_K(x) + \int_0^r \gamma_{\xi x}(x, r), \quad (1)$$

gdje je r - krivocrtna koordinata točke S , s ishodišnom točkom K ; $u_K(x)$ - pomak točke K u smjeru osi x / pomak poprečnog presjeka kao ravnog presjeka; $\gamma_{\xi x}(x, r)$ - kutna deformacija u središnjoj površini štapa; ξ - os koja se poklapa s tangentom na središnju liniju u točki S ; η - os koja je okomita na os ξ u točki S .



Sl.1. Koordinate točke S središnje linije poprečnog presjeka

Fig.1. Coordinates of the cross-section median line point S

Duljinska deformacija u smjeru uzdužne osi štapa određena je izrazom

$$\epsilon_x(x, r) = \frac{\partial u_k(x, r)}{\partial x} = \frac{du_k(x)}{dx} + \int_0^r \frac{\partial \gamma_{\xi x}(x, r)}{\partial x} dr \quad (2)$$

U stijenama štapa pretpostavlja se ravninsko stanje naprezanja:

$$\sigma_\eta = \tau_{\eta x} = \tau_{\eta \xi} = 0, \quad \epsilon_\eta \neq 0, \quad (3)$$

gdje je σ_η - normalno naprezanje u smjeru osi η ; $\tau_{\eta x}$ i $\tau_{\eta \xi}$ - tangencijalna naprezanja u središnjoj ravnini; ϵ_η - duljinska deformacija u smjeru normale η . Hookeov zakon u tom slučaju glasi

$$\epsilon_x(x, r) = \frac{1}{E} [\sigma_x(x, r) - \nu \sigma_\xi(x, r)], \quad \epsilon_\xi(x, r) = \frac{1}{E} [\sigma_\xi(x, r) - \nu \sigma_x(x, r)];$$

$$\gamma_{x\xi}(x, r) = \frac{\tau_{x\xi}(x, r)}{G}, \quad (4)$$

gdje su $\sigma_x(x, r)$ i $\sigma_\xi(x, r)$ - normalna naprezanja u smjeru osi x i ξ ; $\epsilon_x(x, r)$ i $\epsilon_\xi(x, r)$ - duljinske deformacije u smjeru osi x i ξ ; E - modul elastičnosti; ν - Poissonov faktor; G - modul smicanja $\tau_{x\xi}(x, r) = \tau_{\xi x}(x, r)$ - tangencijalno naprezanje u smjeru osi ξ , odnosno x ; $\gamma_{x\xi}(x, r) = \gamma_{\xi x}(x, r)$. Sustav jednačbi (4)₁₋₂, imajući u vidu pretpostavku o deformiranju, može se riješiti približno:

1. rješenje:

$$\sigma_\xi(x, s) = 0: \quad \sigma_x(x, s) = E\epsilon_x(x, s), \quad \epsilon_\xi(x, s) = -\frac{E}{\nu} \sigma_x(x, s); \quad (5)$$

2. rješenje:

$$\nu = 0: \quad \sigma_x(x, s) = E\epsilon_x(x, s), \quad \sigma_\xi(x, s) = E\epsilon_\xi(x, s); \quad (6)$$

3. rješenje:

$$\epsilon_\xi(x, s) = 0: \quad \sigma_x(x, s) = E^* \epsilon_x(x, s), \quad \sigma_\xi(x, s) = \nu \sigma_x(x, s), \quad (7)$$

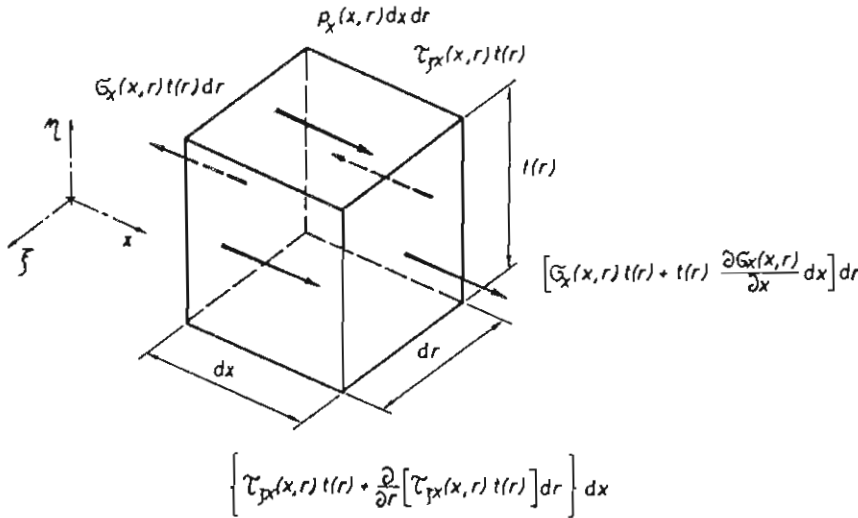
gdje je

$$E^* = \frac{E}{1 - \nu^2}.$$

Prvo i treće rješenje uobičajeno je u klasičnoj teoriji štapova, gdje se zanemaruje utjecaj smicanja [1-2]. Drugo rješenje ne poništava komponente σ_x i ϵ_x ; samo ih ne razmatra [6].

Imajući u vidu (2) i (4)₃ te prvo ili drugo rješenje, (5) ili (6), dobiva se

$$\sigma_x(x, r) = E \frac{du_k(x)}{dx} + \frac{E}{G} \int_0^r \frac{\tau_{\xi x}(x, r)}{\partial x} dr. \quad (8)$$



Sl. 2. Ravnoteža elementa štapa

Fig. 2. Equilibrium of an element of the beam

Veza između normalnog i tangencijalnog naprezanja može se odrediti iz uvjeta ravnoteže elementa štapa (slika 2). Suma svih sila u smjeru osi x daje:

$$\frac{\partial \sigma_x(x, r)}{\partial x} t(r) + \frac{\partial}{\partial r} [\tau_{\xi x}(x, r) t(r)] + p_x(x, r) = 0, \quad (9)$$

gdje je $t(r)$ - debljina stijenke; $p_x(x, r)$ - površinsko uzdužno opterećenje. Izlazi da je

$$\tau_{\xi x}(x, r) = - \frac{1}{t(r)} \left[\int_0^r \frac{\partial \sigma_x(x, r)}{\partial x} t(r) dr + \int_0^r p_x(x, r) dr + f(x) \right]. \quad (10)$$

Ako se pretpostavi da je $\tau_{\xi x}(x, 0)t(0) = 0$, bit će $f(x) = 0$, pa se može (10) prikazati u obliku

$$\tau_{\xi x}(x, r) = \frac{1}{t(r)} \left[\int_{A^*} \frac{\partial \sigma_x(x, r)}{\partial x} dA + \int_{r^*} p_x(x, r) dr \right], \quad (11)$$

gdje je r^* - krivocrtna koordinata koja se mjeri od slobodnog ruba, tj. od ruba na kojem je $\tau_{\xi x}(x, r) = 0$ (slika 1); A^* - površina dijela poprečnog presjeka, određena koordinatom r^* ; $dA = t(r)dr$.

Ako je

$$\frac{\partial^2 \tau_{\xi x}(x, r)}{\partial x^2} = 0, \quad (12)$$

može se izraz za tangencijalno naprezanje (11), imajući u vidu (8), prikazati u obliku

$$\tau_{\xi x}(x, r) = \frac{1}{t(r)} \left[E \frac{d^2 u_k(x)}{dx^2} A^*(r) + \int_{r^*} p_x(x, r) dr \right], \quad (13)$$

gdje je

$$A^*(r) = \int_{A^*} dA. \quad (14)$$

Ako je

$$\frac{d^2 u_k(x)}{dx^2} = - \frac{q_x(x)}{EA}, \quad q_x(x) = \int_L p_x(x, r) dr, \quad (15)$$

A - površina poprečnog presjeka, L - duljina konture poprečnog presjeka, može se tangencijalno naprezanje (13) prikazati u obliku

$$\tau_{\xi x}(x, r) = \tau_{x\xi}(x, r) = \frac{1}{t(r)} \left[\int_{r^*} p_x(x, r) dr - q_x(x) \frac{A^*(r)}{A} \right]. \quad (16)$$

Opterećenje može biti zadano također kao sila na jedinicu duljine $q_x(x)$, u više točaka K . Tada je

$$q_x(x) = \int_L p_x(x, r) dr + \sum_{i=1}^n q_{xi}(x). \quad (17)$$

Uvrštenjem izraza (16) u (8) dobiva se

$$\sigma_x(x, r) = E \frac{du_k(x)}{dx} + \frac{E}{G} \int_0^r \frac{1}{t(r)} \int_{r^*} \frac{\partial p_x(x, r)}{\partial x} dr dr - \frac{E}{GA} \frac{dq_x(x)}{dx} \int_0^r \frac{A^*(r)}{t(r)} dr. \quad (18)$$

Integriranjem naprežanja (18) po poprečnom presjeku dobiva se uzdužna sila $N(x)$:

$$EA \frac{du_k(x)}{dx} - \frac{EA}{G} \times \frac{dq_x(x)}{dx} = N(x), \quad (19)$$

gdje je

$$\chi = \frac{1}{A^2} \int_A \left[\frac{A^*(r)}{t(r)} \right]^2 dA - \frac{1}{A \frac{dq_x(x)}{dx}} \left[\int_A \frac{A^*(r)}{t(r)} \frac{1}{t(r)} \int_{r^*} \frac{\partial p_x(r)}{\partial x} dr dA \right]. \quad (20)$$

Uvrštenjem jednadžbe (19) u izraz za naprezanje (18), dobiva se

$$\begin{aligned} \sigma_x(x, r) = \frac{N(x)}{A} + \frac{E}{G} \times \frac{dq_x(x)}{dx} + \frac{E}{G} \int_0^r \frac{1}{t(r)} \int_{r^*} \frac{\partial p_x(x, r)}{\partial x} dr dr - \\ - \frac{E}{GA} \frac{dq_x(x)}{dx} \int_0^r \frac{A^*(r)}{t(r)} dr. \end{aligned} \quad (21)$$

Deriviranjem jednadžbe (19), imajući u vidu da je

$$\frac{dN(x)}{dx} = -q_x(x), \quad (22)$$

dobiva se

$$EA \frac{d^2 u_M(x)}{dx^2} = -q_x(x) + \frac{EA}{G} \times \frac{d^2 q_x(x)}{dx^2}, \quad (23)$$

diferencijalna jednadžba rastezanja s utjecajem smicanja. Imajući u vidu uvjet (12), drugi član desne strane bi trebao biti jednak nuli, kada bi jednadžba odgovarala pretpostavljenoj jednadžbi (15). Približno, može se uzeti da jednadžba vrijedi bez ograničenja (12).

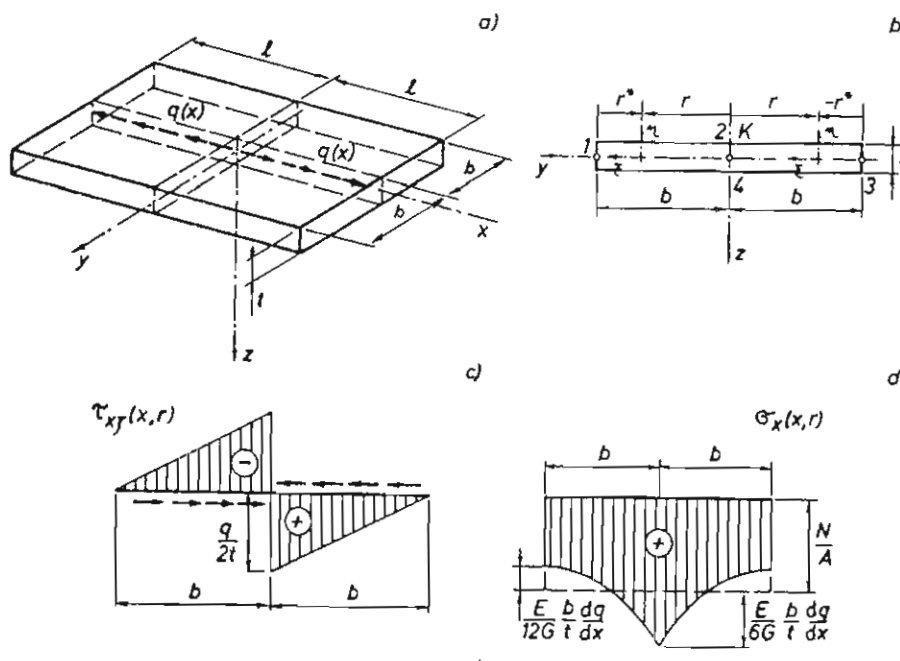
Imajući u vidu vezu između konstanti elastičnosti, koja glasi $E/G = 2(1 + \nu)$, može se izraz (21) prikazati i tako da se umjesto omjera E/G uvrsti 2, u skladu s rješenjem (6). Treći oblik dobio bi se ako bi se umjesto omjera E/G uvrstio omjer E^*/G , u skladu s rješenjem (7).

3. PRIMJERI. Za primjer na slici 3 tangencijalno naprezanje (16) glasi (slika 3b)

$$2-1, 4-3: \tau_{x\xi}(x, r) = \mp \left(1 - \frac{r}{b}\right) \frac{q_x(x)}{2t}, \quad 0 \leq r \leq b. \quad (24)$$

Normalno naprezanje (21) glasi (slika 3c)

$$2-1, 4-3: \sigma_x(x, r) = \frac{N(x)}{A} + \frac{E}{12G} \frac{b}{t} \left[2 - 6 \frac{r}{b} + 3 \left(\frac{r}{b}\right)^2\right] \frac{dq_x(x)}{dx}. \quad (25)$$



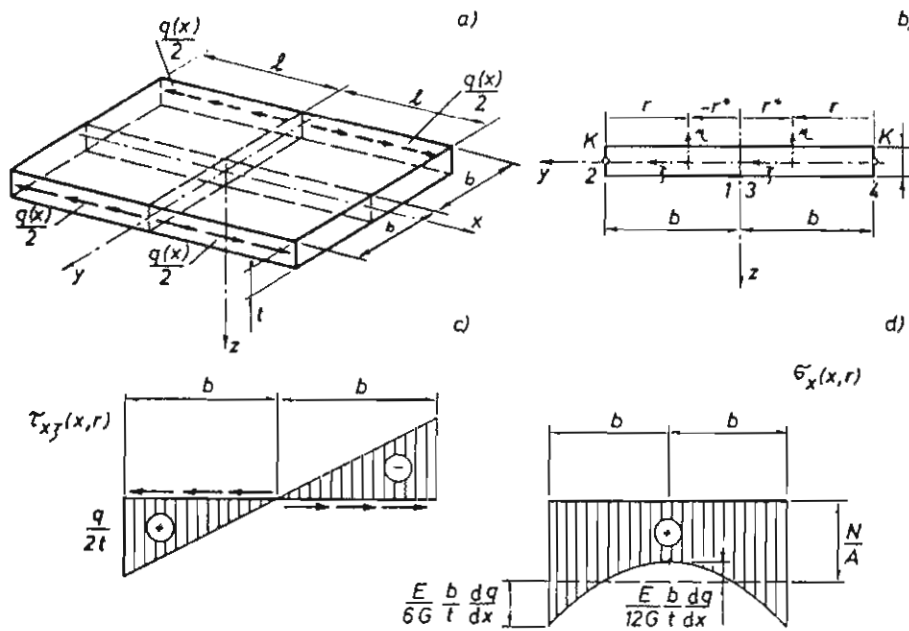
b) Sl.3. Naprezanja za štap opterećen duž uzdužne osi: a) zadani štap; b) dimenzije poprečnog presjeka i koordinate; c) tangencijalno naprezanje; d) normalno naprezanje

d) Fig. 3. Stresses for a beam loaded along the longitudinal axis: a) loaded beam; b) cross-section dimensions and coordinates; c) tangential stress; d) normal stress

Za primjer na slici 4 tangencijalno naprezanje glasi

$$2-1, 4-3: \tau_{x\xi}(x, r) = \pm \left(1 - \frac{r}{b}\right) \frac{q_x(x)}{2t}, \quad 0 \leq r \leq b; \quad (26)$$

Normalno naprezanje glasi formalno, kao u prethodnom primjeru.



Sl. 3. Naprezanja za štap opterećen duž rubova: a) zadani štap; b) dimenzije poprečnog presjeka i koordinate; c) tangencijalno naprezanje; d) normalno naprezanje

Fig. 3. Stresses for a beam loaded along the edges: a) loaded beam; b) cross-section dimensions and coordinates; c) tangential stress; d) normal stress

Neravnomjernost raspodjele normalnog naprezanja po središnjoj liniji poprečnog presjeka može se prikazati faktorom (vrijedi za oba primjera):

$$\psi = \frac{1}{b\sigma_x(x, 0)} \int_0^b \sigma_x(x, r) dr = \frac{1}{1 + \frac{Eb^2}{3G} \lambda}, \quad \lambda = \frac{\frac{dq_x(x)}{dx}}{N(x)}. \quad (27)$$

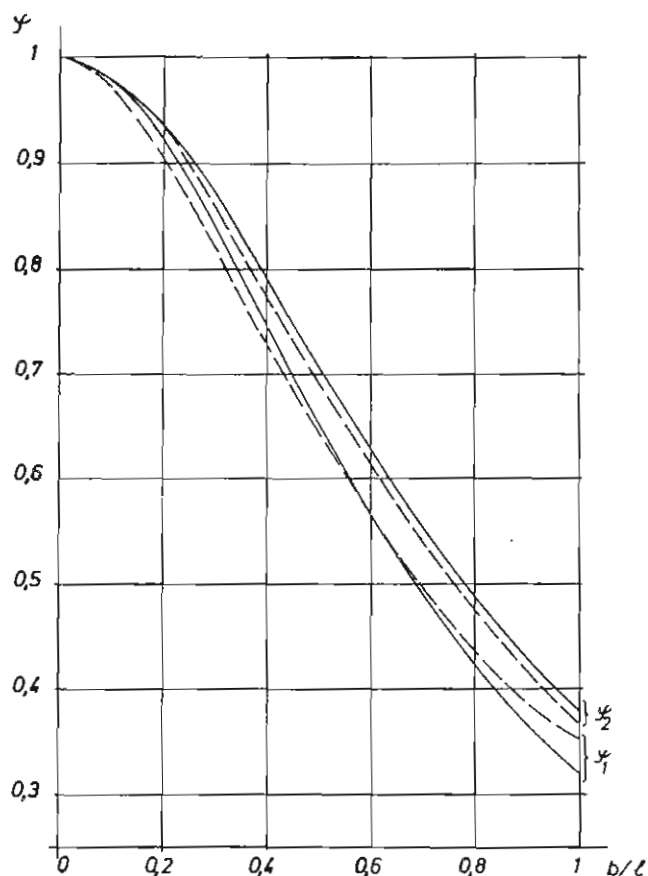
Faktori: \$\psi_1\$ - po prvom rješenju, \$\psi_2\$ - po drugom rješenju i \$\psi_3\$ - po trećem, za \$q_x(x) = q \sin \frac{\pi}{2l} x\$, odnosno \$\lambda = \left(\frac{\pi}{2l}\right)^2\$, glase

$$\psi_1 = \frac{1}{1 + \frac{\pi^2(1+\nu)}{6} \left(\frac{b}{l}\right)^2}, \quad \psi_2 = \frac{1}{1 + \frac{\pi^2}{6} \left(\frac{b}{l}\right)^2}, \quad \psi_3 = \frac{1}{1 + \frac{\pi^2}{6(1-\nu)} \left(\frac{b}{l}\right)^2}. \quad (28)$$

Za primjer na slici 2 naprezanja \$\sigma_\xi\$ jednaka su nuli na uzdužnim rubovima, gdje su i naprezanja \$\sigma_x\$ jednaka nuli (slika 3c), pa se može pretpostaviti da naprezanja \$\sigma_\xi\$ neće utjecati značajnije na naprezanja \$\sigma_x\$, duž čitavog presjeka, odnosno da su jednaka nuli - kao po prvom rješenju (5). Za primjer na slici 4 naprezanja \$\sigma_x\$ maksimalna su na rubovima, pa se može pretpostaviti da će utjecaj naprezanja \$\sigma_\xi\$ biti značajniji, tj. da se naprezanja \$\sigma_x\$ ne mogu odrediti pod pretpostavkom da su naprezanja \$\sigma_\xi\$ jednaka nuli - kao po drugom rješenju (6).

Primjeri na slici 3 i 4 obrađeni su egzaktno u teoriji elastičnosti [3,4], pa se izrazi za faktore neravnomjernosti (28) mogu usporediti s točnim izrazima, npr. za \$\nu = 0,3\$ (slika 5). Usporedba pokazuje relativno veliku točnost izraza. Za \$\psi_1\$ (primjer na slici 3) relativna greška manja je od 4%, do omjera \$b/l = 0,8-1\$.

Za faktor ψ_2 (primjer na slici 4) točnost je relativno velika do omjera $b/l = 1,4$. Po izrazu $(28)_2$ bit će $\psi_2 = 0,237$, a po teoriji elastičnosti $\psi_2 = 0,246$. U [5] dan je izraz za faktor neravnomjernosti, koji je identičan izrazu $(28)_1$. Međutim, daje se samo jedno rješenje, koje je u ovom radu označeno kao prvo rješenje (5). Sa slike 5 očigledno je da prvo rješenje ne odgovara za primjer na slici 4.



Sl.5. Faktori neravnomjernosti: ψ_1 za primjer na slici 3: — teorija štapova, --- teorija elastičnosti; ψ_2 - za primjer na slici 4: — teorija štapova, --- teorija elastičnosti

Fig.5. Factors of inequality: ψ_1 - for the example in Fig.3: — theory of beams, --- theory of elasticity; ψ_2 - for the example in Fig.4: — theory of beams, --- theory of elasticity

4. ZAKLJUČAK. Izrazi za naprezanja pri rastezanju štapova otvorenog tankostjenog presjeka s utjecajem smicanja mogu se primijeniti i na ekstremno kratke štapove. Točnost izraza prihvatljiva je za tehničku praksu. Točnost je provjerena na primjerima za koje postoje egzaktna rješenja u teoriji elastičnosti.

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ANALIZA NAPREZANJA PRI UVIJANJU ŠTAPOVA OTVORENOG TANKOSTJENOG
PRESJEKA S UTJECAJEM SMICANJA

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Sažetak

Razmatrano je naprezanje pri uvijanju štapova otvorenog tankostjenog presjeka s utjecajem smicanja. Pretpostavlja se da se komponenta čistog uvijanja može zanemariti u odnosu na komponentu vitoperenja. Izrazi za naprezanja dani su u analitičkom obliku. Riješen je primjer štapa složenog poprečnog presjeka, s dvije osi simetrije. Rezultati su provjereni metodom konačnih elemenata.

ANALYSIS OF STRESSES IN THIN-WALLED BEAMS OF OPEN CROSS-SECTION
UNDER TORSION WITH INFLUENCE OF SHEAR

Abstract

Stresses for torsion of thin-walled beams of open cross-section with influence of shear are considered. The assumption is that the component of pure torsion is negligible compare to the component of warping. The expressions for stresses are given in an analytic form. An example of the beam with complex cross-section and double axes of symmetry is solved. The results were verified by the finite element method.

1. UVOD. U klasičnoj teoriji štapova otvorenog tankostjenog presjeka (Vlasov) polazi se - pored pretpostavke da se oblik poprečnog presjeka ne mijenja pri deformiranju - od pretpostavke da su kutne deformacije u središnjoj površini štapa jednake nuli [1], što teoriju ograničava na relativno dugačke štapove, kod kojih su pomaci od kutnih deformacija mnogo manji od pomaka uslijed duljinskih deformacija. Kratki štapovi razmatraju se kao poseban slučaj, kad se komponenta čistog uvijanja može zanemariti u odnosu na komponentu vitoperenja:

$$k = l \sqrt{\frac{GI_t}{EI_\omega}} \rightarrow 0;$$

E - modul elastičnosti, G - modul smicanja, I_t - torzijski moment tromosti, I_ω - sektorski moment tromosti, l - duljina štapa. Tada, kao što je poznato [2], postoji potpuna analogija između savijanja i uvijanja.

U novijim radovima isključuje se pretpostavka da su kutne deformacije u središnjoj površini jednake nuli [3-5]. Rješenja su, međutim, najčešće neprikladna za inženjersku praksu, pa se uvode različita ograničenja, kao npr. o načinu opterećenja [3].

U ovom radu polazi se od pretpostavke da se oblik poprečnog presjeka ne mijenja, uz ograničenje da k teži nuli - kad postoji potpuna analogija između savijanja i uvijanja. U [6] pokazano je da u tom slučaju postoji također potpuna analogija između savijanja s utjecajem smicanja i uvijanja s utjecajem smicanja.

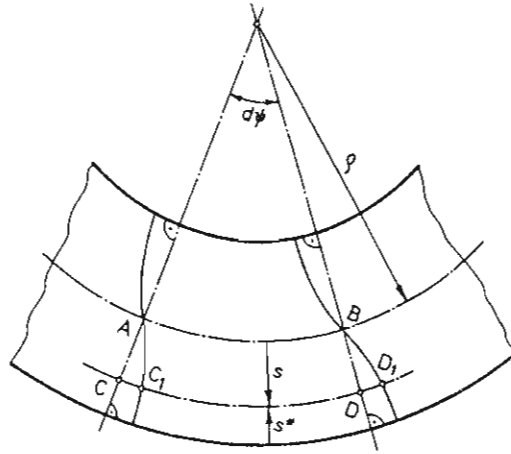
2. DEFORMACIJE I NAPREZANJA. Na slici 1 prikazan je deformirani pojas tankostjenog štapa poprečnog presjeka s dvije osi simetrije, gdje je s $\rho(x)$ označen polumjer zakrivljenosti elastične linije pojasa, koja se poklapa s presjecištem središnjih površina struka i pojasa. Isprekidanom crtom prikazane su središnje linije pojasa zakrenute kao krute linije uslijed vitoperenja presjeka, pri zakretu poprečnog presjeka u ravnini presjeka. Punom crtom prikazane su središnje linije nakon iskrivljenja uslijed kutnih deformacija $\gamma_{x\xi}(x, s)$.

Sa slike je

$$\overline{AB} = dx = \rho(x) d\psi(x), \quad \overline{CD} = [\rho(x) + s] d\psi(x),$$

$$\overline{C_1D_1} = \overline{CD} - \overline{CC_1} + \overline{DD_1};$$

$$\overline{CC}_1 = \int_0^s \gamma_{x\xi}(x, s) ds, \quad \overline{DD}_1 = \int_0^s \left[\gamma_{x\xi}(x, s) + \frac{\partial \gamma_{x\xi}(x, s)}{\partial x} dx \right] ds.$$



Sl.1. Iskrivljenje središnjih linija pojasa štapa

Fig.1. Distortion of the beam strip median lines

Budući da je po Hookeovu zakonu $\gamma_{x\xi}(x, s) = \tau_{x\xi}(x, s)/G$, gdje je $\tau_{x\xi}(x, s)$ tangencijalno naprezanje u smjeru tangente na središnju liniju ξ , bit će

$$\overline{CC}_1 = \frac{1}{G} \int_0^s \tau_{x\xi}(x, s) ds, \quad \overline{DD}_1 = \frac{1}{G} \int_0^s \left[\tau_{x\xi}(x, s) + \frac{\partial \tau_{x\xi}(x, s)}{\partial x} dx \right] ds;$$

$$\overline{C_1D_1} = [\rho(x) + s] d\psi(x) + \frac{1}{G} \int_0^s \frac{\partial \tau_{x\xi}(x, s)}{\partial x} dx ds.$$

Za prizmatičan štap bit će

$$\overline{C_1D_1} = [\rho(x) + s] d\psi(x) + \frac{dx}{G} \int_0^s \frac{\partial \tau_{x\xi}(x, s)}{\partial x} ds.$$

Prema tome, duljinska deformacija u smjeru uzdužne osi štapa x glasi

$$\epsilon_x(x, s) = \frac{\overline{C_1D_1} - \overline{AB}}{\overline{AB}} = \frac{s}{\rho(x)} + \frac{1}{G} \int_0^s \frac{\partial \tau_{x\xi}(x, s)}{\partial x} ds.$$

Budući da je

$$\tau_{x\xi}(x, s) = \frac{M_p(x) S_\omega^*(s)}{I_\omega t(s)}, \quad \frac{dM_p(x)}{dx} = -m_p(x),$$

gdje je M_p moment uvijanja, m_p - moment uvijanja na jedinicu dužine (vanjsko opterećenje), S_ω^* - sektorski statički moment dijela površine presjeka, t - debljina stijenke, bit će

$$\epsilon_x(x, s) = \frac{s}{\rho(x)} - \frac{m_p(x)}{GI_\omega} \int_0^s \frac{S_\omega^*(s)}{t(s)} ds. \quad (1)$$

Ako se prvi član pomnoži i podijeli s udaljenošću razmatrane središnje linije od glavnog pola (središta uvijanja) h_p , dobiva se

$$\epsilon_x(x, s) = \frac{\omega(s)}{\rho(x)h_p} - \frac{m_p(x)}{GI_\omega} \int_0^s \frac{S_\omega^*(s)}{t(s)} ds, \quad (2)$$

gdje je $\omega(s) = h_p s$ - sektorska koordinata. Za normalno naprezanje dobiva se

$$\sigma_x(x, s) = E\epsilon_x(x, s) = E \frac{\omega(s)}{\rho(x)h_p} - m_p(x) \frac{E}{GI_\omega} \int_0^s \frac{S_\omega^*(s)}{t(s)} ds. \quad (3)$$

Integriranjem naprezanja (3) po površini presjeka A dobiva se bimoment:

$$\int_A \sigma_x(x, s) \omega(s) dA = B(x),$$

odnosno

$$\frac{E}{\rho(x)h_p} \int_A \omega^2(s) dA - m_p(x) \frac{E}{GI_\omega} \int_A \omega(s) dA \int_0^s \frac{S_\omega^*(s)}{t(s)} ds = B(x),$$

Budući da je

$$\omega t(s) ds = dS_\omega^*(s), \quad \int_A \omega^2(s) dA = I_\omega,$$

bit će

$$\frac{1}{\rho(x)h_p} = \frac{B(x)}{EI_\omega} + m_p(x) \frac{1}{GI_\omega^2} \int_A \left[\frac{S_\omega^*(s)}{t(s)} \right]^2 dA,$$

odnosno

$$\frac{1}{\rho(x)h_p} = \frac{B(x)}{EI_\omega} + \frac{m_p(x)}{GI_{Pr}}, \quad (4)$$

gdje je

$$I_{Pr} = \frac{I_P}{\chi}, \quad \chi = \frac{I_P}{I_\omega^2} \int_A \left[\frac{S_\omega^*(s)}{t(s)} \right]^2 dA; \quad (5)$$

χ - faktor smicanja, I_{Pr} - reducirani cirkulacijski moment

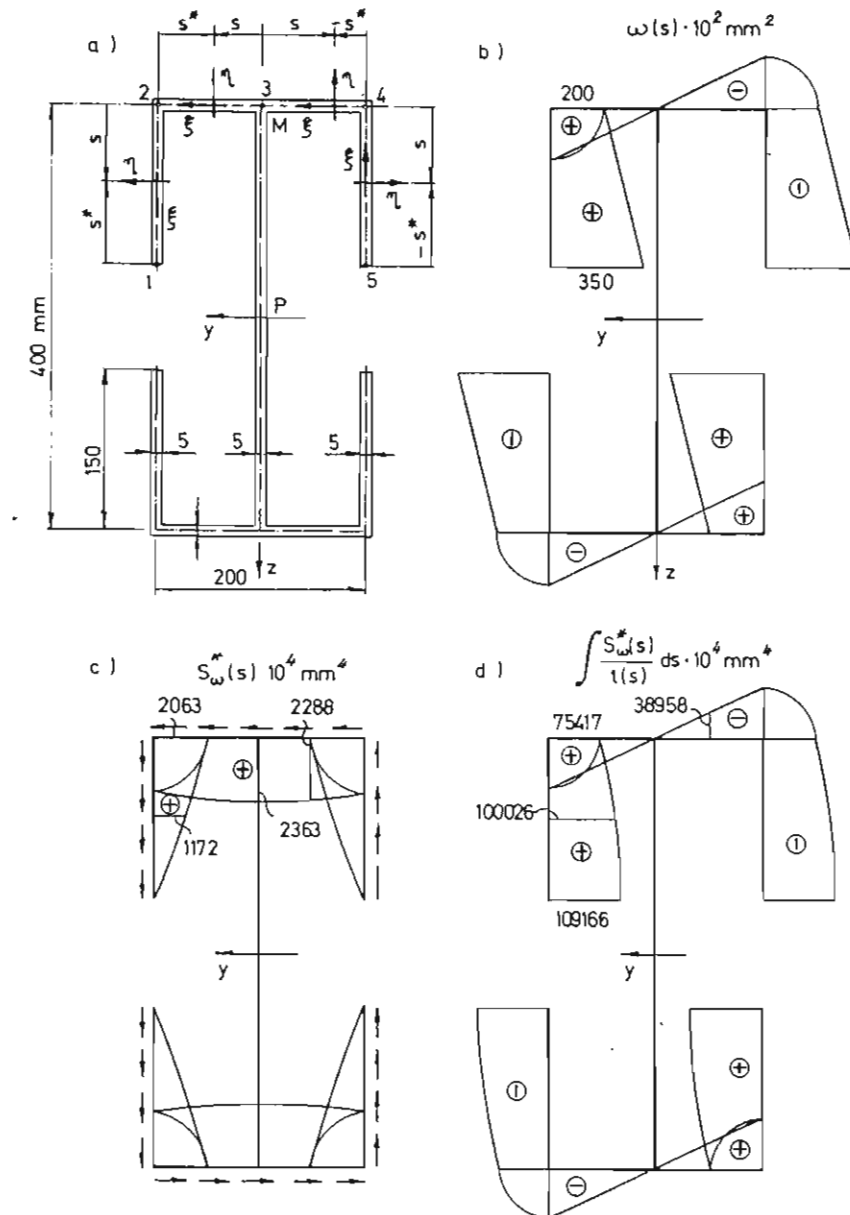
tromosti, I_p - cirkulacijski moment tromosti:

$$I_p = \int_A h_p^2(s) dA. \quad (6)$$

Konačno, uvrštenjem (4) u (3), dobiva se

$$\sigma_x(x, s) = \frac{B(x)}{I_\omega} \omega(s) + m_p(x) \frac{E}{GI_{Pr}} \omega(s) - m_p(x) \frac{E}{GI_\omega} \int_0^s \frac{S_\omega^*(s)}{t(s)} ds. \quad (7)$$

3. PRIMJER



Sl.2. Zadani poprečni presjek: a) dimenzije i krivocrtne koordinate; b) sektorska koordinata; c) sektorski statički moment dijela površine; d) integral sektorskog statičkog momenta dijela površine

Fig.2. Given cross-section: a) dimensions and line coordinates; a) sectorial coordinate; b) sectorial static moment of a portion the area; d) integral of the sectorial static moment of a portion of the area

Na slici 2a zadan je poprečni presjek s dvije osi simetrije, s koordinatama s^* i s . Na slici 2b dana je sektorska koordinata $\omega(s)$, za glavni pol P (središte uvijanja), koji se poklapa s težištem presjeka. Na slici 2c dana je raspodjela sektorskog statičkog momenta, u mm^4 :

$$1-2, 5-4: S_{\omega}^*(s) = (-0,025s^2 - 10s + 2063) \cdot 10^4, \quad 0 \leq s \leq 150;$$

$$2-3, 4-3: S_{\omega}^*(s) = (-0,03s^2 + 2363) \cdot 10^4, \quad 0 \leq s \leq 100.$$

Na slici 2d prikazana je raspodjela integrala sektorskog statičkog momenta, u mm^4 :

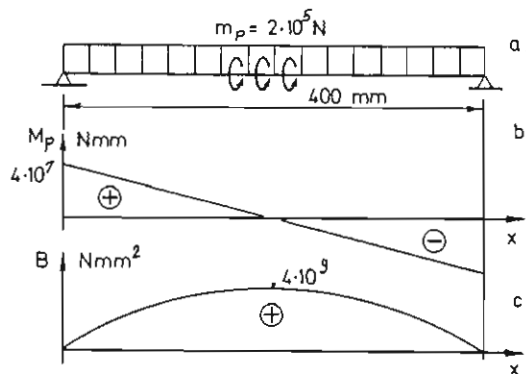
$$3-2, 3-4: \int_0^s \frac{S_{\omega}^*(s)}{t(s)} ds = \pm(-0,00333s^3 + 788s) \cdot 10^4, \quad 0 \leq s \leq 100;$$

$$2-1, 4-5: \int_0^s \frac{S_{\omega}^*(s)}{t(s)} ds = \pm(-0,00167s^3 - s^2 + 413s + 75417) \cdot 10^4, \quad 0 \leq s \leq 150.$$

Izlazi da je

$$I_{\omega} = 2485 \cdot 10^9 \text{ mm}^6, \quad I_P = 78 \cdot 10^6 \text{ mm}^4, \quad \int_A \left[\frac{S_{\omega}^*(s)}{t(s)} \right]^2 dA = 878212 \cdot 10^{11} \text{ mm}^8;$$

$$x = 1,109, \quad I_{Pr} = 7033 \cdot 10^4 \text{ mm}^4.$$



Sl.3. a) zadani štap; b) momenti uvijanja; c) bimomenti

Fig.3. a) given beam; b) moments of twist; c) bimoments

Za slučaj zglobno oslonjenog štapa opterećenog jednolikim momentima uvijanja na jedinicu duljine, na slici 3a, moment uvijanja (slika 3b) i bimoment (slika 3c) glase [4], u Nmm i Nmm^2

$$M_P = m_P \left(\frac{l}{2} - x \right) = 2 \cdot 10^5 (200 - x),$$

$$B = \frac{m_p l}{2} x \left(1 - \frac{x}{l} \right) = 4 \cdot 10^7 x (1 - 0,0025x).$$

Tangencijalna naprezanja ($0 \leq x \leq 400$):

$$3-2: \tau_{x\xi}(x, s) = 0,05366(1-0,005x)(-0,03s^2+2363), \quad 0 \leq s \leq 100;$$

$$2-1: \tau_{x\xi}(x, s) = 0,0322(1-0,005x)(-0,025s^2-10s+2063), \quad 0 \leq s \leq 150.$$

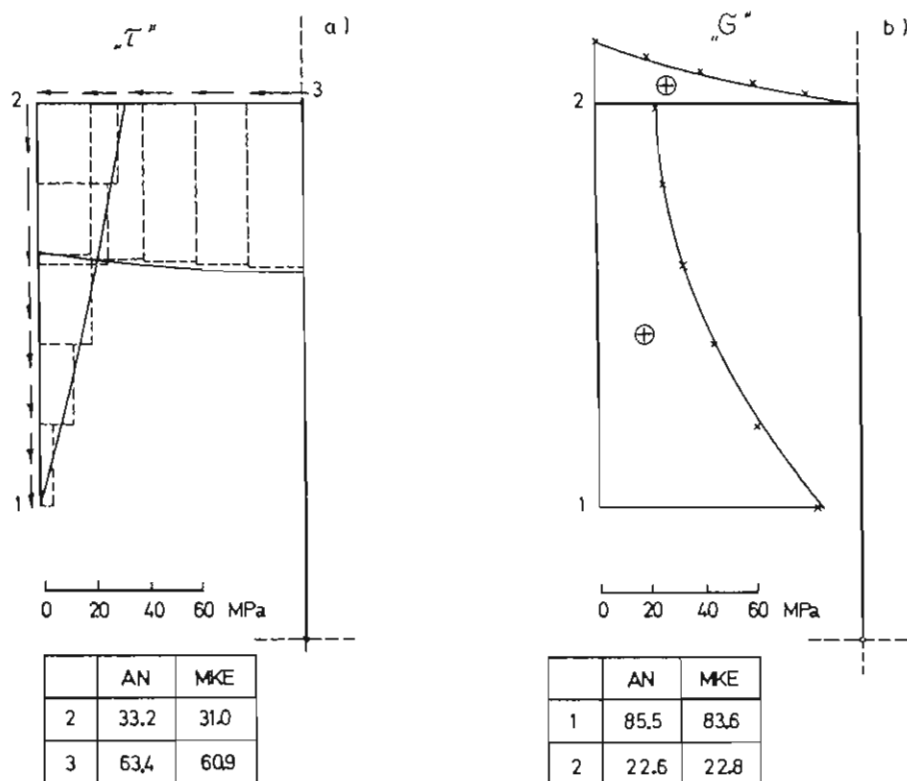
Na slici 4a prikazana je raspodjela za $x = 100$ mm.

Normalna naprezanja ($0 \leq x \leq 400$):

$$3-2: \sigma_x(x, s) = 3,2193x(1-0,0025x) \cdot 10^{-3}s + 6,706 \cdot 10^{-6}s^3 - 0,16271s, \quad 0 \leq s \leq 100;$$

$$2-1: \sigma_x(x, s) = x(1-0,0025x)(0,32193 + 0,16097 \cdot 10^{-2}s) + 3,354 \cdot 10^{-6}s^3 + 2,0121 \cdot 10^{-3}s^2 - 0,11908s - 9,565, \quad 0 \leq s \leq 150;$$

Na slici 4b dana su naprezanja za $x = 200$ mm.



Sl.4. Raspodjela naprezanja po poprečnom presjeku: a) tangencijalna naprezanja za $x = 100$ mm; b) normalna naprezanja za $x = 200$ mm

Fig.4. Distribution of stresses over a cross-section: a) tangential stresses, at $x = 100$ mm; b) normal stresses, at $x = 200$ mm

Na slici 4 prikazani su također rezultati analize metodom

konačnih elemenata (raspodjele označene križićima). Štap je modeliran s 900 membranskih elemenata i 735 čvorova (polovica štapa); neizmjenljivost oblika poprečnih presjeka osigurana je poprečnim stijenama na svakom redu čvorova.

4. ZAKLJUČAK. Naprezanja pri uvijanju štapova otvorenog tankostjenog presjeka s utjecajem smicanja mogu se odrediti pomoću jednostavnih analitičkih izraza, ako se komponenta čistog uvijanja može zanemariti u odnosu na komponentu vitoperenja. Rezultati su zadovoljavajući i za štapove s vrlo malim omjerom duljine prema širini pojaseva. Za omjer 2, greške u procjeni maksimalnih normalnaprezanja nisu veće od 3%, u odnosu na metodu konačnih elemenata.

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ELASTIC STABILITY OF COMPRESSED MULTISPAN BARS

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SUMMARY

The paper deals with the problem of elastic stability of multispans bars with various lengths and flexural rigidities of the spans and with constant compressive longitudinal forces in the spans. Especially the bars with hangovers and fixed ends are studied. The equations which enable the determination of the critical load are derived.

1. Introduction

The analysis of the equilibrium of a buckled bar by the theory of the second order yields for the general case of axial loading the following equilibrium equation

$$[EI(x) w'']'' + F_a(x) w'' + q_x(x) w' = 0 \quad (1)$$

where $F_a(x)$ is the longitudinal force and $q_x(x)$ is the distributed axial load. There are four constants in the general solution of this equation. The constants must be chosen so as to satisfy the boundary conditions what produces a system of four homogeneous linear equations with constants as unknowns. The condition of a non-trivial solution to the system demands that its determinant equals zero. As the terms of the determinant are functions of the axial load any value of this load making the determinant to be equal to zero is a load that causes buckling of the bar. The critical load is the smallest value among these loads. Mathematically the determination of the critical load appears to be an eigenvalue problem. Thus the critical load is the one joined to the smallest eigenvalue.

2. Analysis of the buckled multispans bar

A multispans bar with various lengths and flexural rigidities of the spans is subjected to central axial forces producing constant compressive longitudinal forces varying from span to span (Fig. 1).

$$w' = -Ak \sin kx + Bk \cos kx + C \quad (7)$$

$$w'' = -Ak^2 \cos kx - Bk^2 \sin kx \quad (8)$$

$$w''' = Ak^3 \sin kx - Bk^3 \cos kx \quad (9)$$

Constants A, B, C and D are evaluated from the boundary conditions (5) and (6)

$$A = -D = \frac{M_L}{k^2 EI}, \quad B = \frac{1}{k^2 EI} \left(\frac{M_R}{\sin kl} - \frac{M_L}{\operatorname{tg} kl} \right), \quad C = \frac{M_L - M_R}{k^2 EI l} \quad (10)$$

Substituting (10) in (7) when $x = 0$ yields the slope of the elastic curve at the left support

$$w'_L = \frac{M_L l}{EI} f_1(u) + \frac{M_R l}{EI} f_2(u) \quad (11)$$

and when $x = l$ yields the slope at the right support

$$w'_R = -\frac{M_L l}{EI} f_2(u) - \frac{M_R l}{EI} f_1(u) \quad (12)$$

where $u=kl$ and

$$f_1(u) = \frac{1}{u} \left(\frac{1}{u} - \frac{1}{\operatorname{tg} u} \right), \quad f_2(u) = \frac{1}{u} \left(\frac{1}{\sin u} - \frac{1}{u} \right) \quad (13)$$

Finally the condition of equality of the slopes of elastic curves of the adjacent spans l_{i-1} and l_i at the support i (Fig. 2)

$$w'_{R,i-1} = w'_{L,i}$$

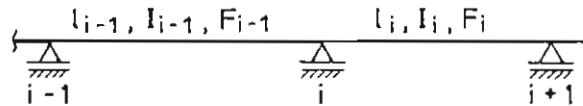


Fig. 2

expressed by (11) and (12) leads to

$$M_{i-1} \frac{l_{i-1}}{I_{i-1}} f_2(u_{i-1}) + M_i \left[\frac{l_{i-1}}{I_{i-1}} f_1(u_{i-1}) + \frac{l_i}{I_i} f_1(u_i) \right] + M_{i+1} \frac{l_i}{I_i} f_2(u_i) = 0 \quad (14)$$

Further the overhanging ends of the bar will be considered. A buckled left overhang is shown in Fig. 3.

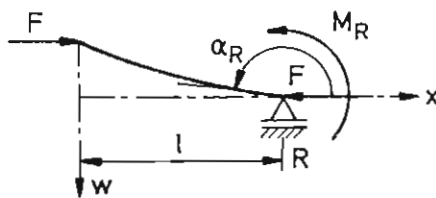


Fig. 3

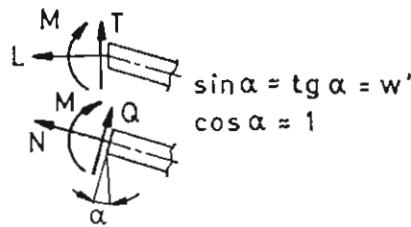


Fig. 4

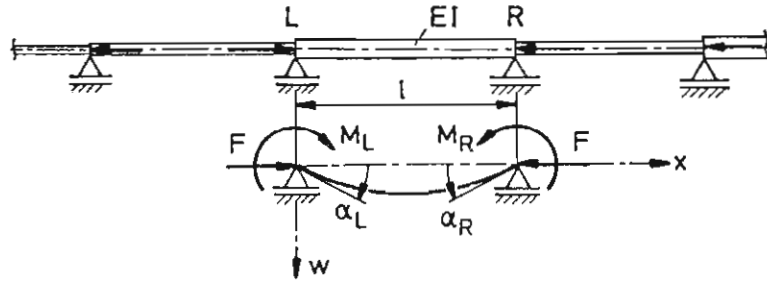


Fig. 1

A span of the bar between the supports L and R can be analysed separately by cutting hinges into the bar over these supports and applying moments M_L and M_R to replace the removed constraints with adjacent spans (Fig. 1). The equilibrium equation for the considered portion of the bar comes out from equation (1) for $EI(x) = EI$, $F_s(x) = F$ and $q_s(x) = 0$ as follows

$$[EI w''']' + F w'' = 0 \quad (2)$$

Double integration of equation (2) gives

$$w'' + k^2 w = C_1 x + C_2 \quad (3)$$

where C_1 and C_2 are the constants of integration and $k^2 = F/EI$. The general solution of this equation is

$$w = A \cos kx + B \sin kx + Cx + D \quad (4)$$

where A, B, C and D are the constants of integration.

As only two boundary conditions are known

$$\text{at } x = 0, w = 0; \quad \text{at } x = l, w = 0 \quad (5)$$

the system of four homogeneous linear equations with the constants as unknowns cannot be obtained. Thus the determination of the critical force cannot be done as it has been described in the introduction. This is understandable because it is obvious that the critical load that causes the loss of stability of the whole bar cannot be determined by considering only one portion of it independently of the others. Therefore the connection of this buckled portion of the bar with the other buckled portions has to be found. The connection follows from the continuity of the elastic curve of the buckled bar. This continuity must also exist at the joints of spans over the supports. This practically means that the slopes of the elastic curves of the adjacent spans must be equal at the joint support. In order to formulate this fact the expressions for the slopes at the left and at the right support has to be determined.

The bending moments M_L and M_R can be expressed as follows:

$$\text{at } x = 0, EI w'' = -M_L; \quad \text{at } x = l, EI w'' = -M_R \quad (6)$$

The derivatives of w given by (4) are

There are three boundary conditions here

$$\text{at } x=l, w=0; \quad \text{at } x=0, M=0; \quad \text{at } x=0, T=0; \quad (15)$$

The bending moment

$$M = -EI w'' \quad (16)$$

and the transversal force (Fig. 4)

$$T = Q + N w' = Q + (L + Q w') w' \approx Q + L w' = M' - F w' = -EI w''' - F w' \quad (17)$$

Substituting (16) and (8) and (17), (7) and (9) in (15) yields $A=0$, $C=0$. Respecting $A=0$ it follows from (8) and $EI w''(x=l) = -M_R$

$$B = \frac{M_D}{k^2 EI} \cdot \frac{1}{\sin kl} \quad (18)$$

Respecting $A=0$, $C=0$ and (18) the slope of the elastic curve at the support follows from (7) when $x=l$

$$w'_R = \frac{M_R l}{EI} f_3(u) \quad (19)$$

where $u=kl$ and

$$f_3(u) = \frac{1}{u} \cdot \frac{1}{\operatorname{tg} u} \quad (20)$$

Analogically the slope of the elastic curve at the support of the right hangover (Fig. 5) is obtained

$$w'_L = -\frac{M_L l}{EI} f_3(u) \quad (21)$$

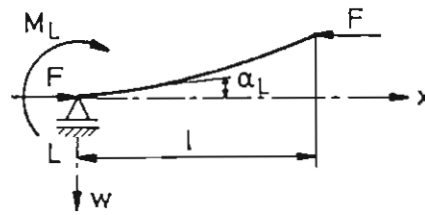


Fig. 5

The condition of continuity of the elastic curve at the first support i (Fig. 6 a) $w'_{R,i-1} = w'_{L,i}$ expressed by (19) and (11) leads to

$$M_i \left[-\frac{l_{i-1}}{I_{i-1}} f_3(u_{i-1}) + \frac{l_i}{I_i} f_1(u_i) \right] + M_{i+1} \frac{l_i}{I_i} f_2(u_i) = 0 \quad (22)$$

and at the last support $i+1$ (Fig. 6 b) $w'_{R,i} = w'_{L,i+1}$ expressed by (12) and (21) leads to

$$M_i \frac{l_i}{I_i} f_2(u_i) + M_{i+1} \left[\frac{l_i}{I_i} f_1(u_i) - \frac{l_{i+1}}{I_{i+1}} f_3(u_{i+1}) \right] = 0 \quad (23)$$

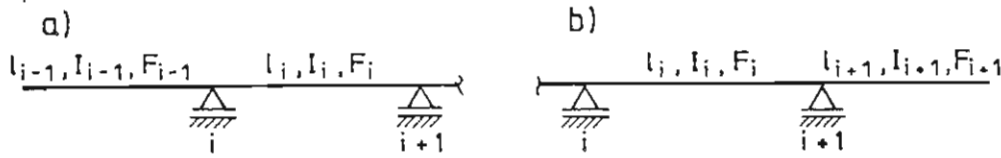


Fig. 6

At last the fixed end of the bar have to be considered. From the boundary condition $w'_{L,i} = 0$ (Fig. 7 a) expressed by (11) it follows

$$M_i \frac{l_i}{I_i} f_1(u_i) + M_{i+1} \frac{l_i}{I_i} f_2(u_i) = 0 \quad (24)$$

and from the boundary condition $w'_{R,i} = 0$ (Fig. 7 b) expressed by (12) it follows

$$M_i \frac{l_i}{I_i} f_2(u_i) + M_{i+1} \frac{l_i}{I_i} f_1(u_i) = 0 \quad (25)$$

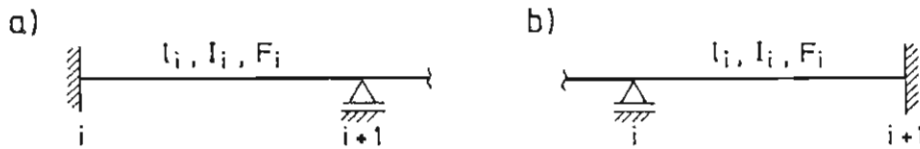


Fig. 7

3. Determination of the critical load

For any multispan bar with n spans equation (14) can be set up for $(n-1)$ intermediate supports. If the bar has overhanging end on the left equation (22) can be written and equation (23) if the bar has overhanging end on the right. If the left end of the bar is fixed equation (24) can be set up and equation (25) if the right end is fixed.

Setting up all these equations yields a system of homogeneous linear equations with moments M_i as unknowns.

The coefficients in the equations are functions of u_i , i.e. of F_i , l_i and I_i . The condition of non-trivial solution to the system demanding that its determinant equals zero produces the equation $\text{Det}(u_1, u_2, \dots, u_i, \dots, u_n) = 0$. If all the variables u_i are expressed by a reference variable u_r relating to a particular span of the bar, i.e.

$$u_i = u_r \frac{l_i}{l_r} \sqrt{\frac{I_r}{I_i}} \sqrt{\frac{F_i}{F_r}} \quad (26)$$

an equation $\text{Det}(u_r)=0$ is obtained. The critical force in the reference span is obtained from

the smallest root u_{r1} of this equation as follows

$$F_{r cr} = u_{r1}^2 \frac{EI_r}{l_r^2} \quad (27)$$

and hence the critical forces in other spans.

4. Conclusion

The procedure of determination of the critical load for multispan bars differs from the one described in the introduction because the homogeneous linear equations with bending moments at the supports as unknowns (and not the constants of the general solution) arise from the condition of continuity of the elastic curve of the bar and not only from the boundary conditions. The determination of the critical load itself does not differ as it also comes to the eigenvalue problem.

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ANALYSIS OF THE PLATES OF VARIABLE STIFFNESS BY THE FINITE DIFFERENCE METHOD

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Summary

A numerical procedure for structural analysis of thin and elastic plates of variable stiffness is described. The procedure is based on the finite difference method and the difference equations are derived from the algebraic equations of local equilibrium. In order to illustrate the described procedure the eight different plates has been analyzed. The calculation results are logical which confirms the accuracy of the described procedure.

Key words: thin plates, plates of variable stiffness, statical analysis, biharmonic equation, finite difference method.

1. Introduction

Mathematical model of thin and elastic plate of constant stiffness is the biharmonic differential equation:

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{K} \quad (1)$$

where:

$w = w(x,y)$	- deflection,
$p = p(x,y)$	- lateral load,
$K = \frac{E\delta^3}{12(1 - \nu^2)}$	- stiffness,
E	- Young's modulus,
ν	- Poisson's ratio,
δ	- thickness of the plate.

The moments m_x , m_y , m_{xy} and the forces q_x , q_y are given by :

$$m_x = -K \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right]; \quad m_y = -K \left[\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right]; \quad m_{xy} = -K(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \quad (2)$$

$$q_x = \frac{\partial}{\partial x} (m_x) + \frac{\partial}{\partial y} (m_{xy}); \quad q_y = \frac{\partial}{\partial y} (m_y) + \frac{\partial}{\partial x} (m_{xy}). \quad (3)$$

The common convention for the positive directions of forces and moments will be used.

Equation (1) is directly derived from the equation:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0 \quad (4)$$

representing the equilibrium of all forces acting on the differential plate element.

2. Finite difference Method

By the finite difference method equation (4) will be solved in the terms of nodes of a regular rectangular mesh (Fig.1). Rows and columns of the mesh are numerated and the node in the i -row and j -column will be denoted by (i,j) . The value of function f at node (i,j) will be denoted by $f(i,j)$ or f_{ij} .

Equation (4) is valid for any point of the plate. Consequently, the following algebraic equation is valid:

$$\frac{\partial q_x(i,j)}{\partial x} + \frac{\partial q_y(i,j)}{\partial y} + p(i,j) = 0. \quad (5)$$

If partial derivations in eq.(5) are approximated by the corresponding central differences and the equation obtained is multiplied by ab , we obtain the equation:

$$bq_x(A) - bq_x(B) + aq_y(C) - aq_y(D) + P_{ij} = 0 \quad (6)$$

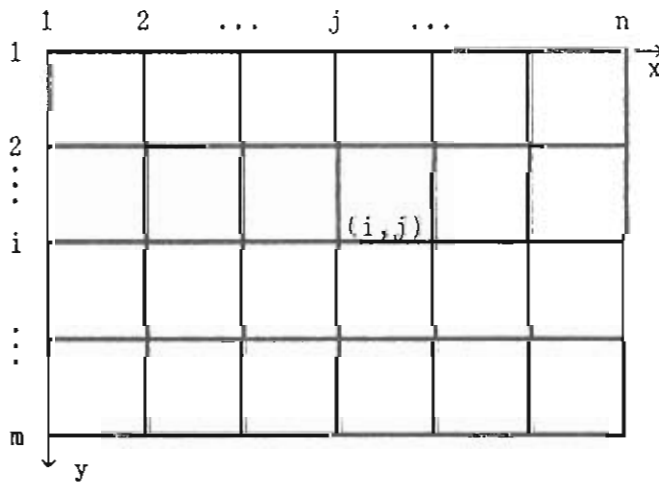


Fig.1 Discretization of the plate

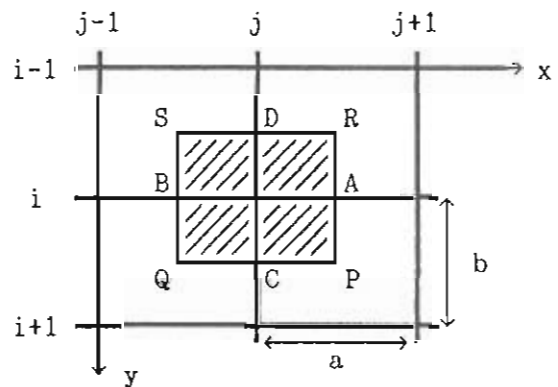


Fig.2 Surrounding of the node (i,j)

The values $q_x(A)$, $q_x(B)$, $q_y(C)$ and $q_y(D)$ are the intensities of internal forces at points A,B,C and D (Fig.2), while $P_{ij} = abp(i,j)$ is the overall lateral force acting to the surrounding of node (i,j) (shaded area in Fig.2).

Relation (6) represents the equation of the equilibrium of all forces acting to the surrounding of node (i,j) and we will call it the equation of local equilibrium. It is a basis for further application of the finite difference method. If we insert expressions (3) into equation (6), the following equation is obtained:

$$\begin{aligned} & b \left[\frac{\partial}{\partial x} m_x + \frac{\partial}{\partial y} m_{xy} \right]_A - b \left[\frac{\partial}{\partial x} m_x + \frac{\partial}{\partial y} m_{xy} \right]_B + a \left[\frac{\partial}{\partial y} m_y + \frac{\partial}{\partial x} m_{xy} \right]_C - \\ & - a \left[\frac{\partial}{\partial y} m_y + \frac{\partial}{\partial x} m_{xy} \right]_D + P_{i,j} = 0. \end{aligned} \quad (7)$$

Due to a simpler writing of the next formulae, a local numeration of nodes in the star of node (i,j) is introduced. (All the numerated nodes in Fig.3 are called the star of node (i,j).) If partial derivations of the moments at points A, B, C and D are approximated by the corresponding central differences and if thus obtained equation is multiplied by ab, we obtain:

$$\begin{aligned} & b^2 [m_x(8) - 2m_x(7) + m_x(6)] + a^2 [m_y(3) - 2m_y(7) + m_y(11)] + \\ & 2ab [m_{xy}(P) - m_{xy}(Q) - m_{xy}(R) + m_{xy}(S)] + abP_{i,j} = 0 \end{aligned} \quad (8)$$

In this equation the moments m_x and m_y are at the nodes 3,6,7,8,11 (Fig.3) and the mixed moments m_{xy} at the points P,Q,R and S (Fig.2). When we introduce the terms:

$$\alpha = \frac{a^2}{b^2}; \quad \beta = \frac{b^2}{a^2} \quad (9)$$

and substitute the partial derivations in expressions (2) with the corresponding central differences and insert them in (8), we obtain the difference equation of node (i,j):

$$\begin{aligned} & K_3 \alpha w_1 + [2K_Q(1-\nu) + K_3\nu + K_6\nu]w_2 - 2[(K_3 + K_7)(\alpha + \nu) + (K_P + K_Q)(1-\nu)]w_3 + \\ & + [2K_P(1-\nu) + K_3\nu + K_8\nu]w_4 + K_6\beta w_5 - 2[(K_6 + K_7)(\beta + \nu) + (K_Q + K_S)(1-\nu)]w_6 + \\ & + [\alpha K_3 + \alpha K_{11} + \beta K_6 + \beta K_8 + 4K_7(\alpha + \beta + 2\nu) + 2(K_P + K_Q + K_S + K_R)(1-\nu)]w_7 - \\ & - 2[(K_7 + K_8)(\beta + \nu) + (K_P + K_R)(1-\nu)]w_8 + K_8\beta w_9 + [2K_S(1-\nu) + K_6\nu + K_{11}\nu]w_{10} - \\ & - 2[(K_7 + K_{11})(\alpha + \nu) + (K_R + K_S)(1-\nu)]w_{11} + [2K_R(1-\nu) + K_8\nu + K_{11}\nu]w_{12} + \\ & + K_{11}\alpha w_{13} = abP_{i,j}. \end{aligned} \quad (10)$$

The stiffnesses of the plate at the nodes 3,6,7,8 and 11 (Fig.3) are denoted by K_3 , K_6 , K_7 , K_8 and K_{11} , while K_P , K_Q , K_R and K_S are stiffnesses at the points P, Q, R and S (Fig.2). The stiffnesses may mutually differ.

Difference equation (10) can be written for any node which is not on the edge of the plate. For node (i,j) which is on the edge of the plate, the difference equation is derived from the corresponding equation of the local equilibrium. For the edge node in Fig.4 it is:

$$\frac{b}{2} q_s(A) - \frac{b}{2} q_s(B) + a q_s(7) - a q_s(D) + \frac{1}{2} abp(7) = 0. \quad (11)$$

the external load is constant in the y-direction while in the x-direction it is being changed after the sine law (Fig.6):

$$p = 100 \sin \frac{\pi}{8} x.$$

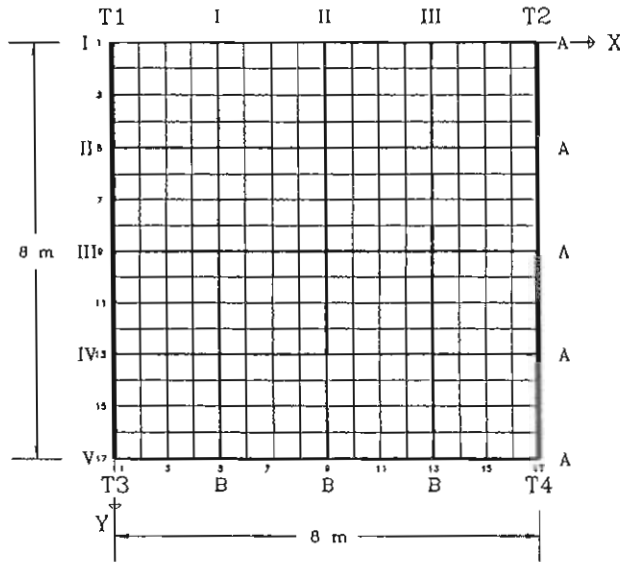


Fig.5 The shape of the plate

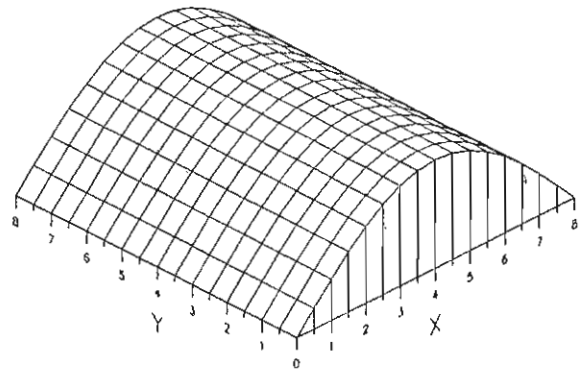


Fig.6 The intensity of loading

Table 1 The thicknesses of plates in mm

Debljina	P r i m j e r							
	1	2	3	4	5	6	7	8
h_1	525	500	475	450	425	525	525	525
h_2	525	500	475	450	425	475	475	475
h_3	525	500	475	450	425	525	475	425
h_4	555	500	475	450	425	475	425	375

Each plate was analyzed for the following four discretization: $a=b=2.0m$, $a=b=1.0m$, $a=b=0.5m$ and $a=b=0.25m$. For the plates of constant stiffness the results have been compared with the correct solutions. For the roughest discretization the errors of moments and deflections were less than 9% and the errors of internal forces less than 14%. For the finest discretization the errors of all values were negligible (less than 2 per mille). By reducing the step of the mesh for twice, the errors have decreased for four times. For the plates of variable stiffness the errors have been determined so that the values obtained by the finest discretization have been taken as the correct ones. In these cases, as well, the extent of errors was within the mentioned range and the errors behaved in the same way when the mesh spacing was reduced.

In Tables 2 and 3 the numerical values of deflection are given in one-hundredths of millimeters for cross section III-A and II-B. These deflections have

been calculated for the finest discretization ($a=b=25$ cm) and written for the discretization in Fig.5. The plotted results are represented in Fig.7 and 8. The deflections of the plates of constant stiffness are represented with dotted line and the deflections of the plates of variable stiffness with full line.

Table 2 The deflections in cross-section III-A in one-hundredths of mm

Row	Col.	1	2	3	4	5	6	7	8
9	1	0	0	0	0	0	0	0	0
9	2	217	251	293	344	409	245	285	335
9	3	425	492	574	675	801	481	560	658
9	4	617	714	833	980	1163	700	816	958
9	5	785	909	1061	1247	1481	894	1042	1224
9	6	924	1069	1247	1467	1741	1056	1231	1446
9	7	1026	1188	1386	1630	1935	1179	1374	1615
9	8	1090	1261	1471	1730	2054	1257	1466	1723
9	9	1111	1286	1500	1764	2094	1288	1503	1767
9	10	1090	1261	1471	1730	2054	1270	1481	1743
9	11	1026	1188	1386	1630	1935	1202	1403	1650
9	12	924	1069	1247	1467	1741	1087	1269	1493
9	13	785	909	1061	1247	1481	928	1084	1276
9	14	617	714	833	980	1163	732	855	1007
9	15	425	492	574	675	801	506	591	696
9	16	217	251	293	344	409	258	302	355
9	17	0	0	0	0	0	0	0	0

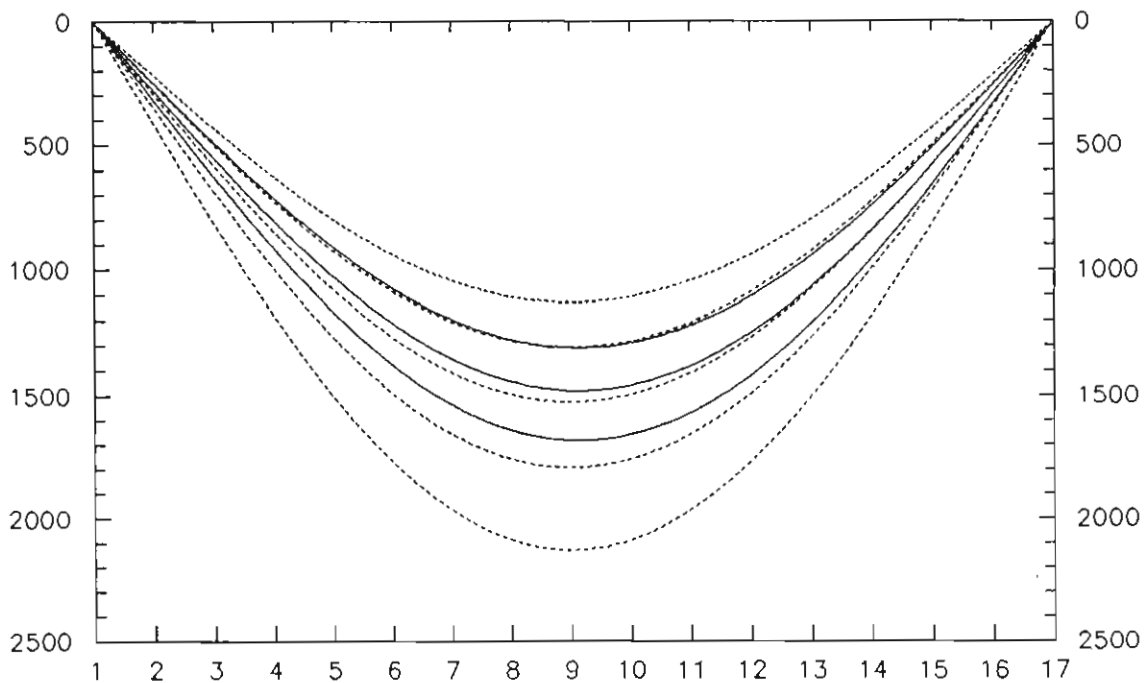


Fig.7 The cross-section III-A

Table 3 The deflections in cross-section II-B in one-hundredths of mm

Row	Col.	1	2	3	4	5	6	7	8
1	9	1209	1400	1633	1920	2279	1402	1548	1714
2	9	1180	1366	1593	1873	2224	1368	1518	1691
3	9	1158	1340	1563	1838	2182	1342	1498	1679
4	9	1141	1321	1541	1812	2151	1323	1486	1676
5	9	1129	1307	1525	1793	2128	1309	1480	1682
6	9	1121	1297	1513	1780	2112	1300	1480	1695
7	9	1115	1291	1506	1771	2102	1293	1484	1714
8	9	1112	1287	1501	1766	2096	1289	1492	1738
9	9	1111	1286	1500	1764	2094	1288	1503	1767
10	9	1112	1287	1501	1766	2096	1289	1516	1800
11	9	1115	1291	1506	1771	2102	1293	1533	1836
12	9	1121	1297	1513	1780	2112	1300	1553	1877
13	9	1129	1307	1525	1793	2128	1309	1577	1921
14	9	1141	1321	1541	1812	2151	1323	1604	1971
15	9	1158	1340	1563	1838	2182	1342	1638	2026
16	9	1180	1366	1593	1873	2224	1368	1679	2091
17	9	1209	1400	1633	1920	2279	1402	1730	2167

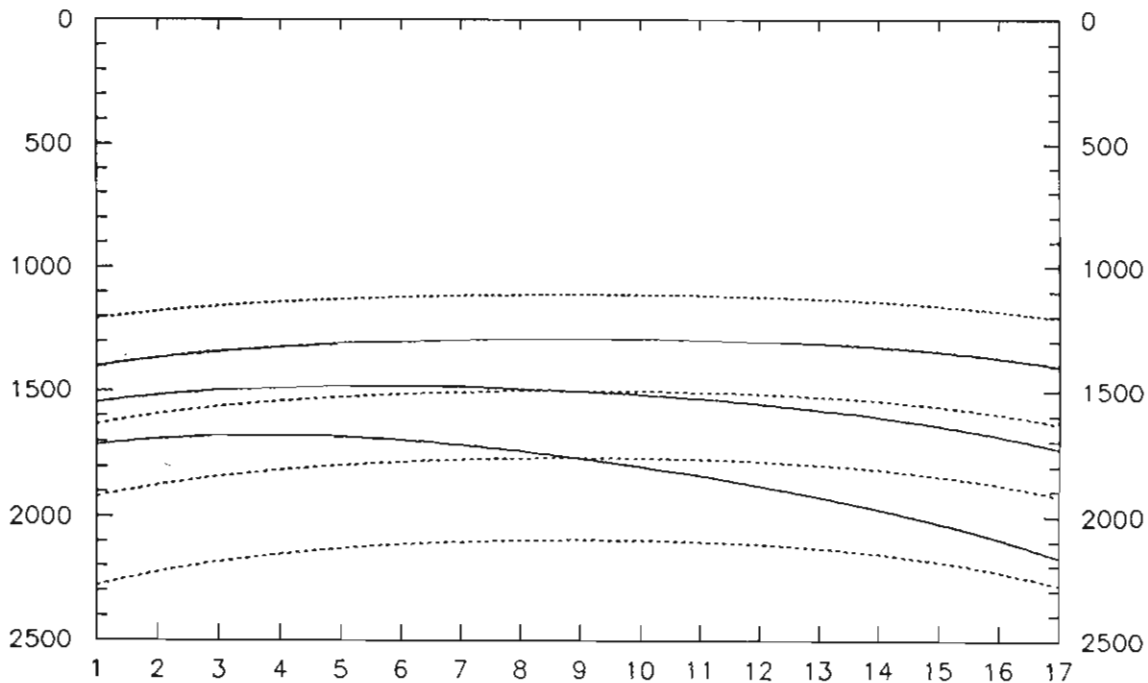


Fig.8 The cross-section II-B

4. Conclusions

On the basis of a good congruence between the deflection approximative and true values for the plates of constant stiffness, it may be concluded that the system of difference equations, obtained by the described procedure, represents a good numerical model for the biharmonic equation. Since the calculated deflection values of the plate of variable stiffness also behave logically, a conclusion can be made that the described procedure gives a good numerical model for such plates as well. Moreover, this model satisfy the global and local equilibrium which is a very significant quality.

The other significant quality of the described numerical model is a simplicity of its algorithmization. Namely, algebraic method [5] can be used for deriving difference equations e.g. the difference equation for any node can be obtained as a linear combination of several fixed vectors. This fact substantially shortens the computer programme and increases the probability of its correctness.

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NUMERIČKI POSTUPAK ZA ODREĐIVANJE SEKUNDARNIH RAVNOTEŽNIH PUTANJA ZA PLOČE BEZ POČETNE IMPERFEKCIJE

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Sažetak

U radu se prikazuje algoritam iterativnog postupka kojim se definira ponašanje tanke elastične ploče pod opterećenjem većim od kritičnog, koji se bazira na metodi konačnih razlika. Prema njemu su sastavljeni kompjutorski programi koji numerički rješavaju navedeni problem. Navode se rezultati nekoliko primjera, koji se dobro slažu s podacima iz literature.

NUMERICAL SOLUTION FOR DETERMINATION OF SECONDARY EQUILIBRIUM PATHS FOR PLATES WITHOUT INITIAL IMPERFECTION

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Summary

In this paper an algorith is shown, which defines the postbuckling behavior of thin elastic plates. Numerical solution is based on the finite difference method. Several computer programs are established, to solve the problem numerically. The shown results agree well with exaples found in the litereture.

1. UVOD

Nagli razvoj kompjutorske tehnike u posljednjih nekoliko desetljeća omogućuje točniju primjenu teorijskih spoznaja u praksi i s tim u vezi, na području građevinarstva npr. , optimalno iskorištenje konstruktivnih svojstava određenih tipova konstrukcija i svojstava materijala. Projektiraju se vizuelno sve neobičnije konstrukcije, premošćuju sve veći rasponi. Pri proračunu takovih "smjelih objekata" dosadašnje u pravilu linearizirane proračunske teorije sve češće se zamjenjuju nelinearnim, koje uključuju materijalnu ili geometrijsku nelinearnost ili oboje. Geometrijska nelinearnost karakteristična je za probleme stabilnosti koji se javljaju kod tlačno opterećenih vitkih konstruktivnih elemenata.

Poznato je naime, da takvi elementi kod određenog nivoa opterećenja naglo prelaze iz početno stabilnog ravnotežnog položaja u drugi bliski ili daleki ravnotežni položaj, koji u eksploataciji konstrukcije može biti, ali najčešće nije prihvatljiv.

Poznato je da se problemi stabilnosti općenito svrstavaju u dvije grupe :

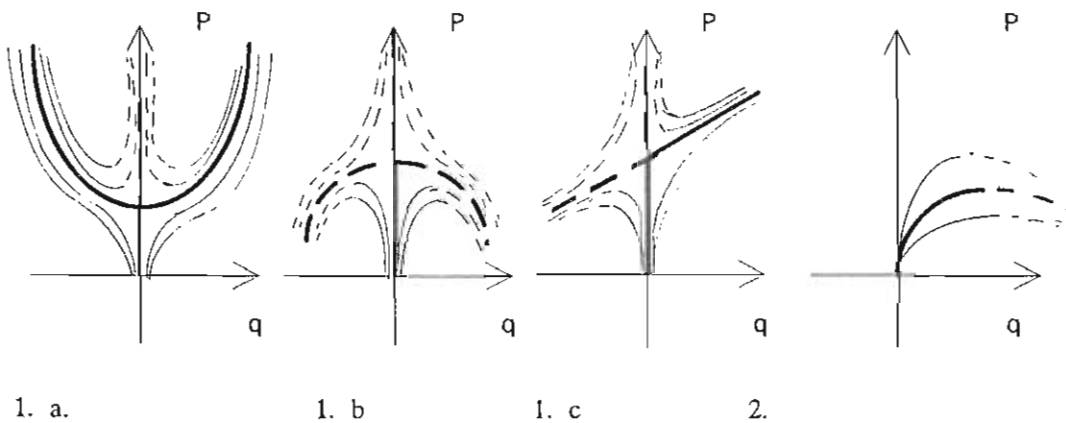
1) Stabilnost sa razgranjenjem ravnoteže (bifurkacijom)

"Razgranjenje ravnoteže" karakterizira pojava da je na određenom tzv. "kritičnom" nivou opterećenja (do kojeg konstrukcija slijedi primarni stabilni ravnotežni put) moguće više bliskih ravnotežnih položaja (sekundarni ravnotežni putevi). Ovisno o karakteru tih sekundarnih putanja razlikuje se nekoliko različitih točaka razgranjenja:

- a) - stabilna simetrična točka razgranjenja
- b) - nestabilna simetrična točka razgranjenja
- c) - nesimetrična točka razgranjenja

2) Stabilnost s graničnom točkom

Drugi tip gubitka stabilnosti karakterizira nagli prelaz iz jednog stabilnog ravnotežnog položaja u drugi također stabilni ali udaljeni ravnotežni položaj, pri čemu se očituje tzv. granična ravnotežna točka, koju definira max. opterećenje za koje je sistem još u prvobitnoj stabilnoj ravnoteži.



Sl.1. Ravnotežne putanje kod različitih tipova stabilnosti
Equilibrium paths for different types of stability

Na sluici 1. pune linije definiraju stabilne ravnotežne putanje, a crtkane labilne. Deblje linije odnose se na sisteme s idealnom geometrijom, a tanje na sisteme s početnom imperfekcijom. Problemi stabilnosti ploča svrstavaju se u područje sa stabilnim simetričnim razgranjenjem. To znači da se njihova moć nošenja ne iscrpljuje na nivou kritičnog opterećenja.

U ovom radu prikazat će se numerički postupak, kojim se određuju sekundarne ravnotežne putanje tankih elastičnih ploča i rezultati proračuna provedenog pomoću programa sastavljenog za tu svrhu, koji se bazira na metodi konačnih razlika. Ograničit ćemo se na ploče bez početne imperfekcije.

2. ODREĐIVANJE KRITIČNOG OPTEREĆENJA

Diferencijalne jednačbe ravnoteže ploče ovise o osnovnim pretpostavkama teorije iz koje se izvode.

Na nivou kritičnog opterećenja može se primjeniti teorija malih pomaka, tako da se diferencijalne jednačbe ravnoteže ploče opterećene u svojoj ravnini mogu prikazati u obliku:

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} = 0 \quad (2.1.)$$

$$\frac{\partial n_y}{\partial x} + \frac{\partial n_x}{\partial y} = 0 \quad (2.2.)$$

$$D\nabla^4 w(x, y) = \lambda(n_{x0} \frac{\partial^2 w}{\partial x^2} + n_{y0} \frac{\partial^2 w}{\partial y^2} + 2n_{xy0} \frac{\partial^2 w}{\partial x \partial y}) \quad (2.3.)$$

n_{x0} , n_{y0} , n_{xy0} membranske sile po jedinici duljine od zadanog opterećenja

λ kritični faktor opterećenja

Prve dvije jednačbe predstavljaju diferencijalne jednačbe ravninskog problema, a kako su u okviru malih pomaka nezavisne od treće, mogu se tako i riješiti. Treća jednačba predstavlja u matematičkom smislu problem vlastitih vrijednosti, za koji postoji niz numeričkih metoda rješavanja.

Ovdje će se ravninski problem riješiti metodom iznijetom u radu [1], a problem vlastitih vrijednosti primjenom metode Stodola-Vianello na sustav algebarskih jednačbi oblika $Kw = \lambda Aw$, koje se dobiju aproksimacijom diferencijalnog operatora diferencijskim u trećoj jednačbi. λ označava kritični faktor opterećenja.

Tok postupka je slijedeći :

1.) Za određeni nivo opterećenja riješi se ravninski problem i odrede unutarnje sile

$$n_{x0}, n_{y0}, n_{xy0}$$

2.) Za početno odabrani vektor pomaka $w_0^{(0)}$ koji se generira generatorom slučajnih brojeva formira se vektor $w^{(0)} = Aw_0^{(0)}$, koji aproksimira desnu stranu jednačbe (2.3).

3.) Riješi se sustav jednačbi $K\bar{w}^{(1)} = w^{(0)}$

4.) Dobiveni vektor pomaka $\bar{w}^{(1)}$ je zapravo $\frac{1}{\lambda} w^{(1)}$. λ definiramo sa $\frac{1}{w_{1,max}}$ gdje je

$w_{i,max}$ maksimalna komponenta vektora $\bar{w}^{(1)}$

5.) Dobiveni vektor pomaka $w^{(1)}$ je novi vektor s kojim se ulazi u slijedeći

iterativni korak $K\bar{w}^{(2)} = \lambda w^{(1)}$

$$\bar{w}^{(2)} = \frac{1}{\lambda} w^{(2)}$$

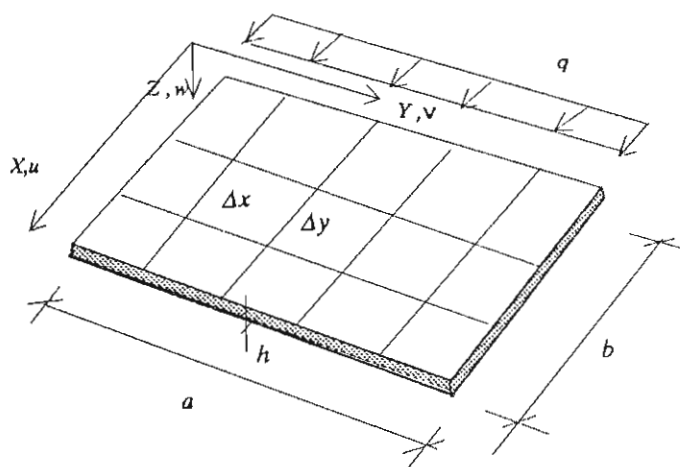
6.) U svakom koraku usporede se komponente novog i starog vektora i definira

max. razlika između njih. Postupak se ponavlja tako dugo dok se ne zadovolje

kriteriji konvergencije, koji se mogu proizvoljno odabrati.

U daljnjem tekstu uspoređuju se rezultati nekih primjera dobivenih ovom metodom sa rezultatima iz literature (D.V.Vainberg : "Plastini , diski , balki-stenki", Gosudarstvenoe izdateljstvo, Kiev 1959. i Voljmir : "Ustoičivost deformiruemih sistem ", Izdateljstvo "Nauka", Moskva, 1967) i rezultatima dobivenim s programom Cosmos.

Koriste se slijedeće oznake :



h - debljina ploče

E - modul elastičnosti gradiva

ν - Poissonov koeficijent

q - kontinuirano opterećenje

(u programu se aproksimira sa konc. silama u čvorovima)

Sl. 2.1.

Rubni uvjeti u ravlini ploče (#) : Rubni uvjeti izvan ravnine ploče (*) :

— - slobodni rub

▨ - nepomični

↔ - pomični u x smjeru

↑↓ - pomični u y smjeru

— - slobodni rub

— — — - slobodno oslonjeni

▨ - upeti

Kritično opterećenje računa se prema Vainbergu iz izraza :

$$\sigma_k = k \frac{\pi^2 E}{12(1 - \nu^2)} \left(\frac{h}{a} \right)^2$$

k je koeficijent koji ovisi o omjeru a/b , rubnim uvjetima i opterećenju i u knjizi je naveden tabelarno za razne omjere a/b

U svim primjerima uzima se ista mreža $\Delta x = \Delta y = 0.2 \text{ m}$, $h=0.005\text{m}$, a kontinuirano opterećenje $q = 12 \text{ kN/m}$, $E = 2.1 \times 10^8 \text{ kN/m}^2$

KRITIČNI FAKTOR λ					
	a	b	KONAČNE RAZLIKE	PREMA VAINBERGU	PREMA COSMOSU
PRIMJER 1 (*)  (#) 	3.2	3.2	0.754	0.773	0.757
	1.6	2.4	3.2	3.47	3.5
	1.6	3.2	3.0		3.27
PRIMJER 2 (*)  (#) 	3.2	3.2	1.27	1.3	1.29
	3.2	2.4	3.67	4.38	3.9
	3.2	1.6	3.34	3.74	3.65
	1.6	2.4	4.04		4.16
	1.6	3.2	3.65		3.84

Tabela 2.1.

3. ODREĐIVANJE SEKUNDARNIH RAVNOTEŽNIH PUTANJA

Kako se u postkritičnom području, pomaci ne mogu smatrati malim, koristi se teorija umjereno velikih pomaka i malih deformacija. Ravnotežno stanje definira se jednažbama :

$$\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} = 0 \quad \frac{\partial n_y}{\partial y} + \frac{\partial n_x}{\partial x} = 0 \quad (3.1)$$

$$D \nabla^4 w = n_x \frac{\partial^2 w}{\partial x^2} + n_y \frac{\partial^2 w}{\partial y^2} + 2n_{xy} \frac{\partial^2 w}{\partial x \partial y} \quad (3.2)$$

koje sada više nisu nezavisne, jer uzimajući u obzir veće pomake w izvan ravnine ne možemo zanemariti njihov utjecaj na deformacije u ravnini.

Funkcijske veze između pomaka i deformacija imaju sada oblik:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \quad \epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

a između pomaka i napona :

$$\sigma_x = \frac{E}{1-\nu^2} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \nu \left(\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right) \right]$$

$$\sigma_y = \frac{E}{1-\nu^2} \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \nu \left(\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right) \right]$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right]$$

Nakon što se odredi točka razgranjenja ravnoteže i kritično opterećenje pri kojem do njega dolazi, sekundarna grana ili bolje reći onaj njen dio između prvog u drugog razgranjenja određuje se navedenim slijedom :

- 1.) U prvom koraku tretiramo problem kao da se radi o malim pomacima i za svaki nivo opterećenja "q" veći od kritičnog rješi se ravninski problem koji je definiran sustavom algebarskih jednažbi oblika $\bar{K}u = q$, kao da ploča nije deformirana izvan ravnine.
- 2.) Dobiveno rješenje međutim ne odgovara stvarnom stanju jer je pod tim opterećenjem ploča deformirana izvan ravnine. Kako točan vektor pomaka nije poznat, pretpostavlja se proizvoljno početni vektor $w^{(0)}$ (obično se uzima proporcionalan formi prvog vlastitog vektora). Međutim, za takvo stanje pomaka unutarnje sile nisu više u ravnoteži. Treba odrediti tzv. skretne sile, koje će djelovati zapravo kao dodatno opterećenje da bi se uspostavila ravnoteža. Dodatne sile imaju oblik :

$$\Delta n_x = \frac{Eh}{1-\nu^2} \left[\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \nu \left(\frac{\partial w}{\partial y} \right)^2 \right]$$

$$\Delta n_y = \frac{Eh}{1-\nu^2} \left[\frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{1}{2} \nu \left(\frac{\partial w}{\partial x} \right)^2 \right]$$

$$\Delta n_{xy} = \frac{Eh}{2(1+\nu)} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

Gornji izrazi aproksimiraju se odgovarajućim diferencijskim izrazima za svaki čvor mreže i tako se dobije dodatni vektor opterećenja q'

- 3.) S njime se ponovo rješava ravninski problem $\bar{K}u = q + q'$
- 4.) S novim vrijednostima membranskih sila definira se desna strana u jednažbi (3.2) koju sada treba riješiti.

Nakon zamjene diferencijalnih operatora diferencijskim, diferencijalna jednažba se zamjenjuje sa sustavom algebarskih jednažbi oblika $Kw^{(1)} = \bar{A}w^{(0)}$, koji se rješava pomoću dekompozicije matrice K na oblik L^TDL i odgovarajućim supstitucijama.

Dobiveni vektor pomaka $w^{(1)}$ bolje aproksimira pomake uzrokovane zadanim opterećenjem od početno pretpostavljenog $w^{(0)}$.

Time je jedan korak iterativnog postupka završen i proces se ponavlja s vektorom $w^{(1)}$. Iteracija se provodi do zadovoljenja kriterija konvergencije koji se može proizvoljno zadati. Isti postupak provodi

se za svaki novi nivo opterećenja, do trenutka kad dolazi do numeričke nestabilnosti, što ukazuje na mogućnost novog razgranjenja.

PRIMJER 1.

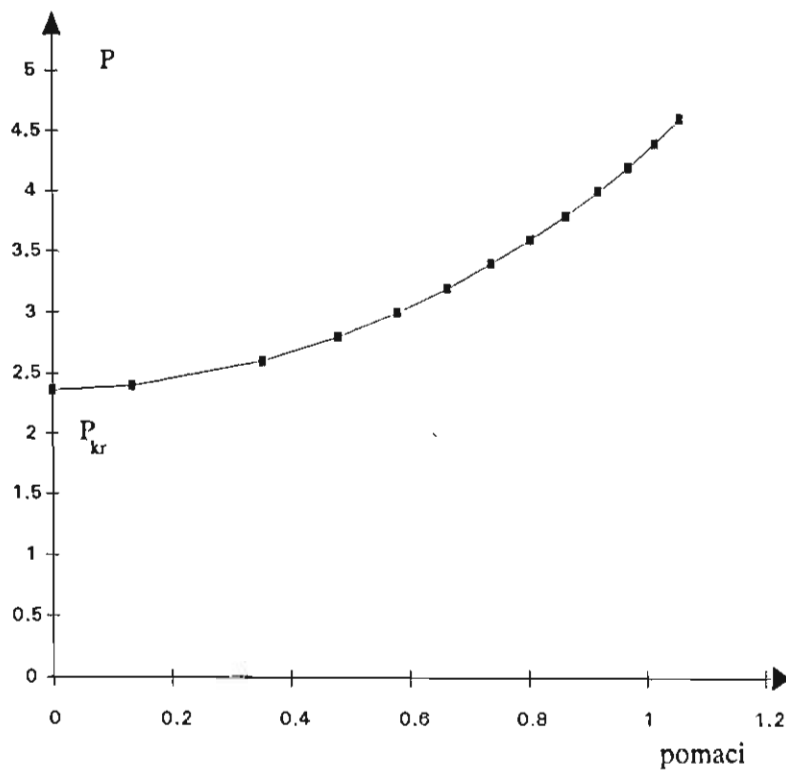
Podaci o ploči su isti kao u primjeru 1. iz tabele 1. ($a=b=2.8\text{m}$)

Navode se pomaci središnje točke ploče (A),

P je koncentrirana sila u čvorovima kojom se aproksimira kontinuirano opterećenje na duljini Δx

Programom koji računa kritični faktor opterećenja dobiveno je kritično opterećenje u čvoru

$P=2.363\text{ kN}$



P	ΔA
2.4	0.13
2.6	0.35
2.8	0.48
3.	0.58
3.2	0.66
3.4	0.74
3.6	0.8
3.8	0.86
4.	0.92
4.2	0.97
4.4	1.0
4.6	1.06

SL. 3. 1. Sekundarna ravnotežna putanja

Pict. 3. 1. Secondary equilibrium path

4. ZAKLJUČAK

Rezultati numeričkih primjera pokazuju dobro slaganje s primjerima iz literature i komercijalnim kompjuterskim programom Cosmos što potvrđuje ispravnost navedenog iterativnog postupka. Primjena metode konačnih razlika, iako ograničena na pravilne oblike ploča, pokazala se relativno jednostavnom i dovoljno točnom.

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THERMAL STRESSES IN THE UTILIZER STEAM BOILER TUBES

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SUMMARY

The artical represents the results of analysis of the utilizer steam boiler side screen tube cracks, as well as deformations and calculations of stresses using a finite element method and measurement of deformations. Actions of the refit of actual condiction are suggested.

1. INTRODUCTION

After repeated crackings on utilizer heat boiler side screen tubes, installed in refinery, a partly cutting of the sheets holding the tubes in a panel has been made to the left upper part of the right side screen, under presumption that it shall minimize stresses and avoid future tube cracking.

To solve this problem, calculations have been performed, using finite element method on models having been given simulations of various design features, as well as measurement of temperature and deformations using strain gauges.

2. DESCRIPTION OF THE STEAM BOILER

The utilizer steam boiler (see Fig.1), with forced circulation for generation of superheated water steam, had been installed in the refinery in 1988.

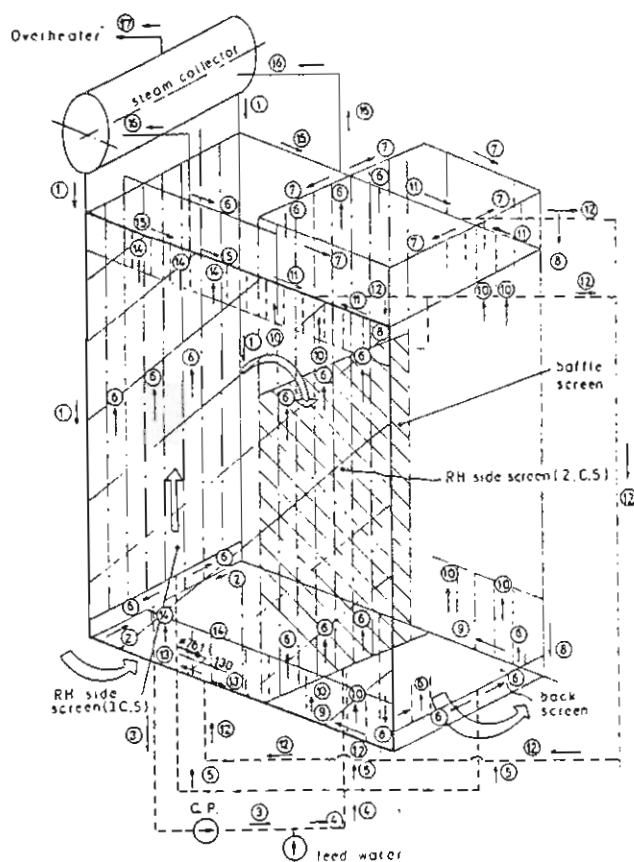


Fig.1 : Sketch of steam boiler, with water and gas circulation

The basic technical data of the boiler are (Lit./1/):

- capacity	18.5 t/h
- operating pressure in steam collector	13.5 bar
- superheated steam temperature	280 deg.C
- temperature of feed water	105 deg.C
- water temperature at screen inlets	182 deg.C

The utilizer steam boiler is consisted of following elements:

- steam collector
- tube system of circulation stage I (front screen, front sides of the left and right side screen, evaporation convex registers)
- tube system of circulation stage II (baffle and back screens, the back part of left and right side screens, economizer register)

The utilizer is of self-standing design, with tube structure consisted of front, mid-part and back downwards tubes, with longitudinally and transversely mounted collecting chambers in its upper and lower part. The tube structure is made of dia. 244.5 mm tubes, being the supporting structure as well as part of the circulation system. Within the tube structure, transversely put diaphragm screens are mounted (front, baffle and back), while the side carries side screens (left and right), of diaphragm design, too (see Fig.2.)

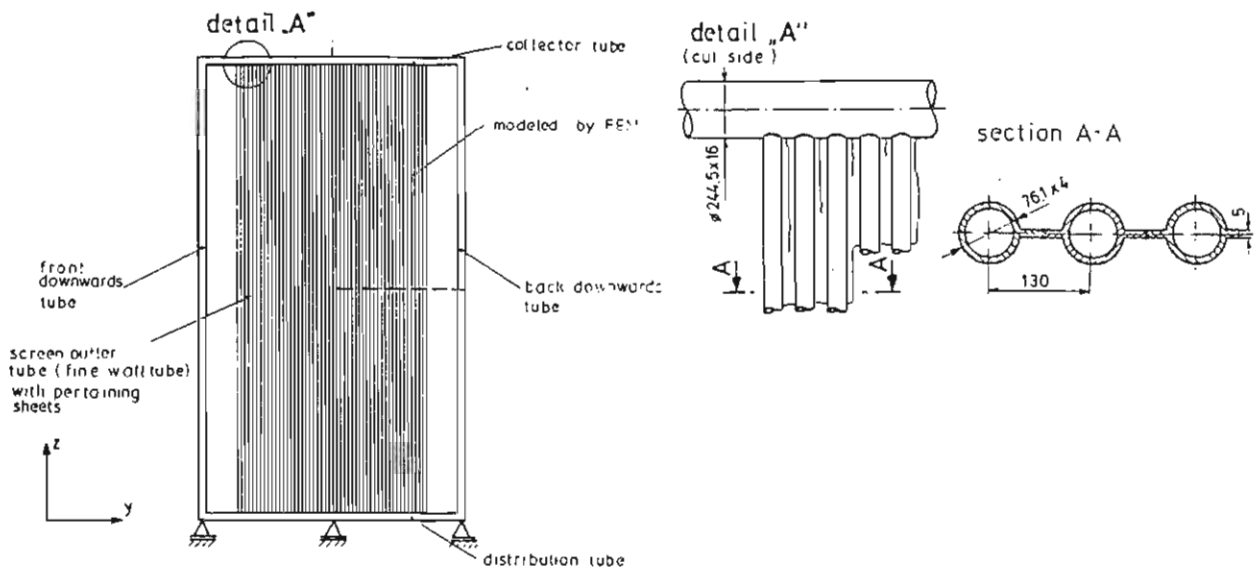


Fig. 2: Sketch of RH side screen

The supporting structure and the side screens of the steam boiler are made of steel marked St 35.8 according to DIN .

3. CALCULATION OF STRESSES

By analysing the cause of crack occurrence to the back tubes (the fine wall tubes) on side boiler screens (where connected to the collecting tube), it has been concluded that the only possible cause are the thermal stresses resulting from uneven heating or cooling of side screens and the rest of the structure, ultimately causing, together with the low-cycle material fatigue, the crackings of tubes. As a critical regime, has been recognized as the one whereby tensile stresses are present, surpassing the fatigue strenght of the steel, and occuring in the "fine wall" tubes of side screens. Tensile stresses in fine wall tubes, as a result of uneven heating, is only possible when the fine wall tube temperature is lower than the one of the boiler supporting structure (collector, downwards and distribution tubes.) Temperature measurements showed that it occurs during the heating of the boiler, as well as during cooling with the circulation pump out of operation. For the calculation of thermal stresses during non-stationary operation, the measured temperature diference between the fine wall and the collecting tube of 11.5 deg.C has been set as a parameter.

Calculations of stationary thermal stresses had been made using the finite element method provided through the ALGOR program package.

Under a presumption that there is an axis-symmetrical problem, a quarter of the side screen has been set as a model, see Fig.2. The side screen, i.e. its one-quarter, has been presented as a two-dimensional model, see Fig.3.

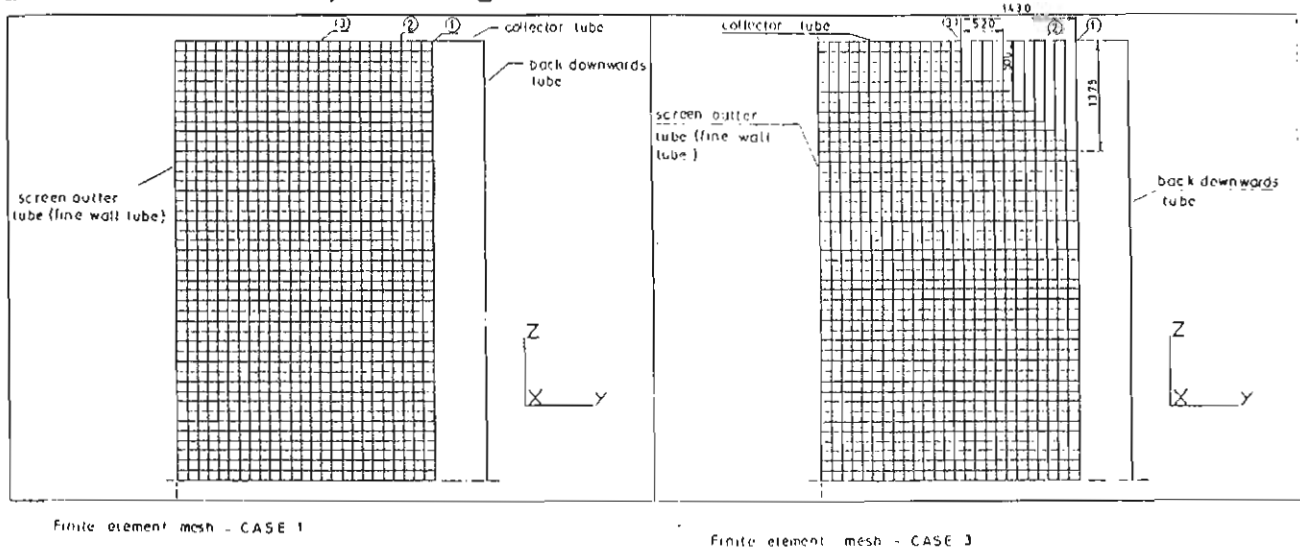


Fig.3: Finite element mesh - CASE 1 and CASE 3

The finite element model is consisted of two types of finite elements: the BEAM elements and the 2-D SOLID ELASTICITY elements. Using BEAM elements and cases of the collecting, the downwards and the fine wall tubes have been supposed, whereby sheets connecting the fine wall tubes in a panel have been provided a model using the SOLID ELASTICITY elements.

The stresses had been set in a manner to assign the panel (i.e. the fine wall tubes and sheets) a temperature 11.5 deg.C lower than the one of the supporting structure.

As for the solution, calculations had been made for five cases:

- CASE 1 - simulation of the "on-site" design of the utilizer steam boiler side screens, with sheets not cut.
- CASE 2 - simulation of cut sheets between the fine wall tubes, as performed in refinery .
- CASE 3 - made on basis of results provided by the two above cases (with a larger surface of cut sheets).
- CASE 4 - simulation of a reconstruction, with the mounting of compensators to the downwards tubes allowing for axial shift.
- CASE 5 - simulation of reconstruction, with mounting of compensator to the downwards tubes allowing for transversal shift only.

Each of the marked nodes in the finite element mesh in Fig.3 represents a typical (critical) point for one of the three primarily described cases.

Comparison of stresses for these points are the stresses resulting from axial and transversal stresses to the tubes.

Table 1 : Comparision of overall stresses in tubes for critical points of particular CASES in accordance with Fig.3

Point	Tube	Overall stress in tubes [MPa]				
		CASE 1	CASE 2	CASE 3	CASE 4	CASE 5
1	outter screen	173	46,5	- 50	44	172,6
	collector	- 53,5	- 34	- 28	- 1	- 54,5
2	outter screen	- 37	83,5	28	- 20,5	- 37
	collector	- 26,5	- 32	- 24	- 9	- 26
3	outter screen	- 8	- 13	56,5	- 10	- 8
	collector	- 11	- 10	- 6,5	- 9	- 11

The table 1 shows that enlarging the cut surface brings to decrease of stress in the screen tubes, while the critical point moves towards the panel center line. Also, one can conclude that the stresses of the structure can be lower if compensators allowing for axial shift are installed, which solution would generate other design problems.

4. DESCRIPTION AND RESULTS OF MEASUREMENT

On the right boiler side screen, measurments of deformations and temperature were performed at locations shown in Fig.4 .

Deformation measurments have been made using strain gauges, at three locations on the left (cut) side, and at three locations on the right (uncut) side of the right side screen, as shown by measurment points numbered 1-6. In both cases, locations were choosen at the end fine wall tubes and the pertaining sheets, immediately to the collecting tube connections.

Measurments of temperature were performed using three probes for temperature measurment, glued over the left side of the right side screen, in such manner that two of the probes were put at the end of fine wall tube (one in the middle section of tube, the other near the connection to collecting tube), and with one probe put to the collecting tube (measuring points 7-9).

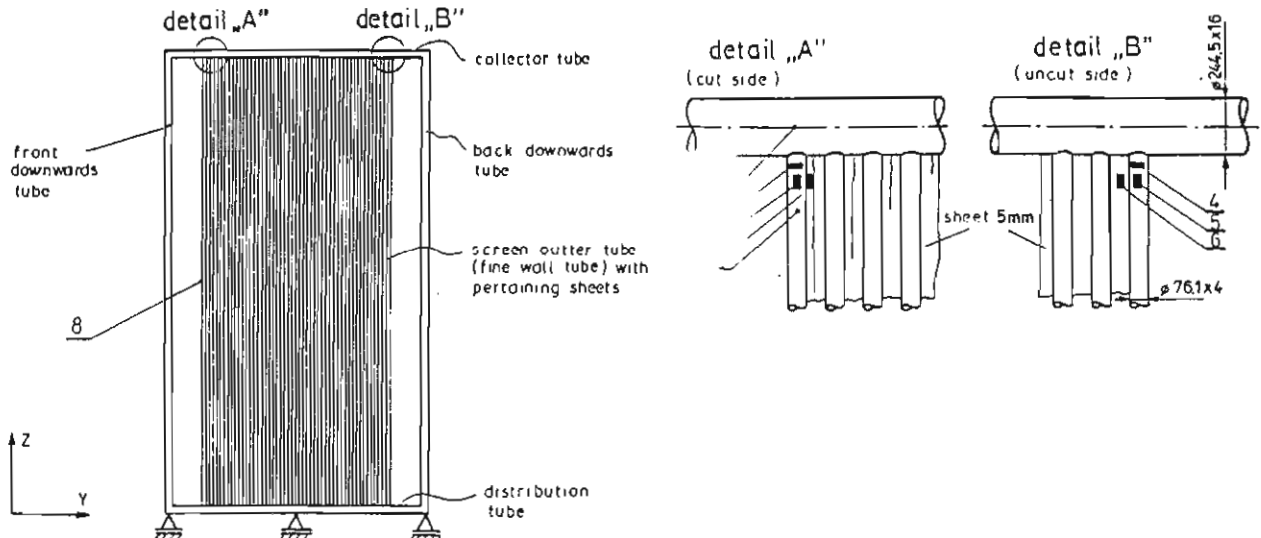


Fig.4: Sketch of RH side screen with measurment locations

Measurments of deformation and temperature have been made for two boiler operational regimes:

The first includes heating-up of the structure, i.e. reach of the operation temperature, the operation in the operation regime, and the cooling with the circulation pump running for a period of 60 minutes.

The second regime includes reaching of the operational temperature and stoppage, i.e. cooling without operating the circulating pump (See Fig.5).

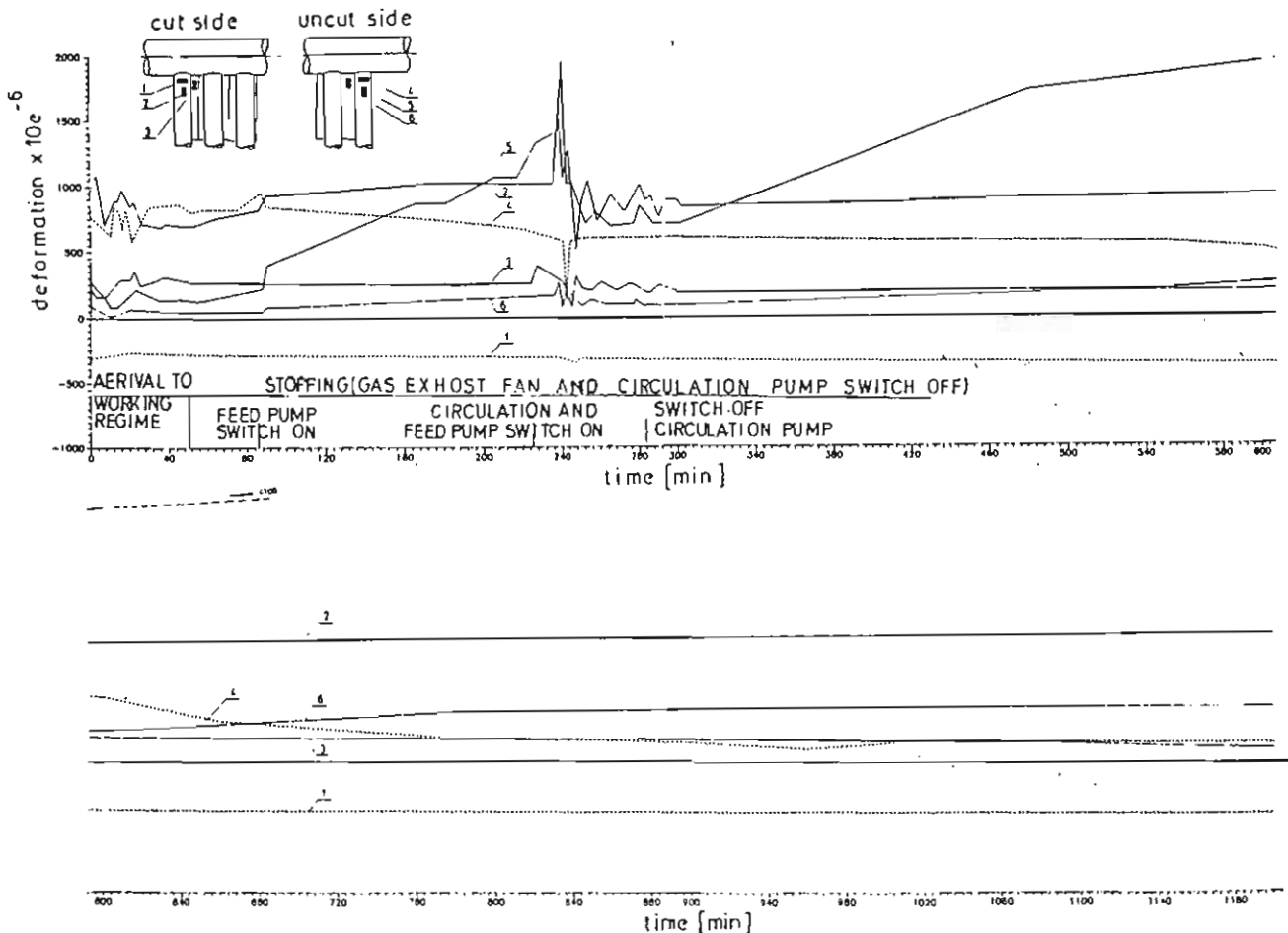


Fig.5: Deformation measurments diagram for the second regime

The stress calculation shows (based on measured deformations) that the stress level for the uncut side (as seen axially by the fine wall tube) remains at approx. 190 MPa during the time of measurements, which corresponds to material yield strength at 200 deg.C. The amount of deformation at the uncut side, during cooling, reached a four times higher level than the one for the cut side, corresponding to stress of 270 MPa, as seen axially. The measurement results could bring to a conclusion that the stress is negatively influenced by the speed of the heating-up and the shut down of the circulation pump, allowing for greater temperature differences, and therefore greater thermal stresses. During heating-up, the greatest temperature difference of 11.5 deg.C between the fine wall tubes and the collecting tube has been measured, while during cooling, 8.7 deg.C. has been measured. Measuring points are situated on places where stress concentration is not present and because of that, special accommodation of measurements are made /2/ and /5/. In addition, using low-cycle fatigue curve of the tube material (Fig 6.), life expectancy of cca 1000 cycles is calculated for non-cut panels and cca 10000 cycles for cut panels.

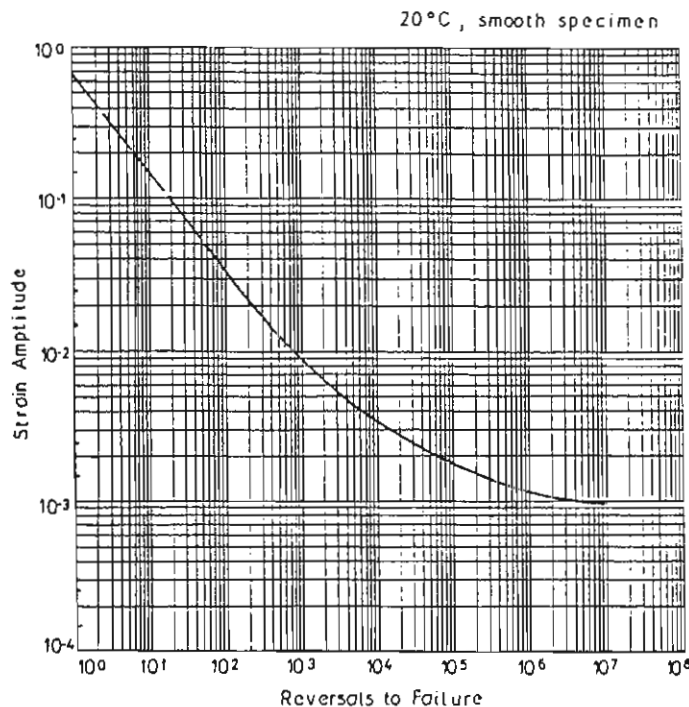


Fig.6: Fatigue curve for boiler tube material

5. CONCLUSION

After repeated crackings on utilizer steam boiler side screen tubes, a partly cutting of the sheet holding the tubes in a panel has been made to the left upper part of the right side screen, under presumption that it shall minimize stresses and avoid the future tube cracking.

By modeling calculations using the finite elements as well as by deformation measurements using strain gauges on the unit itself, it has been found that stresses of the structure can be successfully reduced, by cutting the side screen sheets. Life expectancy can be prolonged from cca 1000 cycles to more than 10000 cycles. The cutting is as much effective as cuts are longer along the tubes and the larger number of sheets are cut toward the center of the panel. Measurements have also proved that as slow as possible heating-up is necessary as the operation of the pump until final cooling of the structure, thus allowing for its even cooling and low thermal stresses occurrence.

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DETERMINATION OF THE STRESS CONCENTRATION FACTORS BY FINITE ELEMENT METHOD

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Abstract

A practical method of determination of the stress concentration factors in design of beam - type structures is proposed in this paper. The proposed method can be applied in the problems of the *Saint - Venant's* torsion and bending with shear. The stress analysis is based on the special quadrilateral finite elements.

1 Introduction

Engineering theory of the beam has been developed and it is the most important theory in the field of solid and structural mechanics.

In the field of structural engineering, it is a common practice to idealize the structure elements as beams and make stress analysis by using the modern engineering theory of beams.

The elements of ships, aircrafts, bridges etc., are typical examples of such structures.

The elementary formulas used in design are based on members having a constant section or section with gradual change of contour.

The presence of grooves, keyways, holes etc., results in modification of the simple stress distribution. In that cases the localized high stresses occur around the exhaustions. This localization of high stresses is known as stress concentration.

The measure of above mentioned stress concentration is the stress concentration factor.

In general it is defined as:

$$k_t = \frac{\sigma_{max}}{\sigma_{nom}}, \quad (1.1)$$

where σ_{max} is maximum stresses in the field of stress concentration and σ_{nom} is nominal stresses, i.e. stresses in the case without exhaustion.

Stress concentration factors are obtained mathematically or experimentally. Experimentally way is by photoelasticity, membrane analogy, strain gage etc. Mathe-

matically way can be done during numerical stress analysis like proposed method in this paper.

2 Theoretical bases

According to the *Saint - Venant's* theory, the displacement field is given by the following equations:

$$\begin{aligned}\mu &= -\theta yz \\ v &= \theta xz \\ w &= \theta \psi(x, y)\end{aligned}\tag{2.1}$$

In the expression (2.1) are:

- θ - angle of twist per unit length
- w - axial displacement
- ψ - *Saint - Venant's* warping function

In the *Saint - Venant's* torsion problem the shear deformation are:

$$\begin{aligned}\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xz} &= 0 \\ \gamma_{zx} &= \frac{\partial w}{\partial x} - \theta y \\ \gamma_{zy} &= \frac{\partial w}{\partial y} + \theta x\end{aligned}\tag{2.2}$$

and therefrom the shear stress:

$$\begin{aligned}\tau_{zx} &= G \left(\frac{\partial w}{\partial x} - \theta y \right) \\ \tau_{zy} &= G \left(\frac{\partial w}{\partial y} + \theta x \right) \\ \tau_{xy} = \sigma_x = \sigma_y = \sigma_z &= 0\end{aligned}\tag{2.3}$$

If we presume the axial displacement in the field of the finite element in following form [4]:

$$w(x, y) = [p(x, y)]\{\alpha\},\tag{2.4}$$

then for the nodal of the finite element is:

$$\{W\} = [p(x, y)^k] \{\alpha\}, \quad (2.5)$$

and finally we have:

$$w(x, y) = [p(x, y)] [p(x, y)^k]^{-1} \{W\} = [N] \{W\} \quad (2.6)$$

The used symbols are:

- $[p(x, y)]$ – polinom matrix
- $\{\alpha\}$ – vector of constants
- $[N]$ – matrix of shape functions

The expression (2.2) and (2.3) according to (2.6) are:

$$\begin{aligned} \gamma_{zx} &= [p_x] [p^k]^{-1} \{W\} - \theta y \\ \gamma_{zy} &= [p_y] [p^k]^{-1} \{W\} + \theta x \end{aligned} \quad (2.7)$$

$$\begin{aligned} \tau_{zx} &= G \left([p_x] [p^k]^{-1} \{W\} - \theta y \right) \\ \tau_{zy} &= G \left([p_y] [p^k]^{-1} \{W\} + \theta x \right) \end{aligned} \quad (2.8)$$

3 Saint – Venant's torsion problem

The total potential energy for construction with unit lenght $\Delta z = 1$, in the *Saint – Venant's* torsion problem is:

$$\Pi = U - W = \int_V \frac{1}{2} \sigma^T \varepsilon dV - M_t \theta = \int_V \frac{G}{2} (\gamma_{zx}^2 + \gamma_{zy}^2) dV - M_t \theta \quad (3.1)$$

According to expression (2.2) now is:

$$\Pi = \iint \frac{G}{2} \left[\left(\frac{\partial w}{\partial x} - \theta y \right)^2 + \left(\frac{\partial w}{\partial y} + \theta x \right)^2 \right] dx dy - M_t \theta \quad (3.2)$$

By minimizing of the total potential energy we have:

$$\begin{aligned}
 \delta \Pi_{(w,\theta)} &= \iint G \left[\left(\frac{\partial w}{\partial x} - \theta y \right) \left(\frac{\partial \delta w}{\partial x} - y \delta \theta \right) + \left(\frac{\partial w}{\partial y} + \theta x \right) \left(\frac{\partial \delta w}{\partial y} + x \delta \theta \right) \right] dx dy - M_t \delta \theta = 0 \\
 \delta \Pi_{(w,\theta)} &= \iint G \left[x^2 \theta \delta \theta + y^2 \theta \delta \theta + x \delta \theta \frac{\partial \psi}{\partial y} \theta - y \delta \theta \frac{\partial \psi}{\partial x} \theta \right] dx dy + \\
 &+ \iint G \left[\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} + \theta x \frac{\partial \delta w}{\partial y} - \theta y \frac{\partial \delta w}{\partial x} \right] dx dy - M_t \delta \theta = 0 \\
 \delta \Pi_{(w,\theta)} &= \iint G \theta \delta \theta \left(x^2 + y^2 + x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) dx dy - M_t \delta \theta + \\
 &+ \iint G \left[\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} + \theta x \frac{\partial \delta w}{\partial y} - \theta y \frac{\partial \delta w}{\partial x} \right] dx dy = \\
 &= (G \theta I_t - M_t) \delta \theta + \iint G \left[\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} + \theta x \frac{\partial \delta w}{\partial y} - \theta y \frac{\partial \delta w}{\partial x} \right] dx dy = 0
 \end{aligned}$$

From the last expression we have:

$$\begin{aligned}
 1. & (G \theta I_t - M_t) \delta \theta = 0 \\
 2. & \iint G \left[\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} + \theta x \frac{\partial \delta w}{\partial y} - \theta y \frac{\partial \delta w}{\partial x} \right] dx dy = 0 \quad (3.3)
 \end{aligned}$$

Equation (2) from (3.3), we can write in the following form:

$$\begin{aligned}
 & \iint G \left[\frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} + \theta x \frac{\partial \delta w}{\partial y} - \theta y \frac{\partial \delta w}{\partial x} \right] dx dy = \\
 & = \delta_{(w)} \iint \frac{G}{2} \left[\left(\frac{\partial w}{\partial x} - \theta y \right)^2 + \left(\frac{\partial w}{\partial y} + \theta x \right)^2 \right] dx dy = 0 \quad (3.4)
 \end{aligned}$$

If we compare the expression (3.4) and (3.2) we arrive to a conclusion that is:

$$\delta \Pi_w = \delta U_w = 0 \quad (3.5)$$

This equation gives the warping distribution during the cross section.

3.1 Stiffness matrix and load vector

On the basis of equations (3.5), (3.2) and (2.7) it is easy to come to stiffness matrix and load vector. So it is:

$$\begin{aligned}
 U_e &= \iint \frac{1}{2} G \left[\left([p_x] [p^k]^{-1} \{W\} - \theta y \right)^2 + \left([p_y] [p^k]^{-1} \{W\} + \theta x \right)^2 \right] \\
 \delta U_{e(w)} &= \iint G \left[\left([p_x] [p^k]^{-1} \{W\} - \theta y \right) \left([p_x] [p^k]^{-1} \{\delta W\} \right) + \right. \\
 &\quad \left. + \left([p_y] [p^k]^{-1} \{W\} + \theta x \right) \left([p_y] [p^k]^{-1} \{\delta W\} \right) \right] dxdy = 0 \\
 \delta U_{e(w)} &= \{\delta W\}^T G \iint \left\{ \left([p^k]^{-1} \right)^T [p_x]^T \left([p_x] [p^k]^{-1} \{W\} - \theta y \right) + \right. \\
 &\quad \left. + \left([p^k]^{-1} \right)^T [p_y]^T \left([p_y] [p^k]^{-1} \{W\} + \theta x \right) \right\} dxdy = 0 \\
 \delta U_{e(w)} &= \{\delta W\}^T G \iint \left\{ \left([p^k]^{-1} \right)^T \left([p_x]^T [p_x] + [p_y]^T [p_y] \right) [p^k]^{-1} \{W\} + \right. \\
 &\quad \left. + \left([p^k]^{-1} \right)^T \left([p_y]^T x - [p_x]^T y \right) \theta \right\} dxdy = 0 \quad (3.6)
 \end{aligned}$$

From the equation (3.6), the stiffness matrix and load vector of the finite element are:

$$[K_e] = G \left([p^k]^{-1} \right)^T \iint \left([p_x]^T [p_x] + [p_y]^T [p_y] \right) dxdy [p^k]^{-1} \quad (3.7)$$

$$\{F_e\} = -G \left([p^k]^{-1} \right)^T \iint \left([p_y]^T x - [p_x]^T y \right) dxdy \quad (3.8)$$

The equation of the finite element is:

$$[K_e] \{\psi_e\} = \{F_e\}, \quad (3.9)$$

and the equation of the construction is:

$$[K] \{\Psi\} = \{F\} \quad (3.10)$$

3.1.1 Quadrilateral element

The quadrilateral element on the figure 1. is represented.

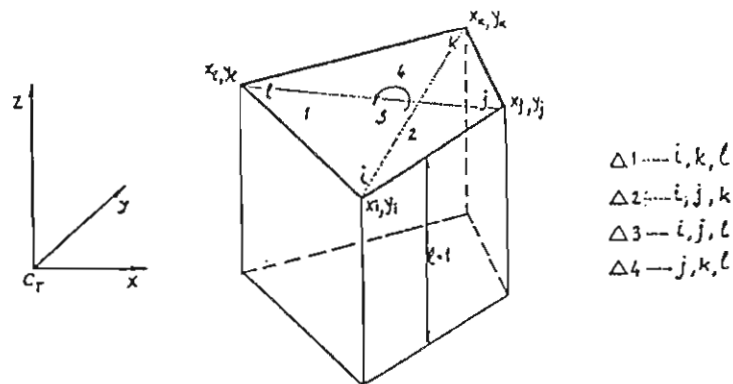


Figure 1.: Quadrilateral element

The stiffness matrix is:

$$[K_e] = G \left([p^k]^{-1} \right)^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_e & 0 & S_x \\ 0 & 0 & A_e & S_y \\ 0 & S_x & S_y & I_x + I_y \end{bmatrix} [p^k]^{-1} \quad (3.11)$$

Presumed:

$$p(x, y) = [1 \ x \ y \ xy]$$

The load vector is:

$$\{F_e\} = -G \left([p^k]^{-1} \right)^T \begin{bmatrix} 0 \\ -S_x \\ S_y \\ I_y - I_x \end{bmatrix} \quad (3.12)$$

In the equation (3.11) and (3.12) are:

- A_e - Finite element area
- I_x, I_y - Moments of inertia of the finite elements
- S_x, S_y - Static area moments

4 Bending with shear

If we presumed:

$$p(x, y) = [1 \quad x \quad y \quad xy],$$

then the shear deformation are:

$$\begin{aligned}\gamma_{xx} &= [p_x] [p^k]^{-1} \{W\} \\ \gamma_{xy} &= [p_y] [p^k]^{-1} \{W\}\end{aligned}\quad (4.1)$$

and the shear stresses:

$$\begin{aligned}\tau_{xx} &= G [p_x] [p^k]^{-1} \{W\} \\ \tau_{xy} &= G [p_y] [p^k]^{-1} \{W\}\end{aligned}\quad (4.2)$$

Load is determined as follows:

$$\bar{F} = \left(\frac{Q_y}{I_x} y + \frac{Q_x}{I_y} x \right) \Delta z, \quad \Delta z = 1 \quad (4.3)$$

4.1 Stiffness matrix and load vector

The stiffness matrix in general form is given by equation (3.7) and load vector is given as:

$$\{F_e\} = \left([p^k]^{-1} \right)^T \int \int F [p(x, y)]^T dx dy \quad (4.4)$$

So it easy to come to the value for stiffness matrix and load vector for quadrilateral element.

4.1.1 Quadrilateral element

The stiffness matrix is given by equation (3.11) and for load vector we have:

$$\{F_z^e\} = \frac{Q_y}{24 I_x} \begin{Bmatrix} A_1(3y_1 + y_i) + A_2(3y_2 + y_i) + A_4(3y_4 + y_i) \\ A_1(3y_1 + y_j) + A_3(3y_3 + y_j) + A_4(3y_4 + y_j) \\ A_1(3y_1 + y_k) + A_2(3y_2 + y_k) + A_3(3y_3 + y_k) \\ A_2(3y_2 + y_l) + A_3(3y_3 + y_l) + A_4(3y_4 + y_l) \end{Bmatrix} \quad (4.5)$$

$$\{F_v^e\} = \frac{Q_v}{24I_v} \begin{Bmatrix} A_1(3x_1 + x_i) + A_2(3x_2 + x_i) + A_4(3x_4 + x_i) \\ A_1(3x_1 + x_j) + A_3(3x_3 + x_j) + A_4(3x_4 + x_j) \\ A_1(3x_1 + x_k) + A_2(3x_2 + x_k) + A_3(3x_3 + x_k) \\ A_2(3x_2 + x_l) + A_3(3x_3 + x_l) + A_4(3x_4 + x_l) \end{Bmatrix} \quad (4.6)$$

where are:

- $A_i, i = 1 \dots 4$ - finite element areas
- $x_1 \dots x_4, y_1 \dots y_4$ - gravity center of the finite element

For justification of the proposed method a numerical example is shown in figure 2..

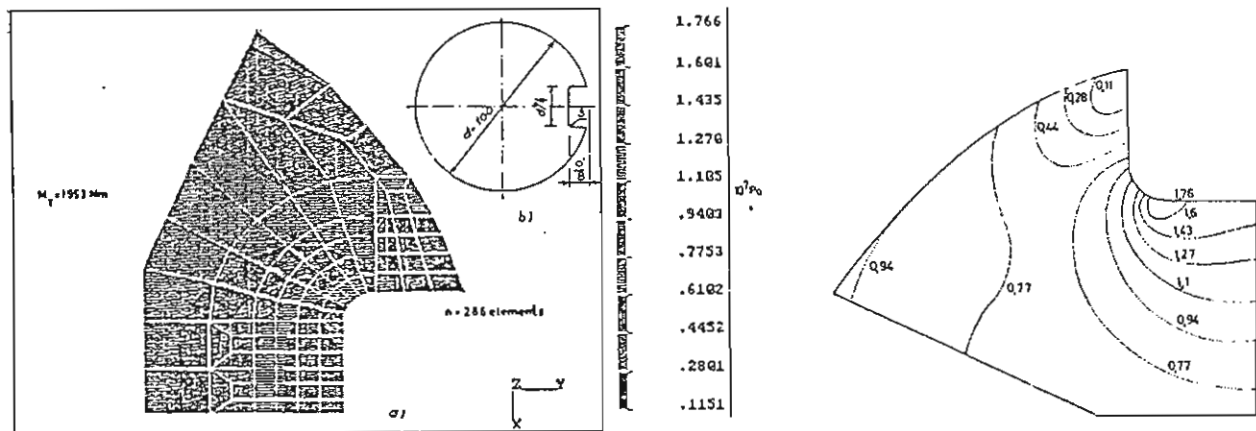


Figure 2.: Stress distribution (stress concentration factor)

Conclusions

The proposed method based on the nodal axial displacements of the quadrilateral elements, is very efficient for the determination of the stress concentration in *Saint - Venant's* torsion problem and bending with shear. A computer program is the most reliable tool on the stress analysis of solid cross section of beam as a thin walled structures too.

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SAVIJANJE KRUŽNIH PLOČA

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Sažetak: U radu se pomoću analitičkog i numeričkog postupka analizira savijanje kružnih ploča. U primjeni se metode konačnih elemenata (MKE) za rješavanje savijanja kružnih ploča u elastično-plastičnom području kod malih deformacija koristi Prandt-Reussov zakon tečenja. Opterećenje kružne ploče kontinuirano - trapezna raspodjela. Rezultati dobiveni MKE pokazuju dobru suglasnost s provedenim eksperimentima. Rezultati važe za deformacije do 2 %.

BENDING OF CIRCULAR PLATES

Summary: Analytic and numeric procedure for circular plate bending is treated in the paper. In the application of finite element method (FEM) in solving the problem of circular plate bending in elastic-plastic area with small deformation Prandt-Reuss's yield law is used. Loading of circular plate is continuous of trapezoidal distribution. The results obtained by FEM agree well with the conducted experiments. The results apply for deformations up to 2 %.

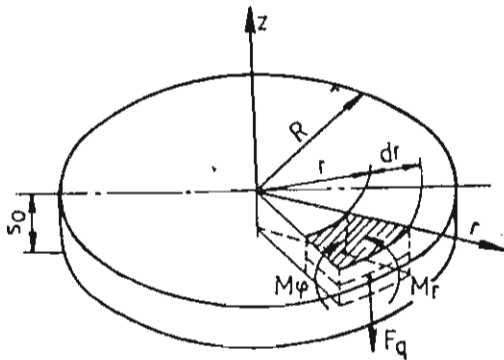
1. UVOD

Rješavanje je zadaće savijanja kružnih ploča za elastično-plastično stanje složeno. Potrebno je izvršiti pojednostavljenja kod značajki materijala [1,2,3]. Ispitivanjem se elasto-plastičnih stanja kod savijanja kružnih ploča pokazuje, da se ona neznatno razlikuje od savijanja grede, kada potpuno prelazi u plastično stanje. Dosadašnja teorijska razmatranja temeljila su se uglavnom na pojednostavljenju proračuna. Rothstein [4] je potvrdio ovo preko istraživanja na savijenim limovima putem vizioplastične metode. Pored postupka simulacije procesa prema Fait i Rothstein [5] pod primjenom infinitezimalnog segmenta modela lima, omogućava primjena MKE proračun pomaka, naprezanja i elastičnog vraćanja. Kod numeričkog je izračunavanja polja elasto-plastičnih deformacija, naprezanja i pomaka doraden program [6], koji je sačinjen po uzoru Morozova i Nikiškova [7] te idejama Zienkiewieza [8,9].

2. ANALITIČKI PRORAČUN

U presjeku kružne ploče $r=\text{konst.}$, $\varphi=\text{konst.}$ djeluju unutarnji momenti savijanja M_r , M_φ izazvani vanjskim opterećenjem F_Q slika 1, koji zajedno stvaraju diferencijalnu jednadžbu ravnoteže:

$$\frac{dM_r}{dr} + \frac{M_r - M_\varphi}{r} - F_q = 0 \quad (1)$$



Slika 1. Savijanje kružne ploče
Figure 1. Circular plate bending

gdje je

$$M_r = \int_{-\frac{s_0}{2}}^{\frac{s_0}{2}} \sigma_r \cdot z \cdot dz \quad (2)$$

$$M_\varphi = \int_{-\frac{s_0}{2}}^{\frac{s_0}{2}} \sigma_\varphi \cdot z \cdot dz \quad (3)$$

pri čemu je z -koordinata neke točke mjerena od sredine poprečnog presjeka ploče. Vanjsko opterećenje F_q iznosi

$$F_q = -\frac{1}{r} \int_0^r q \cdot r \cdot dr \quad (4)$$

Moment savijanja M_r i M_φ mogu se izračunati u trigonometrijskom obliku preko funkcije ψ [10,11].

$$M_r = \frac{2}{\sqrt{3}} M_T \cos \psi \quad (5)$$

$$M_\varphi = \frac{2}{\sqrt{3}} M_T \cos\left(\psi - \frac{\pi}{3}\right) \quad (6)$$

Ako izraz (5) i (6) uvrstimo u izraz (1) slijedi

$$\frac{d}{dr} \left(\frac{2}{\sqrt{3}} M_T \cos \psi \right) + \frac{\frac{2}{\sqrt{3}} M_T \cos \psi - \frac{2}{\sqrt{3}} \cos\left(\psi - \frac{\pi}{3}\right)}{r} - F_q = 0 \quad (7)$$

Sređivanjem izraza (7) dobiva se

$$\frac{dr}{r} = \frac{\sin \psi \cdot d\psi}{\frac{\sqrt{3}}{2} \cdot \frac{F_q \cdot r}{M_T} - \sin\left(\psi - \frac{\pi}{6}\right)} \quad (8)$$

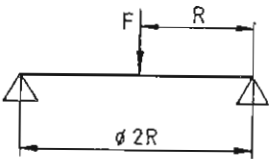
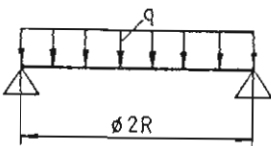
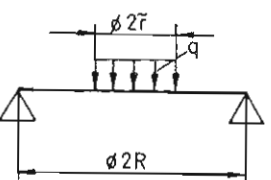
Izraz (8) je lak za integriranje ako je $F_Q=0$ (savijanje čistim momentom) ili kod uvjeta

$$F_Q = \frac{a}{r}, \text{ gdje je } a\text{-konstanta.}$$

Najveća opterećenja kod kružnih ploča za razne slučajeve opterećenja dano je u tablici 1[10], u primjeru kada je nastupila potpuna plastifikacija, gdje je M_T u ovisnosti od debljine materijala s_0 i naprezanja tečenja σ_T dan izrazom:

$$M_T = \sigma_T \frac{s_0^2}{4} \quad (9)$$

Tablica 1. Određivanje opterećenja
Table 1. Determination of loading

OPTEREĆENJE OKRUGLE PLOČE	ODREĐIVANJE OPTEREĆENJA
	$F = 2\pi M_T$
	$q = \frac{6 M_T}{R^2}$
	$q = \frac{6 R M_T}{r^2 (3R - 2r)}$

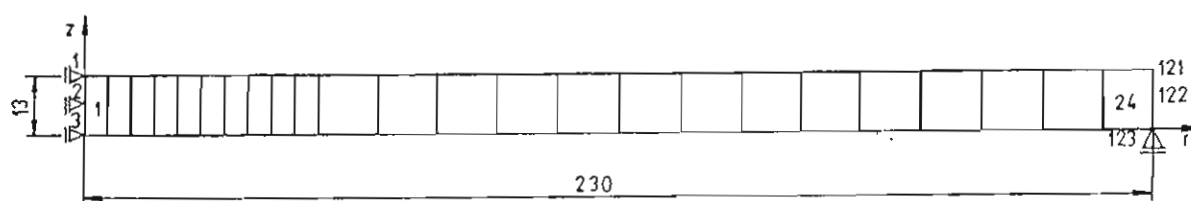
3. METODA KONAČNIH ELEMENATA

Korišteni se program MATCVON, koji primjenjuje MKE, može razložiti u tri dijela: priprema ulaznih podataka, glavni program i obrada izlaznih rezultata [12,13,14]

3.1 Priprema ulaznih podataka

Ulazni podaci koji su potrebni za glavni program MATCVON sadrže [12]: geometriju, podatke o materijalu, podatke o opterećenju i rubnim uvjetima kao i podatke o upravljanju izlaznim rezultatima i načinu proračuna.

Mreža kružnih osmočvornih elemenata osnog presjeka kružne ploče kao aksijalno-simetrične zadaće dana je na slici 2.



Slika 2. Mreža za kružno savijanje ploče
Figure 2. Plate circular bending net

Odabrani je materijal StE 500 proizvođač "AG der Dillinger Hüttenwerke" Zapadna Njemačka. Materijal je proizveden prema DIN 17102/83. Dimenzija table lima je 13x3200x9500 mm. Iz navedene table lima izrađene su epruvete za ispitivanje značajki materijala. Dimenzije epruveta su B6x50 DIN 50125. Iz iste table lima napravljene su i rondle ϕ 460 mm za provedbu eksperimenta. Stroj na kojem su vršena ispitivanja je proizvod tvrtke "Schenck". Stroj ima računalom podržani postupak ispitivanja. Na istom stroju provedeni su i eksperimenti pomoću alata za savijanje kružnih ploča. Točnost je stroja 1 prema DIN 51221.

Ispitivanjem su epruveta dobiveni podaci za:

1. naprezanje plastičnog tečenja $k_f = 588 + 700 \epsilon^{0.4} \text{ Nmm}^{-2}$
2. modul elastičnosti $E = 206454 \text{ Nmm}^{-2}$
3. Poissonov koeficijent $\nu = 0,32$

3.2 Glavni program

Kod programa MATCVON se koristi elasto-plastičan zakon tečenja, koji uzima u obzir plastično ponašanje materijala. On radi pomoću iterativnog postupka koristeći metodu početnih naprezanja. Temeljni algoritam metode početnih naprezanja može se ukratko zapisati u obliku [6]

$$\begin{aligned} \{\Delta u_i\} &= [K_e]^{-1} \{\psi_{i-1}\} \\ \{u_i\} &= \{u_{i-1}\} + \{\Delta u_i\} \\ \{\sigma_i\} &= \{\sigma_{i-1}\} + \{\Delta \sigma_i\} \\ \{\psi_i\} &= \{F\} - \int_V [B]^T \{\sigma_i\} dV \end{aligned} \quad (10)$$

Ovdje indeks "i" označava broj iteracije. Time se rješenje nelinearnog zadatka svodi na postupno rješavanje zadatke elastičnosti s konstantnom matricom krutosti $[K_e]$, kao opterećenje se koristi "greška" $\{\psi_i\}$. Kod prve iteracije dosegnuta vrijednost pomaka $\{u_0\}$ i naprezanje $\{\sigma_0\}$ te:

$$\{\psi_0\} = \{\Delta F\} \quad (11)$$

iteracijski proces završava:

$$\frac{\{\Delta u_i\}^T \{\Delta u_i\}}{\{u_i\}^T \cdot \{u_i\}} \leq \epsilon \quad (12)$$

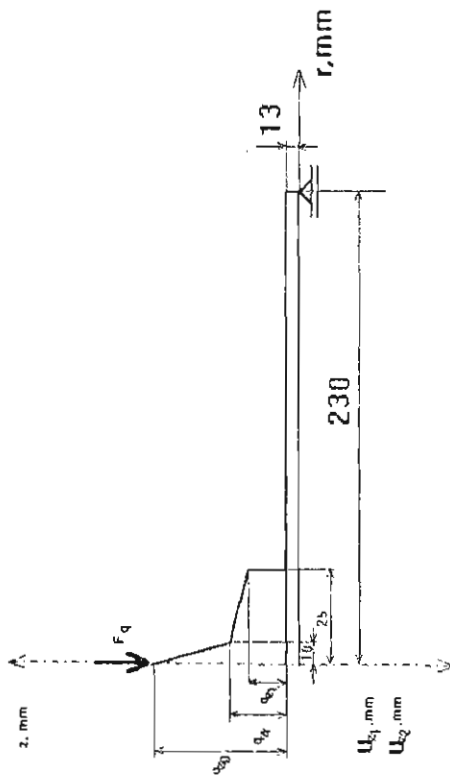
gdje je ϵ - zadana točnost izračunavanja.

3.3 Izlazni rezultati

Rezultati su dobiveni MKE za kružnu ploču ϕ 460 mm i debljine 13 mm prikazani u tablici 2 i na slici 3,4. Opterećenje ploče na promjeru ϕ 50 mm je dvostruko-trapezna raspodjela.

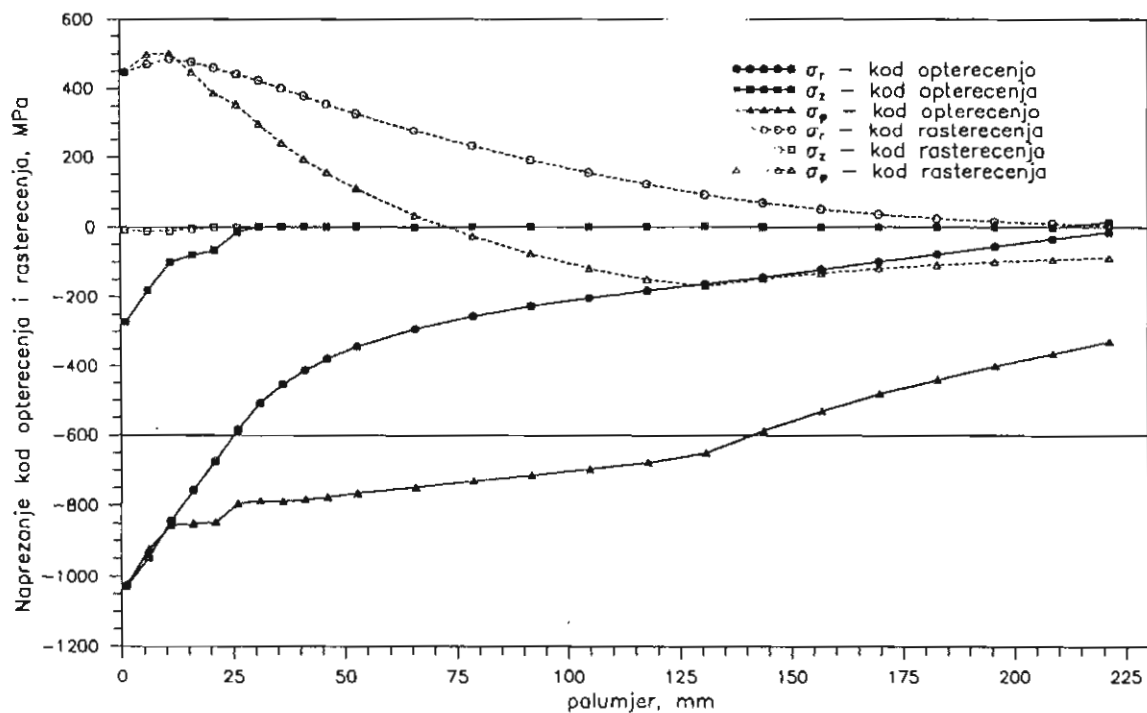
Tablica 2. Naprezanja uz površinu i pomaci sredine kružne ploče ϕ 460 mm na konkavnoj strani opterećene kontinuirano na promjeru ϕ 50 mm - trapezna raspodjela,

Table 2. Stresses on the surface and displacements in the middle of the circular plate continually loaded on its concave side in the diameter-50 mm - trap-distribution



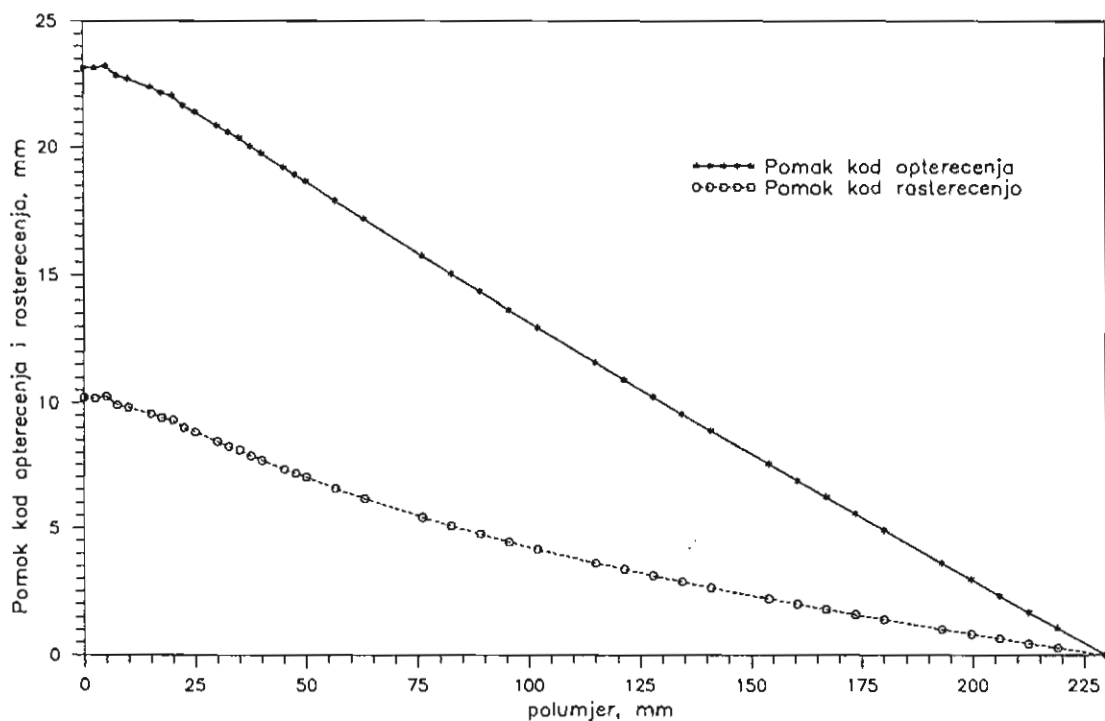
OPTEREĆENJE										RASTEREĆENJE									
q_{z0}	q_{z1}	q_{zr}	F_g	Π	Pos.t.	σ_r	σ_ϕ	σ_z	τ_{rz}	σ_{ekv}	ϵ_{ekv}	Pom. u_{z1}	σ_r	σ_ϕ	σ_z	τ_{rz}	σ_{ekv}	ϵ_{ekv}	Pom. u_{z2}
170	0	1,998976	19634			-194,0	-193,9	-114,1	-1,5	79,9	0	1,25	0	0	0	0	0	0	0
200	0	9,713329	39269			-370,2	-370,1	-195,3	-2,8	174,9	0	2,49	0	0	0	0	0	0	0
300	0	29,99911	58904			-507,2	-507,3	-204,2	-3,6	303,1	0	3,73	0	0	0	0	0	0	0
310	0	50,28489	78539	IS.3	0,251E-07	-641,3	-641,8	-210,9	-4,4	430,7	0	4,93	2,9	2,7	2,3	0,0	0,5	0,0000002	-0,04
320	10	60,28496	98174	IS.5	0,862E-07	-769,5	-769,5	-221,7	-3,9	547,7	0	6,20	2,9	2,8	-0,5	0,9	3,7	0,0000001	0
330	20	63,93693	111809	IS.10	0,932E-07	-841,2	-837,9	-228,7	-4,5	610,9	0,0001941	7,19	17,6	21,3	-0,4	0,5	20,1	0,0003186	0,14
340	30	80,28728	137446	IS.21	0,840E-07	-897,9	-894,0	-237,4	-5,4	658,6	0,0032310	9,58	129,5	133,9	-0,2	0,2	132,0	0,0033591	0,92
350	40	90,28516	157079	IS.31	0,649E-07	-932,7	-929,0	-246,0	-5,9	684,9	0,0071244	12,06	222,4	226,6	-1,0	0,0	225,5	0,0072532	2,17
355	45	95,28574	166897	IS.37	0,861E-07	-943,7	-946,2	-250,3	-6,1	697,7	0,0097276	13,60	269,1	273,3	-1,3	0,1	272,5	0,0098568	3,10
360	50	100,2852	176714	IS.44	0,828E-07	-967,3	-963,8	-254,7	-6,4	711,0	0,0129310	15,43	315,5	319,6	-1,5	0,1	319,1	0,0130605	4,31
370	60	107,0113	193349	M.56	0,105E-06	-998,9	-995,5	-263,4	-6,6	734,0	0,0198672	19,30	391,1	395,1	-2,7	0,1	395,8	0,0199972	7,14
375	65	115,2859	206167	M.57	0,379E-06	-1027,3	-1024,5	-272,8	-6,8	753,2	0,0270741	23,16	446,9	450,5	-7,9	0,2	456,6	0,0272037	10,20

Naprezanja uz površinu na konkavnoj strani kružne ploče $\Phi 460$ mm
opterećene kontinuirano na promjeru $\Phi 50$ mm – trapezna raspodjela
ukupna sila 206,167 kN



Slika 3. Numeričko ispitivanje napr.
Figure 3. Numerical testing of stress

Kontura deformirane kružne ploče $\Phi 460$ mm
opterećene kontinuirano na promjeru $\Phi 50$ mm – trapezna raspodjela
ukupna sila 206,167 kN

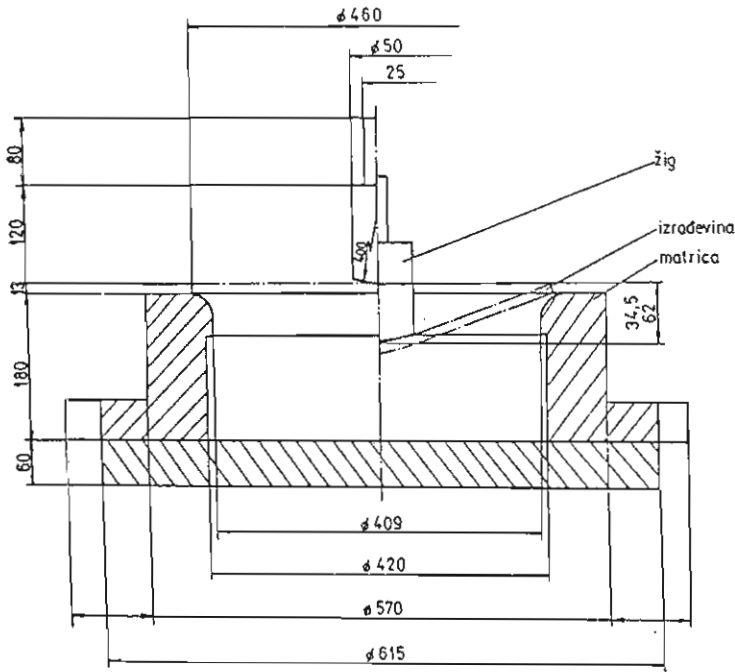


Slika 4. Kontura deformirane kružne ploče
Figure 4. Deformed circular plate contour

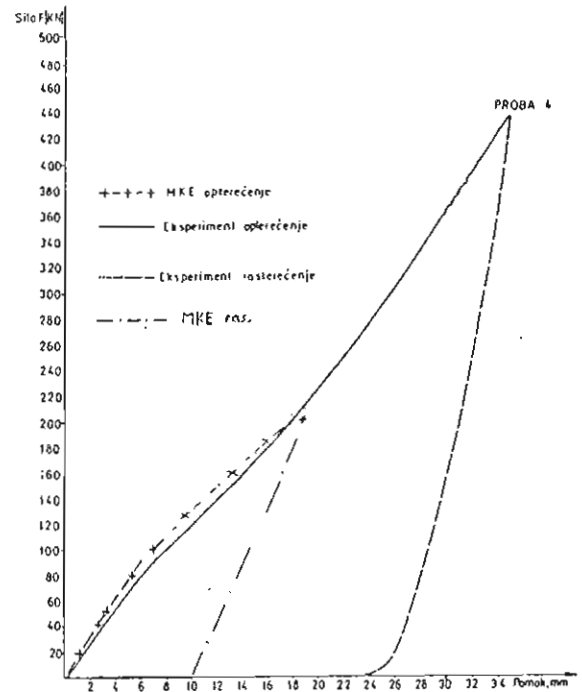
4. EKSPERIMENTALNI REZULTATI

Alat na kojem su sprovedeni eksperimenti prikazan je na slici 5.

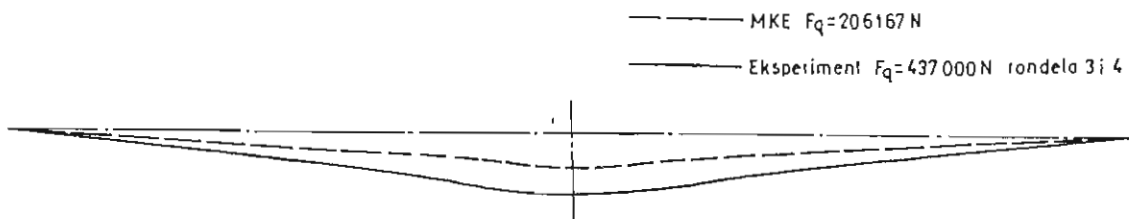
Numerički i eksperimentalni rezultati (sila u ovisnosti o pomaku) prikazan je na slici 6. Kontrola deformirane kružne ploče nakon rasterećenja prikazano je na slici 7. (Sila opterećenja kod MKE je $F_q = 206167 \text{ N}$ a u eksperimentu $F_q = 437000 \text{ N}$).



Slika 5. Alat za aksijalno simetrično savijanje
Figure 5. Tool for axial symm. plate bending



Slika 6. Numeričke i eksperimentalne
vrijednosti
sile u ovisnosti od pomaka
Figure 6. Numerical and experimental values of
force depending on displacement.



Slika 7. Kontura deformirane kružne ploče $\phi 460 \text{ mm}$ i debljine 13 mm nakon rasterećenja
Figure 7. Contour of deformed circular plate of 460 mm diameter and 13 mm thickness after
removal of loading

5. ZAKLJUČAK

Uspoređujući podatke za pomake dobivene MKE s onima dobivenim eksperimentima uočava se dobra suglasnost u području malih trajnih deformacija. Rezultati se gotovo podudaraju za deformacije do 2% . Raspoloživi program MATCVON nije razvijen za područje velikih globalnih pomaka mreže.

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KONSTRUKCIJA BIOMEHANIČKOG MODELA ZA KOMPJUTORSKU SIMULACIJU LJUDSKOG GIBANJA

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Sažetak

Opisan je biomehanički model ženskog tijela i metoda koja omogućuje određivanje dinamičkih momenata tromosti mjerenjem antropomjera. Tijelo je modelirano sa 16 geometrijskih **krutih tijela**. Pomoću kompjutorskog mjernog sustava ELITE snimljeno je 120 ispitanica, a 3D koordinate označenih točaka bile su ulazni podaci za računanje dinamičkih antropomjera pomoću vlastitog programa u MATLAB-u. Rezultati su prikazani u ovisnosti o percentilskoj distribuciji ispitanica.

DETERMINATION OF BIOMECHANICAL MODEL FOR COMPUTER SIMULATION OF HUMAN MOVEMENT

Summary

A biomechanical model of female body and a method which permits the determination of moments of inertia from anthropometric measurements are described. The body is modelled using 16 geometric solids. 120 females subjects are recorded by means of computer measuring system **ELITE** and 3D coordinates of marked points are the input data for calculating dynamical anthropomeasures using our own program in MATLAB. The results are given as a function of percentile distribution of females subjects.

1. UVOD

Točna analiza gibanja čovjeka u prostoru predstavlja složen problem. Ta složenost ostaje i u onim primjerima kada se pojednostave mnogi parametri gibanja. Kompjutorski simulacijski model ljudskog gibanja zahtijeva kao ulazne podatke informacije o promjenama konfiguracije **tijela te o raspodjeli mase tijela** tijekom gibanja [1, 2]. Tako duljine segmenata ljudskog tijela, **segmentalne mase i njihova raspodjela** prema izabranom koordinatnom sustavu te dinamički momenti tromosti segmenata i čitavog tijela predstavljaju temeljne veličine potrebne kod opisivanja gibanja [3]. **Točnije određivanje tih veličina kod analize gibanja** uvjetovat će manja odstupanja izračunatih od izmjerenih vrijednosti.

Precizno poznavanje dimenzija i inercijskih parametara ljudskog tijela te tvorba biomehaničkog modela koji će što bolje simulirati stvarno stanje, prvi je zadatak kod provođenja analize gibanja **te nužan uvjet za primjenu analize**. Zato postoji stalna potreba za **praktičnim metodama dobivanja podataka o antropodinamičkim parametrima čovjeka**, koje bi pored određenih metodoloških pogodnosti osiguravale i pouzdanije rezultate.

Postoji velik broj eksperimentalnih tehnika za određivanje inercijskih parametara ljudskog tijela. Procjene inercijskih svojstava segmenata najčešće se temelje na postupcima i podacima utvrđenim kod proučavanja ljudskih leševa gdje su mjerenja vršena direktno odsijecanjem pojedinih segmenata [4, 5]. Problem je što su takva istraživanja vršena na malom broju leševa starih i bolesnih ljudi, pretežno muškaraca, te takvi podaci najčešće nisu prikladni za subjekte čije se gibanje analizira u biomehaničkim studijama. Nedostaci korištenja podataka utvrđenih na leševima vodili su razvoju

metoda određivanja inercijskih svojstava na živim ispitanicima [6, 7, 8, 9, 10]. Tako su razvijene metode fotogrametrije, skeniranja gama zračenjem, kompjuterizirana tomografija (CT) te matematičko modeliranje. Matematičko modeliranje, kod kojeg su segmenti tijela predstavljeni geometrijskim krutim tijelima, omogućuje uz poznavanje gustoće segmenata, određivanje inercijskih parametara segmenata s većom ili manjom točnosti ovisno o modelu.

Cilj ovog rada je utvrditi što točniji biomehanički model matematičkim modeliranjem te praktičnu metodu koja će omogućiti precizno i jednostavno utvrđivanje antropodinamičkih vrijednosti putem prostornog definiranja antropomjera gdje bi se iz statički snimljenih veličina izvršila automatizirana transformacija u dinamičke veličine. Odnosno, potrebna je ponovljiva metoda kojom bi se uvijek na isti način mogle odrediti statičke i dinamičke antropomjere i to za ženske ispitanike.

2. METODE I ISPITANICI

U ovom radu utvrđivanje antropodinamičkih veličina ograničeno je na tzv. vanjske dinamičke antropomjere, tj. antropomjere koje se određuju na temelju vanjskih granica dijelova tijela, dijeljenjem tijela na odgovarajuće segmente. Međutim, s obzirom na unutrašnju građu tijela koja se zbog funkcionalnih veza organa ne dijeli na isti način kao vanjski dijelovi tijela, postoje unutrašnje dinamičke antropomjere. U ovom slučaju mase segmenata za vrijeme gibanja su promjenljive, a dinamički momenti tromosti segmenata se mijenjaju ovisno o trenutnom položaju segmenta prema tijelu subjekta. Unutrašnji dinamički momenti tromosti nisu predmet proučavanja ovog rada.

2.1 Biomehanički model i ispitanici

Biomehanički model treba obuhvatiti što veći broj stupnjeva slobode gibanja tijela te svojim simulacijskim karakteristikama ostvariti prostorno gibanje bez ograničavanja relativnih gibanja segmenata. Zato nam najviše odgovara model kinematičkog lanca u kojem su zglobovi veze između pojedinih segmenata modela. Dakle, potreban je matematički model utemeljen na biomehaničkim ograničenjima, koji bi uvođenjem nekih parametara direktno određivao vrijednosti antropodinamičkih veličina, tj. dinamičkih momenata tromosti.

Ljudsko tijelo modelirano je korištenjem 16 geometrijskih krutih tijela spojenih zglobovima. Tijelo je podijeljeno na segmente ravninama okomitim na uzdužne osi segmenata, na način koji su utvrdili Donski i Zaciorski što nam je omogućilo upotrebu njihovih podataka za utvrđivanje središta masa pojedinih segmenata i regresijskih jednadžbi za određivanje segmentalnih masa [8]. Glava i vrat smatraju se jednim segmentom, a od gornjeg dijela trupa odijeljeni su ravninom okomitom na uzdužnu os glave koja prolazi grlenom rupom (suprasternale) i drugim vratnim kralješkom. Trup je podijeljen na tri dijela: prsni koš (thorax), trbuh (abdomen) i zdjelicu (pelvis). Granica između prsnog koša i trbuha prolazi kroz vršak prsne kosti, a granica između trbuha i zdjelice kroz pupak. Zdjelica se proteže od pupka do zglobova kukova. Segmenti donjih i gornjih ekstremiteta podijeljeni su ravninama u susjednim zglobovima (kuk, koljeno i gležanj, odnosno rame, lakat i zglob šake).

Mase pojedinih segmenata određuju se pomoću regresijskih jednadžbi Donskog i Zaciorskog [8]. Donski i Zaciorski su primijenili tzv. radioizotopnu metodu na živim ispitanicima tj. na 100 muškaraca starosti 19 - 35 godina i 16 žena između 19 i 27 godina. Na temelju podataka utvrđenih takvom metodom određene su regresijske jednadžbe, različite za muškarce i žene, pomoću kojih je moguće odrediti mase segmenata s poznatim statističkim odstupanjima. Budući da su naše ispitanice pretežno iste starosti te da u literaturi ne postoje druga opsežnija istraživanja segmentalnih masa ženskog dijela populacije, a metode skeniranja svake pojedine ispitanice upotrebom CT (Computer Tomography) ili MRI (Magnetic Resonance Imaging) tehnologije su skupe i složene te još nisu raširene u masovnoj primjeni, mi ćemo u ovom radu koristiti podatke središta masa segmenata te regresijske jednadžbe za utvrđivanje mase segmenata koje su utvrdili Donski i Zaciorski. Tablica 1 pokazuje položaj središta masa segmenata kao postotak duljine segmenta, mjereno od gornje rubne točke segmenta.

Tablica 1. Središta masa segmenata [8] i prosječne gustoće segmenata [11]

segment	udaljenost središta mase od gornje granice segmenta	gustoća [kg/dm ³]
glava i vrat	50,02 %	1,11
gornji dio trupa	50,66 %	0,96
srednji dio trupa	45,02 %	1,00
donji dio trupa	59,59 %	1,03
nadlaktica	44,98 %	1,07
podlaktica	42,74 %	1,125
šaka	36,91 %	1,155
natkoljenica	45,49 %	1,05
potkoljenica	40,49 %	1,09
stopalo	44,15 %	1,095

Mase segmenata računaju se prema regresijskoj jednadžbi:

$$m_i = B_0 + B_1 \cdot M + B_2 \cdot h$$

m_i - masa pojedinog segmenta [kg]

M - ukupna masa ispitanika [kg]

h - stojeća visina ispitanika [cm]

B_0 [kg], B_1 [-], B_2 [kg/cm] - regresijski koeficijenti

Tablica 2 prikazuje frekvencijske koeficijente i standardne devijacije za ženske ispitanike.

Tablica 2. Regresijski koeficijenti prema Donskom i Zaciorskom [8]

segment	B_0 [kg]	B_1 [-]	B_2 [kg/cm]	σ
glava i vrat	2,388	-0,001	0,015	0,49
gornji dio trupa	-16,593	0,140	0,0995	1,47
srednji dio trupa	-2,741	0,031	0,056	1,09
donji dio trupa	-4,908	0,124	0,0272	0,9
nadlaktica	0,206	0,0053	0,0066	0,21
podlaktica	0,295	0,009	0,0003	0,11
šaka	-0,116	0,0017	0,0020	0,03
natkoljenica	5,185	0,183	-0,042	0,81
potkoljenica	-0,436	-0,011	0,0238	0,36
stopalo	-1,207	0,0175	0,0057	0,11

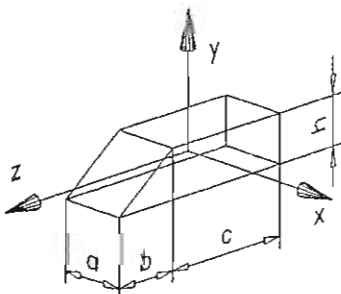
Segmenti tijela zamijenjeni su geometrijskim tijelima ovisno o obliku segmenta te o položaju središta mase segmenta. Za izračunavanje volumena pojedinih segmenata potrebni su podaci o gustoćama. Modeli su pojednostavljeni pretpostavkom jednolike gustoće duž segmenta, a prosječne gustoće (Tablica 1) uzete su iz literature [11].

Glava i vrat modelirani su rotacionim elipsoidom. Pri tome je potrebno mjeriti samo dužu os elipsoida tj. udaljenost od tjemena do grlene rupe, dok se manja os računa iz volumena.

Nadlaktica, podlaktica, natkoljenica i potkoljenica oblikovani su krnjim stošcima. Dovoljno je mjeriti samo duljinu segmenata te se iz izraza o položaju središta mase i izračunatog volumena mogu odrediti polumjeri gornje i donje baze.

Da bi se pojednostavio model, kod oblikovanja šake zanemario se položaj središta mase te je ona oblikovana eliptičnom pločom, a mjere se poluosi elipse dok se debljina ploče računa iz volumena.

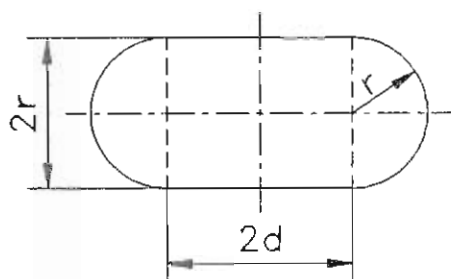
Stopalo je oblikovano paralelepipedom od kojeg je odrezana trostrana prizma kao što se vidi na slici 1. Veličine a i h se direktno mjere, a veličine b i c računaju se iz izraza za težište i volumen.



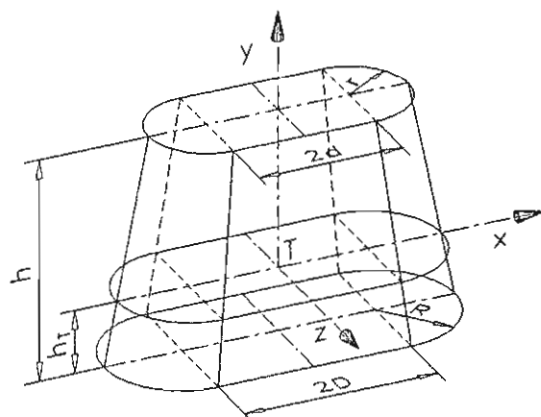
Slika 1. Model stopala

Za modeliranje trupa koriste se geometrijska tijela s poprečnim presjekom u obliku „stadiona” tj. lika sastavljenog od pravokutnika širine $2d$ i visine $2r$ te dva polukruga polumjera r (slika 2).

Na slici 3 prikazano je kruto tijelo čije su baze „stadioni” različitih dimenzija.



Slika 2. „Stadion”



Slika 3. Geometrijsko tijelo s bazama „stadion”

Gornji dio trupa oblikovan je krutim tijelom s bazama „stadion” jednakih dimenzija. Širina baze ($2r + 2d$) je udaljenost ramenih zglobova te se direktno mjeri. Visina segmenta, tj. udaljenost ravnine kroz grlenu rupu i ravnine kroz vršak prsne kosti, također se direktno mjeri. Iz izračunatog volumena segmenta i širine baze, mogu se odrediti parametri r i d za gornji dio trupa. Ti parametri ujedno predstavljaju parametre gornje baze srednjeg dijela trupa.

Kod srednjeg dijela trupa, mjeri se direktno visina segmenta kao udaljenost vrška prsne kosti od pupka. Budući da je volumen i položaj središta mase segmenta poznat, moguće je iz izraza za računanje volumena i položaja središta mase izračunati parametre donje baze srednjeg dijela trupa r_1 i d_1 , a to su istovremeno parametri gornje baze donjeg dijela trupa tj. zdjelice.

Visina zdjelice direktno se mjeri, a iz poznatog volumena i položaja središta mase može se izračunati donja baza tj. r_2 i d_2 .

Slika 4 prikazuje model ženskog tijela od 16 geometrijskih tijela.

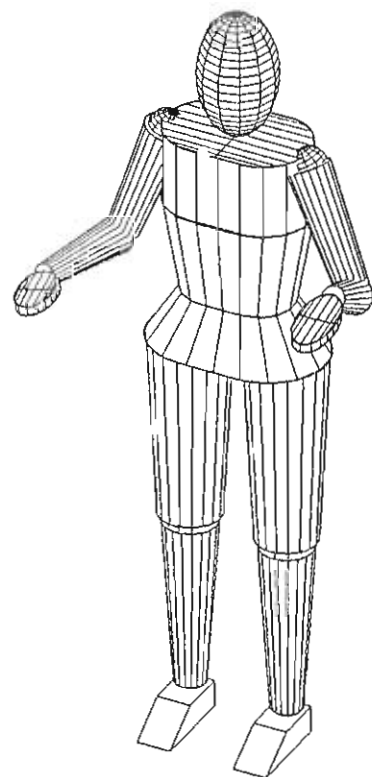
Izrazi za računanje dinamičkih momenata tromosti rotacionog elipsoida, krnjeg stošca, eliptične ploče, paralelepipeda i trostrane prizme nalaze se u literaturi [12, 13].

Izvedeni su izrazi za računanje volumena, položaja središta mase i glavnih središnjih momenata tromosti geometrijskih tijela s bazama „stadion” na slici 3. Da bi se olakšalo korištenje tih izraza definirane su funkcije f_1, f_2, f_3, f_4 i f_5 :

$$f_1(v, w) = 1 + \frac{v + w}{2} + \frac{v \cdot w}{3}$$

$$f_2(v, w) = \frac{1}{2} + \frac{v + w}{3} + \frac{v \cdot w}{4}$$

$$f_3(v, w) = \frac{1}{3} + \frac{v + w}{4} + \frac{v \cdot w}{5}$$



Slika 4. Model žene od 16 geometrijskih tijela

$$f_4(v, w) = 1 + \frac{v + 3 \cdot w}{2} + \frac{3 \cdot v \cdot w + 3 \cdot w^2}{3} + \frac{3 \cdot v \cdot w^2 + w^3}{4} + \frac{v \cdot w^3}{5}$$

$$f_5(v, w) = 1 + v + w + \frac{v^2 + 4 \cdot v \cdot w + w^2}{3} + \frac{v \cdot w(v + w)}{2} + \frac{v^2 w^2}{5},$$

$$\text{gdje je: } v = \frac{r - R}{R}, w = \frac{d - D}{D}$$

r - polumjer polukružnice manje baze

R - polumjer polukružnice veće baze

d - polovica širine pravokutnika manje baze

D - polovica širine pravokutnika veće baze.

Volumen se računa po formuli:

$$V = h \cdot R [4 \cdot D \cdot f_1(v, w) + \pi \cdot R \cdot f_1(v, v)]$$

h - visina geometrijskog tijela

Udaljenost težišta od veće baze:

$$h_T = \frac{h^2}{V} \cdot [4 \cdot R \cdot D \cdot f_2(v, w) + \pi \cdot R^2 \cdot f_2(v, v)]$$

Središnji glavni momenti tromosti

$$I_x = \rho h \left[\frac{4RD^3 f_4(v, w)}{3} + \frac{\pi R^4 f_4(v, v)}{4} + 4h^2 R D f_3(v, w) + \pi h^2 R^2 f_3(v, v) \right] - \rho V h_T^2$$

$$I_y = \rho h \left[\frac{4RD^3 f_4(v, w)}{3} + \pi R^2 D^2 f_5(v, w) + 4R^3 D f_4(w, v) + \frac{\pi R^4 f_4(v, v)}{2} \right]$$

$$I_z = \rho h \left[\frac{8R^3 D f_4(w, v)}{3} + \frac{\pi R^4 f_4(v, v)}{4} + \pi R^2 D^2 f_5(v, w) + 4h^2 R D f_3(v, w) + \pi h^2 R^2 f_3(v, v) + \frac{4RD^3 f_4(v, w)}{3} \right] - \rho V h_T^2$$

ρ - gustoća segmenta

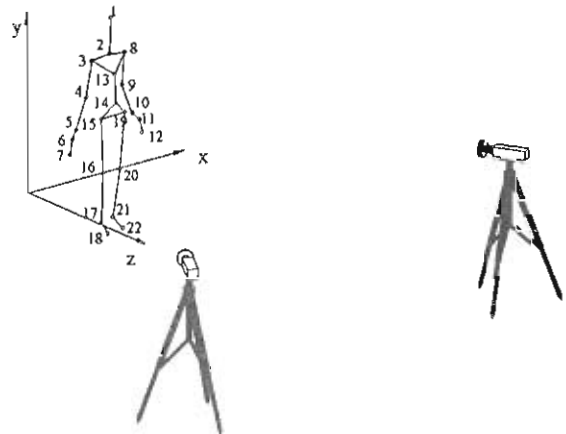
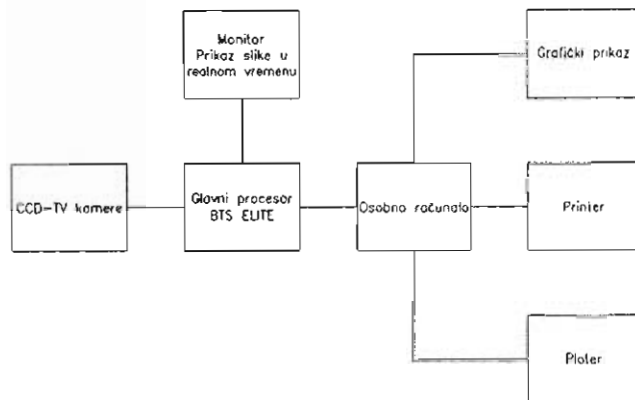
Kod analize gibanja inercijski parametri subjekta određuju se za svakog ispitanika posebno, a kod pokušaja poopćavanja rezultata kompjutorske simulacije inercijski parametri mogu biti odabrani iz statistički obrađenih vrijednosti za prosječnog čovjeka, odnosno za neki od percentila. Naše istraživanje provedeno je na 120 žena starosti 19 - 30 godina, čime je osiguran dovoljan broj za statističku analizu te podjelu u odgovarajuće percentilske skupine. Tako je moguće odrediti odnose dinamičkih veličina prema populacijskoj distribuciji.

2.2 Opis uređaja i mjerenje

U našem istraživanju koristi se kompjuterski mjerni sustav ELITE (slika 5). Sustav ELITE je biomehanički analizator gibanja velike točnosti koji sadrži dvije TV kamere konstruirane tako da budu posebno osjetljive na infracrveni (IC) dio spektra. Na kamere su montirane infracrvene bljeskalice sinkronizirane s kamerama. Kamere otkrivaju infracrveno svjetlo reflektirano s markera te sustav prati gibanje samo posebno označenih točaka na tijelu, a ne čitavog tijela. Glavni procesor

sustava u realnom vremenu analizira sliku svake kamere te se računaju trodimenzijske koordinate svakog markera u inercijalnom koordinatnom sustavu, a podaci se zapisuju na tvrdi disk i mogu se upotrijebiti kao ulazni podaci za druge programe.

Mjerenje se provodi tako da se ispitanice na koje su postavljeni markeri, snimaju u kalibriranom prostoru tj. „radnom volumenu“ u stanju mirovanja ili gibanja (slika 6).



Slika 5. Shema sustava ELITE

Slika 6. Snimanje ispitanice u sustavu kamera

Markeri se postavljaju na karakteristične točke ispitanica prema definiranom biomehaničkom modelu. Osim na zglobove, markeri se postavljaju na točke potrebne zbog određivanja dimenzija segmenata te na točke neophodne zbog određivanja orijentacije segmenata u globalnom koordinatnom sustavu.

Ako je segment zamijenjen rotaciono simetričnim tijelom onda je os rotacije ujedno i jedna glavna os. Na takve segmente dovoljno je postaviti dva markera. Os rotacije predstavlja y_i os lokalnog koordinatnog sustava. Ako segment nije rotaciono simetričan (šake, stopala, gornji, srednji i donji trup) potrebna su tri markera da bi se odredila orijentacija segmenta tj. lokalni koordinatni sustav (x_i, y_i, z_i) postavljen u središte mase segmenta. Lokalni koordinatni sustav za stopalo postavlja se na način prikazan na slici 1, a za trup kao na slici 3. Za šaku, os y_i lokalnog koordinatnog sustava poklapa se s većom poluosi elipse, a os x_i s manjom poluosi.

24 markera postavljena su na karakteristične točke na slijedeći način (slika 6):

marker 1 - vrh glave	marker 9 - lijevi lakat	marker 17 - desni gležanj
marker 2 - suprasternalc	marker 10 - zglob lijeve šake	marker 18 - korijen palca desne noge
marker 3 - desno rame	marker 11 - korijen palca lijeve ruke	marker 19 - lijevi kuk
marker 4 - desni lakat	marker 12 - srednji prst lijeve ruke	marker 20 - lijevo koljeno
marker 5 - zglob desne šake	marker 13 - vršak prsne kosti	marker 21 - lijevi gležanj
marker 6 - korijen palca desne ruke	marker 14 - pupak	marker 22 - korijen palca lijeve noge
marker 7 - srednji prst desne ruke	marker 15 - desni kuk	marker 23 - mali prst desnog stopala
marker 8 - lijevo rame	marker 16 - desno koljeno	marker 24 - mali prst lijevog stopala

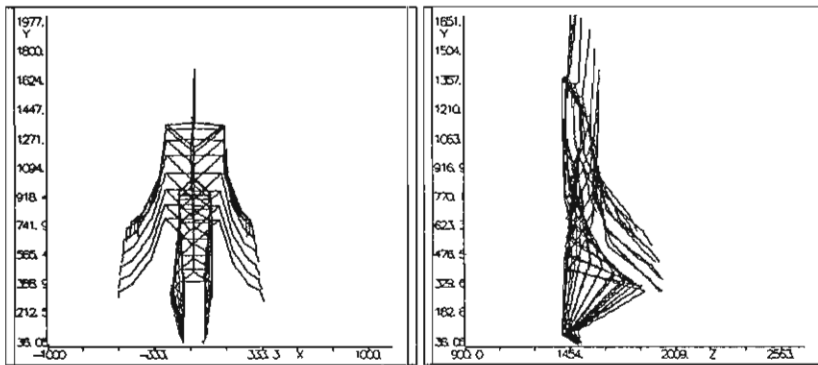
Markeri 23 i 24 nisu povezani linijama u kinematički lanac jer služe samo za određivanje orijentacije i širine stopala.

Programska podrška sustava ELITE nadograđena je vlastitim programom u programskom paketu MATLAB. Taj program, za svaki međupoložaj snimljenog gibanja, na temelju trodimenzijskih koordinata markera računa dimenzije segmenata te određuje orijentacije segmenata u globalnom koordinatnom sustavu. Kod računanja dimenzija segmenata, vrijednosti za desne i lijeve udove se uprosječu da bi se dobile simetrične vrijednosti dinamičkih momenata tromosti. Budući da se markeri postavljaju na površine tijela, program aproksimira položaje zglobova. Nakon toga određuju se segmentalne mase i položaji središta masa pojedinih segmenata te se izračunava središte mase cijelog tijela. Središta mase pojedinih segmenata te središte mase cijelog tijela određeni su vektorima

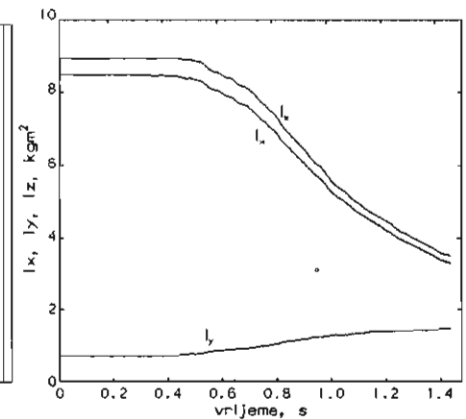
u globalnom koordinatnom sustavu (x, y, z). Zatim program računa dinamičke momente tromosti segmenta tijela prema lokalnom koordinatnom sustavu (x_i, y_i, z_i) kroz središte mase svakog segmenta. Tako dobiven tenzor tromosti transformira se u sustav paralelan s globalnim sustavom (x, y, z). Zatim se, pomoću Steinerovog pravila, računaju ukupni dinamički momenti tromosti tijela s obzirom na osi kroz središte mase cijelog tijela paralelne s globalnim koordinatnim sustavom (x, y, z).

4. REZULTATI

Kao primjer, na slici 7 nalazi se grafički prikaz modela jedne ispitanice koji daje sustav ELITE u x - y i z - y ravnini. Ispitanica mase 55 kg i visine 171 cm spustila se iz uspravnog položaja u čučajući položaj, a promjene središnjih dinamičkih momenata tromosti cijelog tijela za poprečnu os I_x , uzdužnu os I_y i sagitalnu os I_z , prikazane su na slici 8.

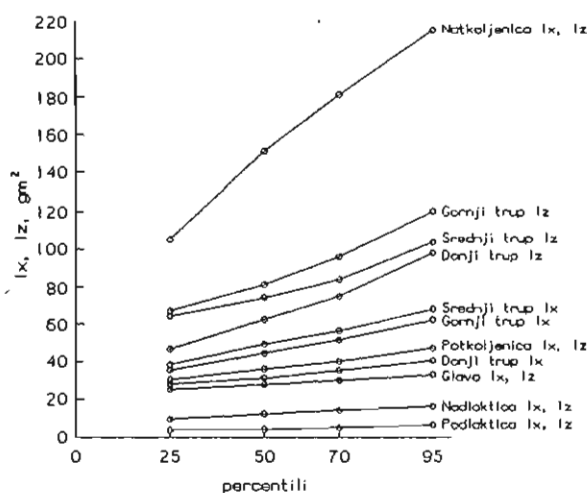


Slika 7. Grafički prikaz modela u xy i zy ravnini

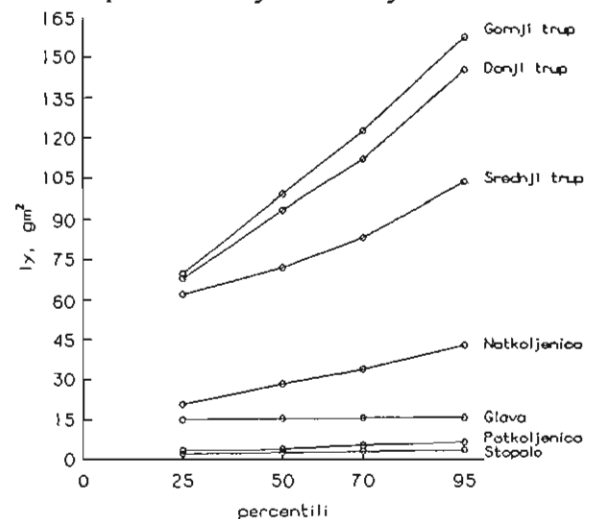


Slika 8. Promjena središnjih dinamičkih momenata tromosti

Opisanom metodom provedeno je mjerenje na 120 žena što je osiguralo dovoljan broj za statističku analizu i podjelu u odgovarajuće percentilske skupine. Na slikama 9 i 10 je grafički prikaz dinamičkih momenata tromosti segmenata tijela u ovisnosti o percentilskoj distribuciji.



Slika 9. Središnji dinamički momenti tromosti segmenata I_x i I_z



Slika 10. Središnji dinamički momenti tromosti segmenata I_y

Na slici 9 nisu prikazani momenti tromosti I_x i I_z za šaku i stopalo. I_x stopala mijenja se od 2,1 do 3,8 gm^2 ovisno o percentilskoj distribuciji, a I_z od 0,52 do 0,87 gm^2 . Momenti tromosti I_x šake, i to za ispruženi dlan, mijenjaju se od 0,62 do 0,89 gm^2 , a I_z šake od 0,71 do 0,90 gm^2 .

Na slici 10 nisu ucrtani dinamički momenti tromosti I_y za nadlakticu (1,26 do 1,5 gm^2), podlakticu (0,39 do 0,58 gm^2) i šaku (0,09 do 0,29 gm^2).

5. ZAKLJUČCI

Utvrđen je biomehanički model za ženske ispitanike te metoda koja omogućuje određivanje dinamičkih antropomjera za svaki međupoložaj tijela tijekom gibanja izravnim snimanjem subjekata pomoću mjernog sustava ELITE. Programska podrška sustava ELITE nadograđena je vlastitim programom te je moguće pomoću snimljenih trodimenzijskih koordinata markera, zapisivanjem „zamrznutog pokreta“, izračunati dimenzije segmenata, segmentalne mase, položaje središta masa te odgovarajuće dinamičke momente tromosti. Ova metoda predstavlja praktički primjenljivu metodu za kompjuterizirano praćenje i opisivanje gibanja, prije svega hodanja, trčanja i skakanja čovjeka, a time i mogućnost za optimiranje športskih pokreta te analizu ortopedskih i protetičkih problema. Netočnosti koje se mogu pojaviti mogu imati uzrok u lošoj kalibraciji prostora u kojem se vrši snimanje te u pogrešnom postavljanju markera na ispitanice.

Također, doprinos ovog rada je što su istraživanja izvršena za ženske ispitanike jer su u literaturi istraživanja uglavnom provedena za muškarce. U radu je izvršeno mjerenje 120 ispitanica, a veličine središnjih dinamičkih momenata tromosti segmenata prikazane su dijagramski za pojedine vrijednosti postotaka razdiobe ženskog dijela populacije.

Utvrđen biomehanički model sastoji se od 16 segmenata te za razliku od modela iz literature omogućuje gibanje u ručnom zglobu, gležnju i gibanje glave u odnosu na trup, a rezultati statičkih i dinamičkih antropomjera dobivaju se u realnom vremenu. Za usporedbu, u literaturi [10] istraživanje je provedeno na jednom ženskom ispitaniku, model se sastojao od 11 segmenata podijeljenih u 40 podsegmenata bez mogućnosti gibanja u ručnom zglobu, gležnju i vratu, a mjerenje se vršilo pomoću antropometrijskih kalipera i mjernih traka, a obuhvaćalo je 95 izmjera što je zahtijevalo utrošeno vrijeme od 30 minuta.

Točnost našeg modela povećana je upotrebom novog oblika za poprečni presjek trupa u obliku „stadiona“ koji je matematički prikladniji te bolje aproksimira poprečni presjek trupa nego eliptični presjek do sada korišten u literaturi.

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FLOW STUDIES IN ELASTIC Y-MODELS USING NEWTONIAN AND NON-NEWTONIAN FLUIDS

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Abstract

Flow parameters were studied in elastic rubber model, in a 30°-Y-junction. Velocity components were carried out using a one component LDA system. Fluid have to be transparent, so that the beams of the laser were not reflected. Solution has a refraction index, $n = 1.41$ in fact the same refraction index as the, silicon rubber wall of the model. Particle-particle, particle-vessel wall interactions were evaluated using a low shear viscometer. The velocity distribution of the non-Newtonian flow is different than Newtonian flow. The elasticity of the wall also influence on the flow behaviour, special in the flow separation regions, where the flow separates from the wall. The flow was visualized with dyes and a birefringent solution using a photoelasticity apparatus. Analysis of the flow regions were made by a laser-Doppler-anemometer. The most important effects which are influences on flow behaviour are the geometry, the pulse form, wall elasticity and non-Newtonian behaviour of fluid.

1. Introduction

Fundamental studies of flow of Newtonian and non-Newtonian fluids through elastic Y models by means of LDA are important to analyse the hemodynamic parameters. These studies must consider following factors:

- In vivo and in vitro laser Doppler anemometer measurements are not possible with blood since laser light is absorbed by the red cells.
- Blood vessels are not transparent so that the laser beam can not go through the vessel wall. Blood vessel material is inhomogenous, porous and curvilinearly anisotropic. We performed our tests on transparent, elastic, silicon rubber models. The elastic Y model was made of two-components silicon rubber with an elasticity similar to the blood vessel wall.
- We used a glycerine water solution with a dynamic viscosity $\eta = 9.71$ mPas and a density of $\rho = 1150$ kg/m³ at a temperature $t = 20$ °C and a non-Newtonian fluid (DMSO-Separan) with a representative viscosity $\eta_{rep.} = 6.162$ mPas and a density $\rho = 1050$ kg/m³ at a temperature $t = 20$ °C. The second fluid shows characteristics corresponding to the physical and rheological characteristics of blood.

2. Experimental methods

We used an elastic silicon rubber Y model, Figure 1. The diameter of the main pipe was 10 mm while the diameter of each branch was 8 mm. Thickness of the model wall was 1 mm,

while the bifurcation angle was 30° . For the purpose of this investigation we used two-component silicon rubber 601A and 601B made by Wacker. Our model had an elasticity similar to the vessel wall. Cross sections observed in the model were 15 mm in front of the branching point; at the branch itself (0 mm); and 2.5 mm, 5 mm, 10 mm, 15 mm and 25 mm downstream of the branches. We divided one cross section into 9 diameters and measured the velocity over these nine diameters. LDA measurements were done at 9 measuring points over one diameter, step by step, Figure 1. The material constants of this material are comparable to the material constants of the carotid artery which enables determination of velocity rate and the law of the flow rate loss by means of the vessel deformation law.

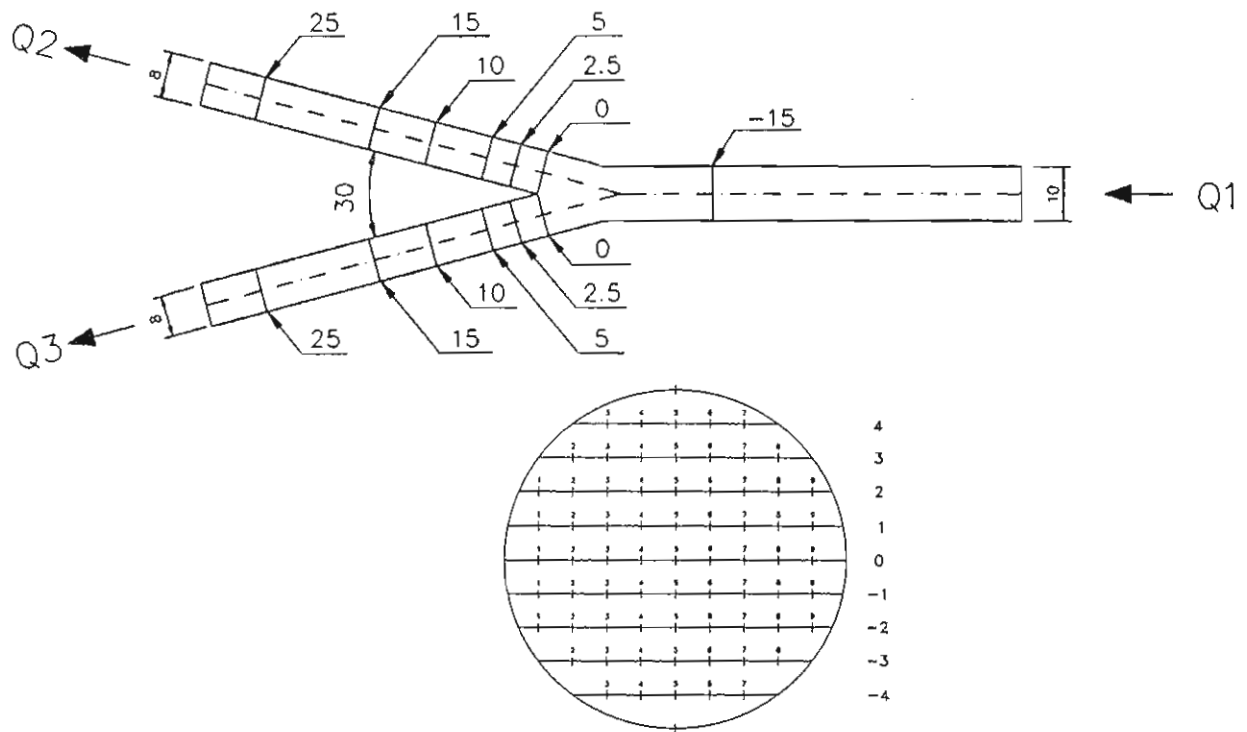


Figure 1. Elastic silicone rubber Y-shaped model. Position of measurement cross sections.

The material constants of this material are comparable to the material constants of the carotid artery which enables determination of velocity rate and the law of the flow rate loss by means of the vessel deformation law. Model fluid used in the elastic Y model, corresponding to Newtonian fluid streaming, was a glycerin water solution mixed 1.38 : 1 (58% glycerin and 42% water). Haake rotational viscosimeter RV 100 showed its dynamic viscosity to be $\eta = 9.71$ mPas and a density of $\rho = 1150$ kg/m³ at the temperature of $t = 21^\circ\text{C}$. The refraction index of the fluid was $n = 1.4096$ to match the refraction index of the silicon rubber wall of the model. Steady and pulsatile flow of the model fluid were observed with the average Reynolds number $Re = 250$. With the aforementioned parameters of the model fluid and the diameter $d = 10$ mm of the main model tube, Figure 1, we obtained the flow rate through the main tube of $Q1 = 59.683$ l/h. Flow through branches Q2 and Q3 was 70 : 30. The experiments were repeated for steady and pulsatile flow of non-Newtonian fluids through the elastic Y model. The model fluid which imitated non-Newtonian streaming was a 51.7% aqueous Dimethylsulfoxide solution mixed 1 : 1 with a mixture of 0.005% AP45 and 0.007% AP302 aqueous polyacrylamide solution.

Representative dynamic viscosity obtained for the DMSO-Separan was $\eta_{\text{rep.}} = 6.162 \text{ mPas}$ at the density of $\rho = 1050 \text{ kg/m}^3$ and temperature of $t = 21^\circ\text{C}$. Measurements were taken by means of rotational viscometer CV 100/RV 100 made by Haake.

The refraction index corresponding to the refraction index of the silicon rubber model wall was $n = 1.41$. The whole model was embedded in a glycerin water solution whose refraction index was $n = 1.41$ so that the laser beams could pass the model without any refraction. Measurement were taken for the average Reynolds number $Re = 250$ and flow ratios in the branches Q2/Q3 (70 : 30) and the entrance tube diameter $d = 10 \text{ mm}$. For unsteady flow we used a piston pump and superimposed a pulse wave over the steady flow, Figure 2. The curve was similar to the flow curve measured in vivo with an ultrasound. With these parameters the entrance pipe flow rate was $Q_1 = 41.48 \text{ l/h}$.

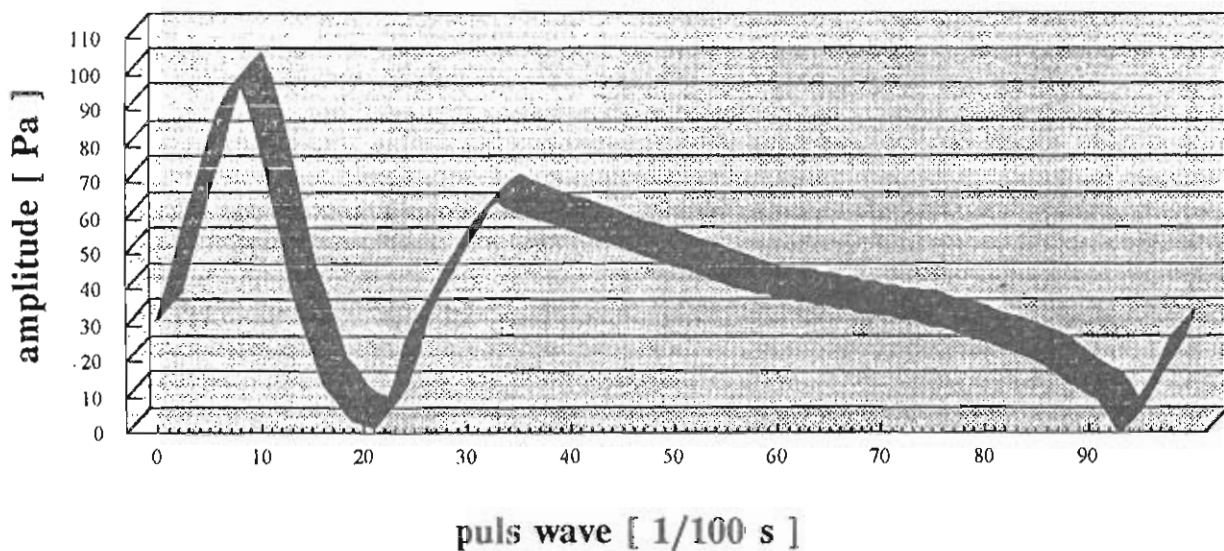


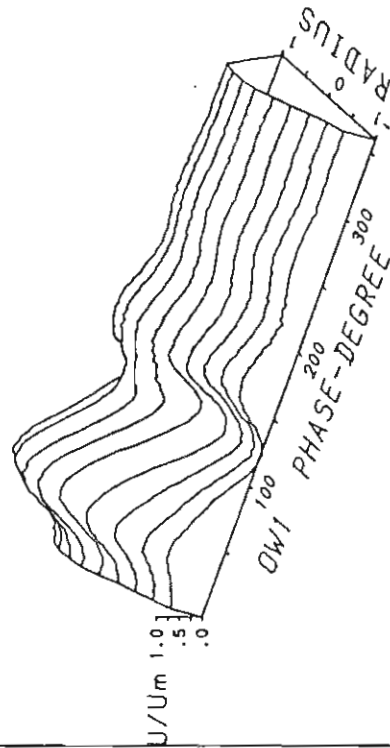
Figure 2. Velocity profiles over a whole pulse cycle.

3. Results and Discussion

Figure 3. shows velocity profiles at particular phase angles of the branching point itself (0 mm) in the branch (Q2) through which 70% of the fluid flow. The pulsatile wave for the phase angle from 0° to 360° was divided into a hundred measurement units, i.e. measurement points. The average Reynolds number was $Re = 250$ and the average velocity rate was calculated by means of the continuity equation. Using a glycerin water solution with dynamic viscosity of $\eta = 9.71 \text{ mPas}$ and density $\rho = 1150 \text{ kg/m}^3$, with pulsatile wave frequency $f = 1.045 \text{ Hz}$ and the model entrance pipe radius 5 mm, we obtained the Womersley parameter $\alpha = 4.4$. The velocity was measured perpendicular to the plane.

The profile velocity analysis in horizontal crosssection (Figure 3a) and vertical crosssection shows higher velocities on internal wallside of the model (IW). At the points 2.5 mm, 5 mm, and 10 mm secondary flows, but not the reverse flow areas, can be seen. Figure 3b. shows velocity profiles at the branching point of the branch (Q3), i.e. 0 mm. As in the branch (Q2) velocity rates are higher on the inner model wall (IW), but we can also notice initial flow separations. On these places it is possible to see one of flow separation or zone recirculation.

30-DEGREE-Y: AXIAL VELOCITY



RE = 250

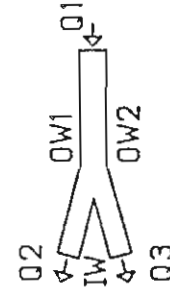
FLUID = GLYCERIN-WATER

LOCATION = 0 mm Q2

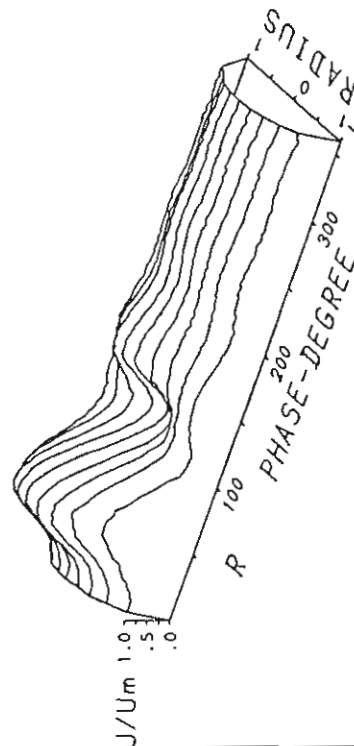
RADIUS = 4 mm

 $U_m = 0.213 \text{ m/s}$

PLANE = CENTER VER.

 $Q2/Q1=0.7$ $\alpha = 4.3$

30-DEGREE-Y: AXIAL VELOCITY



RE = 250

FLUID = GLYCERIN-WATER

LOCATION = 0 mm Q2

RADIUS = 4 mm

 $U_m = 0.213 \text{ m/s}$

PLANE = CENTER HOR.

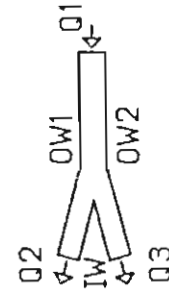
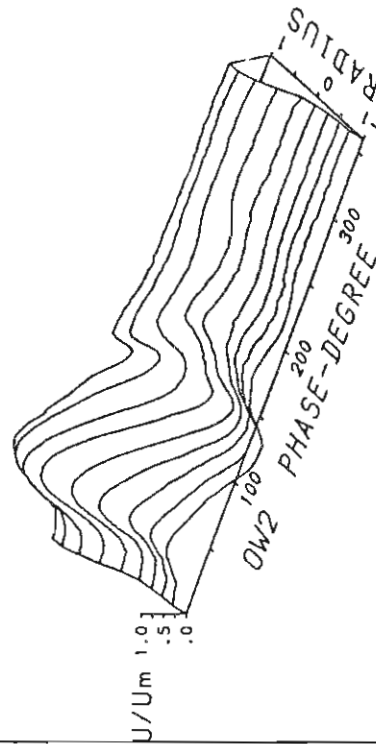
 $Q2/Q1=0.7$ $\alpha = 4.3$

Figure 3a. Local velocity (u/mm) over a whole pulse cycle in the branch (Q2), at the branching point (0 mm) in an Y model.

30-DEGREE-Y: AXIAL VELOCITY



RE = 250

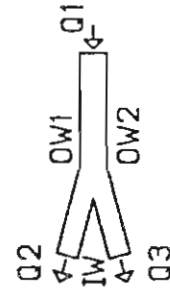
FLUID = GLYCERIN-WATER

LOCATION = 0 mm Q3

RADIUS = 4 mm

$U_m = 0.213 \text{ m/s}$

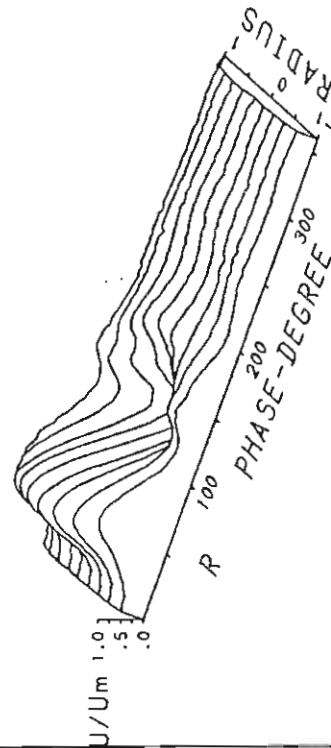
PLANE = CENTER VER.



$$Q2/Q1=0.7$$

$$\alpha = 4.3$$

30-DEGREE-Y: AXIAL VELOCITY



RE = 250

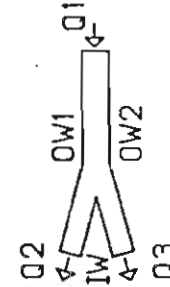
FLUID = GLYCERIN-WATER

LOCATION = 0 mm Q3

RADIUS = 4 mm

$U_m = 0.213 \text{ m/s}$

PLANE = CENTER HOR.



$$Q2/Q1=0.7$$

$$\alpha = 4.3$$

Figure 3b. Local velocity (u/mm) over a whole pulse cycle in the branch (Q3), at the branching point (0 mm) in an Y model.

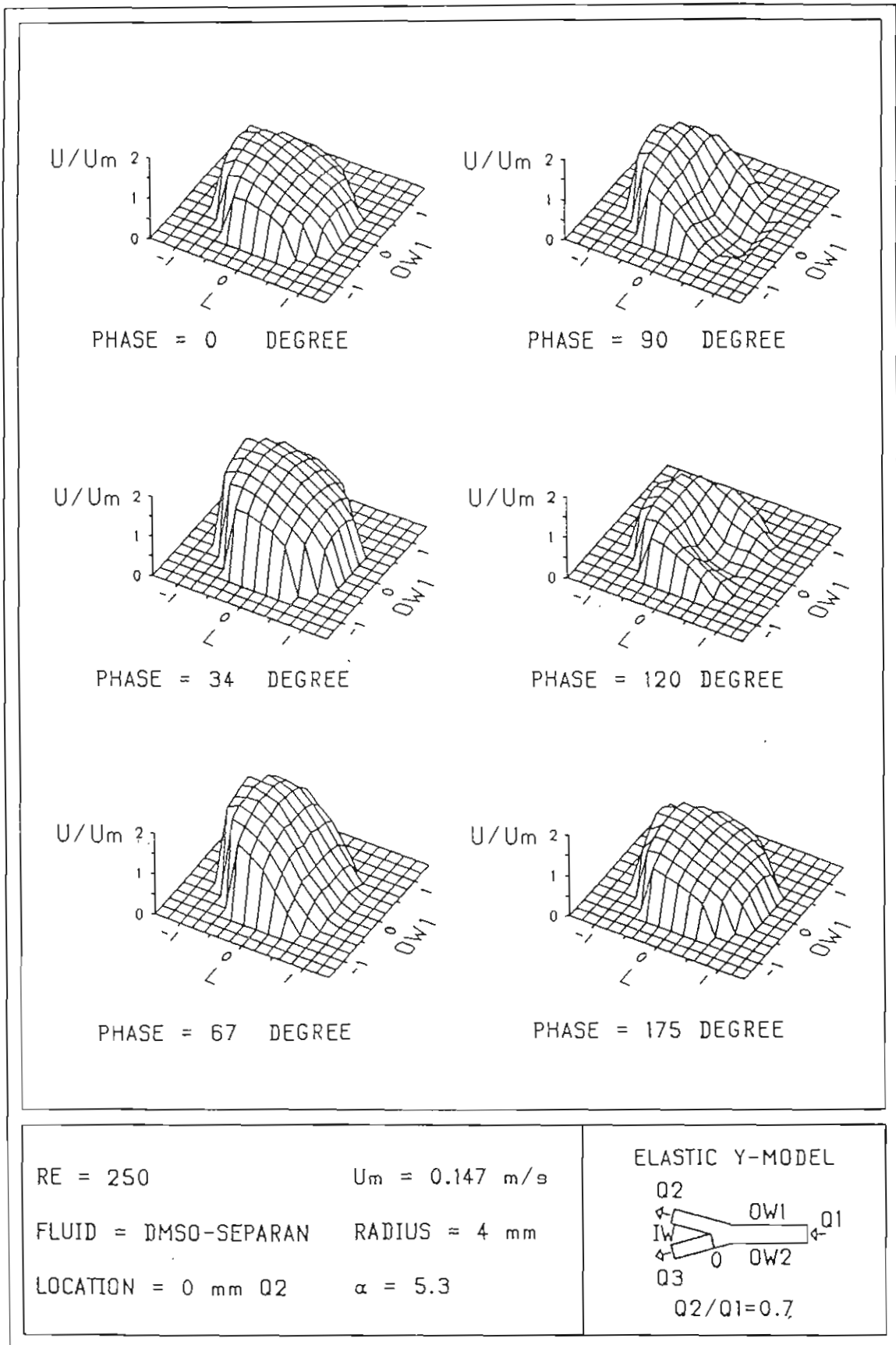


Figure 4. Velocity profiles over a whole pulse cycle (phase $\omega T = 0^\circ, 34^\circ, 67^\circ, 90^\circ, 120^\circ$ and 175°) in the branch (Q2), at the branching point (0 mm), recorded with a LDA in the elastic Y-model. The fluid used is a Newtonian glycerine water solution. A Womersley parameter $\alpha = 4.3$ and Reynolds number $Re = 250$.

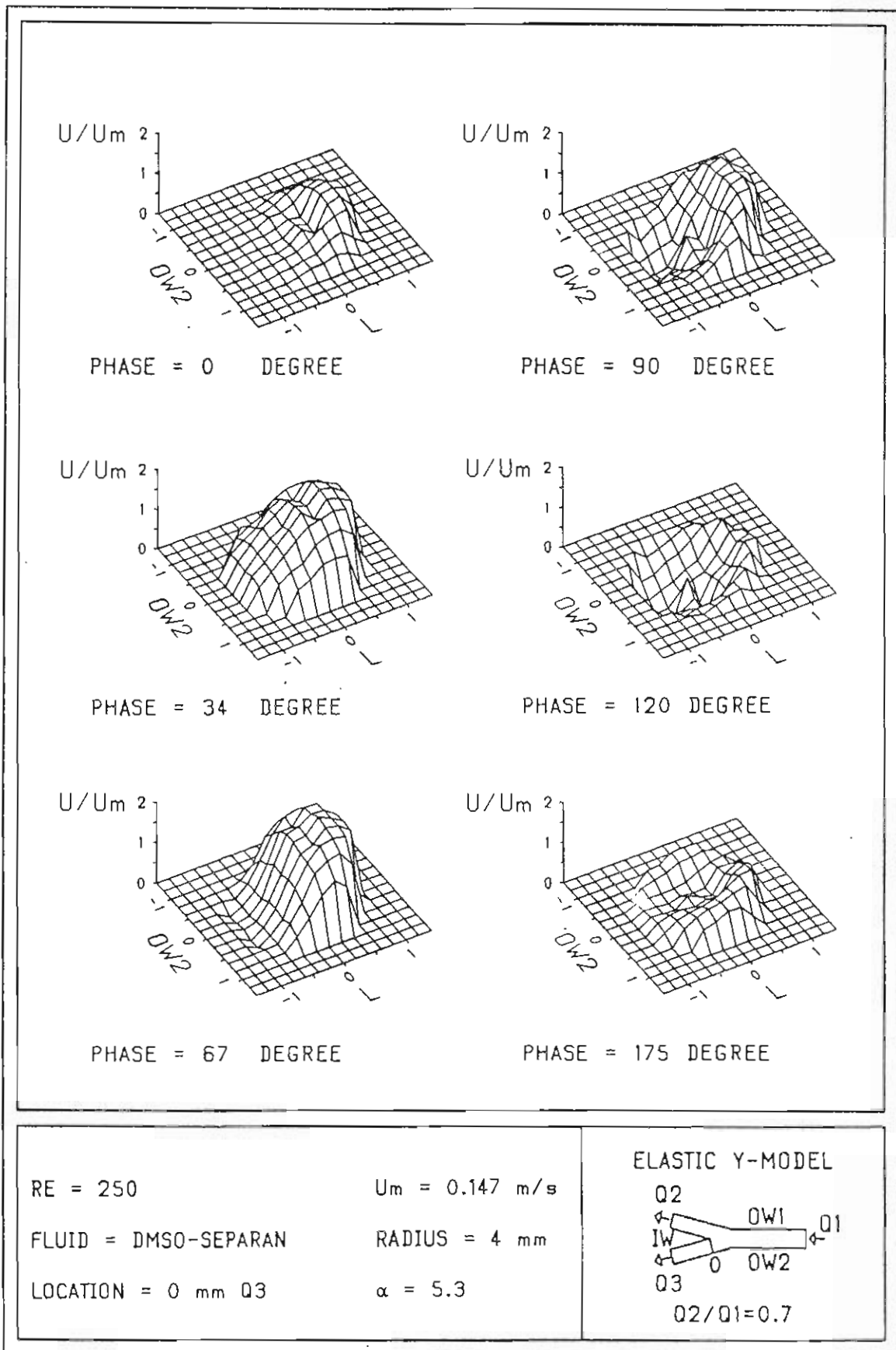


Figure 5. Velocity profiles over a whole pulse cycle (phase $\omega T = 0^\circ, 34^\circ, 67^\circ, 90^\circ, 120^\circ$ and 175°) in the branch ($Q3$), at the branching point (0 mm), recorded with a LDA in the elastic Y-model. The fluid used is a Newtonian glycerine water solution. A Womersley parameter $\alpha = 4.3$ and Reynolds number $Re = 250$.

On the part of the branch (Q2) with 70% flow on place bifurcation (0 mm) we have velocity profile of the non-newtonian fluid (Figure 4.). We selected phases 0° , 34° , 67° , 90° , 120° i 175° and we can see that the velocity is lower on outer wall side (OW1), and higher on the inner (IW) wall side (Q2). Reason for that is maximum velocity on internal wall side with secondary flow. We also measured velocity profiles at 0 mm, 2.5 mm, 5 mm, 10 mm, 15 mm and 25 mm downstream of the branching point in the part of the branch (Q2). We detected higher velocity on internal wall side with secondary flow. Analysis of six velocity profiles, represented in the Figure 5., on bifurcation place (0 mm) on the branch (Q3), clearly shows changes in the velocity profile shapes on the outer wall (OW2). A small turbulence region, i.e. a small separation region can be seen, as illustrated by the Figure 5.

We received by measurement that the most significant reverse flow at phase angles 2.5 mm and 5 mm, which indicates that the separation region is moved from the point 0 mm downstream of the branching point.

On the described result and above discussion we can conclude:

- Differences in velocity distribution between Newtonian and non-Newtonian fluids were observed in the flow separation regions in both branches of the Y model.
- The highest reverse flow area occurred between 2.5 mm and 5 mm in the branch with 30% of the flow. This indicates that the separation region moved downstream of the branching point beyond the 0 point (0 mm).
- Elastic Y model was made of two-component silicon rubber with an elasticity similar to the blood vessel wall. Material constants of this material are comparable with material constants of a carotid artery which helped us to determine velocity rate and loss of flow strenght by applying the blood vessel deformation law.
- The experiment on the model showed that geometry, fluid characteristics, wall elasticity and velocity have an impact on both linear and separation flow formation.

Acknowledgement

The experimental model measurements were made in the Laboratory for Biofluid Mechanics at the Technical University of Munich and at the Fachhochschule Munich.

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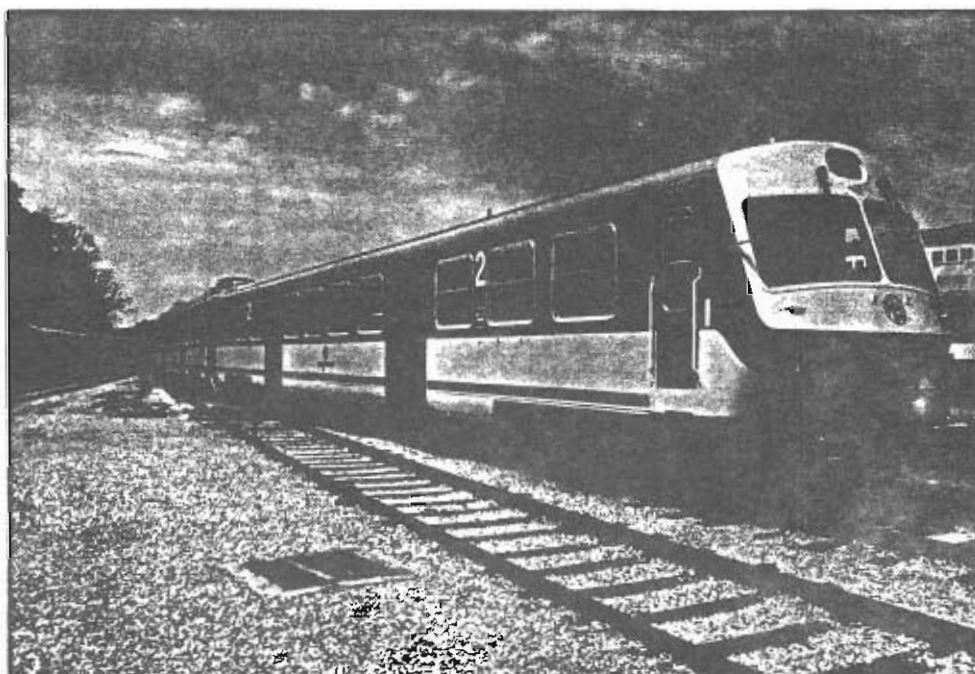
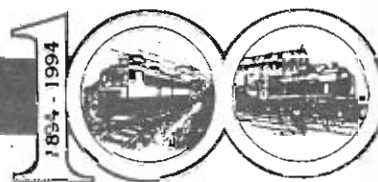
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