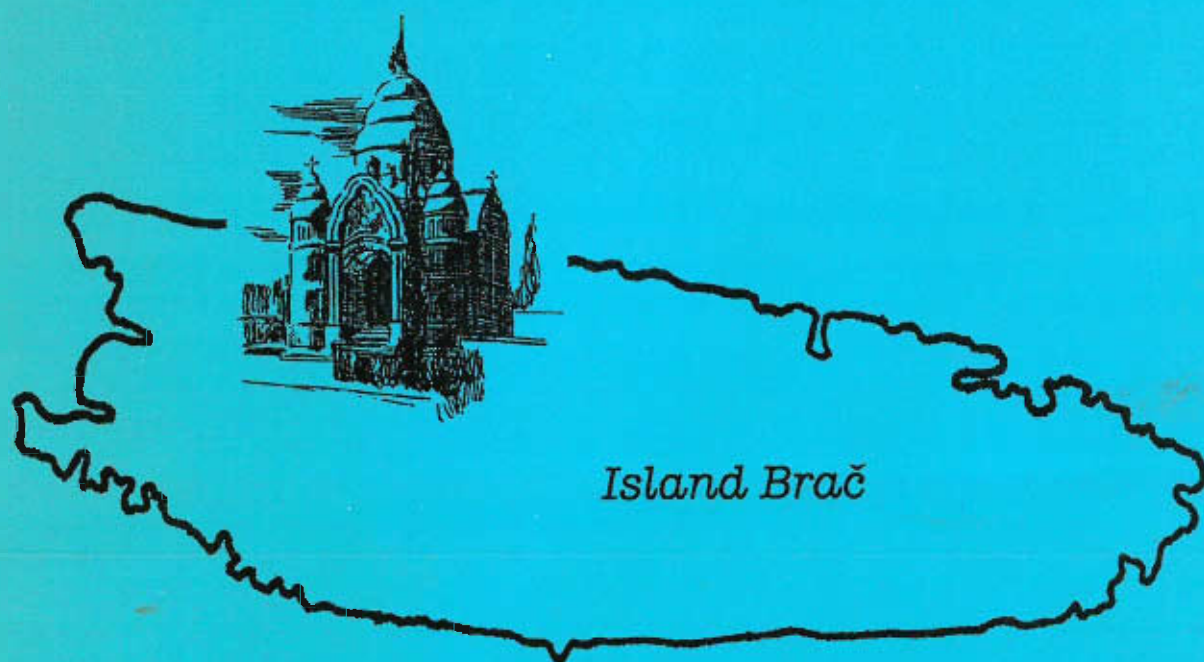


PROCEEDINGS
OF THE 2nd CONGRESS OF CROATIAN
SOCIETY OF MECHANICS



Supetar - Croatia
18 - 20 September, 1997

**PROCEEDINGS
OF THE 2nd CONGRESS OF CROATIAN
SOCIETY OF MECHANICS**

**held in Supetar, Croatia
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Edited by

*Pavao Marović
Jurica Sorić
Nikola Vranković*

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P R E F A C E

The 2nd Congress of Croatian Society of Mechanics, which is organized by Croatian Society of Mechanics and Central European Association for Computational Mechanics (CEACM), is the second in the series of organized gathering of scientists and researches in the fields of general mechanics, mechanics of solids and mechanics of fluids.

This Proceedings contains three invited lectures and 84 session papers presented at the Congress in Supetar during the period September 18-20, 1997. The overwhelming majority of papers deal with topics of computational mechanics but the papers from other areas of mechanics are included, too.

All contributions are printed from camera - ready copies supplied by the authors and the editors cannot accept responsibility for any errors or inconsistencies contained in the papers.

Editors are very grateful to all participants for their contributions in making this Congress a success. We also wish to thank our sponsors for their financial support and all those who helped in any way in organizing this Congress.

*P. Marović, J. Sorić, N. Vranković
Zagreb, September 1997.*

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INVITED LECTURES



Computer Simulations of Reinforced Concrete Structures including Structural Damage

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Abstract: The paper describes simulation techniques for damage processes in reinforced concrete structures, especially shell structures. After a brief sketch of the virtual work incrementation, classical and higher-order layered finite elements are introduced. The material description of reinforced concrete then is followed by a description of the multi-level analysis concept which is finally illustrated by simulations of crack-damage processes of cooling tower shells.

1 Introduction

The simulation of nonlinear reinforced concrete structural responses including damage evolution processes requires a computational treatment in a 3-dimensional environment $\{\sigma^{ij}, \gamma_{ij}; i, j = 1, 2, 3\}$ in which damage phenomena used to be formulated. Thus the application of classical structural models of reduced internal dimensions, such as rods or shells, is restricted to the kinematic mapping, further discretized into finite elements. Since the material description remains in the E3, the computational strategy calls for a multi-dimensional, multi-level concept as described in the following paper.

2 Multi-Layered Shell Continua

2.1 General Nonlinear Shell Theories

There exist different levels of nonlinear shell theories capable for the simulation of reinforced concrete structural damage processes [1,2]. In general, the necessary materially nonlinear description is supplemented also by a geometrically nonlinear one. Suitable shell theories possess the following physical structure. Loads $\mathbf{p}$ and stress resultants $\boldsymbol{\sigma}$ are connected by the dynamic equilibrium conditions

$$-\mathbf{p} = \mathbf{D}_e \cdot \boldsymbol{\sigma} = (\mathbf{D}_{eL} + \mathbf{D}_{eN}(\mathbf{u})) \cdot \boldsymbol{\sigma} , \quad \{\mathbf{p}, \boldsymbol{\sigma}\} \in F , \quad (2.1)$$

in which $\mathbf{D}_{eL}$ describes a linear, $\mathbf{D}_{eN}$ a nonlinear partial differential operator and linear functional of $\mathbf{u}$. $\mathbf{p}$ may include D'ALEMBERT forces. Displacement variables $\mathbf{u}$ and strain measures $\boldsymbol{\epsilon}$ are linked by the kinematic relations

$$\boldsymbol{\epsilon} = \mathbf{D}_k \cdot \mathbf{u} = (\mathbf{D}_{kL} + \mathbf{D}_{kN}(\mathbf{u})) \cdot \mathbf{u} , \quad \{\mathbf{u}, \boldsymbol{\epsilon}\} \in F \quad (2.2)$$

with the kinematic operator $\mathbf{D}_k$ consisting of the linear part $\mathbf{D}_{kL}$ and the nonlinear $\mathbf{D}_{kN}$, the latter also a linear functional of $\mathbf{u}$. Prescribed displacements $\mathbf{r}$ are related to $\mathbf{u}$ by the kinematic boundary conditions and prescribed external force variables $\mathbf{t}$ by the dynamic boundary conditions [3]:

$$\mathbf{t} = \dot{\mathbf{t}} = \mathbf{R}_t \cdot \dot{\boldsymbol{\sigma}} , \quad \mathbf{r} = \dot{\mathbf{r}} = \mathbf{R}_r \cdot \dot{\mathbf{u}} , \quad \{\mathbf{t}, \mathbf{r}\} \in C, \quad \dot{\mathbf{t}} \in C_t, \quad \dot{\mathbf{r}} \in C_r . \quad (2.3)$$

For general nonlinear material behaviour the standard constitutive law is based due to the physical framework of 1st and 2nd law of thermodynamics on rate formulations in the E3, which have to be transformed into shell deformation space by numerical integrations yielding

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= \mathbf{C} \dot{\boldsymbol{\epsilon}} + \mathbf{d} \rightarrow \boldsymbol{\sigma} = \int \dot{\boldsymbol{\sigma}} dt = \int (\mathbf{C} \dot{\boldsymbol{\epsilon}} + \mathbf{d}) dt , \\ \boldsymbol{\epsilon} &= \int \dot{\boldsymbol{\epsilon}} dt = \int \mathbf{C}^{-1} \cdot (\dot{\boldsymbol{\sigma}} - \mathbf{d}) dt . \end{aligned} \quad (2.4)$$

In time-independent processes this relation is largely substituted by incremental relations. It should be pointed out that rods or 3-dimensional solids can be described by sets of equations with identical structure, but with different variables and operators of course.

2.2 Shell Modelling in the Thickness Direction

In order to capture the materially nonlinear behaviour of reinforced concrete in a manner sufficient for the simulation of complete structural responses the concept of layered models is widely accepted. This concept will be repeated for shells in the following.

The REISSNER-MINDLIN shell theory as the most conventional analysis concept consists of a transversely uniform kinematic model, in which the strains in each sub-layer over the thickness can be derived from the 1st ($\alpha_{\alpha\beta}$) and 2nd ($\beta_{\alpha\beta}$) strain tensors and from the shear distortion ($\gamma_{\alpha 3}$) of the reference surface:

$$\gamma_{\alpha\beta} = \alpha_{\alpha\beta} + \Theta^3 \beta_{\alpha\beta} , \quad \gamma_{\alpha 3} = \gamma_{\alpha} . \quad (2.5)$$

In this case of a singlelayer, singledirector representation, the deformations of all sub-layers of the shell are solely controlled by variables of the reference surface.

However, if large differences in thickness or material properties of sub-layers

occur or if 3-dimensional damage effects have to be assessed, so-called multi-director theories are in use [4]. Their most simple representation is a multilayer model with single directors each [5, 6, 7], with or without director stretch, but also multilayer models with several directors for each sub-layer have been derived [8, 9] leading to higher-order strain approximations than customary in shell theory.

2.3 Total and Incremental Principle of Virtual Work

The principle of virtual work

$$-\int_F \bar{\rho} \ddot{\mathbf{u}}^T \delta \mathbf{u} d\bar{F} + \int_F \bar{\mathbf{p}}^T \delta \mathbf{u} d\bar{F} + \int_{C_t} \bar{\mathbf{t}}^T \delta \mathbf{r} d\bar{C}_t - \int_F \bar{\boldsymbol{\sigma}}^T \delta \boldsymbol{\varepsilon} d\bar{F} = 0 \quad (2.6)$$

with the mass density $\bar{\rho}$ of the undeformed reference surface $\bar{F}$ with boundary $\bar{C}$ governs the damage-inducing load response of the structure. Aiming at a later incremental-iterative solution process, all fields of response variables, such as

$$\mathbf{u} = \bar{\mathbf{u}} + \bar{\mathbf{u}} + \dot{\mathbf{u}} \quad (2.7)$$

due to Fig. 1 with $\bar{\mathbf{u}}$ as imperfections or prestresses, $\bar{\mathbf{u}}$ as fundamental state variables and $\dot{\mathbf{u}}$ as incremental changes from the fundamental to the adjacent state, have to be incremented and substituted into (2.6). After some tedious transformations we receive the following incremental principle of virtual work:

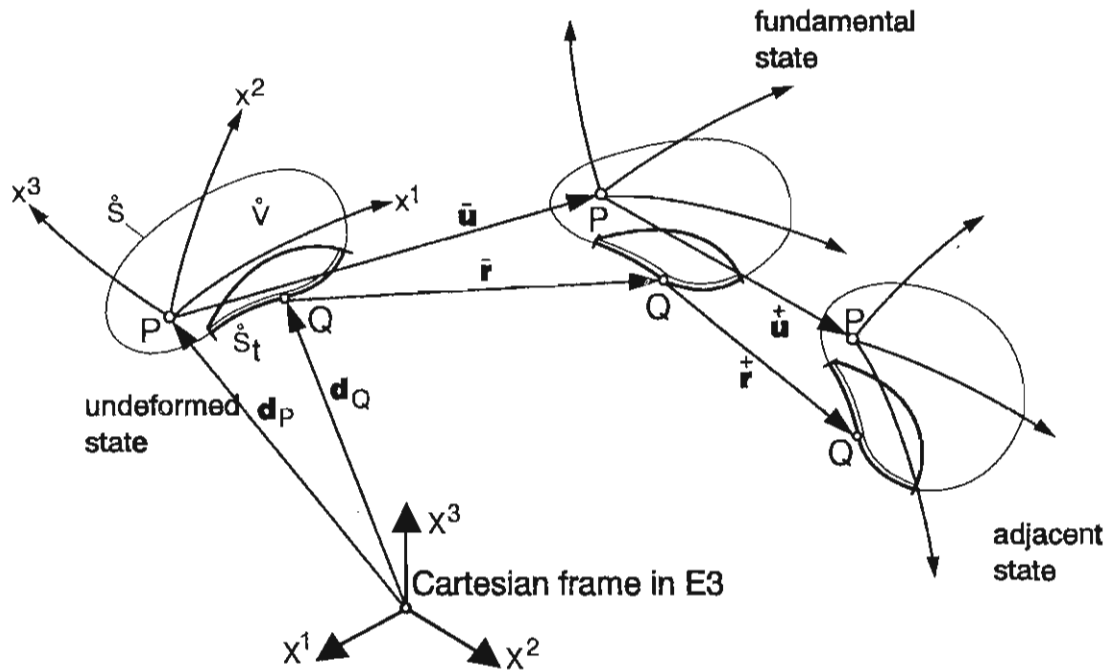


Fig. 1 Different states of deformation

$$\begin{aligned}
& \int_F \bar{\rho} \ddot{\mathbf{u}}^T \delta \mathbf{u} d\bar{F} + \int_F \left[(\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\sigma}})^T \frac{1}{2} \delta \bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\sigma}}^T \delta \bar{\boldsymbol{\epsilon}} \right] d\bar{F} \\
& = \delta \left[\int_F \bar{\mathbf{p}}^T \bar{\delta} \bar{\mathbf{u}} d\bar{F} + \int_{C_t} \bar{\mathbf{t}}^T \bar{\delta} \bar{\mathbf{r}} d\bar{C}_t - \int_F \bar{\rho} \ddot{\mathbf{u}}^T \bar{\delta} \bar{\mathbf{u}} d\bar{F} - \int_F (\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\sigma}})^T \bar{\delta} \bar{\boldsymbol{\epsilon}} d\bar{F} \right] \\
& \quad + \int_F \bar{\mathbf{p}}^T \delta \bar{\mathbf{u}} d\bar{F} + \int_{C_t} \bar{\mathbf{t}}^T \delta \bar{\mathbf{r}} d\bar{C}_t = 0
\end{aligned} \tag{2.8}$$

with $\delta \bar{\boldsymbol{\epsilon}} = \mathbf{D}_{kN}(\bar{\mathbf{u}}) \cdot \bar{\mathbf{u}}$

as basis of all future simulations.

3 The Discretized System

3.1 Displacement Discretization

In order to identify material points in a reinforced concrete shell, a layerwise concept of steel and concrete components due to Fig. 2 is adopted as mentioned previously. If the shell is modelled as a single layer, single director continuum, the displacement discretization process for the p-th element is standard:

$$\mathbf{u}^p = \boldsymbol{\Omega}^p \cdot \mathbf{v}^p, \quad \boldsymbol{\epsilon}^p = \mathbf{D}_k \cdot \mathbf{u}^p = \mathbf{D}_{kL}^p \cdot \mathbf{v}^p + \mathbf{D}_{kN}^p(\mathbf{v}^p) \cdot \mathbf{v}^p. \tag{3.1}$$

The resulting 5 parametric theory delivers at least 5 degrees of freedom at each element node of the reference surface, from which the kinematic variables of all sub-layers can be evaluated. Also for KIRCHHOFF-LOVE-type shell theories

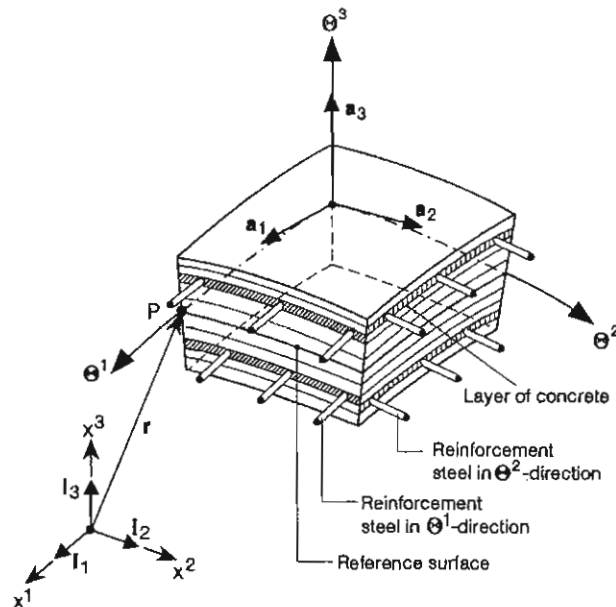


Fig. 2 Layered reinforced concrete shell model

the number of 5 degrees of freedom is maintained, but 2 internal constraints restrict the deformability.

For a multilayer model with single directors each due to Fig. 3, each layer adds 2 rotational degrees of freedom to those of the reference surface for suppressed transverse stretching, since the inplane displacements are coupled at the layer bounds. If transverse stretching shall be taken into account, each layer adds another transverse displacement degree of freedom.

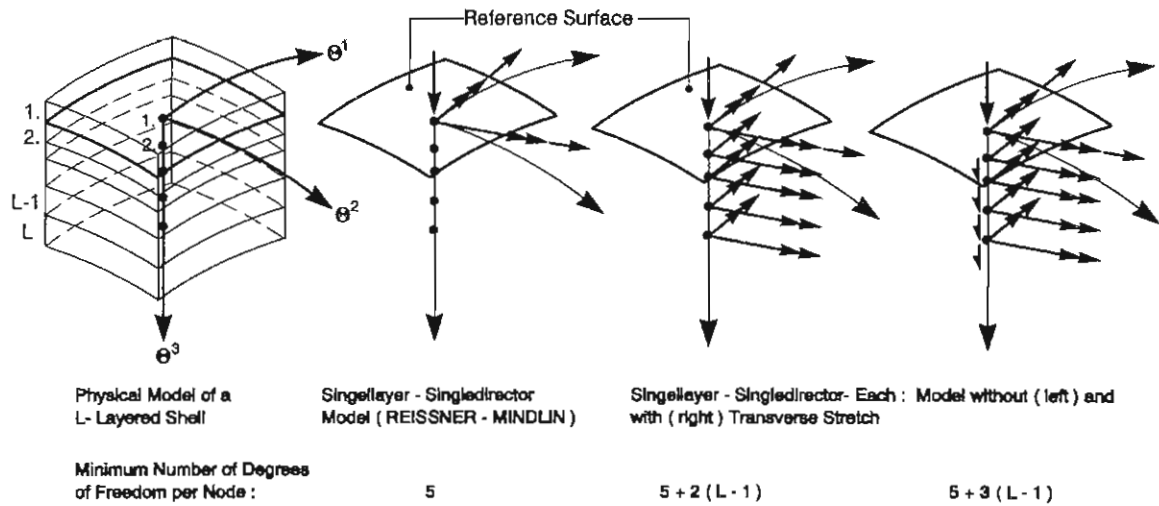


Fig. 3 Possible discretized shell representations

3.2 Tangential Equation of Motion or Stiffness Equation

Accomplishing the discretization process for all elements and assembling all element contributions [3] together, the global tangential equation of motion in the time-dependent case is received as follows:

$$\mathbf{M} \cdot \ddot{\mathbf{V}} + \mathbf{C}_T \cdot \dot{\mathbf{V}} + \mathbf{K}_T \cdot \mathbf{V} = \mathbf{P} - \mathbf{F}_i. \quad (3.2)$$

With the mass matrix $\mathbf{M}$, the tangential damping $\mathbf{C}_T$ and stiffness $\mathbf{K}_T$ matrices, with the total applied load $\mathbf{P}$ and the vector of internal forces $\mathbf{F}_i$, and with the increments $\mathbf{V}$, $\dot{\mathbf{V}}$, $\ddot{\mathbf{V}}$ of the nodal degrees of freedom, their velocities and their accelerations, (3.2) describes the global response properties of the considered loading process $\mathbf{P}(t)$ in a tangential subspace to the fundamental one. For all time-independent response simulations ($\mathbf{M} = \mathbf{C}_T = \mathbf{0}$), (3.2) degenerates to the tangent stiffness equation:

$$\mathbf{K}_T \cdot \mathbf{V} = \mathbf{P} - \mathbf{F}_i. \quad (3.3)$$

Both equations (3.2) and (3.3) form the basis for all future nonlinear structural simulations.

4 Damage-Included Material Modelling

4.1 Standard Material Description

While the later incremental-iterative solution techniques are applied to (3.2), (3.3) and thus controlled on the structural level, all material properties are defined in points of the respective layer level. The corresponding constitutive laws have to be nonlinear in order to treat damage processes properly. In contrast to state-dependent elastic behaviour, inelastic materials deliver path-dependent responses. Applying descriptions like

$$\boldsymbol{\varepsilon} = \mathbf{f}(\boldsymbol{\sigma}) \quad \text{or} \quad \boldsymbol{\sigma} = \mathbf{g}(\boldsymbol{\varepsilon}) , \quad (4.1)$$

complicated implicate functional operators enter the computational strategy. Advantageously, most engineering materials can be properly described by 1st order differential equations. If we introduce the following rate-abbreviations

$$\dot{\boldsymbol{\varepsilon}}^* = \begin{bmatrix} \dot{\gamma}_{ij} \\ \dot{h}_i \end{bmatrix} , \quad \dot{\boldsymbol{\sigma}}^* = \begin{bmatrix} \dot{\sigma}^{ij} \\ \dot{h}_i \end{bmatrix} , \quad \dot{\boldsymbol{\varepsilon}} = [\dot{\gamma}_{ij}] , \quad \dot{\boldsymbol{\sigma}} = [\dot{\sigma}^{ij}] , \quad (4.2)$$

in which h_i denotes a set of suitable internal variables and the dot a material time derivative, engineering materials fit into rate-form laws, as [10] or [20] explains:

$$\dot{\boldsymbol{\varepsilon}}^* = \mathbf{I} \cdot \dot{\boldsymbol{\sigma}}^* + \mathbf{b} , \quad \dot{\boldsymbol{\sigma}}^* = \mathbf{C} \cdot \dot{\boldsymbol{\varepsilon}} + \mathbf{d} . \quad (4.3)$$

(4.3) form initial value problems in each GAUSS point of each sub-layer; for time-invariant responses the material time-derivatives in general are substituted by increments.

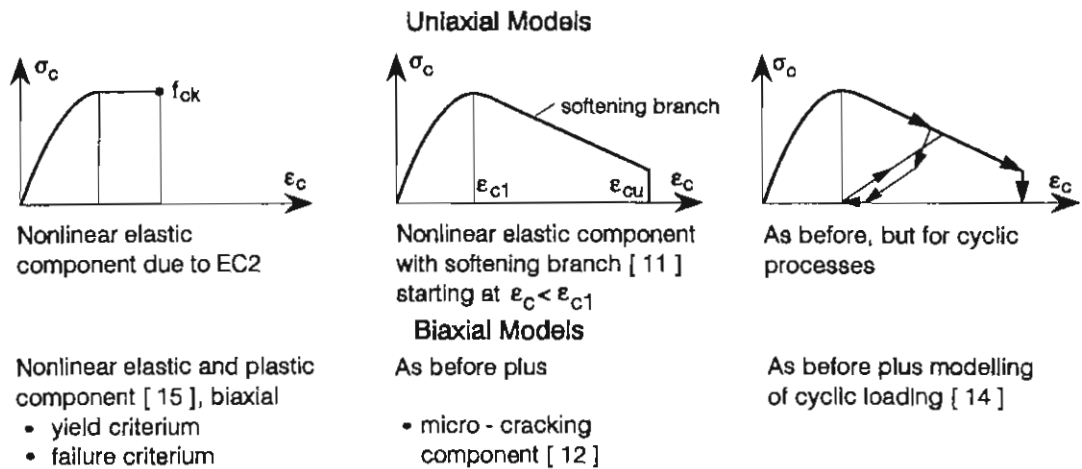
4.2 Material Modelling of Reinforced Concrete

Reinforced concrete is a highly nonlinear reacting composite, which fits well into the above mentioned scheme. Its response is characterized by the following nonlinear phenomena [11], which can be modelled in many different degrees of exactness as pointed out on Fig. 4:

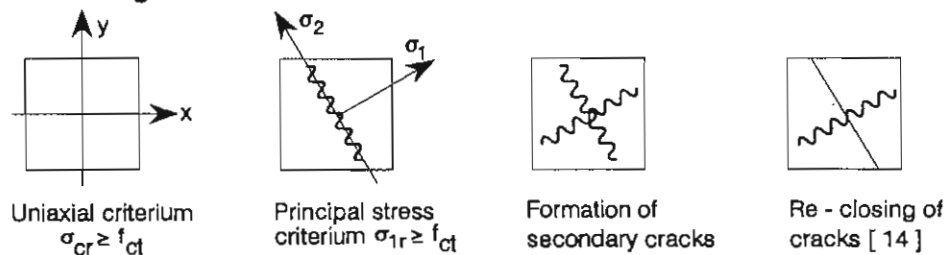
- nonlinear stress-strain relation in compression,
- tension cracking at low stress level,
- elasto-plastic behaviour of the reinforcement,
- nonlinear bond between reinforcement and concrete.

Each of these phenomena can be responsible for stress redistributions, connected with essentially irreversible material processes, contributing to the damage evolution. Our present simulations are based on the following material models.

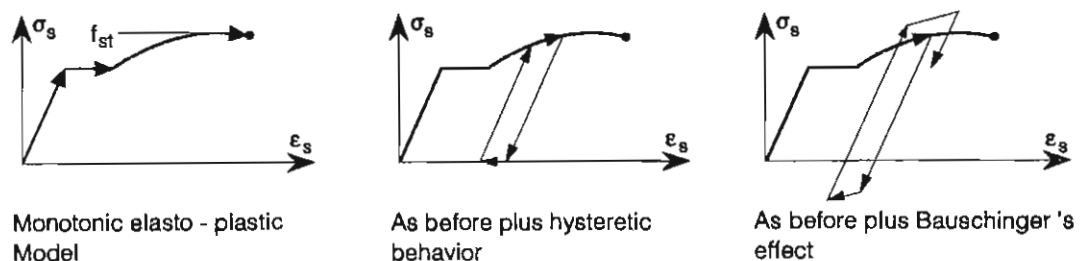
Nonlinear Compression Behavior of Concrete



Tension Cracking of Concrete



Elasto - plastic Behavior of Reinforcement Steel



Inelastic Bond

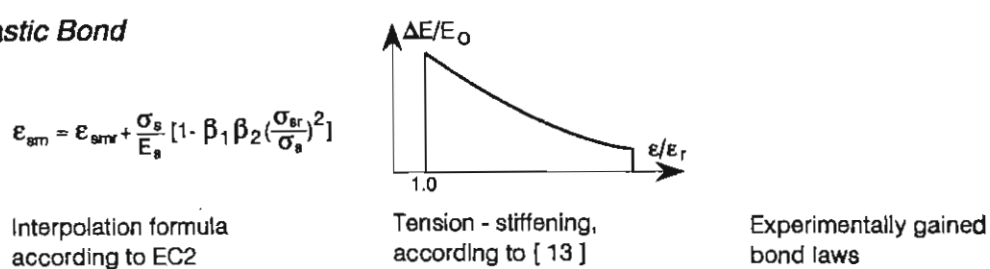


Fig. 4 Nonlinear components of reinforced concrete simulations in different degrees of accuracy

The nonlinear behaviour of concrete in compression is simulated due to the elasto-plastic-fracturing theory [12]. This model subdivides each stress increment resulting from a certain incremental strain into an elastic stress increase (el), a plastic stress decrease (pl) and a decrease due to microfracturing (fr):

$$\dot{\sigma}^{ij} = \dot{\sigma}_{el}^{ij} + \dot{\sigma}_{pl}^{ij} + \dot{\sigma}_{fr}^{ij} = (C_{el}^{ijkl} + C_{pl}^{ijkl} + C_{fr}^{ijkl}) \dot{\gamma}_{kl} . \quad (4.4)$$

A series of required material parameters [12] as well as the failure-envelope – see Fig. 5 – is fitted to well established material data.

The reinforced steel model covers elastoplastic behaviour in a multi-linear way for cyclic processes, under special attention of the BAUSCHINGER effect. To describe the nonlinearly fading bond, the tension-stiffening concept due to [13] is applied.

Concrete under tension fails in a brittle manner leading to stiffness reductions in the material points affected. For the present analysis, cracking is modelled in a smeared manner neglecting localization phenomena. The analysis employs a principal stress criterion, by which cracks oriented at an arbitrary angle in a given frame of reference can be described. After exceedance of the tension strength, all stiffness coefficients of (4.4) or-thogonal to an assumed crack are removed. Shear stiffnesses are partly retained due to aggregate interlocking. Closing and re-opening of cracks as well as the formation of secondary cracks can also be modelled [14]. In order to detect damage processes, the most important material modelling aspect seems to lie in the proper simulation of cyclic response processes [12,15].

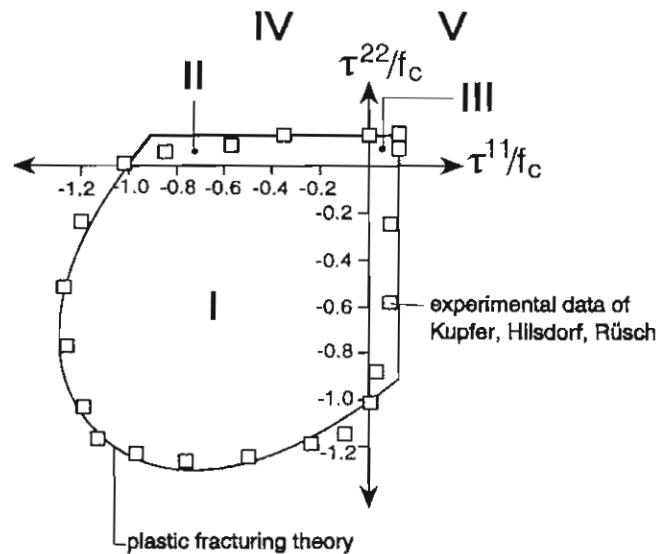


Fig. 5 Applied plane failure envelope compared with experimental data

5 Simulation Strategy

After all, the final simulation strategy for time-invariant responses is summarized on Fig. 6. Incremental-iterative strategies are applied to the global tangential stiffness equation (3.3) for time-independent problems. With a new increment $\dot{V}_n$ of the global degrees of freedom, one enters each finite element on all GAUSS points and proceeds with the kinematic model (2.5) to each material sub-point of the L sub-layers. Here, the incremental constitutive law (4.3) is evaluated and its result are integrated numerically over all sub-layers arriving back at the original GAUSS point of the (layered) shell level, e.g.:

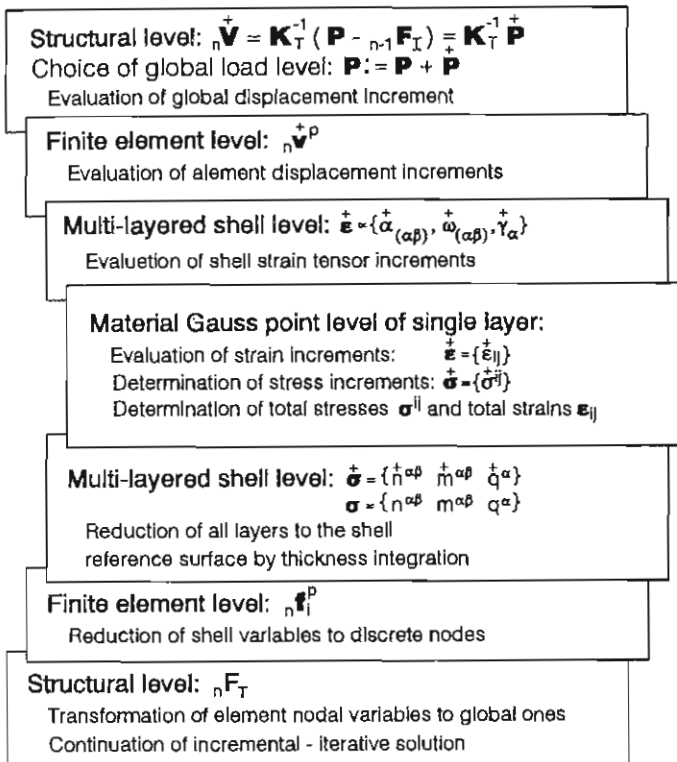
$$\begin{aligned}\dot{n}^{\alpha\beta} &= E^{\alpha\beta\lambda\mu(1)} \dot{\alpha}_{\lambda\mu} + E^{\alpha\beta\lambda\mu(2)} \dot{\beta}_{\lambda\mu}, \\ \dot{m}^{\alpha\beta} &= E^{\alpha\beta\lambda\mu(1)} \dot{\alpha}_{\lambda\mu} + E^{\alpha\beta\lambda\mu(3)} \dot{\beta}_{\lambda\mu}\end{aligned}\quad (5.1)$$

with

$$E^{\alpha\beta\lambda\mu(0)} = \int_h C^{\alpha\beta\lambda\mu} (\Theta^3)^{l-1} d\Theta^3 \quad (5.2)$$

from the original material law (4.3):

$$\dot{\sigma}^{ij} = C^{ijkl} \dot{\gamma}_{kl}, \quad d^{ij} \equiv 0. \quad (5.3)$$



This step requires the integration of the imbedded initial value problem (4.3) in best possible quality, solving for the stress-increments, then for the total strains, stresses and damage parameters. From here we proceed further to the element level, and then to the global level for the evaluation of the next increment $\dot{\mathbf{V}}_{n+1}$.

Fig. 6 Multi-level simulation strategy

6 Wind and Temperature Damage of Cooling Tower Shells

6.1 Cooling Tower K at Gersteinwerk Power Station

Fig. 7 portrays the cooling tower shell of the VEW power plant GERSTEINWERK, substation K, to which all further investigations will be applied. Necessary material parameters can be found in [16]. The reinforced concrete shell of 123.90 m of height is composed of two different hyperboloidal middle surfaces

$$r = r_0 + \frac{a}{b} \sqrt{b_2 + (H_T - z)^2} \quad (6.1)$$

with $r_o = 29.407$ m respectively 2.579 m
 $a = 3.873$ m 30.701 m
 $b = 30.270$ m 85.230 m
 $H_T = 98.900$ m 98.900 m

for the lower respectively upper part.

The cooling tower shell is loaded due to BTR-regulations [17] by dead weight g , wind w and temperature action t [18].

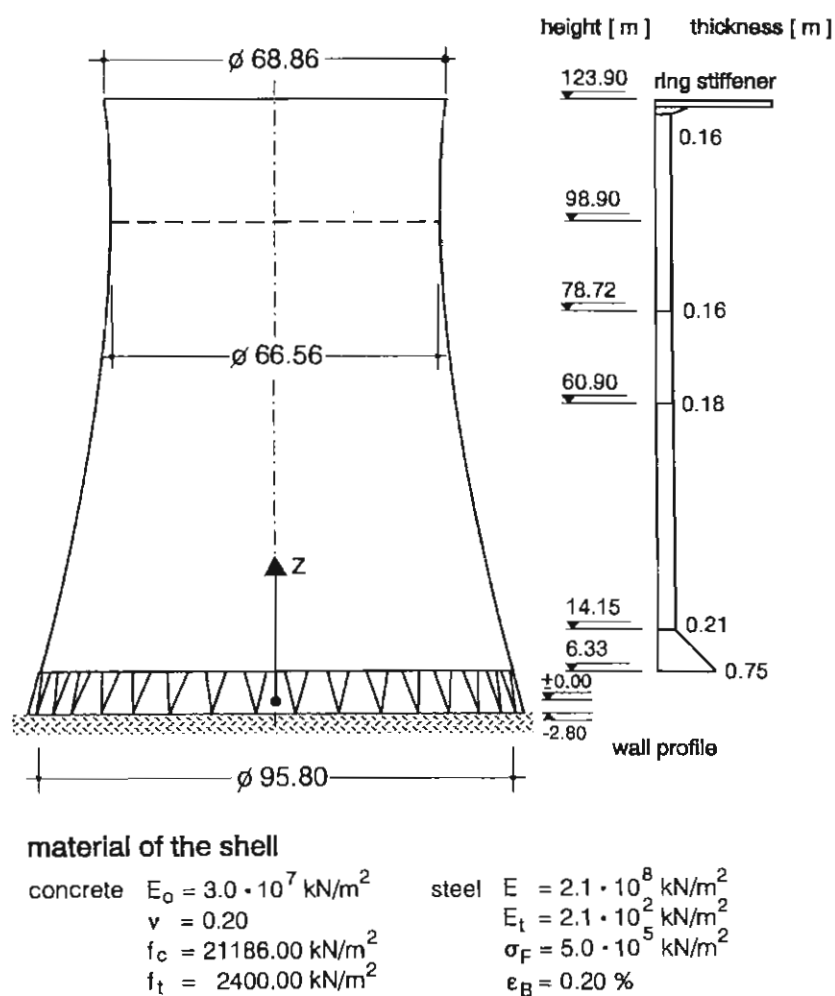


Fig. 7 Geometry of cooling tower shell

6.2 Influence of Different Service Temperature

Fig. 8 studies the influence of different temperature actions given in the figure on the collapse load level, related to wind by the wind-load factor λ . The study demonstrates that the collapse level related to wind (from 2.25 to 2.30) does not differ considerably for varying service (temperature) conditions. The shapes of single response curves however deviate because of considerable

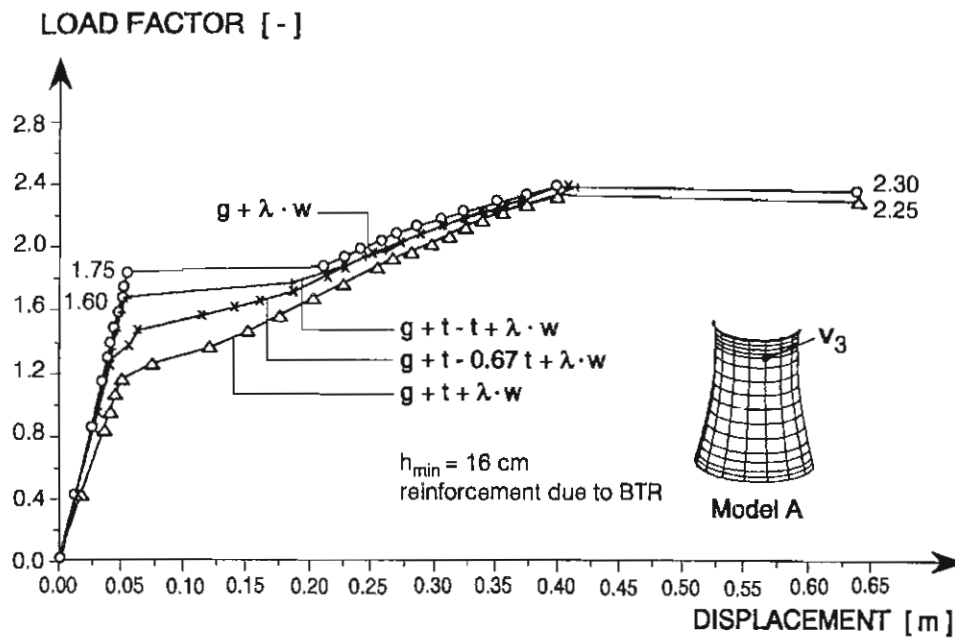


Fig. 8 Influence of temperature actions

pre-damage of the shell, especially in the case of extreme temperature conditions [19], but obviously, crack-damage does influence the ultimate load level only to a small degree.

6.3 Repeated Medium Storms

Because of the widely used linear analysis concepts in structural design analysis, the damage evolution in cooling tower shells, accessible only through physically nonlinear simulations, is largely unknown. We continue our studies of this behaviour with the cooling tower of Fig. 7 in extreme winter service conditions $g+t$ ($t: \Delta t = 45 \text{ K}$), subjected to 20 repeated storms up to $\lambda = 1.00$, storms which possess a return period of approximately 10 years in Germany. As Fig. 9 demonstrates, wind actions of this intensity damage the structure only marginally. Compared to the response of a monotonic wind increase $g + t + \lambda w$ up to collapse at $\lambda = 2.25$, the additional damage is initiated by temperature. At the first of 20 repeated storms the shell is largely cracked by temperature effects, but only small damage contributions remain after temperature removal. Thus, additional wind damage of low intensities is of negligible order.

6.4 Gale Cycles of High Intensity

This reported behaviour changes fundamentally for higher storm intensities, as portrayed in Fig. 10 by example of 4 repeated storms $\lambda = 2.00$ corresponding

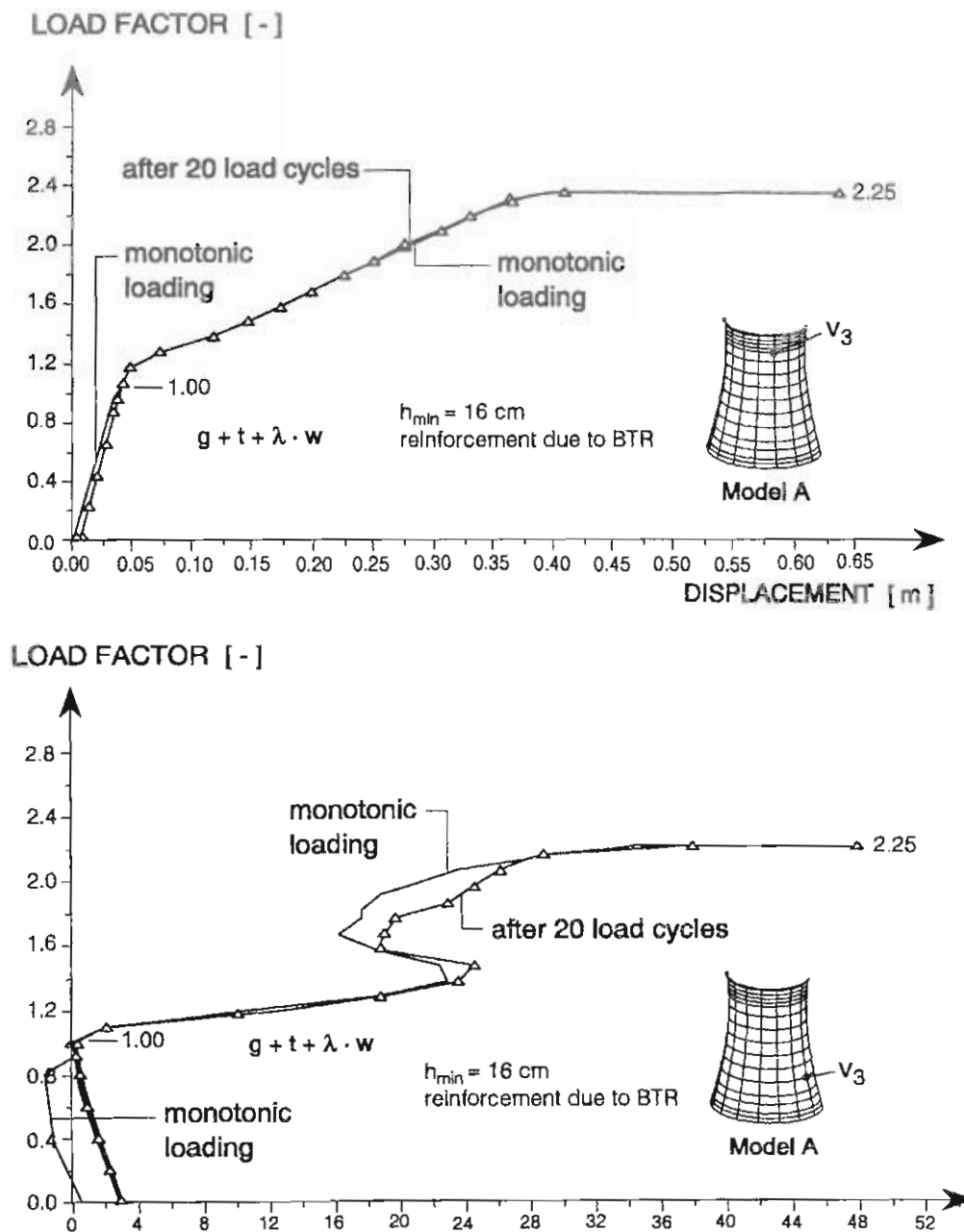


Fig. 9 Several medium wind actions

to a return period of around 500 years. In comparison to the monotonic response curve $g + t + \lambda w$ with collapse at $\lambda = 2.25$, each storm cycle damages the shell to a new damage level by a shift of the response curve to the right. Finally, the 5th load cycle is increased up to collapse level at $\lambda = 2.15 < 2.25$. The computed damage is pictured in Fig. 10 in terms of the integrating damage measure of permanent shell-displacements, as can be observed in particular in the lower part of Fig. 10. In the reality of cooling tower shells this damage would lead to wide-spread crack-pattern, as demonstrated on Fig. 11 for different wind-load intensities of another, much more avariciously reinforced tower shell.

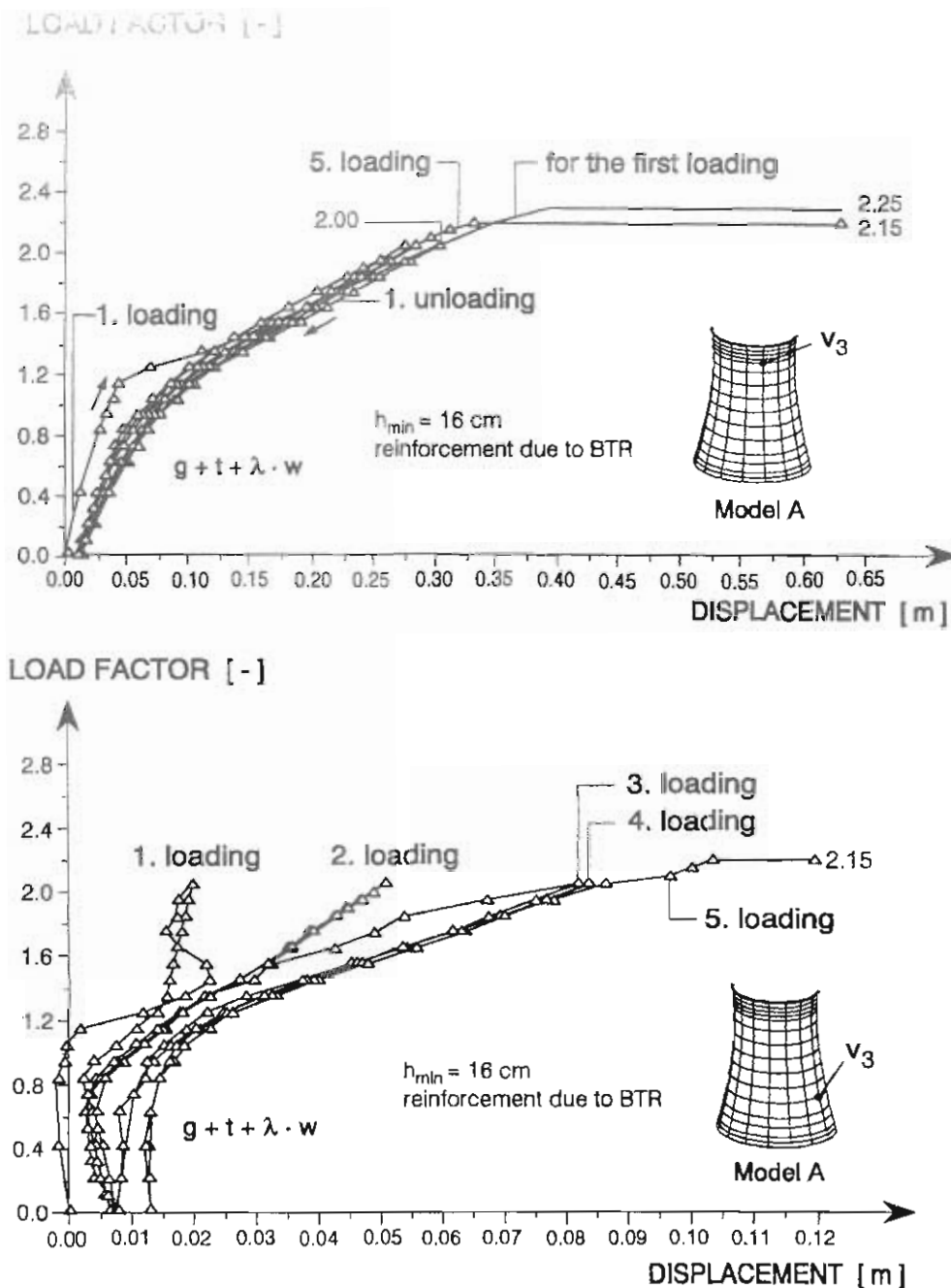


Fig. 10 Damage due to five repeated gales

7 Conclusions

The paper has demonstrated the ability of numerically nonlinear simulations of damage processes for large reinforced concrete shells. The arising damage has been expressed in terms of inelastic deformations and crack-simulations. Obviously, virgin cooling towers remain undamaged until extreme winter conditions and gales initiate a damage process. After this initiation, also events of medium intensity contribute to further damage evolution [19].

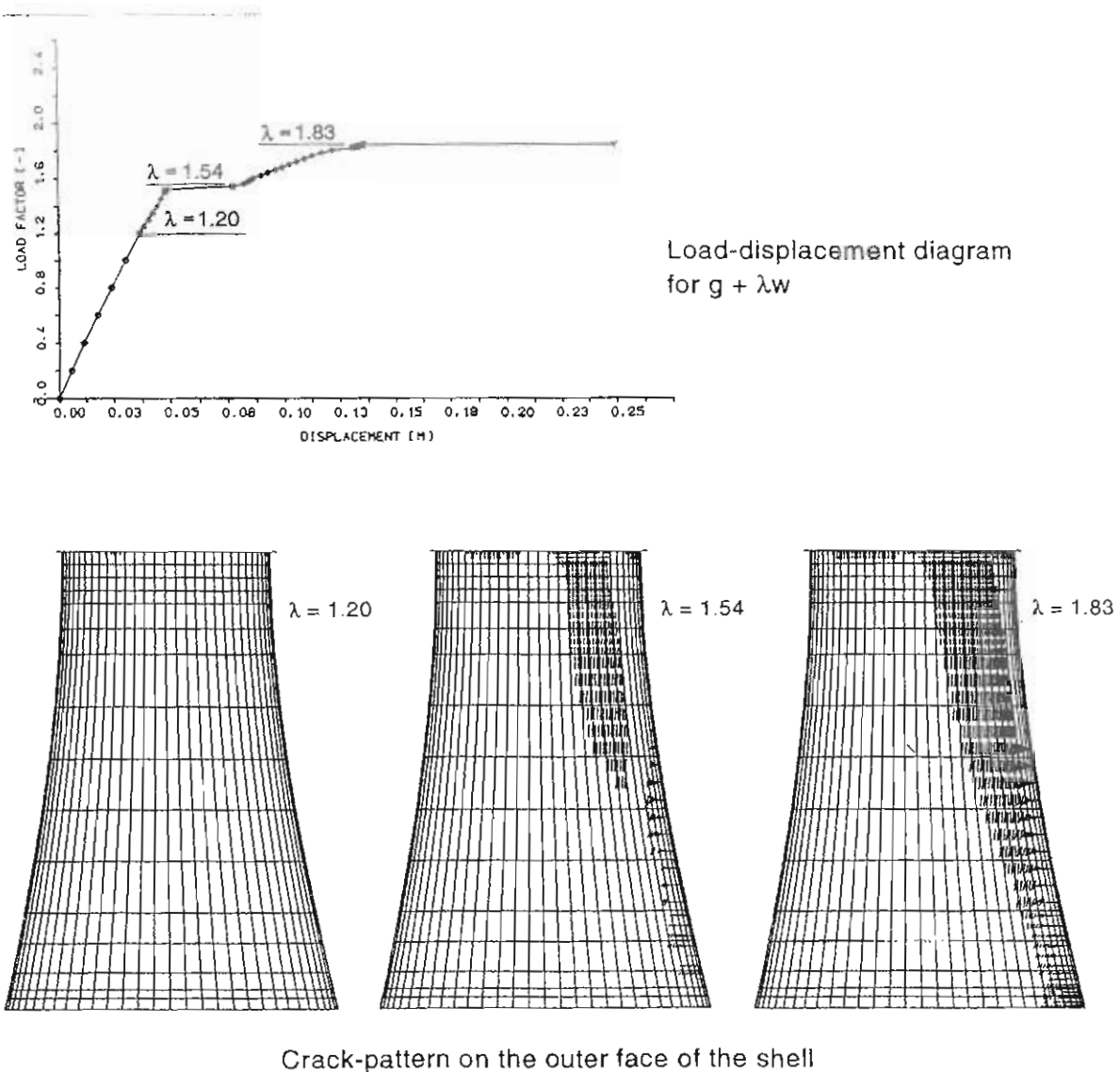


Fig. 11 Simulated crack-pattern

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SUMMARY

The paper addresses finite element formulations for automobile tires considering nonlinear material behavior, large strains and finite deformations while using efficient computational strategies for realistic large-scale 3D-FE simulations. A new 3D finite element model for cord-reinforced rubber composites is employed for the numerical representation of the reinforcing plies. A hyperelastic Mooney material law in conjunction with a hybrid finite element formulation is used for the modelling of the rubber material. A strategy for the coupling of two FE-meshes with different degrees of discretization is developed. Use of an iterative solver is proposed to improve the computational efficiency of the simulations. A 3D simulation of frictional static contact of a tire on a rigid road surface is presented.

INTRODUCTION

Using realistic and efficient numerical models for rubber and the reinforcing cords is of great importance for the feasibility and the reliability of tire simulations. Therefore, rubber is represented by means of a hybrid finite element formulation. A frequently used method for the 3D numerical representation of cord-reinforced composites is the formulation of an "equivalent homogeneous medium", resulting in anisotropic constitutive laws for the cord layers and the surrounding matrix material, see Tabaddor (1985) [10], or in orthotropic material laws for each of the cord layers and the surrounding rubber material (Rothert (1986) [8]). As a result, a discretization-dependent error with respect to the representation of the mechanical influence of the discrete cord layers is induced. To avoid this, the concept of rebar elements for geometrically and physically nonlinear analysis of cord-reinforced rubber composite materials (Helnwein et al. (1993) [1], Meschke and Helnwein (1994) [4]) is employed for the representation of the cord plies and the carcass.

A local mesh refinement technique is developed for the coupling of two originally incompatible finite element meshes with different degrees of refinement. This algorithm allows an efficient generation of 3D meshes of tires, characterized by a realistic discretization of the tread in the anticipated contact zone and its vicinity

¹This paper has been presented previously at ICES 97, May 4-9, 1997, San Jose, Costa Rica

and a coarser discretization of the rest of the tire.

An iterative solver with preconditioning is employed for the solution of the resulting systems of linear algebraic equations instead of a direct solver. The iterative solution makes large-scale tire simulations feasible concerning the required disc space and the CPU-time. This is demonstrated by means of a large-scale analysis of a tire in the last Section of this paper.

NUMERICAL MODEL FOR RUBBER

Rubber is numerically represented by the Mooney material model in the context of a hybrid formulation based on the three-field Hu-Washizu variational principle (Simo (1985) [9]). The Mooney-Rivlin form of the free energy function W is given as

$$W = A_1(I_1 - 3) + A_2(I_2 - 3), \quad (1)$$

where A_1 and A_2 are material constants and

$$I_1 = \text{tr } \mathbf{C}, \quad I_2 = \frac{1}{2}(\text{tr } \mathbf{C} \text{tr } \mathbf{C} - \text{tr } \mathbf{C}^2) \quad (2)$$

are the invariants of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$, where $\mathbf{F}$ denotes the deformation gradient. For the case of an incompressible material, characterized by $J = \text{Det } \mathbf{F} = 1$, W only contains a volume-preserving part.

In the reduced form of the three-field Hu-Washizu variational principle, the displacements $\mathbf{u}$, the dilatation θ and the hydrostatic pressure p are treated as independent variables. Making use of the separation of the strain energy density into a volume-changing part $U(J)$ and a volume-preserving part $W(\bar{\mathbf{C}})$, the potential energy is obtained as

$$\Pi(\mathbf{u}, \theta, p) = \int_V [U(\theta) + W(\bar{\mathbf{C}}) + p(J - \theta)] dV + \Pi^{\text{ext}}(\mathbf{u}), \quad \bar{\mathbf{C}} = J^{-2/3} \mathbf{C}, \quad (3)$$

where V is the volume of the body in the reference configuration. The Lagrange multiplier p , introduced to enforce equality (in a weighted integral average sense) between J and θ , will be identified as the hydrostatic pressure. $\Pi^{\text{ext}}(\mathbf{u})$ is the potential of the external forces. Setting the variation of (3) with respect to $\mathbf{u}$, θ and p equal to zero yields three nonlinear equations. For a solution by means of a Newton iteration technique these equations have to be linearized. The three equations resulting from the linearization can be reduced to a generalized displacement formulation by employing (identical) approximations for θ and p which are discontinuous across the boundaries of the finite elements (Liu (1994) [2]).

NUMERICAL REPRESENTATION OF REINFORCING CORDS

The numerical representation of the carcass and the different layers of reinforcing cords with different spatial orientation is based upon the so-called "rebar layer

concept". The computational effectiveness of this approach stems from the fact that several cord layers may be represented within one single overlay element ("rebar" element) without increasing the number of degrees of freedom. In addition, it allows separate characterization of the different constituents of the composite. The variational formulation of a rebar element together with the exact linearization, suitable for implementation in a 3D FE code, and a comparative evaluation of several constitutive formulations, is given in [4].

A one-dimensional single fibre ("rebar") can be defined with its orientation in the reference configuration by the unit vector $\mathbf{T}$. The corresponding unit vector $\mathbf{t}$ in the deformed (spatial) configuration is given by

$$\lambda_R \mathbf{t} = \mathbf{F} \mathbf{T}, \quad (4)$$

where λ_R is the axial stretch of the fibre and $\mathbf{F}$ denotes the deformation gradient. The stretch λ_R may be expressed in terms of the right Cauchy Green tensor $\mathbf{C}$ and of $\mathbf{T}$ as

$$\lambda_R^2 = \mathbf{T}^T \mathbf{C} \mathbf{T}. \quad (5)$$

In the reference configuration, the variational equation is given as

$$\delta_u \Pi = \int_V S_R \cdot \mathbf{T}^T \mathbf{F}^T \text{GRAD} \delta \mathbf{u} \mathbf{T} dV + G^{ext}(\mathbf{u}), \quad (6)$$

where $S_R = \mathbf{T}^T \mathbf{S} \mathbf{T}$ is the Second Piola-Kirchhoff stress, transformed to the direction of the rebar fibre. The uniaxial constitutive behavior of the rebar is expressed in the case of hyperelasticity in terms of a function of stored energy, W , as

$$S_R(\lambda_R, \lambda_L) = 2 \mathbf{T}^T \partial_{\mathbf{C}} W(\lambda_R, \lambda_L) \mathbf{T}, \quad (7)$$

where λ_L is the lateral stretch of the fibre. In order to solve (6) under the use of (7) by means of a Newton iteration technique, the resulting equation has to be linearized [4].

TYING ALGORITHM FOR LINKING OF TWO FE-MESHES

The contact zone of a tire tread generally requires a considerably higher degree of mesh refinement than the rest of the tire. To reduce the required computer resources, the part of the tire tread, which contains the anticipated contact region, and its vicinity is discretized by a fine mesh. The remaining part of the tire is discretized by a coarse mesh which is generated automatically from data related to the cross-sectional structure of the tire [5].

The coupling of two geometrically incompatible FE-meshes with different degrees of refinement is characterized by the enforcement of continuity between the two meshes at their common boundary and by specification of displacement constraints ("tying") for the nodes located on this boundary. The tying algorithm is characterized by enforcing geometric compatibility between the two meshes through modification of the coordinates of nodes located on their common boundary and

by specifying displacement constraints for these nodes in order to achieve geometric compatibility of the deformed configuration [3].

ITERATIVE SOLVER WITH PRECONDITIONING

Since the resulting coefficient matrices of the linear systems of equations are symmetric, the method of conjugate gradients (CGM) (Papadrakakis (1993) [6], Payer (1995) [7]) in conjunction with preconditioning is used for the iterative solution. The linear system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{b} \quad (8)$$

is transformed into

$$(\bar{\mathbf{L}}^{-1}\mathbf{A}^*\bar{\mathbf{L}}^{-T})(\bar{\mathbf{L}}^T\mathbf{u}^*) = \bar{\mathbf{L}}^{-1}\mathbf{b}^* \quad (9)$$

before applying the CGM. $\mathbf{u}$ and $\mathbf{b}$ are the incremental vectors of displacements and nodal forces, respectively. The expressions associated with the superscript $*$ are derived from equation (8) by scaling the diagonal elements of the tangent stiffness matrix $\mathbf{A}$ to 1. The preconditioning matrix $\bar{\mathbf{L}}$ is obtained by means of an incomplete Cholesky decomposition [6], [7]. The lower triangular matrix $\bar{\mathbf{L}}$ approximates the triangular matrix $\mathbf{L}$ of the full Cholesky decomposition, characterized by

$$\mathbf{A}^* = \mathbf{L}\mathbf{L}^T. \quad (10)$$

To minimize computation time and disc space, certain elements of the Cholesky decomposition are set equal to zero during the decomposition. Neglecting certain elements of the Cholesky decomposition matrix may result in numerical instabilities. Scaling down the off-diagonal-elements of $\mathbf{A}^*$ before performing the incomplete decomposition, proved to be the best strategy for stabilizing the decomposition.

NUMERICAL EXAMPLE

The usefulness of the proposed computational model for automobile tires is demonstrated by means of a 3D FE - analysis of an automobile tire with a realistic discretization of the tread pattern. The employed FE-mesh is shown in Figure 1. 52664 8-node trilinear 3D elements (16536 rebar elements and 36128 hybrid elements) were used for discretizing the whole tire, resulting in a total of 139693 DOFs. The constitutive models and the iterative solver described in the previous Sections were implemented in the multipurpose FE program MARC (Version K6.1). The analysis was performed on a DEC 8200 5/300.

The mounting of the tire onto the rim was simulated by a horizontal displacement of the bead toe of 7.69 mm simultaneously to the application of the internal pressure of 2.0 bar. Subsequently, the rigid plane was moved incrementally towards the tire axis. Frictional effects were considered in this process, assuming the coefficient of friction as 0.5.

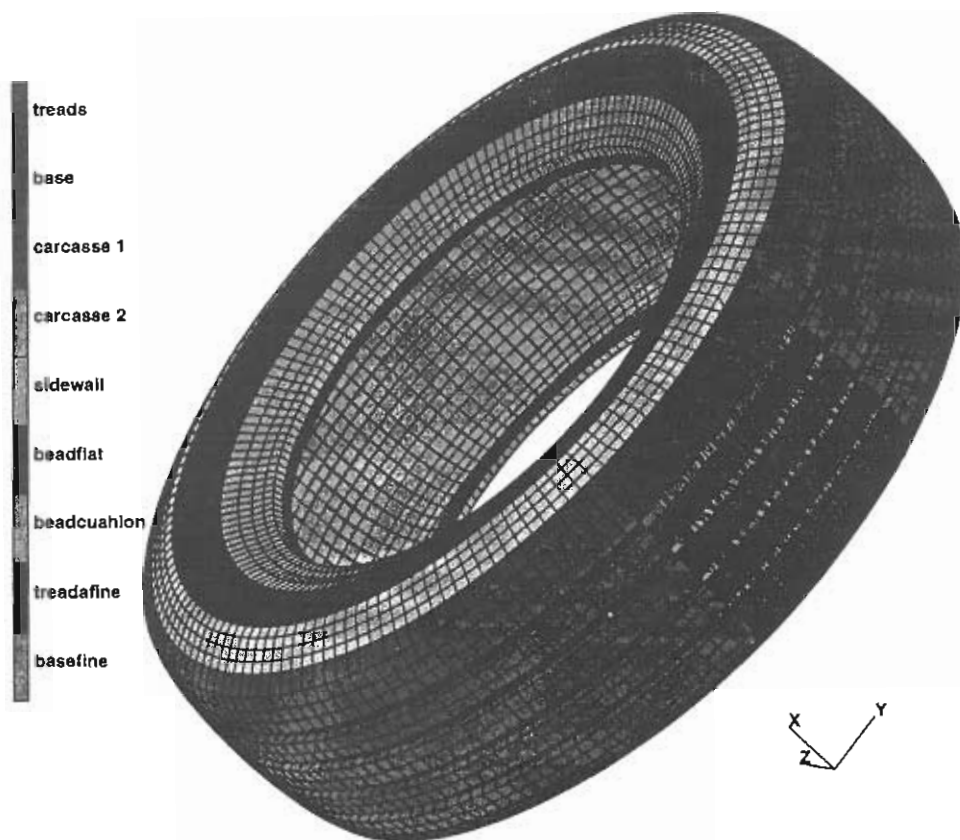


Figure 1: Discretization of the automobile tire

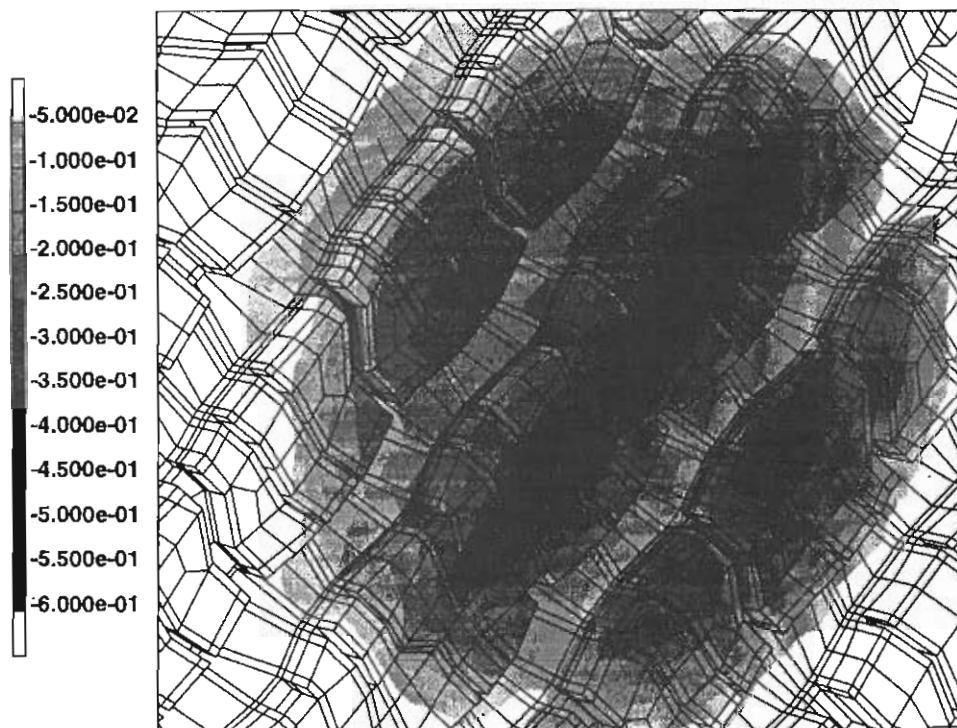


Figure 2: Pressure distribution (N/mm^2) at a contact displacement of 6.9 mm

Figure 2 shows the distribution of the contact pressure at a contact displacement of 6.9 mm. The solution of the resulting systems of linear algebraic equations was performed by means of the method of conjugate gradients. The incomplete Cholesky decomposition was employed for computing the preconditioning matrices. An average of 2925 seconds was necessary for one FE iteration step (matrix assembly: 220 seconds, solution step: 2570 seconds, stress recovery: 135 seconds).

293 FE iteration steps had to be performed in order to reach the situation illustrated in Figure 2. In this simulation the direct solver provided by MARC could actually not be used because of size limitations set by the employed file system. The size of the disc space required for the out-of-core storage of the stiffness matrix exceeded the limit of two Gigabyte.

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An Approach for Parallel Computing in the Explicit Dynamic Analysis

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Abstract. Explicit dynamic algorithms are commonly used for numerical simulation of sheet stamping or impact studies because they reduce the memory requirements. They still require large CPU times and parallel processing offers the solution of overcoming this obstacle. The present paper briefly describes the parallel program Stampack used for the solution of industrial sheet stamping problems and its version Simpact, for impact analyses. The methodologies used for the parallelisation are described as well as an example for each case with timings that show the advantage of the proposed approach.

1 Introduction

The simulation of industrial sheet stamping problems or impact studies require the combination of many numerical methods in order to achieve an acceptable result. Industrial problems tend to be geometrically complex and the large deformation, large strain, material plasticity, friction and contact properties of the problem must be considered.

Explicit dynamic programs are more efficient, require less memory and contain simpler algorithms than their implicit counterparts. The disadvantage is that the analyses require a big number of time steps, due to the conditional stability of the methods. Therefore is to be expected the intent of distributing the calculus over various computers which are supposed to work simultaneously with smaller parts of the general input data file. This is actually the basic idea of parallel processing. The parallelisation of explicit dynamic software is obviously not new [1, 2] but the tools available for parallelisation have improved and the new facilities of computer networking are stimulating further and continuously developments.

The work described in this paper relates to the Stampack/Simpact software programs developed by the authors [3] which utilises explicit dynamic algorithms for the analysis of sheet stamping and impact problems. The developments were made to run on a parallel processing platform envisaged to be a heterogeneous network of computers including PC's, workstations and/or distributed memory machines. The methodologies employed in the parallelisation exploit the advantages of explicit dynamic techniques and split the finite element mesh into a number of domains based on the mesh nodes by taking into account weighting criteria upon the elements.

To obtain maximum performance from Stampack/Simpact various aspects of the code have been addressed: the computation of the critical time step (computed as a global minimum over all the elements), the triangular shell or BST elements (that utilise a surrounding patch of elements to compute the bending tensor in terms of displacement variables only) and the contact algorithms (which require searching for contact zones between distinct bodies).

The effectiveness of the parallel program Stampack is shown by means of an example from the most recent NUMISHEET conference [4]. The Simpact testing example is presented for an impact analysis, compared with the results given by [11]. Both examples show that almost linear speed-up can be achieved with an efficiency of about 80%.

2 Numerical Formulation

The dynamic equilibrium equations are written in terms of the principle of virtual work using an updated Lagrangian formulation. The discretized form of equilibrium is obtained in the standard manner [5]:

$$M\ddot{\mathbf{r}}^t + C\dot{\mathbf{r}}^t = \mathbf{R}^t - \mathbf{F}^t - \mathbf{F}_c^t \quad (1)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ the damping matrix, $\ddot{\mathbf{r}}$ the vector of nodal accelerations, $\dot{\mathbf{r}}$ the vector of nodal velocities, $\mathbf{R}$ the vector of external loading, $\mathbf{F}$ the vector of internal forces and $\mathbf{F}_c$ the vector of contact forces, all evaluated at time t . The displacement vector is also determined and denoted by $\mathbf{r}$.

An explicit dynamic numerical procedure integrates the equations of motion (1) from time step n to time step $n + 1$:

1. At time step n the vectors $\mathbf{r}^n$, $\dot{\mathbf{r}}^{n-1/2}$, $\mathbf{R}^n$, $\mathbf{F}^n$, $\mathbf{F}_c^n$ are known at the nodes and the strain and stress tensors, $\boldsymbol{\epsilon}^n$ and $\boldsymbol{\sigma}^n$ respectively, are known for each element.
2. Compute accelerations at all nodes:
 $\ddot{\mathbf{r}}^n = \mathbf{M}_D^{-1}[\mathbf{R} - \mathbf{F} - \mathbf{F}_c - \mathbf{C}\dot{\mathbf{r}}]^n$ where $\mathbf{M}_D = \text{diag } \mathbf{M}$ and $\mathbf{C} = 2\alpha\mathbf{M}$
3. Compute velocities and displacements at all nodes:
 $\dot{\mathbf{r}}^{n+1/2} = \dot{\mathbf{r}}^{n-1/2} + \ddot{\mathbf{r}}^n \Delta t^{n+1/2}$, $\mathbf{r}^{n+1} = \mathbf{r}^n + \dot{\mathbf{r}}^{n+1/2} \Delta t^{n+1}$
where $\Delta t^{n+1/2} = \frac{1}{2}[\Delta t^n + \Delta t^{n+1}]$
4. Compute strains and stresses for all elements: $\boldsymbol{\epsilon}^{n+1}$, $\boldsymbol{\sigma}^{n+1}$
5. Compute internal force vector at all nodes of each element and assemble:
 $\mathbf{F}^{n+1} = \int_V \mathbf{B}^T \boldsymbol{\sigma}^{n+1} dV$
6. Check for frictional contact conditions and compute contact force vector $\mathbf{F}_c$
7. Compute external force vector $\mathbf{R}^{n+1}$
8. GOTO step 1 and repeat the process for the next time step

Conditional stability of the explicit numerical time integration requires that the time step size must not exceed a critical value related to the lowest period of vibration of the elements in the mesh. A special subroutine of the code is evaluating the value of the critical time step. This routine can be called upon the user request.

3 Parallel Processing

Detailed profiling of the sequential explicit dynamic software [6] on industrial size problems revealed that approximately 90-95% of the CPU time was used in the calculation of the internal nodal force vector $\mathbf{F}$; element strains, stresses and internal forces are computed here.

The advantages of parallel processing for explicit dynamic programs are thus obvious : no system of equations needs to be solved and hence almost all the CPU time is spent at the element level. This means that good speed-ups can be attained by decomposing the mesh into domains and carrying out the computations in parallel.

3.1 The Stampack/Simpact System

The basic philosophy of the parallel system is that each processor carries out exactly the same function but on a different domain (there are no master-slave relationships). This is effectively a SPMD (Single Program Multiple Data) structure. To do this effectively and with as much transparency to the user as possible it is necessary to carry out the domain decomposition and analysis steps as automatically as possible.

The use of standard coding and PVM communication tools ensures that Stampack/Simpact can run on a heterogeneous network of computers that may include PC's, workstations and/or shared memory machines.

3.2 Domain Decomposition

Domain decomposition methods have proved to be effective for the solution of explicit dynamic applications since they exploit the coarse grained parallelism and are thus suitable for Distributed Memory Multiprocessors (DMM). Effective domain decomposition ensures load balancing and the minimisation of communications.

The finite element domain is decomposed within Stampack/Simpact in terms of nodes that are assigned to the various processors [7]. Various domain decomposition methods are utilised: strip partitioning, recursive bisection and multilevel partitioning. However, for most of the industrial sheet stamping applications considered a 1.5D strip partitioning method seems to be most effective.

This method consists of 2 steps: in the first step the domain is divided into strips with perfect load balancing. In the second step each strip is divided into boxes using an orthogonal direction to the first one.

3.3 Parallel Calculus

With the domain decomposed in this nodal manner all element computations may be carried out independently on each processor. At the end of these computations each processor contains contributions from all of its nodes to the global vectors. For the internal nodes these vectors are complete and the solution may be readily obtained. However, for the domain interface nodes an exchange of information is necessary so that each processor may assemble the nodal information for the interface nodes within its domain. This is a data communication point when all processors communicate (send and receive) data relating to their external and local interface nodes. Now each processor contains all the necessary information for all the nodes in its domain to carry out the force/mass vector divide and compute the acceleration at each node (see step 2 in section 2).

3.4 Observations

All parts of a program need to be addressed in order to effectively parallelise any software (see Amdahl's law [8]). Aspects that contain an interdependency between processors result in processor waiting, bad synchronisation and reduced performance. The aspects falling into this category that have been identified and addressed in Stampack/Simpact are:

- (1) The computation of the critical time step that involves the computation of a global minimum over all the elements.
- (2) BST elements have been developed that include membrane and bending deformation effects defined only in terms of displacement degrees of freedom [9]; each element requires information from the surrounding patch of elements.
- (3) The computation of the contact force vector F_c involves contact search algorithms to locate the zones of contact between distinct bodies. In sheet stamping many surfaces come into contact so this is a very critical part of the Stampack/Simpact code.

4 Numerical Examples

4.1 Simpact: Impact between two cylindrical bodies

This example analyses the deformation of one cylindrical tube impacting another at right angles.

The initial radius of the cylinders is $R = 0.1\text{ m}$, the initial length is $L = 0.46\text{ m}$ and the thickness is $h = 3\text{ mm}$.

Each cylinder was modelled with 2560 3-node triangle shell elements (a total of 5120 triangles and 2624 nodes).

A velocity of 30 m/s , remaining constant with the time, was applied to all the nodes of both cylinders directed towards the opposite cylinder.

The analysis uses 1848 timesteps with an approximate average time increment of 4.33 nanoseconds for a whole analysis time of 8 milliseconds . The timestepping is automatically calculated by the program. The contact is activated throughout the analysis, with 12 levels of extended searching using a nonsymmetric treatment, which means that penetration is only checked for the nodes of the slave surfaces against the master ones.

The initial meshes and the deformed ones can be seen in Figure 1 and Figure 2 respectively.

Results agree with the ones published in the technical literature (see [11])

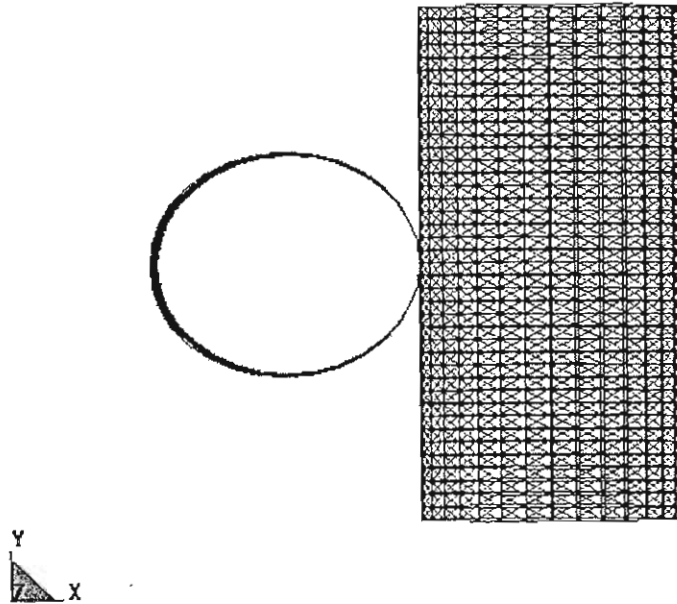


Fig. 1. Initial meshes for the tubes

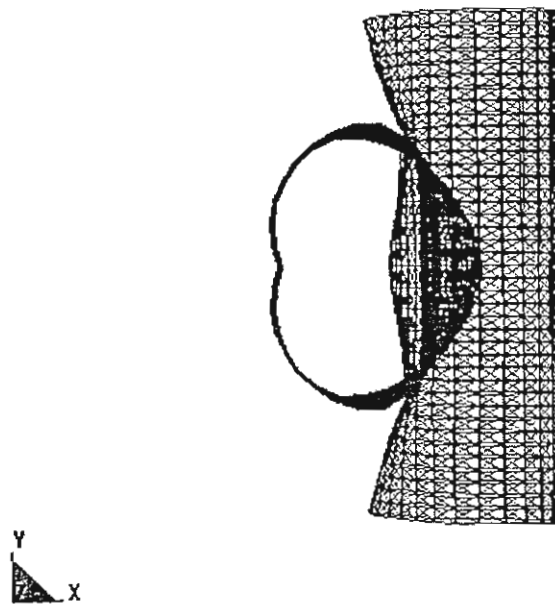


Fig. 2. Deformed tubes after impact.

4.2 Stampack: Deep Drawing of a S-rail

The Stampack program has been utilised to analyse the deep drawing of a steel S-rail defined in the NUMISHEET'96 conference [4]. This problem simulates the stamping and elastic springback of the rail (wrinkling and springback cause many manufacturing problems). The complete mesh for the S-rail stamping in Figure 3 shows, from top to bottom, the die, the sheet (12,000 elements using Hill's plasticity model [10]), blankholders and punch (note that the stamping direction is upwards); the total mesh contains 14,654 elements with 27,879 degrees-of-freedom.

The final stamped results agree very favourably with those presented in NUMISHEET'96 (see [4]): the front view of the deformed shape after springback is shown in Figure 4 where the buckling

of the surfaces is clearly seen (a comparison with experimental results indicated the good agreement that was obtained).

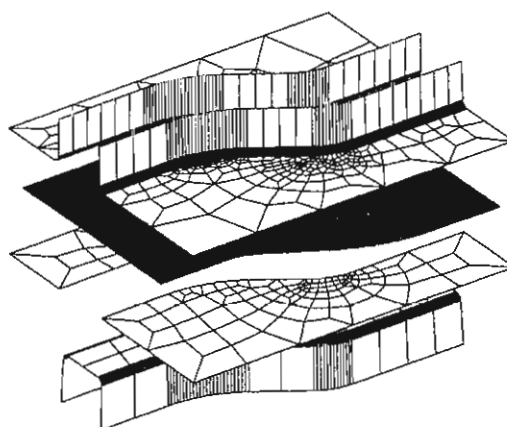


Fig. 3. S-rail initial set-up from top to bottom: dies, sheet, blankholders and punch.

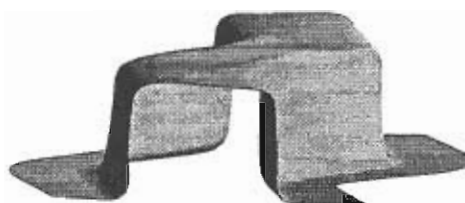


Fig. 4. Deformed shape of S-rail after springback.

4.3 Performances

The stamping and springback of the S-rail was run using Stampack on a network containing various numbers of SG workstations and using the 1.5D strip domain decomposition method. With Simpack was made the impact analysis. The speed-up results are shown in Figure 5 where it can be seen that superior speed-ups were achieved when the contact communications were overlapped with the element computations: the speed-up for less than 12 processors is almost linear with about 83% efficiency (with more processors this reduces to about 76% since communications start to affect the overall performance). This deviation point would, in general, be different for different problems. When no overlapping was used the speed-up reduced to about 62% due again to the communications affecting the performance of the system in general. The speed-up curve is smoother however with no obvious deviation point.

5 Conclusions

Stampack/Simpack, parallel softwares for sheet stamping and impact analyses have been presented. The accuracy and effectiveness of the software have been validated and the efficiency when running on a parallel platform has been proved. These programs, running on parallel platforms give speed ups of about 80% over the sequential versions of the software, which is approximately the theoretical speed-up limit for programs of this nature.

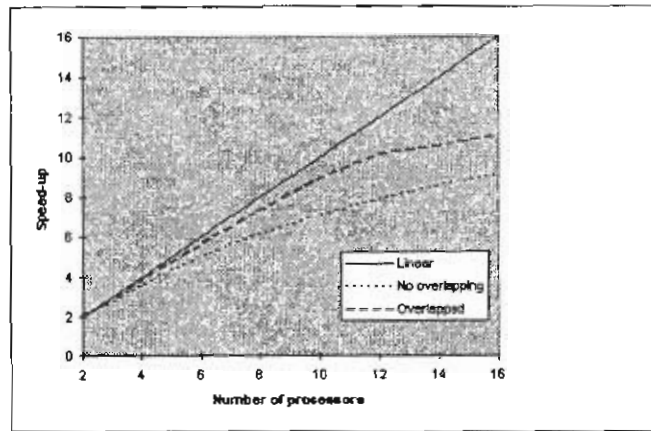


Fig. 5. Medium speed-up

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SESSION PAPERS

ASSESSMENT AND REPAIR EVALUATION OF MASONRY STRUCTURES BY THE DISCRETE ELEMENT METHOD

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ABSTRACT

Masonry, in its various forms, represents one of the oldest methods of engineering construction. Yet predictive modelling of the behaviour of such structures, particularly in the non-linear range, remains a challenge, due predominantly to their semi-discrete and composite nature. This paper presents an approach to computational modelling of masonry structures by considering the composite action of individual components. Emphasis is made on repair strategy by means of anchor stitching thus introducing a computational evaluation method by which the efficiency of such remedial actions can be assessed.

1 INTRODUCTION

An adequate computational model for the treatment of masonry in the non-linear range must include the fundamental mechanisms that characterise the composite action:

- sliding along a bed or head joint at low values of normal stress,
- cracking of the masonry units (bricks, blocks, etc.) in direct tension,
- diagonal tensile cracking of masonry units at values of normal stress sufficient to develop frictional behaviour in the joints, and
- splitting of units in tension as a result of mortar dilatancy at high values of normal stress.

Further aspects which may require consideration include the treatment of reinforcement and/or prestressing in composite construction techniques or repaired structures. A summary of the most recent work accounting for both computational modelling and comparison with experimental studies can be found in [3,6,7].

The loading which may cause the failure of masonry structures can be either dynamic or static in nature. The most common causes of structural cracking under quasi-static conditions are foundation settlement and restrained shrinkage due to thermal or hygroscopic effects. The dynamic loading leading to structural failure is usually related to seismic action, impact loading or explosive blast effects.

To date, most computational predictions have been based on a macro-scale modelling approach in which attempts are made to incorporate some or all of the phenomena described above within a continuum description; employing homogenisation concepts to produce a smeared representation of the brick/joint action and classical plasticity concepts to model the tensile/compressive failure of the resulting composite. Such a modelling strategy has been dictated both by the limits of available computing power and the lack of maturity of other discrete computational techniques.

Earlier work on discrete element techniques was based on the assumption that each element was rigid, but later extension to include local deformation has permitted a more rigorous treatment of both the contact conditions and fracture requirements.

The incorporation of deformation kinematics into the discrete element formulation has also lead naturally to a combined finite/discrete element approach in which the problem is analysed by a combination of the two methods [1,4,5].

In a finite/discrete element analysis the main issues which require consideration, for both dynamic and quasi-static behaviour, are :

- appropriate modelling of the continuum with a view to incorporating the relevant deformation mechanisms for solid material that include treatment of the progressive fracturing of solids,
- detection procedures for monitoring contact between discrete elements, and
- representation of frictional/adhesive contact conditions for contacting elements.

The substantial computational times associated with large scale discrete element problems make the use of parallel processing techniques essential. The degree of natural algorithmic concurrency in the solution of transient dynamic problems by central difference explicit time integration is high and relatively simple parallel processing strategies can be employed which yield high efficiencies and speedups on both shared memory and distributed memory hardware configurations. When a large degree of inter-element contact occurs, as is the case with discrete elements, parallelisation becomes more difficult, with inter-processor communication and data storage requirements continually evolving as contact conditions change. For distributed memory machines this poses considerable difficulties, but the problem becomes far more tractable within a shared memory environment. The solution of quasi-static problems through use of central difference explicit time integration and dynamic relaxation procedures on parallel platforms follows an identical procedure.

2 METHODOLOGY

Whilst considerable fundamental information on the behaviour of masonry structures can be derived from solutions based on a continuum approach, a more natural treatment of this class of problem is offered by use of *discrete element* methods. The use of discrete elements originated in geotechnical and granular flow applications and are based on the concept that individual material elements are considered to be separate and are (possibly) connected only along their boundaries by frictional/adhesive contact. With present day computational power large scale discrete element models can be considered and for industrial applications in the field of rock blasting, etc. 10-50,000 elements are routinely employed.

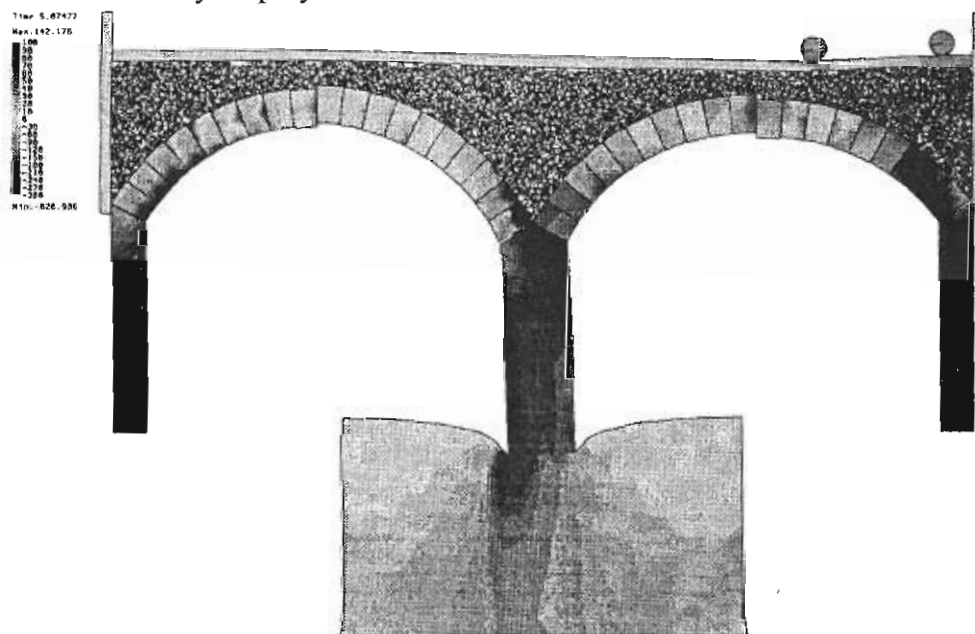


Figure 1: Discrete element modelling of masonry arch failure

The power of the method, and its potential applicability to masonry problems, is illustrated in *Figure 1* where the idealised (and hypothetical) problem of the failure of a two span bridge is illustrated as a feasibility study. The problem is treated as a combined finite discrete/element problem with the masonry blocks being represented by deformable discrete elements in frictional contact, the fill material by spherical discrete elements and the foundation medium beneath the central pier modelled as a Mohr-Coulomb material. The failure mode brought about by settlement of the central pier and the traversing of an abnormal two point load is indicated. Typical data provided by such solutions include; stress and displacement distributions throughout the model (masonry units and fill material), the relative movement and forces transmitted at each individual joint and the progressive cracking of the masonry units.

3 COMPUTATIONAL FRAMEWORK

The discrete element method (DEM) is a family of related numerical techniques designed to solve those problems in applied mechanics that exhibit large discontinuous geometrical and material behaviour. To simulate the complete behaviour of systems of discrete interacting bodies, the applied numerical technique should consider three important aspects:

- the behaviour of solid material,
- the detection of contacts, and
- the representation of contact interaction.

3.1 BEHAVIOUR OF SOLID MATERIAL

The discrete elements themselves represent distinct solids - deformable bodies whose deformations can be determined using the classic concepts of continuum mechanics. The numerical method which is applied to solve problems in solid mechanics should satisfy the postulate of conservation of mass which requires that

$$\dot{\rho} + \rho \operatorname{div}[\dot{\mathbf{u}}] = 0 \quad (1)$$

where ρ is the current material density and $\mathbf{u}$ is the displacement vector field. In other words, the deformation is followed by a change in the material density which should be taken into account in order to preserve the initial total mass of the body under consideration. Furthermore, the momentum balance must be satisfied at each material point $\mathbf{p}$ of the deformed body $\mathcal{B}$ which is defined by its boundary $\partial\Omega$. This can be expressed as follows

$$\left. \begin{array}{l} \operatorname{div}[\boldsymbol{\sigma}] + \rho \mathbf{b} = \rho \ddot{\mathbf{u}} \\ \boldsymbol{\sigma}^T = \boldsymbol{\sigma} \\ \boldsymbol{\sigma} \mathbf{n} = \boldsymbol{\tau} \end{array} \right\} \begin{array}{l} \forall \mathbf{p} \in (\Omega) \\ \\ \forall \mathbf{p} \in (\partial\Omega) \end{array} \quad (2)$$

where $\boldsymbol{\sigma}$ is the stress tensor, ρ is the material density, $\mathbf{b}$ is the body force vector field, $\mathbf{u}$ is the displacement vector field, $\mathbf{n}$ is the outward vector normal to the deformed boundary $\partial\Omega$ and $\boldsymbol{\tau}$ is the boundary traction vector field, thereby representing the dynamic equilibrium of the differential volume. This strong (differential) formulation of the momentum balance equation is derived from the translational and rotational equilibrium equations of a general body. The equations represent local (pointwise) spatial equilibrium and the symmetry of the Cauchy stress tensor. The third part of the above momentum balance equation follows from the definition of the Cauchy stress tensor and represents the local equilibrium on the boundary $\partial\Omega$. The momentum balance equation represents the governing equation for transient dynamic problems in solid mechanics.

The weak (variational) formulation of the momentum balance equation (2) describes the equilibrium of the body $\mathcal{B}$ in terms of the virtual work equation

$$\delta W = \int_{\Phi_i(\Omega)} \rho \ddot{\mathbf{u}} \cdot \delta \mathbf{v} \, d\mathbf{v} + \int_{\Phi_i(\Omega)} \boldsymbol{\sigma} : \delta \mathbf{d} \, d\mathbf{v} - \int_{\Phi_i(\partial\Omega)} \boldsymbol{\tau} \cdot \delta \mathbf{v} \, d\mathbf{a} - \int_{\Phi_i(\Omega)} \rho \mathbf{b} \cdot \delta \mathbf{v} \, d\mathbf{v} = 0 \quad (3)$$

which is approximated by applying the finite element discretisation as follows

$$\begin{aligned} \delta W = & \sum_{e=1}^{NELEM} \int_{\Phi_i(\Omega_e)} \left(\rho \left(\sum_{I=1}^{NNODE} N_I \ddot{\mathbf{u}}_I \right) \right) \left(\sum_{I=1}^{NNODE} N_I \delta \mathbf{v}_I \right) d\mathbf{v} \\ & + \sum_{e=1}^{NELEM} \int_{\Phi_i(\Omega_e)} \boldsymbol{\sigma} : \left(\sum_{I=1}^{NNODE} \delta \mathbf{v}_I \otimes \nabla_x N_I \right) d\mathbf{v} \\ & - \sum_{e=1}^{NELEM} \int_{\Phi_i(\partial\Omega_e)} \boldsymbol{\tau} \cdot \left(\sum_{I=1}^{NNODE} N_I \delta \mathbf{v}_I \right) d\mathbf{a} \\ & - \sum_{e=1}^{NELEM} \int_{\Phi_i(\Omega_e)} \rho \mathbf{b} \cdot \left(\sum_{I=1}^{NNODE} N_I \delta \mathbf{v}_I \right) d\mathbf{v} \end{aligned} \quad (4)$$

The finite element semi-discretisation of the momentum balance partial differential equation results in a system of ordinary differential equations in time. The full discretisation can be obtained by applying an appropriate integration method with respect to time. In general, most multibody contact problems fall into the category of wave propagation problems. Normally, in order to solve numerically this type of problems explicit time integration schemes are employed which decouple the system of equations

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{f}^{int}(\mathbf{u}(t)) = \mathbf{f}^{ext}(t) \quad (5)$$

into a set of independent equations for each discretised degree of freedom which can be written as follows

$$m_{ii}\ddot{u}_i(t) + c_{ii}\dot{u}_i(t) + f_i^{int}(u_i(t)) = f_i^{ext}(t) \quad (6)$$

for each degree of freedom i .

Generalised elasto-plastic or elasto-visco-plastic material models, combined with several different failure models, are applied in order to numerically simulate the fracturing phenomenon [4]. The solid media fracturing technique is based on a strain softening formulation which utilises standard fracture mechanics concepts of energy release required for crack opening. Alternative techniques are being currently considered in order to control the evolution of the failure surface via tensile and/or shear fracture energy densities.

3.2 DETECTION OF CONTACTS

The solution of the equilibrium equations requires the definition of boundary conditions on the surface of a body under consideration. For that purpose the surface $\partial\Omega$ of the body $\mathcal{B}$ is divided into two regions:

- surface region with defined displacements $\partial\Omega_u$, and
- surface region with defined surface tractions $\partial\Omega_\tau$.

In order to determine the surface tractions arising from contacts between interacting bodies, it is necessary to detect these contacts beforehand. The problem of detecting the bodies that interact with each other, known as the *geometric intersection search*, becomes a serious computational task in multibody analysis. A naïve method would require the checking for contact between each body from a given set and each and every other body listed after the body under consideration. The number of operations required to detect all contacts between N bodies would then be

$$O = N \frac{N-1}{2} \quad (7)$$

In this work the *alternating digital tree* (ADT) algorithm [2] is used to reduce the number of operations required to determine the contacts between bodies under consideration. This algorithm, developed initially to solve the problem of mesh generation using the *advancing front technique*, has been adapted for the creation of short lists of potential *contactors* for each *target* body thus reducing the number of operations required to detect all contacts between N bodies to

$$O = N \log_2(N) \quad (8)$$

3.3 REPRESENTATION OF CONTACT INTERACTION

The contact interaction of solid bodies generates surface tractions at the interface, thus causing the deformation and subsequent concentration of stresses in the affected boundary regions. In the central difference explicit time stepping scheme, which is applied to solve the equations of motion, a contact between two discrete elements is detected when these elements are found to be partially overlapping. One method of evaluating the induced contact tractions would be to adopt the physical assumption that there is no penetration between the two bodies in contact. On the other hand, if the chosen method is expected to model successfully wave propagation through discontinuous media, the contact stiffness should be taken into account in order to obtain the correct stress wave propagation speed. In this work a *penalty* method based on the deformation of contact surface regions is devised to determine the contact tractions in question. The overlap - which is physically inadmissible - is assumed to represent the deformation of the contacting domains. Furthermore, it is assumed that the size of this overlap is sufficiently small for *small deformation* theory to be applicable when describing the deformation of the contacting regions. The surface tractions may be calculated to a good approximation by considering each contacting body to be a semi-infinite deformable solid. The surface tractions can then be determined by solving the equilibrium and compatibility equations at the interface.

In the penalty method, a penetration function defined as follows

$$u_n = \begin{cases} (u_1 - u_2) \cdot n & \text{for } (u_1 - u_2) \cdot n > 0 \\ 0 & \text{for } (u_1 - u_2) \cdot n \leq 0 \end{cases} \quad (9)$$

is applied in order to define which parts of the surfaces are in contact as well as to determine the intensity of tractions arising from contact. Two contacting bodies in motion can also experience some relative tangential displacement which can be expressed as follows

$$u_t = (u_1 - u_2) \cdot t \quad (10)$$

where t is the vector in the plane previously defined by its normal n , while u_1 and u_2 represent the displacements of the points P_1 and P_2 respectively (*Figure 2*).

The decomposition of the total relative displacement between the two points on interacting boundaries determines a convenient decomposition of the total contact traction τ^c as follows

$$\tau^c = \tau^c \cdot n + \tau^c \cdot t = \tau_n^c(u_n) + \tau_t^c(u_t) \quad (11)$$

In view of the finite element method which is applied to describe the behaviour of the solid material, the resultant force transmitted from one surface to another through the area of contact is obtained by integrating the contact tractions as follows

$$P(u) = \sum_e P_{(e)}(u) = \sum_e \left(\int_{\Gamma_{(e)}\tau^c} N_I \tau_i^c d\Gamma_{(e)\tau^c} \right) \quad (12)$$

where N_I represents the element shape functions and $\Gamma_{(e)\tau^c}$ determines the part of the contact surface associated with element e .

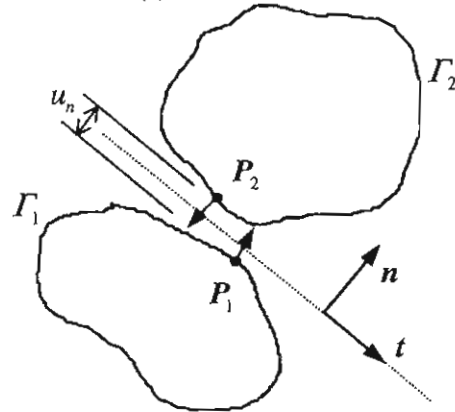


Figure 2: Contact between two bodies

4 REPAIR EVALUATION METHOD FOR MASONRY STRUCTURES

Masonry structures which have been damaged can be repaired by the introduction of stitching anchors. An appropriate evaluation method for this type of strengthening should involve a preliminary analysis by means of finite element method, which would allow the modelling of progressive failure of the bond between the anchor and the injected grout, as well as failure of surrounding masonry components. This paper describes the results of such an analysis performed recently to provide the parameters for discrete element modelling of damage and repair of Strathmashie masonry arch bridge in Scotland.

The actual insitu anchor pullout tests were modelled using an axisymmetric finite/discrete element representation as illustrated in Figure 3. Material properties of the steel anchor were specified as elasto-plastic (Von Mises, linear hardening): Young's modulus $E=2.1 \times 10^5$ MPa, Poisson's ratio $\nu=0.2$, Yield stress $\sigma_{y0}=235$ MPa, Hardening modulus $H=2.1 \times 10^4$ MPa. The grout and masonry were modelled as a Rankine material with the following properties: Young's modulus $E=2.8 \times 10^4$ MPa, Poisson's ratio $\nu=0.17$, Yield stress $\sigma_{y0}=1.35$ MPa, fracture energy $G_f=0.08$ Nm.

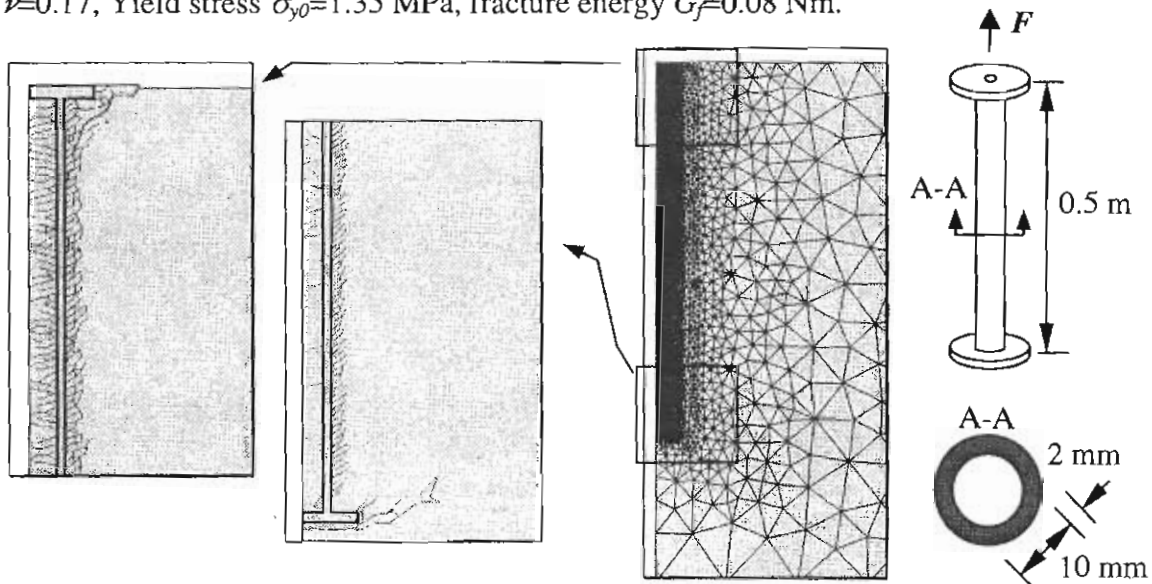


Figure 3: Finite/discrete element model for anchor pullout test and failure patterns

The results of the experimental insitu anchor pullout tests were stored in the form of force-displacement curves which have been closely matched by the results obtained from numerical analysis (Figure 4).

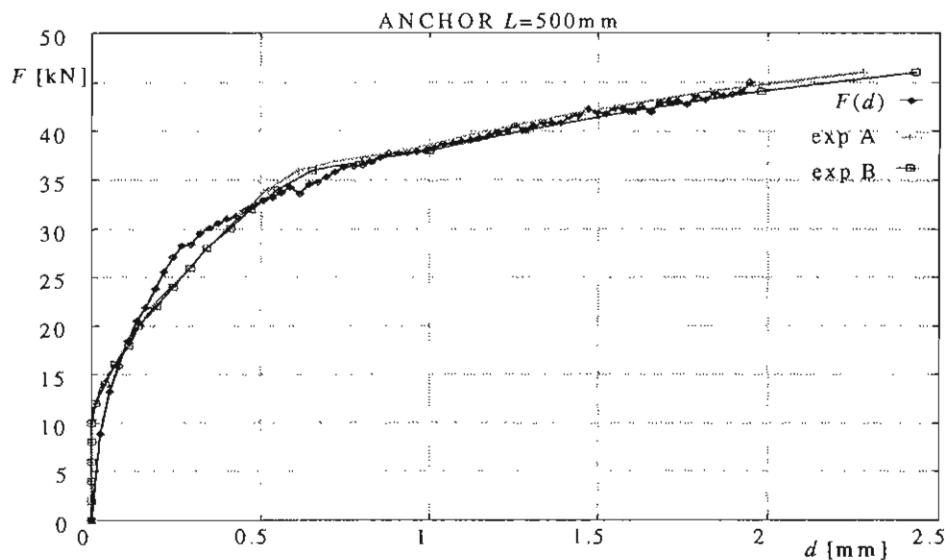


Figure 4: Comparison of experimental and numerical results for anchor pullout test

Preliminary numerical simulation of the failure of the Strathmashie masonry arch bridge has provided the deformation patterns illustrated in Figure 5. The inclusion of a discrete element based anchor stitching system (currently under development) will provide the ability to predict the potential of such techniques for the repair of masonry structures.

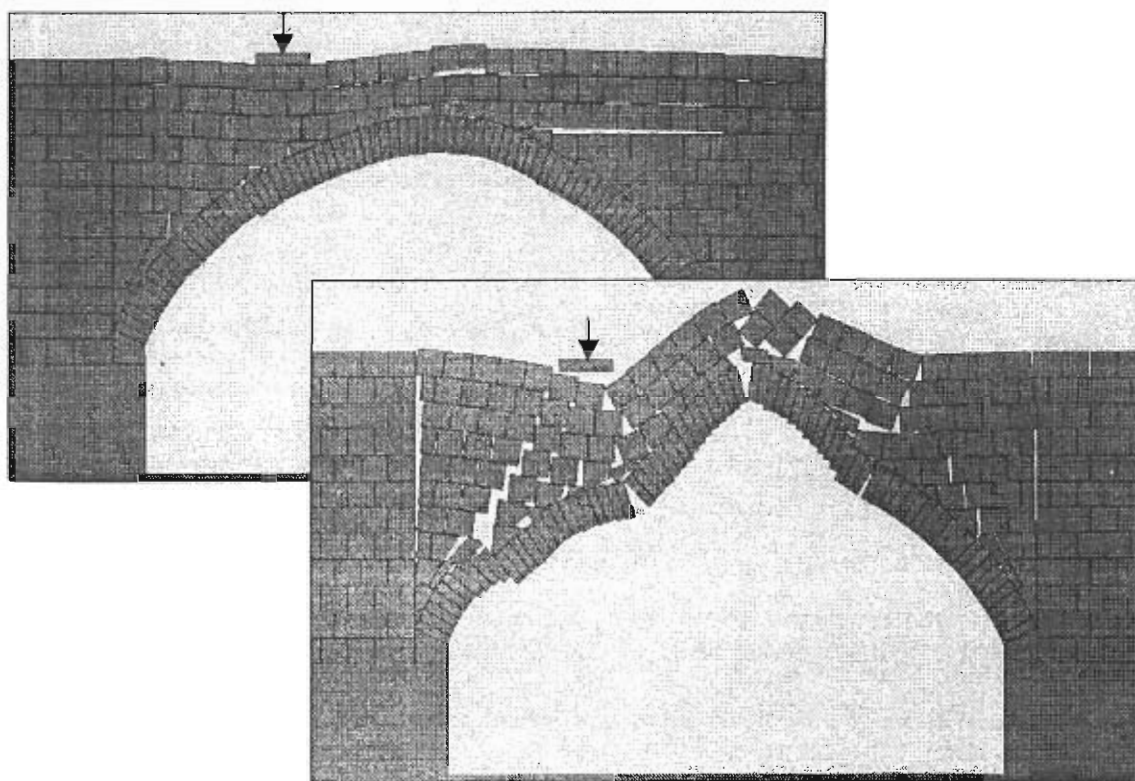


Figure 5: Deformation pattern of Strathmashie masonry arch bridge

The finite/discrete element modelling of the pullout test illustrated above provided the necessary input data to describe the non-linear response of individual stitching anchors for large scale simulation.

5 CONCLUSIONS

The combined finite/discrete element method developed offers an alternative approach to predictive modelling of the failure (and repair) of masonry structures. The combination of efficient automatic contact detection and interaction techniques with well tested finite element techniques for the simulation of material behaviour provides an approach to modelling which considers the composite action of individual masonry components. The same technology can be successfully extended to repair evaluation with respect to anchoring reinforcement systems.

ACKNOWLEDGMENTS

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A Discontinuity Alignment Meshing Strategy For Localised Failure Analysis of Concrete

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Abstract

Localised failure predictions for concrete and other quasi brittle materials have often been found to be influenced by the adopted discretisation space, both in terms of the finite element mesh, as well as the finite element type. A simple mesh alignment strategy is presented, whereby a smooth internal mesh boundary is continuously traced to reflect the development of a potential discontinuity band. The construction and the orientation of the internal band is found by tracing a ridge of the scalar field corresponding to the magnitude of incremental total strain. Procedure is illustrated on a benchmark pull-out model problem, indicating that the mesh alignment can improve the control of the discretisation effects with respect to the peak load as well as the post peak response.

INTRODUCTION

Various studies indicate that the localised failure predictions of concrete can be influenced by the adopted finite element mesh or finite element type chosen for the analysis, especially when lower order elements are employed. Just as an 'unbiased' finite element mesh may not be capable of resolving the localised failure mode, or it may have the effect of smearing the localisation over a number of elements so resulting in an apparently more diffuse failure, it is also possible that such an arbitrary mesh may artificially influence the numerically predicted failure mode by providing a kinematically more preferable solution.

Mesh densification and alignment are frequently used for improving the ability of a finite element mesh to realise the correct failure mode. The increased resolution provided by densification only does not necessarily reduce the diffuse nature of a localisation zone with respect to the number of elements across which it is spread. Correct alignment of the finite elements, however, along a discontinuity direction can allow a proper localisation within a finite band of elements, for simple, uniform states of stress with an imperfection. Such proper alignment of the finite element mesh is generally not possible in advance and therefore it is necessary to improve the mesh successively as the solution progresses. Adaptive meshing strategies, whereby each new mesh is controlled by the solution process, have been the subject of extensive research in various fields over recent years.

Experience of the authors indicate that sometimes alignment of the mesh via an oriented stretching of the finite elements can be a very powerful approach to the problem of capturing localised failure, see also [1]. However, the method appears to be less amenable for general cases, and can be dependent on both the specific problem and the mesh generation tools available.

Here, an alternative strategy for the alignment of the mesh will be explored, whereby smooth discontinuity lines will be constructed to form internal boundaries, along which the finite element mesh can be aligned. The process by which these discontinuity lines may be constructed can vary. Some researchers, [2, 3], have utilised the acoustic tensor analysis to predict the direction of possible discontinuity and therefore construct internal 'slip lines'. A more simple approach is explored here, whereby the discontinuity band is found by tracing the ridge of an appropriate scalar control field.

THE RANKINE MODEL

All presented analyses utilise the well known Rankine tension cut-off yield criterion in a generalised three dimensional form which can be expressed as

$$\Phi_i(\sigma, \kappa) = \sigma_i - hf_t = 0 \quad i = 1 \text{ to } 3 \quad (1)$$

where σ_i are the three principal stresses σ_1, σ_2 and σ_3 (it is assumed that $\sigma_1 \geq \sigma_2 \geq \sigma_3$) and f_t the uniaxial tensile strength. Hardening/softening is introduced via the parameter $h = h(\kappa)$. This model and algorithmic issues related to it have recently been reported in [4] and will only be outlined here.

An exponential softening law is adopted and to regularise the inherent problem of mesh sensitivity with respect to the energy release, the fracture toughness G_f is introduced as a material parameter and the softening slope becomes a function of the mesh size δ . Thus h varies with the internal variable κ (here taken as the effective plastic strain) as

$$h = e^{-\frac{\delta f_t}{G_f} \kappa}$$

The key feature of the algorithmic implementation [4] is the analytical solution to the plastic corrector problem in its backward Euler form which was possible as a result of assumed material isotropy, where the principal stress directions at the elastically predicted trial stress state are preserved throughout the stress return mapping. Moreover, the implicit form of the integration of the constitutive equations of elastoplasticity allow the computation of a consistent tangent operator.

RIDGE TRACING

Several suitable scalar control fields have been explored for ridge tracing - the effective plastic strain, the current tensile strength and incremental total strain. In order to determine the extent and orientation of a localised failure zone, the 'ridge' of a suitable scalar control field ζ is topologically traced. Starting from the node corresponding to the maximum ζ , the local minimum curvature of the ridge is numerically obtained. From a 'patch' of the surrounding nodes a quadratic surface of the scalar field is constructed via a least squares fit; the principal curvatures of this surface are consequently determined and the eigenvector $\mathbf{n}'$ corresponding to the minimum principal curvature is taken as the local direction of the ridge.

The next step is to search along this direction $\mathbf{n}'$ for the nearest node p_i within the patch which is in some sense closest to the search direction. Here the new node along the ridge corresponds to the maximum of

$$e_i = \frac{1}{(\mathbf{t}_i \cdot \mathbf{n})(\mathbf{t}_i \cdot \mathbf{n}')}$$

where $\mathbf{t}_i$ is the direction vector of the new node p_i with respect to the previous node p_{i-1} and $\mathbf{n}$ is the unit eigenvector corresponding to the maximum principal curvature at node p_{i-1} , locally orthogonal to $\mathbf{n}'$, see Fig 1.

Once the next node is located the process is repeated until no new node can be identified. The resulting set of nodes which comprise the ridge are then used to construct a smoothed discontinuity surface; within this work the discontinuity line was restricted to a double cubic spline fit. A parallel line was also constructed in order to produce an internal discontinuity band of a given thickness, see Fig 2.

In the analysis of concrete structures, often multiple crack paths may develop, but it is specifically the propagation of the dominant fracture to which attention must be paid. It is for this reason that in the following the scalar field ζ is taken as the intensity of the incremental strain tensor i.e. $\zeta = \Delta \epsilon : \Delta \epsilon$.

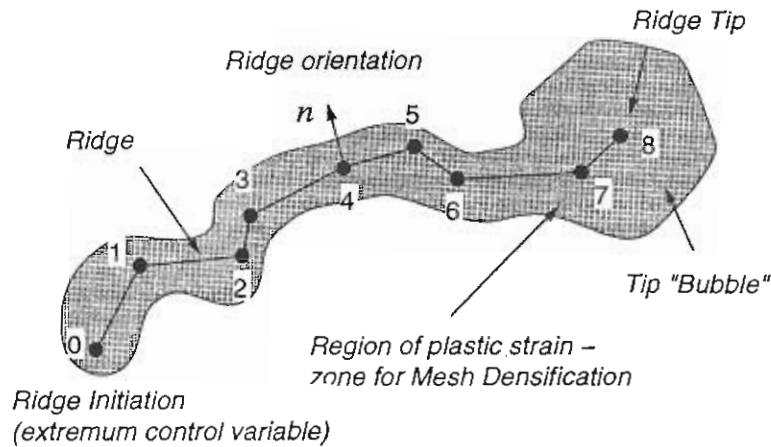


Figure 3. Numerically determined ridge and associated zone of mesh densification.

ILLUSTRATIVE EXAMPLE - RILEM ROUND ROBIN ANCHOR PULL-OUT

This example broadly follows the problem described by Feenstra [6] of the pull-out of an anchor from mass concrete, Fig 4(a), (some corresponding experimental results have been reported by Vervuurt et al. [7]). A plain strain analysis under displacement control has been undertaken as opposed to the plane stress, load control analysis of Feenstra. The initial mesh and the geometry are shown in Fig 4(a) and it should be noted that the anchor itself has been omitted and controlled displacement applied along the upper edge of the notch. The meshes are composed of three noded, constant strain, plane strain elements.

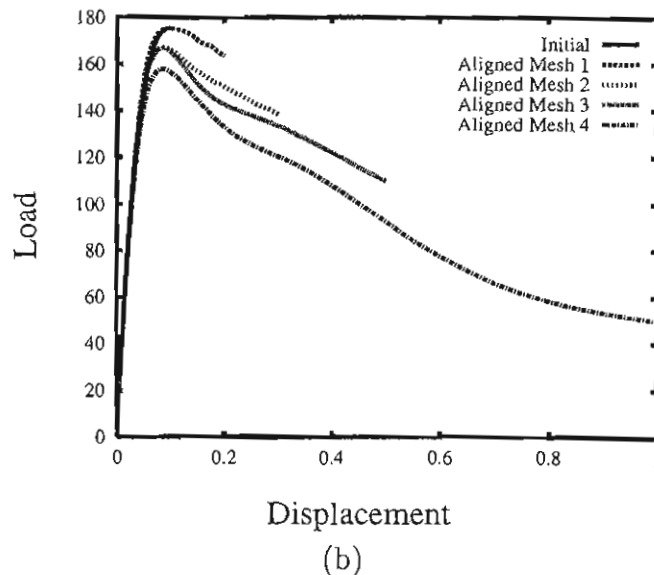
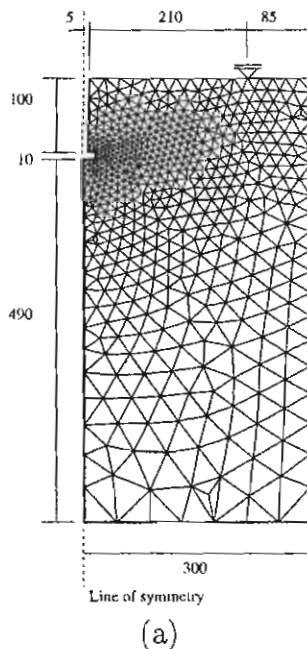


Figure 4. Anchor Pull-Out. (a) Initial mesh and geometry (dimensions in mm). (b) Load-displacement curves for various aligned meshes. Material properties: $E = 37000 \text{ N/mm}^2$, $\nu = 0.3$, $f_t = 2.5 \text{ N/mm}^2$, $G_f = 0.09 \text{ N/mm}$.

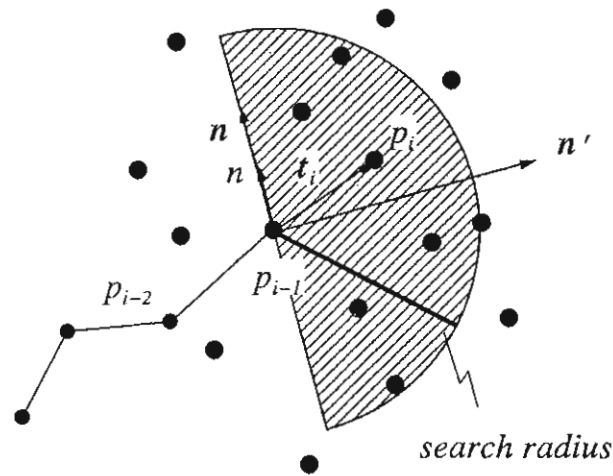


Figure 1. Ridge Tracing. Searching for the next node which lies within the search radius along the direction of the minimum principal curvature.

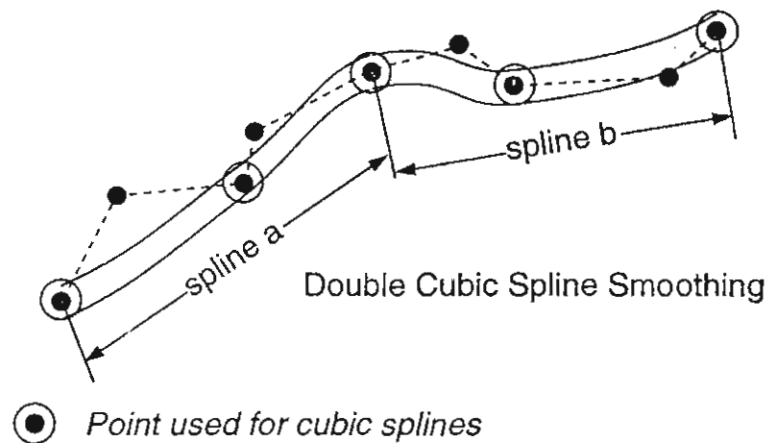


Figure 2. Discontinuity band constructed from a double cubic spline smoothing.

MESH DENSIFICATION

The aim of the construction and the alignment of the discontinuity band is to reflect the current, dominant failure mode of the analysis. However, any such mesh should not overly influence further analysis by favouring only a single mode of failure. It is therefore suggested that in order to lower the alignment bias and to adequately provide further freedom for the analysis to reflect other active or previously active crack paths, the resulting mesh densification should in addition allow for any alternative paths to develop. This is achieved by inversely relating the new mesh size to the intensity of the accumulated plastic strain in all damaged regions, Fig 3.

The mesh generation is performed using an advancing front mesh generator NeGe [5], an unstructured mesh generator, where the mesh density and orientation are controlled by the background mesh. In algorithmic terms the old mesh is passed to the mesh generator as a background mesh with the corresponding information about mesh size.

Several aligned meshes have been analysed in the context of an incremental remeshing strategy, i.e. internal discontinuity band and associated mesh alignment is produced at various load increments, followed by a complete re-analysis with a newly aligned mesh, see Fig 5. Note that the mesh densification, including the discontinuity front bubble zone is adopted in addition to the actual discontinuity band.

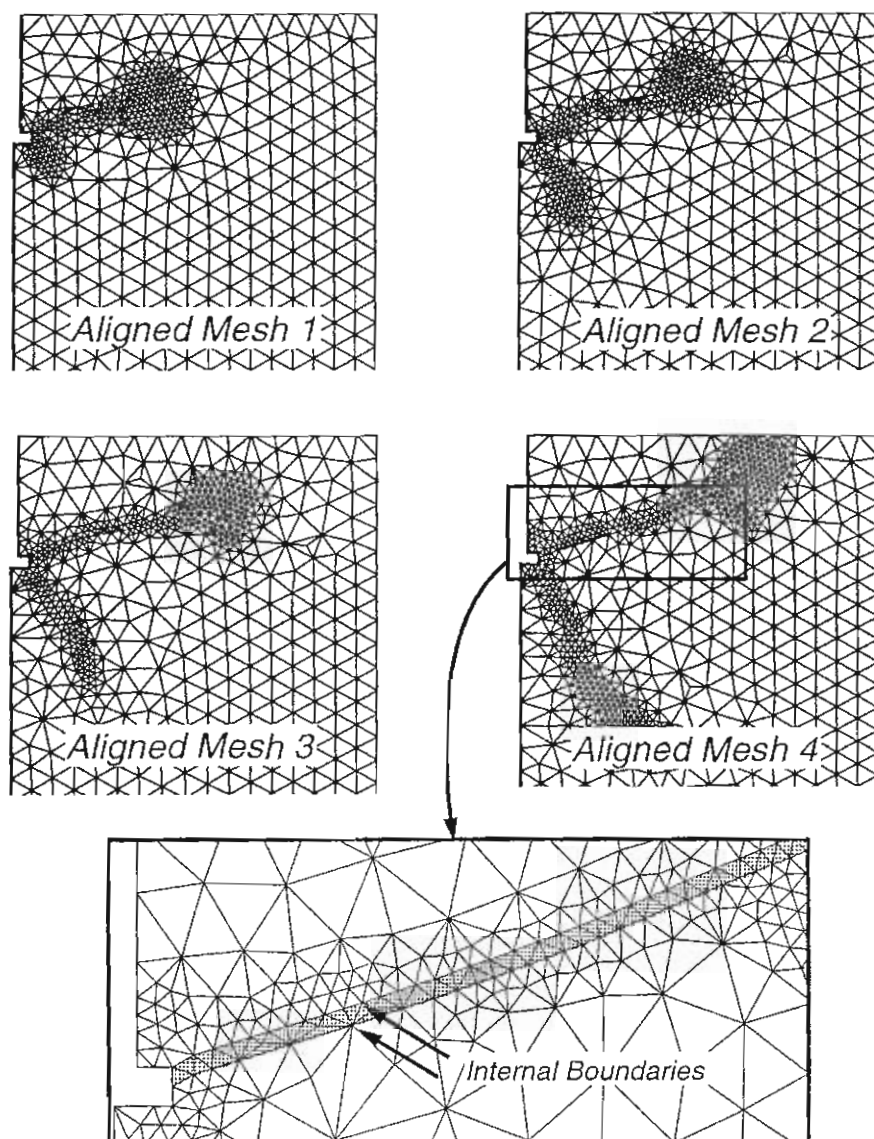


Figure 5. Consecutive meshes showing mesh densification and aligned discontinuity band (including close up of the fourth aligned mesh).

Aligned meshes and associated scalar field (effective plastic strain presented here) clearly indicate the effectiveness of the proposed procedure to force localised failure into a narrow zone, Fig 6. However, it is worth noting that the lower, secondary localisation band is modelled quite well while relying on mesh densification only (only a single, dominant discontinuity band is traced by the above strategy, in this case clearly the upper branch). Finally, force displacement diagrams, Fig 4(b), indicate that both the peak load as well as predictions of the softening branch response are influenced by mesh alignment, and that an improved alignment allows for a better control of discretisation effects.

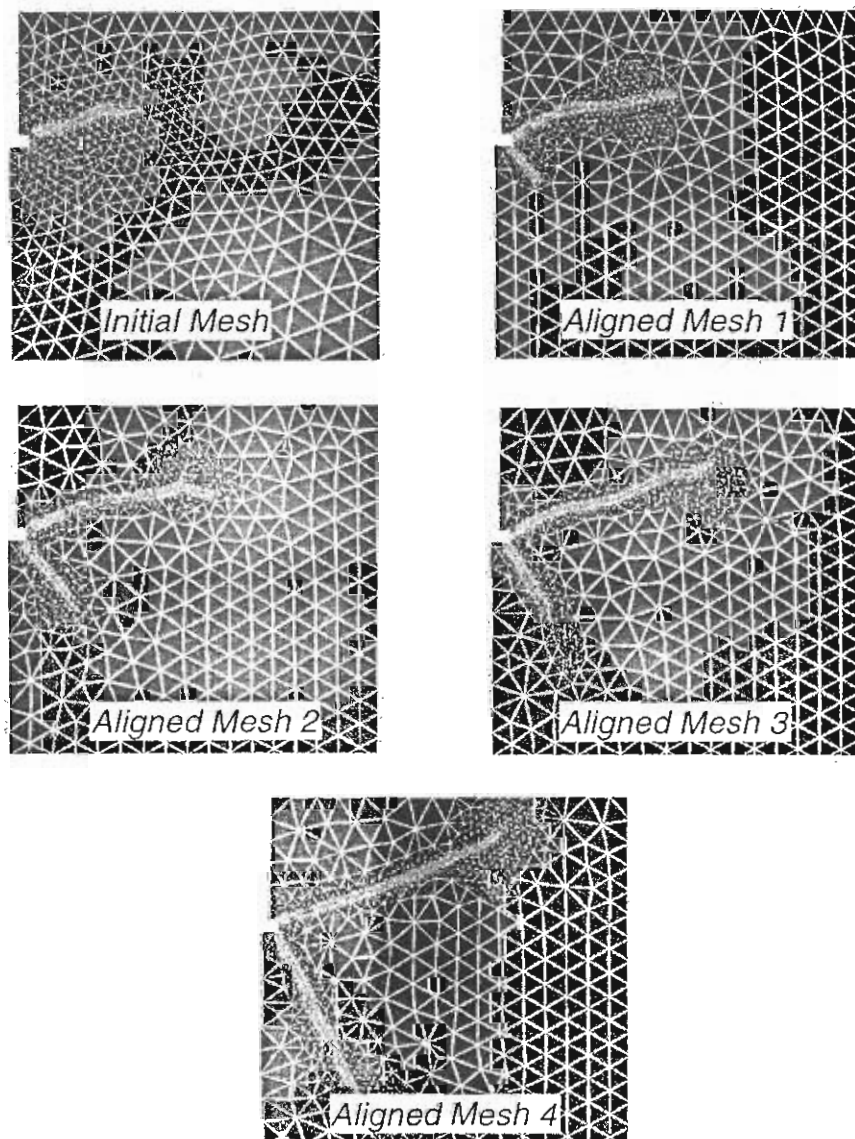


Figure 6. Consecutive meshes showing plots of effective plastic strain.

CONCLUSIONS

A simple discontinuity alignment meshing strategy for localised failure analysis of concrete is presented, whereby the discontinuity band is constructed by tracing a ridge of a suitable scalar control field. The present format includes staggered remeshing, where every mesh alignment is followed by a complete reanalysis to a higher load level than previously achieved, thereby avoiding difficulties in mapping the stress state and internal variables incrementally from one mesh to another. The adopted alignment strategy allowed for a highly localised resolution of a pull-out model problem failure mode.

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Simulation of aeroelastic wind force processes on flexible bridge girders

by

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ABSTRACT: Aeroelastic wind force processes on a streamlined bridge deck are modelled for use within time-domain computations. The F.E. method requires a continuous form of the force model, which permits an arbitrary discretization in time. The model is based on continuous step response functions of Wagner-type, taking advantage of modern advances of that theory. Special emphasis is put on well defined limit values for very high frequencies and for very high wind speeds. A realistic force model must include effects from the wind turbulence. For that reason, the functions are calibrated in wind tunnel tests on section models in grid turbulence and boundary layer flow. Turbulence effects on the aeroelastic coefficients are treated in mean sense. Since aeroelastic experiments can be more easily quantified in the frequency domain, mathematical equivalent models for the time and the frequency domain are shown, which use same parameters.

1. INTRODUCTION

Analysis in time domain makes it possible to model the structural restoring forces in full physical detail, so far it can be expressed in material laws, aiming at the activation of inherent load bearing capacities. The direct integration of the stochastic response of structures works in four main steps (e.g. Borri, Höffer & Zahlten, 1995):

1. Generation of multi-correlated wind field in terms of time-histories of both, the longitudinal $u+u'(t)$ and the vertical $w'(t)$ components
2. Load modelling with transformation of the turbulent flow around the bridge deck into buffeting forces and evaluation of interaction forces due to characteristic response quantities of the bridge girder (i.e. heaving, rotation of the girder)
3. FE model for geometrical and physical nonlinearities of the structural behaviour
4. Return of a fraction of the output process to the input one for simulating full interaction conditions, which are based on actual flow velocities due to the body-fixed coordinates

For the generation of wind speed records numerical methods, such as discrete, multidimensional, recursive algorithms (AR-methods) or the continuous superpositions of generic, parametric functions are applied, taking into account the cross-correlation of wind components (Facchini & Höffer 1995). Experiences exist in applying results from such simulations for the structural design according to guidelines and design rules (Kovacs & Andrä 1994).

Wind force models in time domain are commonly based on the simplified, quasi-stationary aerodynamics for both, the flow-structure interaction and the buffeting forces. The quasi-steady aeroelastic approach is limited to low reduced frequencies, loosing accuracy above ca. $K=0.1$, $K = 2\pi f_0 B / \bar{u}$, f_0 is the modal, or in case of non-linear oscillations, prevailing frequency of the response, B is a characteristic aerodynamic width of the structure (s. Fig. 1), and $\bar{u}$ is the mean flow. Long-span bridges have prevailing frequency of the response of typically up to 3 Hz, so that K has much higher values. Additionally, the spectrum of response quantities can show a broad-banded character in the state of serviceability or near the ultimate limit state. A flow force model should hold over the whole range of spectral components of wind-bridge interaction. Basic concepts are given

e.g. by Scanlan (Scanlan et al. 1971, Scanlan 1993) to take into account non-stationarity of aeroelastic forces and buffeting for time domain analysis.

2. NON-STATIONARY FLOW FORCES

2.1 Modeling memory effects of motion-induced forces

The instationarity of motion-induced flow forces occurs since an harmonical motion of a cross-section generates a vortex trail in the wake of the section. A single vortex separates from the section each time at the turning positions of the oscillation and moves in flow direction; since the vortex was generated from a certain cross-sectional motion it carries information about that motion. The vortex trail affects the flow forces over the time it needs to move a certain distance from the cross-section until it dissipates, which can be understood as a limited memory of the motion. The dissipation occurs in a distance of several λ downstream of the section. The length λ is the wavelength u/B of the motion.

In the idealized situation in potential flow, where the vortex trail cannot dissipate, the memory is infinite. The vertical force F_Z (upright positive), generated by the translational oscillation V_Z (heaving mode) of the section (comp. Fig. 1), is written for a potential flow as

$$\frac{F_Z(t)}{qB} = -\frac{\pi}{2} \cdot \frac{\ddot{V}_Z(t) \cdot B}{u^2} + 2\pi \cdot C(k) \cdot \frac{-\dot{V}_Z(t)}{u}, \quad C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + i \cdot H_0^{(2)}(k)} \quad (1,2)$$

q is the dynamic pressure, the dots (·) and (̈) indicate time derivatives, i.e. velocity and acceleration of the motion, k is the reduced frequency $\pi \cdot f \cdot B / u$, and $C(k)$ is the (complex) Theodorsen circulation function, $H_n^{(2)}$ are the Bessel Functions of third type and n th order, also called Hankel Functions. Fig. 2a) shows the real and the imaginary parts of the complex Theodorsen Function as functions of k . Eq. 1 consists of an in-phase and an out-phase component. Since the vertical position of the section cannot affect the flow forces, the in-phase component is only governed by acceleration terms; the factor of $\ddot{V}_Z(t)$ is called aerodynamic or added mass. The out-of-phase component consists of the reduced translational velocity $\dot{V}_Z(t)/u$ and a frequency dependant term, where 2π is the theoretical slope of the lift force coefficient on streamline sections. The Theodorsen function $C(k)$ (Eq. 2) theoretically predicts the fraction of the sectional velocity returning as a source of driving or damping forces on the cross-section. With $k = \pi \cdot f / \lambda$, the fraction is influenced by the wavelength of the vortex trail.

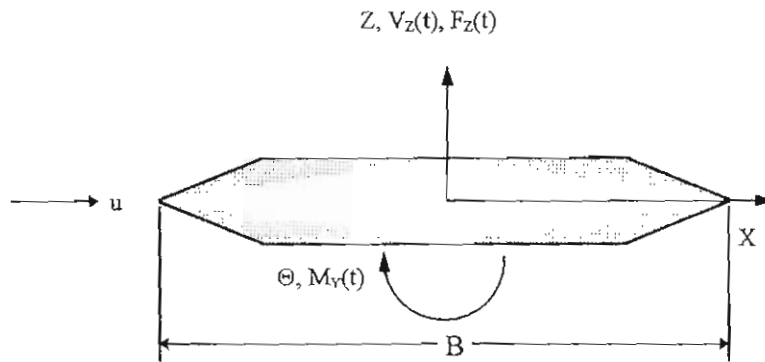


Figure 1: Section of a bridge girder, coordinate system and definitions

As wind turbulence always generates broad-banded processes of pressures and forces, the idealisation of assuming a single frequency oscillation must be abandoned. Instead, the force processes are described as stochastic processes in the frequency domain. The linear model of aeroelastic forces expresses the vector of the aerodynamic lift force and the torsional moment, generated by a vector of a translational oscillation in Z-directions $V_Z(t)$ and a torsional rotation $\Theta(t)$, as given in Eq. 3. The coordinate system and force components are sketched in Fig. 1. The processes $V_Z(t)$ and $\Theta(t)$ are the deformations of the bridge-section under turbulent wind and aeroelastic action.

$$\begin{bmatrix} X_{F_Z}(iK) \\ X_{M_Y}(iK) \end{bmatrix} = \underbrace{\begin{bmatrix} G_{F_Z, V_Z}(iK) & G_{F_Z, \Theta}(iK) \\ G_{M_Y, V_Z}(iK) & G_{M_Y, \Theta}(iK) \end{bmatrix}}_{G_{F,V}(iK)} \underbrace{\begin{bmatrix} \frac{X_{V_Z}(iK)}{B} \\ X_{\Theta}(iK) \end{bmatrix}}_{X_V(iK)} \quad (3)$$

X denotes Fourier-transforms of the indicated quantities, M_Y is the torsional moment, Θ is the torsional rotation, abbreviation $X_F(iK)$ for the vector of Fourier transforms of the forces F_Z (in kN/m) and M_Y (in kNm/m), $X_V(iK)$ for the vector of Fourier transforms of the dimensionless displacements V_Z/B and Θ (in rad), and $G_{F,V}(iK)$ for the matrix of transfer functions in the frequency domain. The transfer matrix $G_{F,V}(iK)$ is given, after Scanlan (e.g. Scanlan & Tomko, 1971), as

$$G_{F,V}(iK) = \begin{bmatrix} qK^2 \left[H_4^*(2\pi/K) + i \cdot H_1^*(2\pi/K) \right] & qBK^2 \left[H_3^*(2\pi/K) + i \cdot H_2^*(2\pi/K) \right] \\ qBK^2 \left[A_4^*(2\pi/K) + i \cdot A_1^*(2\pi/K) \right] & qB^2K^2 \left[A_3^*(2\pi/K) + i \cdot A_2^*(2\pi/K) \right] \end{bmatrix}, \quad (4)$$

where the coefficients $H_1^*(\lambda/B)$, $A_1^*(\lambda/B)$, $H_2^*(\lambda/B)$, and $A_2^*(\lambda/B)$ are the coefficients of aerodynamic damping and $H_3^*(\lambda/B)$, $A_3^*(\lambda/B)$, $H_4^*(\lambda/B)$, and $A_4^*(\lambda/B)$ are aerodynamic stiffness or added mass coefficients, $2\pi/K = \lambda/B$.

The following relations can be deduced (comp. Scanlan 1993) in the theoretic case of a streamline section (i.e. a thin plate or an airfoil, which is free of drag resistance and flow separations) in a potential flow (i.e. a incompressible flow, free of viscosity). As an example, heave induced quantities are treated in the Eq. 5.

$$H_1^*(\lambda/B) = -\frac{2\pi}{K} \cdot C_R(K/2) \quad , \quad H_4^*(\lambda/B) = \frac{\pi}{2} \cdot \left(1 + \frac{4 C_I(K/2)}{K} \right) \quad (5a,b)$$

$$A_1^*(\lambda/B) = -\frac{\pi}{2K} \cdot C_R(K/2) \quad , \quad A_4^*(\lambda/B) = -\pi \cdot \frac{C_I(K/2)}{2K} \quad (5c,d)$$

In the case of a real, bluff bridge deck in natural wind, the coefficients for added masses and aerodynamic damping are wavelength functions and must be determined in wind tunnel experiments (v. Chap. 3).

2.2 Time-domain model for non-stationary, motion-induced forces

In the time domain the motion-induced forces can be expressed by convolution integrals. Using the normalisations after Lin & Li (1993) the transfer functions of Eq. 4 are modeled as a parameter model

$$G_{F,V}(iK, a_k) = qB \frac{dc_F}{d\psi} \left[a_1 + iKa_2 + \sum_{n=3}^N A^{(1)}(a_n, K) \right] \quad (6)$$

$A^{(1)}$ represents a linear filter of first order with the parameters a_n . The corresponding time domain representation

$$\frac{F_{F,V}(s)}{qB} = \frac{dc_F}{d\psi} \left[a_1 V(s) + a_2 \frac{dV(s)}{ds} + \int_{-\infty}^s \frac{dV(\sigma)}{d\sigma} \cdot \{ \Phi_{F,V}(s-\sigma) - 1 \} d\sigma \right] \quad (7)$$

s and σ are reduced times ut/B , can be achieved performing an integral transformation and partial integrations (s. eg. Höffer 1997).

The model in Eq. 8 separates a quasi-stationary part from the convoluted, non-stationary impulse response. The force memory is modeled by a series of N linear filters of first order, following the recommendations of Beddoes (1982) for aircraft wings.

$$\Phi(M, s) = a_1 - \sum_{i=3}^N a_i \cdot e^{-a_{i+2} \cdot \beta^2 \cdot s} \quad , \quad \beta = \sqrt{1 - M^2} \quad (8, 9)$$

β is the Prandtl-Glauert Transformation, which leads to values of nearly unity in the considered conditions, eg. $u=33$ m/s yields $M=0,1$ and $\beta=0.995$. Neglecting the influence of the Mach number and limiting N to 4 leads to the original structure of a series of two exponential functions and a DC-value, i.e. a constant representing the stationary limit of the transient force growth, as it was formulated by Jones (1940) to approximate the basic Wagner function.

$$\Phi(s) = a_1 - \sum_{i=3}^4 a_i \cdot e^{-a_{i+2} \cdot s} \quad (10)$$

Wagner (1925) resolved theoretically the transient behaviour of the circulatory (non-impulsive) lift force on a flat plate in potential flow. The lift force can be considered to be initiated by a jump of the plate's position from 0° to Θ in a constant flow. The parameters for the approximation of Eq. 10 are given by Jones (1940) as $a_1=1$, $a_3=0.165$, $a_4=0.335$, $a_5=0.041$, $a_6=0.52$. Fig. 2b shows the theoretic solution. In a potential flow the curve approximates unity for the reduced time $s=\infty$.

In theory, the aerodynamic lift and the torsional moment are linearly coupled, since a) the moment is influenced by only the lift force, acting in a constant distance from the half-chord width, and b) because drag is not considered in potential flow theories. From that point of view, the Wagner-function would model the transient behaviour for the aeroelastic force component. In real flow, different Wagner-type functions must be found for each aeroelastic force component in an aerodynamic strip. The non-stationary interaction forces F_Z and M_Y for a MDOF aeroelastic system, induced by arbitrary heave and torsional motion, write

$$\frac{F_{Z,V_Z}(t)}{qB} = \frac{dc_L}{d\psi} \left[c_2 \frac{\dot{V}_Z(t)}{u} + \frac{1}{B} \int_{-\infty}^t \Phi_{F_Z,V_Z}(t-\tau) \frac{dV_Z(\tau)}{d\tau} d\tau \right] \quad (11a)$$

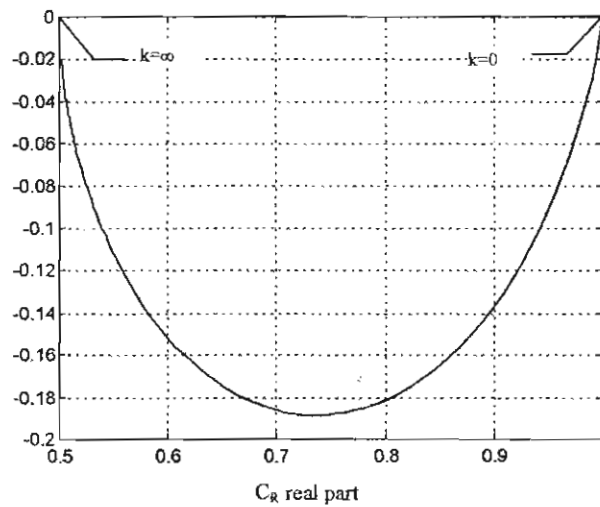
$$\frac{F_{Z,\Theta}(t)}{qB} = \frac{dc_L}{d\psi} \left| \psi \left[e_2 \frac{B \cdot \dot{\Theta}(t)}{u} + e_1 \cdot \Theta(t) + \int_{-\infty}^t \left\{ \Phi_{F_{Z,\Theta}}(t-\tau) - 1 \right\} \frac{d\Theta(\tau)}{d\tau} d\tau \right] \right| \quad (11b)$$

$$\frac{M_{Y,\Theta}(t)}{qB^2} = \frac{dc_L}{d\psi} \left| \psi \left[g_2 \frac{B \cdot \dot{\Theta}(t)}{u} + g_1 \cdot \Theta(t) + \int_{-\infty}^t \left\{ \Phi_{M_{Y,\Theta}}(t-\tau) - 1 \right\} \frac{d\Theta(\tau)}{d\tau} d\tau \right] \right| \quad (11c)$$

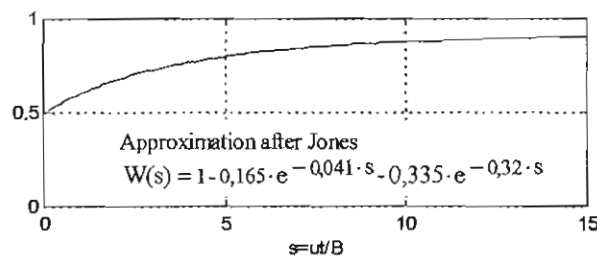
$$\frac{M_{Y,V_Z}(t)}{qB^2} = \frac{dc_M}{d\psi} \left| \psi \left[r_2 \frac{\dot{V}(t)}{u} + \frac{1}{B} \int_{-\infty}^t \Phi_{M_{Y,V_Z}}(t-\tau) \frac{dV_Z(\tau)}{d\tau} d\tau \right] \right| \quad (11d)$$

There exist no DC-terms (associated with parameters c_1 or r_1) in the Eqs. 11a,d because the stationary vertical position of the bridge deck has no effect on the flow forces in the upper range of the shear layer where bridges are commonly situated.

C_I imaginary part



a) Theodorsen circulatory function $C(k) = C_R(k) + i C_I(k)$



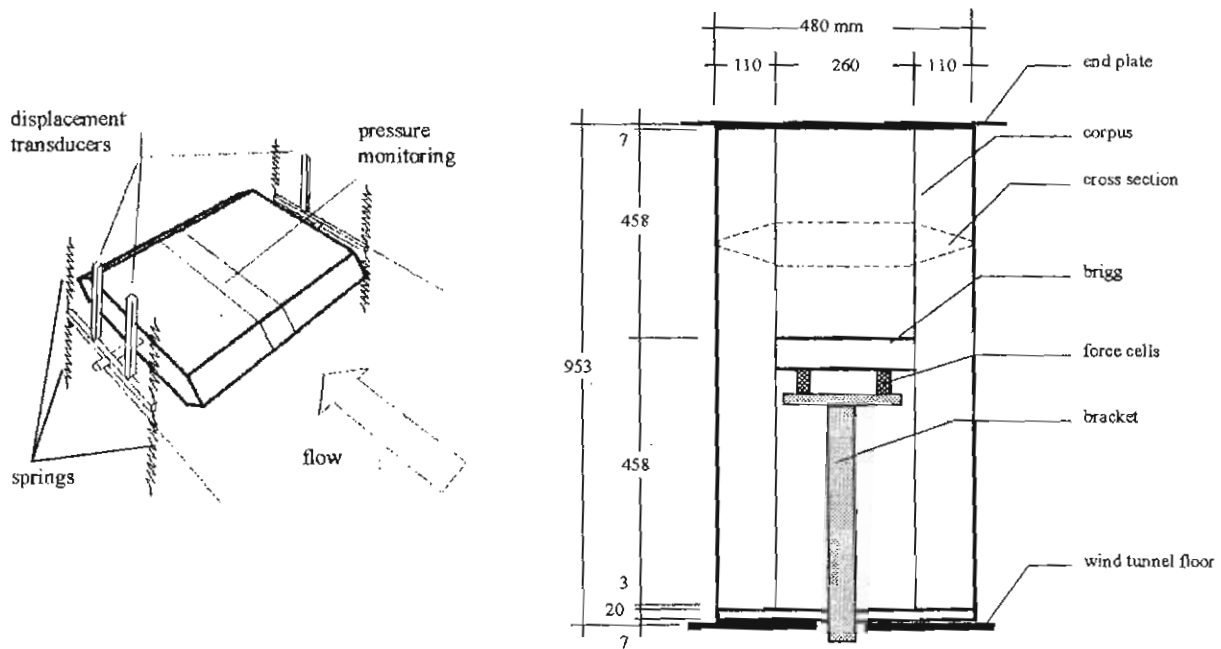
b) Wagner-function; $W(s) = \frac{1}{i2\pi} \int_{-\infty}^{\infty} \frac{C(k)}{k} e^{iks} dk, k = \frac{\pi f B}{u}$

Figure 2: Circulatory lift forces in frequency and time domains; classical representations

3. CALIBRATION IN WIND TUNNEL EXPERIMENTS

3.1 Wind tunnel experiments

The experiments are conducted on an example bridge cross-section (s. Fig. 2) which has been well studied; results are reported in literature (e.g. Scanlan & Tomko 1971, Herlach 1974). The tests have been carried out in the boundary layer wind tunnel at the Building Aerodynamics Laboratory (AIB) at the Ruhr-Universität in Bochum, Germany, and in Wind Tunnel No. 2 of the Danish Maritime Institute (DMI), Lyngby, Denmark. Both are open circuit tunnels of Eiffel-type, the test sections have 1.6 m x 1.8 m and 2.6 m x 1.8 m, respectively. The flow conditions in the wind-tunnel of DMI are low, isotropic turbulence of 0.1% and 3% intensity, approximately isotropic turbulence of $I_u=11.3\%$ (longitudinal) and $I_w=9.4\%$ with the integral length scale $L_{ux}=B/6$. The boundary layer flow generated for this experiments in the wind-tunnel of AIB at Bochum has an anisotropic turbulence of $I_u=14.6\%$ and $I_w=9.8\%$, $L_{ux}=4B$. All experiments are described in detail by Höffer (1997).



a) Elastically mounted section model in boundary layer turbulence (tests at AIB)

b) Forced-oscillation model in low and isotropic turbulence (tests at DMI)

Figure 3: Principle set-up of the conducted experiments on the bridge girder models

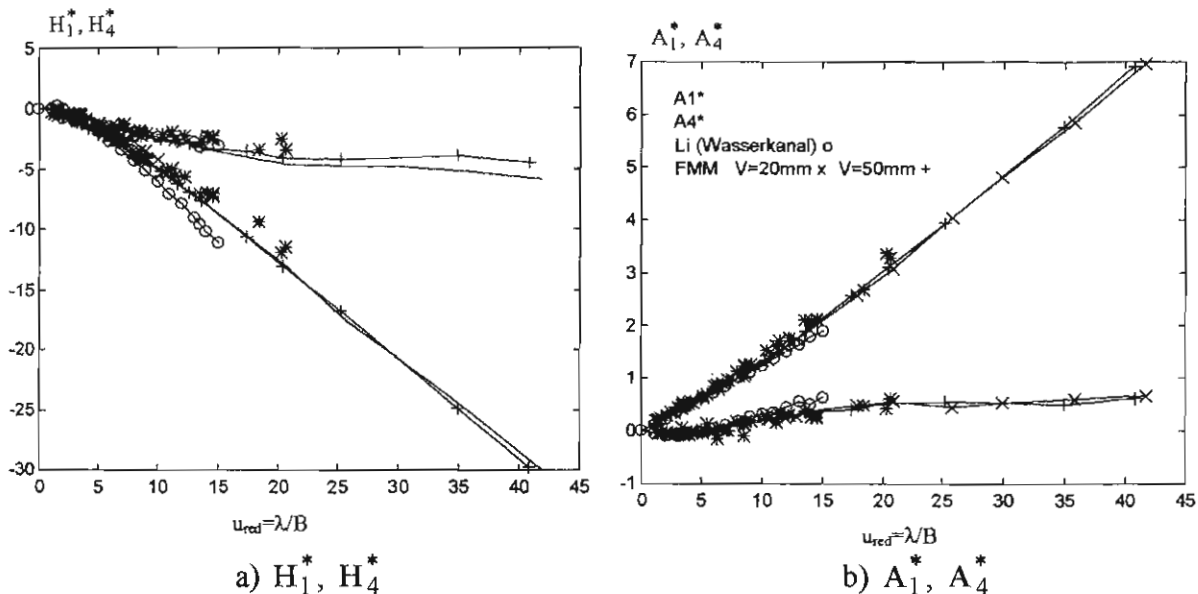
The model's chord-to-thickness ratio are $B/D=6.00$ (DMI) and 6.25 (AIB). Static tests using a force balance and pressure measurements have been made to measure the quasi-stationary lift and moment coefficients in both nearly smooth and turbulent flow. The set of experiments at Bochum uses an elastically mounted SDOF-section placed between endplates which models the torsional oscillations of a deck (s. Fig. 3a).

In the set of experiments at Lyngby the forced-motion method has been applied. A highly accurate sinusoidal heave and torsional oscillation can be generated by putting into action a forced motion mechanism, developed by the DMI for aerodynamic and marine tests. Fig. 3b) shows a sketch of the model, which is uprightly mounted between the wind-tunnel floor and an end plate. The model moves orthogonal to the drawing plane, the flow is from the left.

3.2 Frequency-dependant coefficients for heaving motion as an example

The Figs 4a,b show the derivatives H_1^* , H_4^* , A_1^* , A_4^* in low turbulence, which are measured in a heave motion. The symbols mark the results of single experiments; scattering is very low. The evaluated frequency dependant coefficients H_4^* and A_4^* compare well to results reported by Li (Li 1995) which are obtained in a water tunnel, and those found by Scanlan&Tomko (all results by Li and Scanlan & Tomko are multiplied by factor 2 with respect to the normalizations using 2B and 2B2 instead of B and B2 in the present study). It is more difficult for higher λ/B to measure apparent mass effects in air (i.e. H_4^* , A_4^*) since the fluid density is low compared to the structural density; small disturbances of the experiments are found for H_4^* in the range of λ/B of larger than 15, visible in Fig. 5a as distant curves for the amplitude $\hat{V}_Z=20$ mm and 50 mm. This is not the case for A_4^* .

The out-of-phase (i.e. damping) coefficients H_1^* and A_1^* are very well defined over the whole range of reduced velocities up to ca. 43.



smooth or low turbulence $I_u=I_w<1\%$: Li 1995, waterchannel (o), Scanlan & Tomko 1971 (x), Höffer 1995, FMM $\hat{V}_Z=20$ mm (· —), $\hat{V}_Z=50$ mm (+ —). turbulent: $I_u=11,3\%$, $I_w=9,3\%$: Höffer 1995, FMM $\hat{V}_Z=50$ mm (*)

Figure 4: Measured aerodynamic derivatives H_1^* , H_4^* , A_1^* , A_4^*

Only small effects are found from the applied grid turbulence. It should be expected an earlier reattachment of the separated flow lines and a reduced wake width due to small scale turbulence (SST). The influence on the coefficients remain small, since separated flow lines will reattach on the flanks of the investigated cross-section ($B/D=6.0$) also in case of smooth flow. In turbulent flows, the empiric coefficients exhibit a moderate scattering using the FMM-approach (Figs 5a,b). The coefficients associated to the torsional moment due to heave motion (A_1^* , A_4^*) fairly coincide with the results from experiments in low turbulence, whereas the coefficients associated with the lift force due to heave motion (H_1^* , H_4^*) show lower absolute values in turbulent flow.

4. AEROELASTIC STRIP FORCES IN WIND TURBULENCE

4.1 Continuous step response functions

The required relationship between Eq. 7 in time domain and Eq. 6 in frequency domain is found by an integral/LAPLACE-transformation. Using that result the aerodynamic derivatives can be written in terms of the model parameters c_i , e_i , g_i , and r_i , $i=1,2,\dots,6$ of the step response functions (comp. Eq. 8). The procedure and examples are presented in detail by Höffer (1997).

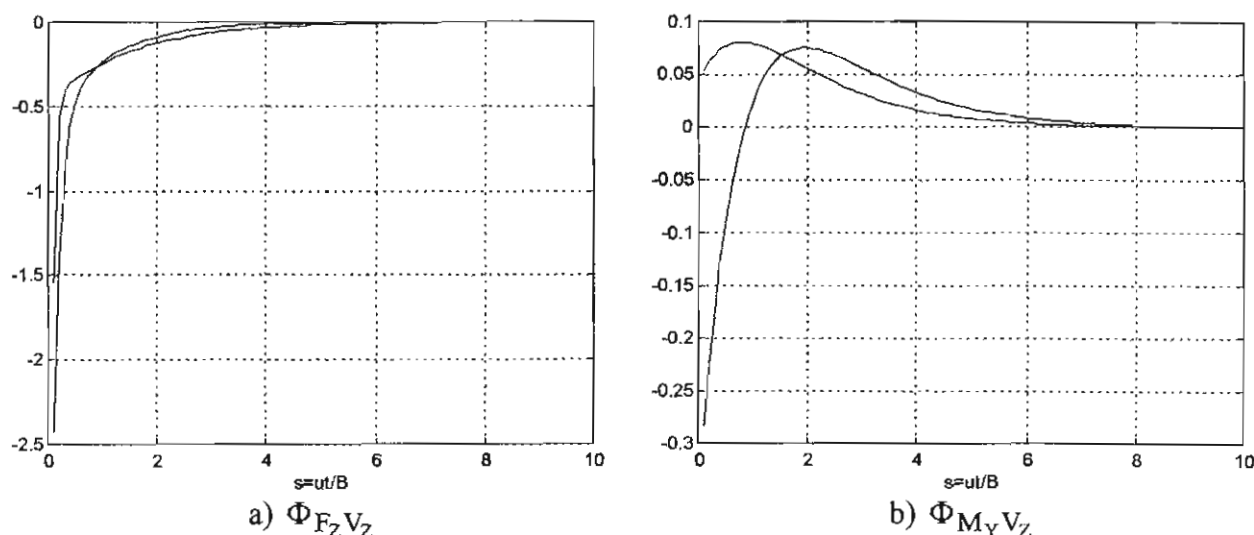


Figure 5: Step response functions after Eq. 8, parameters after Tab. 1

$\Phi_{FZ,VZ}$	c_1	c_2	c_3	d_3	c_4	d_4	$\Delta c_L/\Delta\psi$
low turb. 1%	0	0,116	-8,060	19,938	-0,471	0,663	4,68
isotr. turb.	0	0,321	-3,918	7,394	-0,616	0,979	4,17
$\Phi_{MY,VZ}$	r_1	r_2	r_3	s_3	r_4	s_4	$\Delta c_M/\Delta\psi$
low turb. 1%	0	0,913	-5,214	0,964	4,862	0,882	1,04
isotr. turb.	0	0,930	-5,614	0,952	5,658	0,924	0,911

Table 1: Identified parameters of the normalized step response functions of the heaving model in low turbulence ($I_u < 1\%$) and isotropic turbulence ($I_u = 11.3\%$, $I_w = 9.3\%$)

The step response functions for turbulent conditions are presented in Figs 5a,b. The functions are well defined over the whole range from $s=0$ to the transition to quasi-stationary behaviour. The latter is marked by a horizontal tangent where the convolutions do not contribute to the force memory for $(d\Phi/ds) \rightarrow 0$. Stationarity is practically reached in all cases for $s_{qs} < 10$, the index (qs) indicates the transition to quasi-stationary. The DC-coefficients for heave-induced force components are 0 since the heave position of a bridge deck will not affect the forces. The moment growth functions due to heave $\Phi_{MY,VZ}$ in smooth and turbulent flows present an overshooting behaviour. The impulsive

component acting together with the flow-induced moment growth can generate a step response function which is the typical behaviour of a low pass filter perceptible to instability.

4.2 Effects of compressibility

The original Wagner approach assumes a value of minus infinity for the step response in incompressible flow and its decay in infinite time (infinite speed of pressure propagation) to the circulatory component of the transient lift (Wagner function). The equivalent quantity to be considered in the frequency domain is the limit $f \rightarrow \infty$ of the transfer function G_{FV} . Potential flow theory predicts

$$\lim_{K \rightarrow \infty} \{G_{FZ} V_Z(iK)\} = \lim_{K \rightarrow \infty} \left\{ q \frac{\pi}{2} \left[K^2 + 4K \cdot C_I(K/2) - i2\pi K \cdot C_R(K/2) \right] \right\} = \infty \quad (12a)$$

$$\lim_{K \rightarrow \infty} \{G_{MY} V_Z(iK)\} = \lim_{K \rightarrow \infty} \left\{ qB \frac{\pi}{2} \cdot \left[-\frac{K \cdot C_I(K/2)}{2} - i \cdot \frac{K \cdot C_R(K/2)}{2} \right] \right\} = -\infty \quad (12b)$$

for the force response during the virtual step of a displacement quantity. As a consequence of the assumed incompressibility and absence of dissipation of the fluid, Wagner (1925) assumes a decay in infinite time from the infinite, impulsive force response to the value $W(s=0)=0.5$ of the Wagner-function, which is the initial value of the circulatory part of the step response.

Measurements in real flow, show that the impulsive component in air is finite and consists of added mass contributions and an acoustic perturbation, which can be physically interpreted as the energy of a acoustic wave system generated by the initial jump. Leishman (Leishman 1988) describes the initial value and the decay of the impulsive component by use of a single exponential term, where the amplitude and the decay exponent are functions of the (subsonic) Mach number.

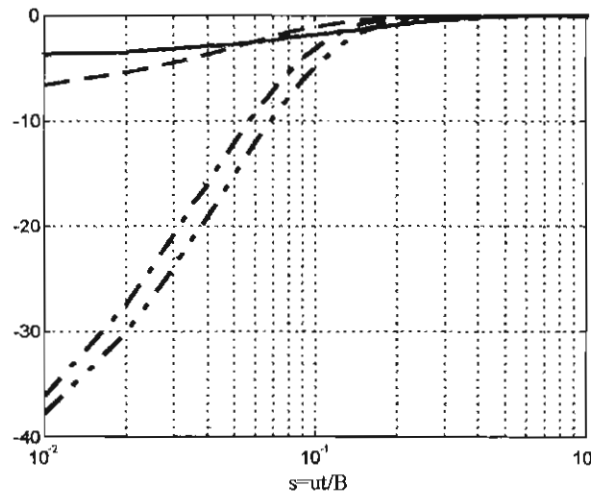


Figure 6: Impulsive component of the step response function; identified low turbulence (---), isotropic turbulence (—), and after Leishman (1981) for low turbulence (- · -)

For a uniform perturbation velocity orthogonal to the secant of the chord the impulsive estimator is after Leishman

$$\Phi^{\text{imp}}(M, s) = \left[\frac{dc_N}{d\psi} \Big|_{\psi_0} \right]^{-1} \frac{4}{M} e^{-\frac{s}{T'(M)}}, \quad T'(M) = \frac{2M}{(1-M) + \pi\beta^{(\gamma-1)} M^2 \sum_{i=3,N} a_i b_i} \quad (13a,b)$$

a_i, b_i are coefficients after Eq. 6, γ is a free parameter between 0.7 and 1.0 and set to 1 here. A comparison of this prediction and the identified impulsive component of $\Phi_{F_Z V_Z}$ for $s < 0.2$ is shown in Fig. 6 The fading out of $\Phi_{F_Z V_Z}^{\text{imp}}$ is well predicted by Eq. 13a whereas the predicted initial value cannot be identified from the experimental data.

4.3 Finite convolution and discretization of strip forces

The complete set of continuous, non-stationary interaction forces is introduced in Eq.s 11. The presented empirical results confirm the assumption, based on the viscosity of a real fluids, that the range of integration of the convolutions can be reduced from $[-\infty, t]$, for the theoretic case of a potential flow, to $[t - s_{qs} \cdot B/(u \cdot t), t]$. In this study, s_{qs} is found to be less than 10.

Fig. 7 explains the application of the principle of exchanging arguments in the convolution integrals, for which the limit s_{qs} permits a regressive convolution.

$$M = \frac{s_{qs}}{\Delta s} + 1$$

$$\sum_{m=1}^{\Delta s} \Phi_{F,V}((n+1-m) \cdot \Delta t) \cdot \Delta V((m-1) \cdot \Delta t) \quad (14)$$

In Eq. 14 $M = s_{qs}/\Delta s + 1$ is the order of the convolution, ΔV is a discrete step of a reduced displacement, n is the time step number, Δt is the time increment. The ordinates Φ_i of the step response function are the regression coefficients of the convolution.

The marked area in Fig. 8 can be looked at as a measure for the memory of the transient force due to the preceding movements of the aerodynamic strip. Applying this model, Eq.s 11 is given in discrete representation as

$$\frac{F_{Z,V_Z}(n \cdot \Delta t)}{qB} = \frac{dc_L}{d\psi} \Big|_{\psi} \left[c_2 \frac{\dot{V}_Z(n \cdot \Delta t)}{u} + \frac{1}{B} \sum_{m=1}^{M=\frac{\hat{s}_{asy}}{\Delta s}+1} \Phi_{F_Z V_Z}((n+1-m) \cdot \Delta t) \cdot \Delta V_Z((m-1) \cdot \Delta t) \right]$$

$$\frac{F_{Z,\theta}(n \cdot \Delta t)}{qB} = \frac{dc_L}{d\psi} \Big|_{\psi} \left[e_2 \frac{B \cdot \dot{\theta}(n \cdot \Delta t)}{u} + e_1 \theta(\Delta t) + \sum_{m=1}^{M=\frac{\hat{s}_{asy}}{\Delta s}+1} \Phi_{F_Z \theta}((n+1-m) \cdot \Delta t) \cdot \Delta \theta((m-1) \cdot \Delta t) \right]$$

$$\frac{M_{Y,\Theta}(n \cdot \Delta t)}{qB^2} = \frac{dc_M}{d\psi} \Big|_{\psi} \left[g_2 \frac{B \cdot \dot{\Theta}(n \cdot \Delta t)}{u} + g_1 \cdot \Theta(n \cdot \Delta t) + \sum_{m=1}^{M=\frac{\hat{s}_{asy}}{\Delta s}+1} \Phi_{M_{Y,\Theta}}((n+1-m) \cdot \Delta t) \cdot \Delta \Theta((m-1) \cdot \Delta t) \right]$$

$$\frac{M_{Y,F_z}(n \cdot \Delta t)}{qB^2} = \frac{dc_M}{d\psi} \bigg|_{\psi} \left[v_2 \frac{\dot{V}_z(n \cdot \Delta t)}{u} + \frac{1}{B} \sum_{m=1}^{M=\frac{\hat{s}_{asy}}{\Delta s}+1} \phi_{M_{Y,F_z}}((n+1-m) \cdot \Delta t) \cdot \Delta V_z((m-1) \cdot \Delta t) \right]$$

(15a,b,c,d)

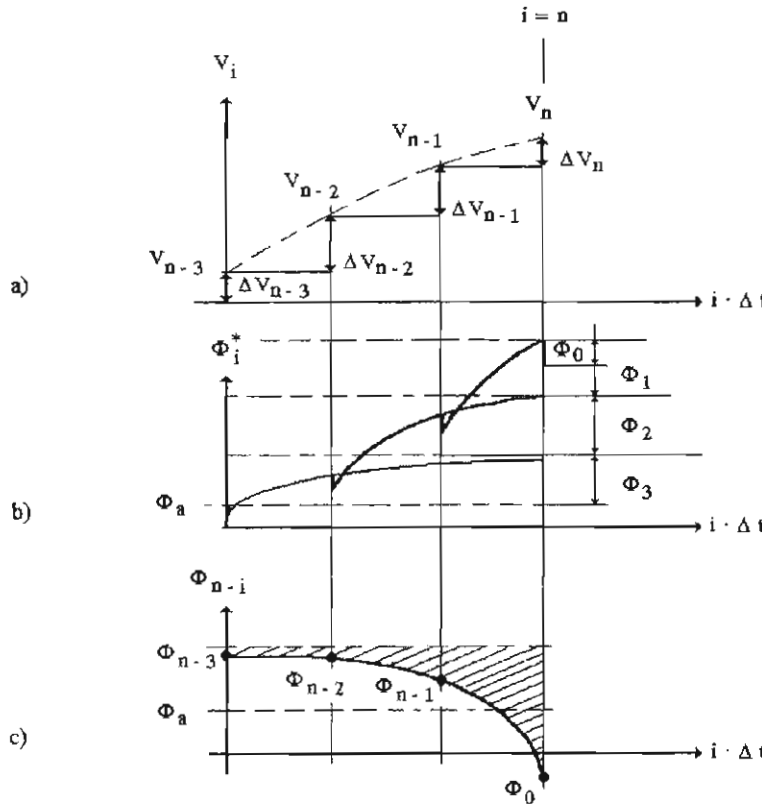


Figure 7: Regressive convolution scheme

The equations consist of a quasi-stationary component of viscose damping, for $\alpha l=1$ follows a quasi-stationary in-phase component for aerodynamic stiffness or mass, and a (convoluted) regressive component of mixed phases.

5. CONCLUSIONS

A time domain model of non-stationary, aeroelastic forces is adapted for use in the F.E. simulation of the stationary fluid-structure interaction of bridge girders in turbulent wind. A symmetric, streamlined deck section has been considered as an example. The procedure is adjustable for problematic cases, e.g. of intermediate instability of non-streamlined bridge sections, by choosing suitable parameter models for the step response functions.

Following main results have been achieved:

1. The model's representations in time and frequency domains are mathematically equivalent and non-ambiguous. The experimental identification of aeroelastic damping, mass and stiffness can be performed in the frequency domain.

2. The discretizability in time for use of the force equations within F.E. computations is ensured by
 - a) continuous step response functions for the arbitrary choice of time increments Δt
 - b) well defined initial values of the step response functions, essential for the case of small Δt
 - c) highly-accurate experimental methods, which permit to indicate realistic aeroelastic coefficients, and respective step responses, in turbulent flows
3. The model can be considered as a development of the previous approach (Borri, Höffer & Zahlten 1996) implemented in a F.E. code, which was restricted to quasi-stationary aerodynamics.

6. ACKNOWLEDGMENTS

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ADVANCED NON-LINEAR SHELL MODELLING OF THE HAMILTONIAN PROBLEMS

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SUMMARY

Some aspects of continuum and numerical formulation for geometrically non-linear dynamics of elastic shells and its temporal discretisation are considered. The main features of the approach: (i) the finite rotation shell theory, (ii) the energy and momentum conserving time-integration algorithm, (iii) the finite element implementation, and (iv) the consistent linearisation of the weak form of the equation of motion - are very briefly presented and discussed.

The shell theory is based on derived approach. It is developed from the three-dimensional continuum theory by employing standard assumptions on the distribution of the displacement field in a shell body. A single director model for thin shells is obtained by approximation of terms describing the shell geometry. Using this description, a non-linear shell model, which is equivalent to the two-dimensional Cosserat continuum, is derived. The kinematic fields are exactly described. In particular, finite rotations of the director field are described by a rotation vector formulation, which is free of singularities. Temporal discretisation of the weak form of the equations of motion is performed using an implicit numerical time-integration scheme. In fact, an algorithm, which may be regarded as a special form of the mid-point rule, is employed. It preserves the fundamental constants of the autonomous motion: the total linear and angular momentum as well as, for a Hamiltonian case, the total energy. A finite element formulation is based on the four noded isoparametric elements. To avoid locking, a so-called assumed natural approach is adopted. Particular attention is devoted to the consistent linearisation of the time discretised weak form of the equation of motion, in order to achieve a quadratic rate of asymptotic convergence typical for the Newton-Raphson based solution procedure. A linearisation is performed after the spatial element discretisation. The performance of the model is here illustrated with a numerical example.

1. INTRODUCTION

Various finite element models for simulation of behaviour of flexible shells are now established and efficient solutions can be obtained using a variety of appropriate elements and formulations. Here, we consider a non-linear dynamics of flexible elastic laminated composite shells. The shells under consideration may be composed of a generally elastic material with through thickness variable mass density and degree of orthotropy. It is common practice to represent mechanical behaviour of a composite laminate on a macroscopic level. In this case the shell structure is treated as a composition of perfectly bonded layers and each layer is treated as a homogeneous and anisotropic continuum. At small strains the behaviour of a particular layer can be assumed as purely elastic and described with an orthotropic material law (e.g. Basar (1993); Brank, Perić and Damjanić (1995)). More details related to the composite formulation can be found in Brank, Briseghella, Tonello and Damjanić (1996). Here, due to a limited space, our main aim is to present a numerically stable finite element formulation for the description of long-term response of shells, which may be undergoing large deformations and rotations, and large overall motion. The formulation has a lot in common with the works of Simo, Rifai and Fox (1992) and Simo and Tarnow (1994). The differences are related to the derivation of the continuum dynamic formulation and the implementation procedure.

The dynamic shell model is developed from the three-dimensional continuum theory by employing standard assumptions on the distribution of the displacement field in the shell body (e.g. Büchter and Ramm (1992)). Single director model for thin shell, which is equivalent to the two-dimensional Cosserat continuum presented by Simo and Fox (1989), is obtained by approximating the terms describing the shell geometry. Rotational vector based on large rotation formulation is used to describe the motion of the shell director field. Starting point for the numerical formulation is the continuum weak form of the initial boundary value problem. Time and spatial discretisation of the weak form are subsequently applied. The implicit numerical time-integration scheme follows the works of Simo and Tarnow (1992, 1994) and Simo, Tarnow and Doblare (1995), who constructed the so-called energy-momentum conserving algorithms for various mechanical systems, i.e. solids, shells and rods. The algorithm, adopted in this work, may be regarded as a special form of the mid-point rule. It is second order accurate and characterized by exact conservation of total energy for Hamiltonian shell systems. It has been shown by many authors (see for example Simo and Tarnow (1992)), that the energy conserving time-integration schemes demonstrate unconditional numerical stability. This is in contrast to conventional time integration schemes of the Newmark family, which may exhibit a loss of stability due to the sudden increase in the higher frequency strain energy components. The energy conserving algorithms are therefore suitable for the analysis of long-term response. If the motion of a shell is driven only by specified initial conditions, and no displacement and rotation boundary conditions are prescribed, the adopted algorithm also conserves both the total linear and the total angular momentum of the shell. The spatial discretisation is based on the four noded isoparametric elements. To avoid locking, a so-called assumed natural approach is adopted. Particular attention is devoted to the consistent linearisation of the time discretised weak form of the equation of motion, in order to achieve a quadratic rate of asymptotic convergence typical for the Newton-Raphson based solution procedure.

In the following sections the attention is restricted to some key issues of the formulation. More details can be found in Brank, Perić and Damjanić (1995) as well as in new papers from the authors which are currently in press.

2. CONCEPTS OF THE CONTINUUM FORMULATION

The classical one-director shell theory is completely defined in terms of the two objects: (i) a surface M , parametrised by curvilinear coordinates ξ^1, ξ^2 , and (ii) a vector field, called shell director field, defined on M . Let the time interval of the interest be $[0, T]$. The position vectors of a material point at the initial time $t=0$, and at time $t>0$ are defined as

$$\mathbf{X} = \mathbf{R} + \xi \mathbf{A}_3, \quad \text{and} \quad \mathbf{x} = \mathbf{r} + \xi \mathbf{a}_3 \quad (1)$$

respectively, where $\xi \in [-h/2, h/2]$ and h is the constant initial shell thickness. The shell director $\mathbf{A}_3$ coincides with the normal to the surface M at the point $\mathbf{R} \in M$. It follows from eq. (1) that the displacement vector is a linear function of ξ coordinate: $\mathbf{u} = (\mathbf{r} - \mathbf{R}) + (\mathbf{a}_3 - \mathbf{A}_3)$. To avoid ill conditioned equations in the thin shell, an assumption of the inextensibility of the shell director is further introduced, $\|\mathbf{a}_3\| = \|\mathbf{A}_3\| = 1$, which allows an introduction of the rotation tensor $\mathbf{Q}$. Relation $\mathbf{a}_3 = \mathbf{Q} \mathbf{A}_3$ enables two-parametric representation of $\mathbf{Q}$. Straining (eq.(2)) is described by the Green-Lagrange strain tensor, which is evaluated with respect to the thickness coordinate ξ , ignoring functions of higher order than one.

$$\mathbf{E} = (\alpha_{ij} + \xi \beta_{ij}) \mathbf{A}^i \otimes \mathbf{A}^j \quad (2)$$

$\mathbf{A}_i$ represents generally non-orthogonal covariant base vectors, spanned by the tangents to the curves ξ^α and ξ , while $\mathbf{A}^i \cdot \mathbf{A}_j = \delta^i_j$.

The dynamic formulation necessitates the definition of kinetic and potential energy. Then the weak form of the canonical Hamilton's equations may be constructed, which further leads to the weak form of momentum equations. Finally, the weak form of the initial boundary value problem for non-linear elastic shells (together with the initial conditions in eq.(4)) can be expressed as

$$\mathbf{G}_{dyn} = \mathbf{G}_{ine} + \mathbf{G}_{int} - \mathbf{G}_{ext} = 0 \quad (3)$$

$$(\mathbf{r}, \mathbf{a}_3)|_{t=0} = (\mathbf{R}, \mathbf{A}_3), \quad (\mathbf{p}, \pi)|_{t=0} = (\mathbf{P}, \Pi) \quad (4)$$

where explicit expressions for inertial, internal and external parts are defined as follows

$$\mathbf{G}_{ine} = \int_M [\mathbf{p}' \cdot \delta \mathbf{v} + \pi' \cdot \delta \mathbf{a}_3] dM, \quad \mathbf{G}_{int} = \int_M (n^{ij} \delta \alpha_{ij} + m^{ij} \delta \beta_{ij}) dM \quad (5)$$

$$\mathbf{G}_{ext} = \int_M (\mathbf{q} \cdot \delta \mathbf{x} + \mathbf{l} \cdot \delta \mathbf{a}_3) dM + \int_{\partial M} (\mathbf{N}^s \cdot \delta \mathbf{x} + \mathbf{M}^s \cdot \delta \mathbf{a}_3) ds. \quad (6)$$

Here, $\mathbf{p} = A \mathbf{r}'$ and $\pi = I \mathbf{a}_3'$, with A and I defining the 0th and 2nd moment of mass density about the surface M , n^{ij} and m^{ij} define the 0th and 1st order second Piola-Kierchhoff stress resultants and $\mathbf{q}$, $\mathbf{l}$, $\mathbf{N}^s$ and $\mathbf{M}^s$ represent external force resultants related to surface M and its boundary ∂M .

3. CONCEPTS OF THE NUMERICAL PROCEDURE

The numerical formulation is based on time and spatial discretisation of the weak form of the initial boundary value problem. Let a time interval of interest be partitioned into N subintervals $n=0 \dots N-1$ $[t_n, t_{n+1}]$. A solution is obtained at discrete points by an algorithmic approximation. Considering a typical time subinterval, the algorithmic problem is: Assuming that initial conditions at t_n are given, define algorithmic approximation of configuration and velocities at t_{n+1} . As already mentioned, a special form of the mid-point rule is employed in this work. Structure of the algorithm is based on a two-step procedure. First, the weak form of the initial boundary value problem is evaluated at the middle of a time subinterval, with the stress resultants left undefined

$$[G_{ine} + G_{int} - G_{ext}]|_{t_{n+1/2}} = 0. \quad (7)$$

Second, the algorithmic stress resultants are defined in such the way, that the total energy is conserved for the Hamiltonian case. If a stored energy function is a quadratic form of Green-Lagrange strain tensor, the energy is conserved with the average of strains at time t_n and t_{n+1} . The algorithmic components of the second Piola-Kierchhoff stress tensor are therefore

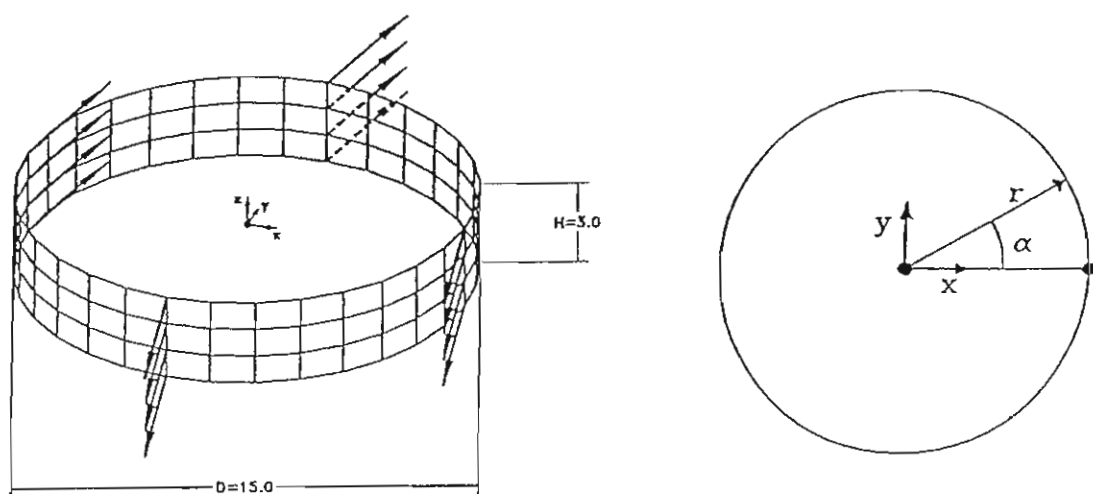
$$S_{ij} = C^{ijkl} \cdot \frac{1}{2} [(E_{kl})_n + (E_{kl})_{n+1}] ; \quad ij \neq 33, kl \neq 33 \quad (8)$$

with C^{ijkl} being the components of the fourth order constitutive tensor.

The time discretisation is followed by spatial discretisation, which is based on the four noded isoparametric finite element. The locking is avoided by the assumed natural strain approach for the transverse shear part (e.g. Brank, Perić and Damjanić (1995)). While the displacement based formulation is used for the membrane and bending parts. Appropriate linearisation of the eq.(7) is performed in order to achieve a quadratic rate of asymptotic convergence of a Newton-Raphson based solution procedure. The tangent operator is obtained by the directional derivative of the fully discrete weak form in the direction $(\Delta r_{n+1}, \Delta a_{3n+1})$.

4. NUMERICAL EXAMPLE

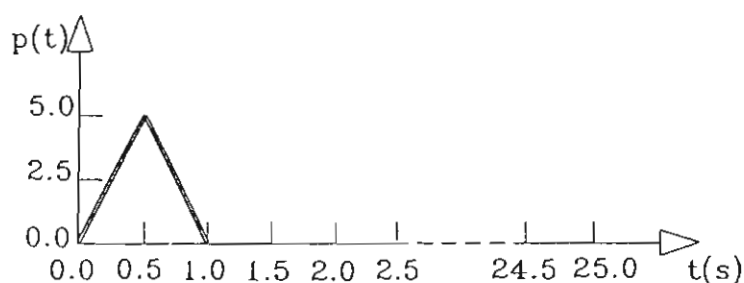
Several examples have been solved in order to validate the capability of the approach developed. However, due to the limited space the results of only one example are here presented: the motion of a short cylinder under impulsive point loads. This example was already solved by Simo and Tarnow (1994) in order to compare the performance of different numerical time integration schemes. The geometry, finite element mesh, material properties and loading conditions are given in Fig.1 and 2. At the reference positions the initial conditions are prescribed to be zero. The cylinder is subjected to short impulsive external loads up to 1 second as shown in Fig. 2. The cylinder motion is computed and traced during a time period of 25 seconds divided into 1250 time steps of $\Delta t=0.02$ sec. The complex motion of the cylinder, from two different views, is shown in Fig. 3, where deformed shapes are drawn for every 2 seconds. Time histories of three vector components of total linear and total angular momentum are presented in Fig. 4 and 5 respectively. It can be seen that all the vector values remain constant during the free motion. As expected, the total input energy (see Fig. 6) of the undamped system is also preserved. However, periodic



$$\begin{aligned} E &= 2 \times 10^8 \text{ N/cm}^2 \\ \nu &= 0.5 \\ \rho_0 &= 1.0 \text{ kg/cm}^3 \end{aligned}$$

$$\begin{aligned} r &= 7.5 \text{ cm} \\ H &= 3.0 \text{ cm} \\ t &= 0.02 \text{ cm} \end{aligned}$$

Figure 1. Geometry, finite element mesh, material properties and loading conditions of the short cylinder.



Angle α	0	$\pi/2$	π	$3/2\pi$
Nodal loads	$[0, -1, -1]^T p(t)$	$[1, 1, 1]^T p(t)$	$[1, 1, 1]^T p(t)$	$[0, -1, -1]^T p(t)$

Figure 2. Time history of external loads

oscillations of internal and kinetic energy can be seen. The total energy computed is slightly higher than the value obtained by Simo and Tarnow (1994). This difference is due to different constants taken for the rotational inertia I_{p0} (here: $\rho_0 H^3/12 = 6.66 \times 10^{-7}$; in the above reference: 3.33×10^{-5}). Throughout the solution process the norms of residuals less than 1.0×10^{-6} were achieved after 6th or 7th iteration exhibiting the quadratic rate of asymptotic convergence. The convergence at various time steps is graphically presented in Fig. 7.

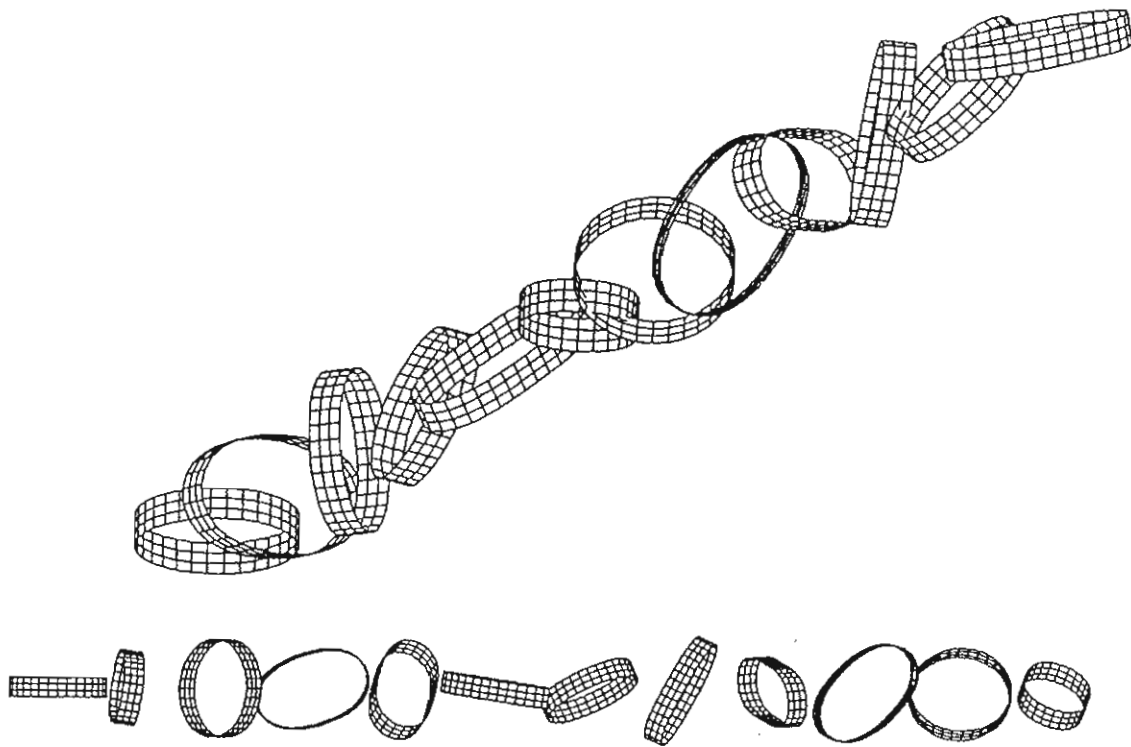


Figure 3. Sequences of deformed finite element mesh for the time interval [0,25 sec] (meshes are drawn for every 2 seconds).

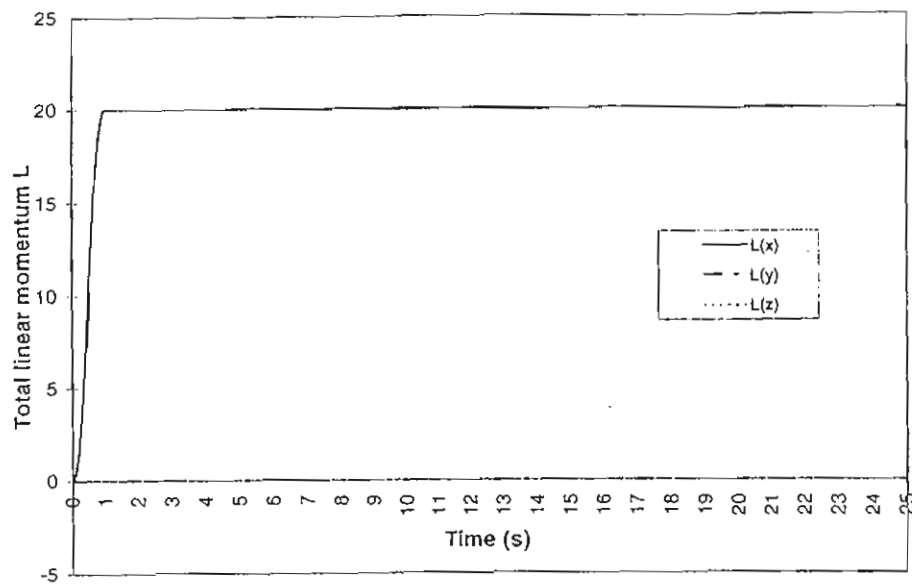


Figure 4. Time history of the vector components of the total linear momentum.

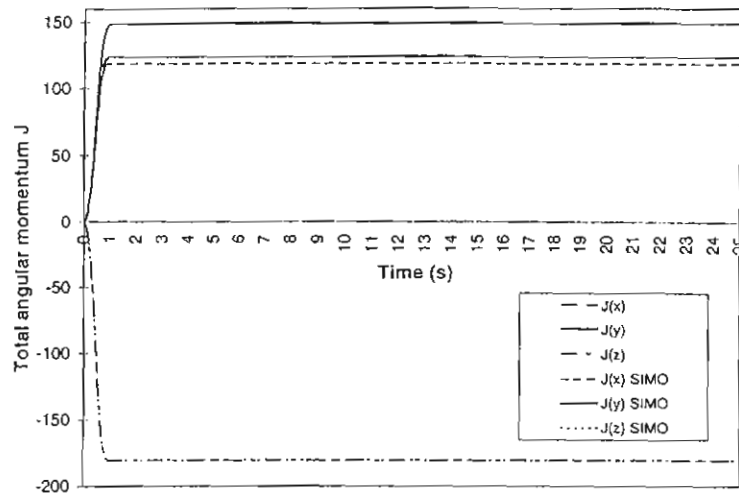


Figure 5. Time history of the vector components of the total angular momentum.

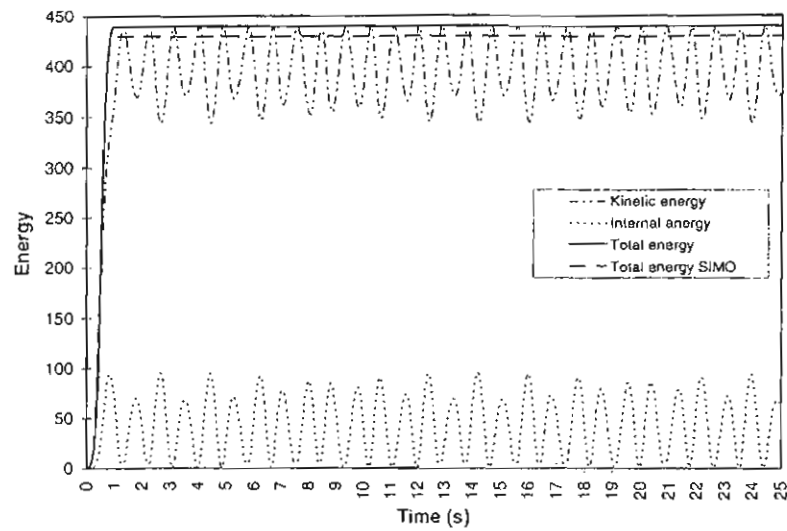


Figure 6. Time history of the kinetic, internal and total energy.

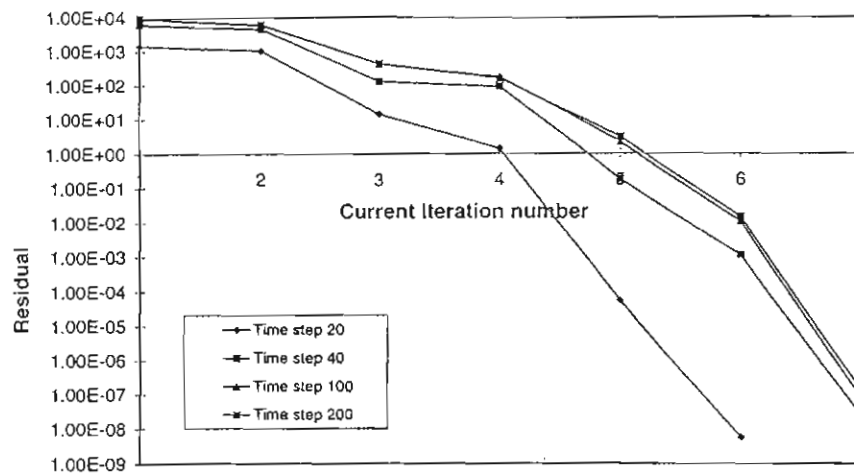


Figure 7. Convergence at various time steps

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EFFICIENT MODELLING OF NONLINEAR HARDENING RESPONSES FOR ELASTOPLASTIC SHELL ANALYSES

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Abstract. An efficient computational strategy for modelling of highly nonlinear hardening responses in elastoplastic analysis of shell structures has been proposed. A closest point projection algorithm employing the Reissner-Mindlin type kinematic model, completely formulated in tensor notation, is presented. A consistent elastoplastic tangent modulus is derived which is found to be unsymmetrical. By using its symmetrized form the quadratic convergence rate in Newton Raphson iteration approach is exhibited, as the numerical examples illustrate. The integration algorithm has been implemented into a layered assumed strain isoparametric finite element, which also permits simulation of geometrical nonlinearity including finite rotations.

1. Introduction

Nonlinear structural analysis requires a great number of load steps and rather large computation time, especially when a complex material model is employed. Fast, accurate and numerically stable algorithms should be available. During recent years an efficient computational strategy based on the closest point projection algorithms in combination with a consistent tangent operator [1,2,3] has been adopted for the simulation of nonlinear material responses. A complex material model including nonlinear kinematic and isotropic hardening has mostly been employed for two-dimensional plane stress and plane strain problems [4,5]. An algorithm for modelling of realistic elastoplastic shell responses is proposed in [6] where the nonlinear kinematic hardening is linearized.

The present contribution is concerned with numerical modelling of nonlinear isotropic and kinematic hardening responses in the elastoplastic analyses of shell structures employing the Reissner-Mindlin type kinematic model. The elastoplastic material model is taken from [7] under the assumptions of small strain, associativity of the flow rule and isothermic material behaviour. The material functions employed in the formulation have been obtained experimentally in [8] for mild steel. In contrast to the formulation presented in [6], the second term in the kinematic hardening equation is considered, which makes the hardening law non-associated. It contributes to accuracy of realistic material modelling, especially when cyclic behaviour is studied. The integration algorithm is based on the closest point projection strategy [9] and consistent elastoplastic tangent operator is derived, which is found to be unsymmetrical due to the assumed nonlinear kinematic hardening model. In order to preserve the numerical efficiency of the global iteration procedure the symmetrized tangent operator is defined.

The computational algorithm has been implemented into the four-noded isoparametric, assumed strain layered finite element [10] based on the Reissner-Mindlin type shell theory, which allows geometrically nonlinear analyses considering finite

rotation. Efficiency of the proposed algorithm is demonstrated by numerical examples, all performed within the finite element system FEMAS [11] developed at the Institute for Statics and Dynamics, Ruhr-University Bochum.

2. Material model

The present elastoplastic material model employs an associative flow rule and the evolution laws of hardening variables proposed by Lehmann [7] with the material functions determined by experimental studies in [8]. The associative expression for the plastic strain rate may be written as

$$\dot{\gamma}_{ij}^p = \lambda \frac{\partial F(\sigma^{ij}, \rho^{ij}, a, T)}{\partial \sigma^{ij}}, \quad (1)$$

where $F(\sigma^{ij}, \rho^{ij}, a)$ is a von Mises-type yield function, λ is the plastic multiplier and σ^{ij} denotes the stress tensor components. ρ^{ij} and a are the internal variables describing the kinematic and isotropic hardening, respectively. T denotes the process temperature.

The kinematic hardening is expressed by the following nonlinear evolutionary equation [7]

$$\dot{\rho}^{ij} = \varsigma \dot{\gamma}^{pij} - \chi \rho^{ij} \sqrt{\dot{\gamma}_i^{pk} \dot{\gamma}_k^{pl}}, \quad (2)$$

with ς and χ as material functions. ρ^{ij} are the deviatoric components of the back stress tensor ρ^{ij} . The material function χ is expressed in dependence on the process temperature by the polynomial in [8], while the value of ς may be obtained by the relation

$$\varsigma = c_1 + c_2 e^{-c_3 A}, \quad (3)$$

where A denotes the second invariant of the back stress deviator and c_1, c_2, c_3 are the temperature dependent coefficients.

The internal variable describing the isotropic hardening is assumed in the form

$$\dot{a} = (S^{ij} - \rho^{ij}) \dot{\gamma}_{ij}^p, \quad (4)$$

where S^{ij} stands for the deviatoric components of the stress tensor. According to [8], the following isotropic hardening model is adopted

$$k^2(a, T) = b_1 + b_2 a + b_3 (1 - e^{b_4 a}), \quad (5)$$

where b_1, b_2, b_3 and b_4 are the coefficients depending on the temperature.

Finally, the yield condition has the form

$$F(\sigma^{ij}, \rho^{ij}, a, T) = (S^{ij} - \rho^{ij})(S_{ij} - \rho_{ij}) - k^2(a, T) = 0. \quad (6)$$

In the present contribution an isothermic material behaviour is assumed. All the temperature - dependent coefficients can be calculated for each temperature level by the polynomial expressions presented in [8].

3. Integration algorithm for Reissner-Mindlin type kinematic model

According to the Reissner-Mindlin type shell kinematic model, the stress and strain measures are described by $\sigma \in \mathbb{R}^8$ and $\gamma \in \mathbb{R}^8$. $\rho \in \mathbb{R}^8$ portrays the back stress tensor describing the kinematic hardening response. In contrast to the standard matrix notation, all deviatoric components of the stress and back stress tensor, $S \in \mathbb{R}^9$ and $\rho' \in \mathbb{R}^9$ are included in the present formulation.

By using equation (1) the plastic strain $\gamma^p \in \mathbb{R}^9$ is expressed by the following tensor components

$$\begin{aligned}\gamma_{\alpha\beta}^p &= 2\lambda \mu_{\alpha\beta\gamma\delta} \eta^{\gamma\delta}, \\ \gamma_{\delta 3}^p &= 2\lambda a_{\delta\epsilon} \eta^{\epsilon 3}.\end{aligned}\quad (7)$$

The last component γ_{33}^p is computed from the incompressibility condition $\gamma_{ij}^p = 0$. In equations (7), $\eta^{\gamma\delta}$ and $\eta^{\epsilon 3}$ represent components of the relative stress tensor $\eta = \sigma - \rho$, while $\mu_{\alpha\beta\gamma\delta}$ and $a_{p\epsilon}$ are the transformation tensor [6] and metric tensor components [12], respectively. The Greek indices take the values 1 and 2 in contrast to the Latin indices which represent numbers 1, 2 and 3. By means of (4) the isotropic hardening variable at time t may be written as

$$^i a = {}^{i-1} a + 4 {}^i \lambda {}^i J_2, \quad (8)$$

where J_2 is the second invariant of the relative stress deviator $\eta' = S - \rho'$. According to (2) and after using the consistency condition (6), the back stress tensor components are updated by the relations

$$\begin{aligned}{}^i \rho^{\alpha\beta} &= {}^i R {}^{i-1} \rho^{\alpha\beta} + 2 {}^i R {}^i \lambda {}^{i-1} \zeta {}^i \eta^{\alpha\beta}, \\ {}^i \rho^{\delta 3} &= {}^i R {}^{i-1} \rho^{\delta 3} + 2 {}^i R {}^i \lambda {}^{i-1} \zeta {}^i \eta^{\delta 3},\end{aligned}\quad (9)$$

where ${}^i R$ is the auxiliary variable given by

$${}^i R = \frac{1}{1 + 2\chi {}^i \lambda {}^i k}, \quad (10)$$

with ${}^i k$ as the isotropic hardening function ${}^i k = k({}^i a)$. It is to note that kinematic hardening variable ζ is computed at the end of the previous time step, at ${}^{i-1} t$. The yield condition (6) may be rewritten in the following form

$$F = 2J_2 - k^2(a) = 0. \quad (11)$$

To integrate the elastoplastic constitutive model a closest point projection scheme [9] is proposed where the predictor phase is expressed as

$$\begin{aligned}{}^i \eta_{\text{trial}}^{\alpha\beta} &= {}^i \sigma_{\text{trial}}^{\alpha\beta} - {}^i R {}^{i-1} \rho^{\alpha\beta}, \\ {}^i \eta_{\text{trial}}^{\delta 3} &= {}^i \sigma_{\text{trial}}^{\delta 3} - {}^i R {}^{i-1} \rho^{\delta 3},\end{aligned}\quad (12)$$

with ${}^i \sigma_{\text{trial}}^{\alpha\beta}$ and ${}^i \sigma_{\text{trial}}^{\delta 3}$ as elastic trial state [9]. Employing the additive decomposition of the strain tensor, the stress state at the end of time step after plastic corrector phase is expressed as

$$\begin{aligned} {}^i\sigma^{\alpha\beta} &= {}^i\sigma_{trial}^{\alpha\beta} - C^{\alpha\beta\gamma\delta} {}^i\gamma_{\gamma\delta}^p, \\ {}^i\sigma^{\delta 3} &= {}^i\sigma_{trial}^{\delta 3} - 2Ga^{\delta e} {}^i\gamma_{\epsilon 3}^p, \end{aligned} \quad (13)$$

where $C^{\alpha\beta\gamma\delta}$ is the elastic material tensor [12], and G denotes shear modulus. By means of equation $\eta = \sigma - \rho$ and using equations (9), (12) and (13), and after inversion, the following expressions for the relative stress tensor components are obtained

$$\begin{aligned} {}^i\eta^{\alpha\beta} &= {}^iA_{\gamma\delta}^{\alpha\beta} {}^i\eta_{trial}^{\gamma\delta}, \\ {}^i\eta^{\delta 3} &= \frac{1}{1 + 2 {}^i\lambda ({}^iR^{i-1} \zeta + 2G)} {}^i\eta_{trial}^{\delta 3}, \end{aligned} \quad (14)$$

where the transformation tensor ${}^iA_{\gamma\delta}^{\alpha\beta}$ can be broken down in the form

$${}^iA_{\gamma\delta}^{\alpha\beta} = \frac{1}{1 + 2 {}^i\lambda ({}^iR^{i-1} \zeta + 2G)} \left[\delta_{\gamma}^{\alpha} \delta_{\delta}^{\beta} + \frac{4 {}^i\lambda G(1-2\nu)}{4 {}^i\lambda G(1+\nu) + 3(1-\nu)(1 + 2 {}^i\lambda {}^iR^{i-1} \zeta)} a_{\gamma\delta} a^{\alpha\beta} \right]. \quad (15)$$

On inserting (14) into the consistency condition (11), nonlinear scalar equation is obtained, which is to be solved for ${}^i\lambda$. In this contribution the Newton iteration method is applied. During the iteration process, the unknown variable ${}^iR$ is also computed numerically by means of the following nonlinear equation obtained by using equations (5), (8) and (11)

$${}^{i-1}a - a + 2\lambda [b_1 + b_2 a + b_3 (1 - e^{b_4 a})] = 0. \quad (16)$$

For a given value of λ this equation is to be solved for isotropic hardening variable a in each iteration step by applying a local iteration scheme. After determination of a , the values of the auxiliary variable ${}^iR$ may be computed by relation (10).

After determination of the plastic multiplier, the updated values of the stresses as well as all internal variables can be calculated. During the computation process the loading/unloading criterion is expressed by the Kuhn-Tucker condition [13]. In order to preserve the numerical efficiency of the global iteration strategy, the elastoplastic tangent modulus consistent with the integration algorithm presented should be derived.

4. Consistent elastoplastic tangent modulus

By forming the derivative of the updated relations presented in the previous section, by means of differentiation of the consistency condition and after some manipulations according to [9], the following differential relationships between the stress and strain components are obtained

$$\begin{aligned} d\sigma^{\alpha\beta} &= C_{ep}^{\alpha\beta\delta\epsilon} d\gamma_{\delta\epsilon} + 2C_{ep}^{\alpha\beta\zeta 3} d\gamma_{\zeta 3}, \\ d\sigma^{\kappa 3} &= C_{ep}^{\kappa 3\delta\epsilon} d\gamma_{\delta\epsilon} + 2C_{ep}^{\kappa 3\zeta 3} d\gamma_{\zeta 3}, \end{aligned} \quad (17)$$

which delivers the expressions for the tensor components of the elastoplastic tangent modulus

$$\begin{aligned}
C_{ep}^{\alpha\beta\delta\epsilon} &= B_{\zeta\eta}^{\alpha\beta} \left[C^{\zeta\eta\delta\epsilon} - \frac{k_1 \mu_{\rho\tau\vartheta\kappa} B_{\psi\omega}^{\rho\tau} C^{\psi\omega\delta\epsilon} D^{\zeta\eta} \eta^{\vartheta\kappa} \right], \\
C_{ep}^{\alpha\beta\zeta^3} &= -\frac{k_1 k_4}{k_3 + T_1 + 2T_2} B_{\gamma\delta}^{\alpha\beta} D^{\gamma\delta} \eta^{\zeta^3}, \\
C_{ep}^{\kappa^3\delta\epsilon} &= -\frac{k_1 k_4}{k_3 + T_1 + 2T_2} \mu_{\alpha\beta\zeta\vartheta} B_{\nu\xi}^{\alpha\beta} C^{\nu\xi\delta\epsilon} H^{\kappa^3} \eta^{\zeta\vartheta}, \\
C_{ep}^{\kappa^3\zeta^3} &= \frac{k_4}{2} \left[a^{\kappa\zeta} - \frac{2k_1 k_4}{k_3 + T_1 + 2T_2} \right] H^{\kappa^3} \eta^{\zeta^3},
\end{aligned} \tag{18}$$

with the following abbreviations

$$\begin{aligned}
B_{\zeta\eta}^{\alpha\beta} &= \frac{1 + 2\lambda(i^{-1}\zeta + \chi k)}{4G\lambda(1 + 2\chi k\lambda) + 2\lambda(i^{-1}\zeta + \chi k) + 1} \left[\delta_{\zeta}^{\alpha} \delta_{\eta}^{\beta} + \frac{4G\lambda(2\nu - 1)k_1}{4G\lambda k_1(3\lambda - 4\nu - 1) - 3(\lambda - 1)} a_{\zeta\eta} a^{\alpha\beta} \right], \\
D^{\zeta\eta} &= \frac{2\mu_{\alpha\beta\gamma\delta} C^{\zeta\eta\gamma\delta}}{1 + 2\lambda(i^{-1}\zeta + \chi k)} \left[(1 + 2\chi k\lambda) \eta^{\alpha\beta} + 2\chi\lambda k_2 \rho^{\alpha\beta} \right], \\
H^{\alpha^3} &= \frac{2}{1 + 2\lambda(i^{-1}\zeta + \chi k)} \left[(1 + 2\chi\lambda k) \eta^{\alpha^3} + k_5 \rho^{\alpha^3} \right], \\
T_1 &= \frac{\mu_{\alpha\beta\gamma\delta} \eta^{\gamma\delta}}{1 + 2\lambda(i^{-1}\zeta + \chi k)} \left[(1 + 2\chi\lambda k) B_{\epsilon\zeta}^{\alpha\beta} D^{\epsilon\zeta} + 2i^{-1}\zeta \eta^{\alpha\beta} - 2\chi k_2 \rho^{\alpha\beta} \right], \\
T_2 &= \frac{a_{\alpha\beta} \eta^{\beta^3}}{1 + 2\lambda(i^{-1}\zeta + \chi k)} \left[k_4(1 + 2\chi\lambda k) H^{\alpha^3} + 2i^{-1}\zeta \eta^{\alpha^3} - 2\chi k_2 \rho^{\alpha^3} \right], \\
k_1 &= 1 - \frac{2\lambda i^{-1}\zeta}{1 + 2\lambda(i^{-1}\zeta + \chi k)}, \\
k_2 &= \frac{2k_3\lambda^2}{k} (b_2 - b_3 b_4 e^{b_4 a}) + k(1 + \lambda b_2 - \lambda b_3 b_4 e^{b_4 a}), \\
k_3 &= \frac{2J_2(b_2 - b_3 b_4 e^{b_4 a})}{1 - 2\lambda(b_2 - b_3 b_4 e^{b_4 a})}, \\
k_4 &= \frac{4G\lambda(i^{-1}\zeta + \chi k) + 2G}{4G\lambda(1 + 2\chi k\lambda) + 2\lambda(i^{-1}\zeta + \chi k) + 1}, \\
k_5 &= \frac{2\chi\lambda^2 k(b_2 - b_3 b_4 e^{b_4 a}) - 2\chi\lambda k}{2\lambda(b_2 - b_3 b_4 e^{b_4 a}) - 1}.
\end{aligned} \tag{19}$$

In these expressions, for notational simplicity, the left upper index "i" referring to all state variables at time t is omitted. It is not difficult to verify that the tensor components C_{ep}^{ijkl} are unsymmetrical with respect to the couple of indices (i j) and (k l), which is related to the assumed nonlinear kinematic hardening response. For $\chi = 0$ the tangent modulus derived is reduced to the symmetrical operator which corresponds

to the expressions presented in [6] for linearized kinematic hardening. For the computation of the stiffness matrix the unsymmetrical tangent operator is to be symmetrized according to the following relation

$$\text{sym } C_{ep}^{ijkl} = \frac{1}{2} (C_{ep}^{ijkl} + C_{ep}^{klij}). \quad (20)$$

The integration algorithm together with the consistent tangent modulus derived in this contribution has been implemented into the four-noded isoparametric finite element formulation [10] based on the assumed strain concept, which permits the simulation of material nonlinearity combined with geometrically nonlinear response assuming finite rotations. For the tracing of nonlinear deformation paths Newton-Raphson and Riks-Wempner-Wessels [14] iteration schemes enhanced with a line search procedure [15] may be applied. The following examples employing Newton-Raphson approach and line search strategy show the numerical efficiency of the algorithm.

5. Numerical examples

The perforated strip under uniaxial extension is considered first. Due to the symmetry, only one quarter is discretized, as shown in Figure 1a. The material model described in the previous sections with the material functions, calculated according to [8] for a German mild steel St 37 at process temperature of 25°C, is applied. The plastic zones for two load levels, 107 and 119 N/mm², depicted on the undeformed configuration are shown in Figure 1b. The numerical values of the residual energy norm in the global iteration process for the load levels corresponding to the plastic zones presented are displayed in Table 1.

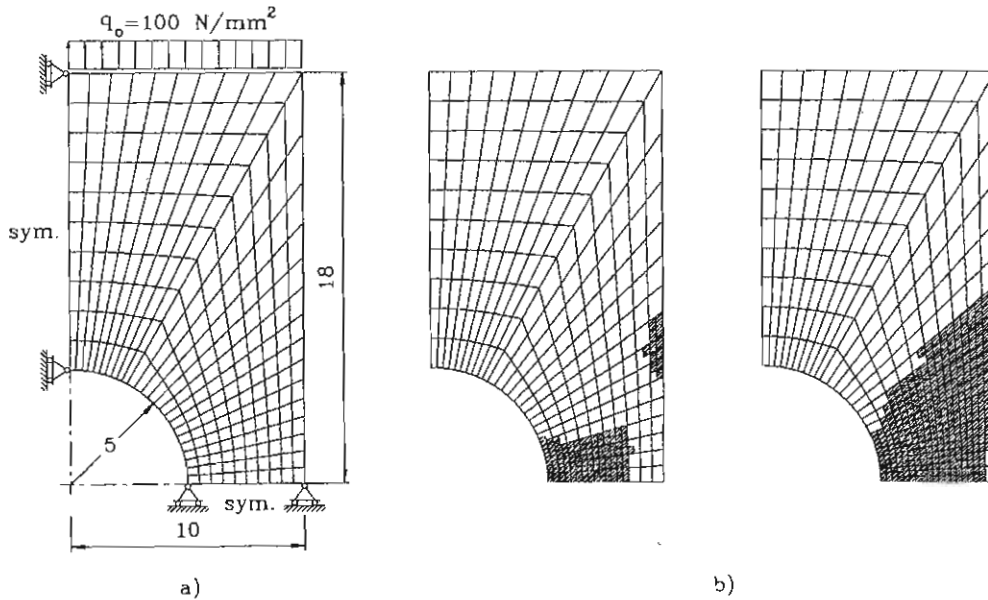
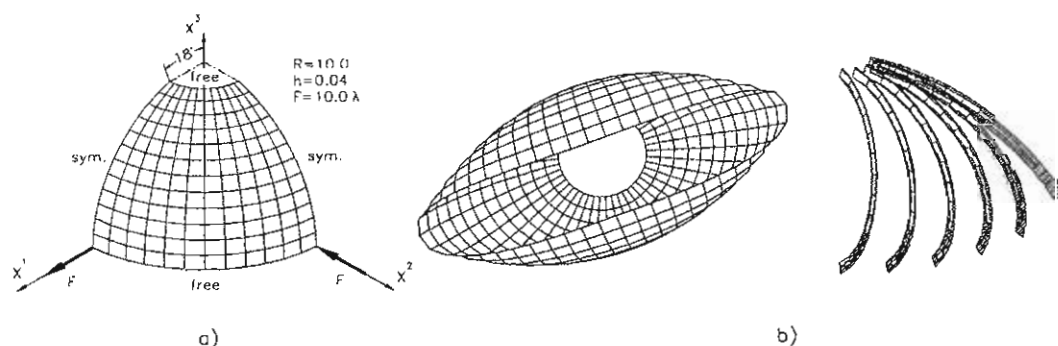


Figure 1. a) Perforated strip geometry; b) Plastic zones at load levels of 107 N/mm² and 119 N/mm²

Table 1. Convergence of global energy error norms for perforated strip

Iteration	Load factor 1.07	Load factor 1.19
1	2.012201D-02	9.401334D-03
2	8.258396D-03	1.703055D-04
3	2.374420D-04	1.854651D-08
4	3.533904D-06	4.621656D-10
5	4.892508D-09	1.748092D-12
6	6.360877D-13	2.855969D-14
7	3.108154D-16	2.145565D-16
8	9.814994D-18	2.426946D-18

As a second example, the pinched hemispherical shell with a hole of 18° subjected to two pairs of inward and outward forces is analysed. Employing the symmetry again, one quadrant is modelled by a finite element mesh of 12×12 , Figure 2a. This shell geometry has been used in [10] in order to demonstrate the finite element capability in strongly geometrically nonlinear range. In addition to geometrical nonlinearity, this example is concerned with the elastoplastic analysis assuming the material model as in the previous computation. In order to predict the propagation of the plastic zones through the shell sections in a precise manner, numerical integration over the thickness is performed by using 10 layers. The deformed configuration and the spreading of plastic zones through the sections traced by the meridional planes containing the axis of rotation at load factor $\lambda = 25$ are presented in Figure 2b. The convergence rate for the global iteration scheme, expressed by the energy norms for the load level corresponding to the presented deformed pattern is recorded in Table 2.

Figure 2. a) Geometry of hemispherical shell; b) Deformed configuration and spread of plastic zones through shell sections at load factor $\lambda = 25$ Table 2. Convergence of global energy error norms for hemispherical shell at load factor $\lambda = 25$

Iteration		Iteration	
1	2.728529D-01	5	3.269695D-07
2	7.814503D-02	6	8.415102D-11
3	1.023230D-02	7	2.829330D-14
4	2.837153D-04	8	9.656304D-18

6. Conclusion

An efficient numerical simulation of highly nonlinear isotropic and kinematic hardening responses in elastoplastic analysis of shell structures has been performed. A closest point projection algorithm for Reissner-Mindlin type kinematic model completely derived in tensor formulation has been presented. The tensor approach allows all of the

nine stress deviator components to be explicitly included in the formulation which makes the advantage over the classical matrix notation.

On applying consistent linearization of the projection algorithm, the elastoplastic tangent modulus is derived, which is found to be unsymmetrical due to the nonlinear kinematic hardening model. By the use of arithmetic mean the symmetrized tangent operator is defined which does not cause loss of the quadratic rate of convergence in global solution procedure. A quadratic convergence rate in the Newton-Raphson iteration approach is clearly demonstrated by numerical examples. Robustness and numerical stability of the proposed algorithm are also exhibited when material nonlinearity is combined with highly geometrically nonlinear responses.

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Trace Function Based Damage Identification

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ABSTRACT

A trace function based method is developed for identifying structural damage on the basis of a number of measured eigenfrequencies of the damaged structure. The number of trace functions considered in an iterative Newton-Raphson scheme is equal to the number of available measured eigenvalues. Knowledge of the damaged eigenvectors is not required. Scalar reduction of element stiffnesses is used to represent damage. Several exclusion criteria are adopted to reduce a number of elements to be included in the iteration process. Feasibility of the method is demonstrated using a simple model problem.

1 INTRODUCTION

A number of numerical techniques have been proposed for the estimation of structural damage on the basis of modified spectral properties, which typically include a finite number of both modified eigenvalues and eigenvectors or modified eigenvalues only, as they are much easier to establish. Various methods differ in the way the underlying dynamic equations are utilised and whether they require the original (reference) eigenvalues and eigenvectors, whether they require full modal data etc. The inverse problem is inherently ill-conditioned and is often in practical situations associated with non unique solutions. The procedure presented here explores the suitability of utilising the trace theorem as the mathematical framework to develop a damage detection procedure.

2 TRACE FUNCTION AND DAMAGE FACTORS

Trace theorem approximations [1], [2] are based on the well-known iterative methods for finding roots of the characteristic polynomial. These approximations are remarkable in that the coefficients of the characteristic polynomial need not be calculated explicitly. Traditionally trace theorem based methods were used to approximate the modified eigenvalues λ_{ia} associated with a modification of the stiffness matrix $\mathbf{K}_0 + \Delta\mathbf{K}$ without having to solve a modified eigenvalue problem

$$\lambda_{ia} = \lambda_{0a} + \frac{1}{\text{trace of } \left[\left[\mathbf{K}_0 + \Delta\mathbf{K} - \lambda_{0a} \mathbf{M} \right]^{-1} \mathbf{M} \right]}$$

In the context of the damage identification from the several known modified eigenvalues, the situation is reverse, where a number of modified eigenvalues are known (from measurements) and a modified stiffness matrix needs to be established. Mass matrix is usually assumed to remain unchanged.

Modified stiffness matrix is represented as a difference between the reference (original, undamaged) stiffness matrix and a scalar change denoted by damage factors (α_j) of the expanded element stiffness matrices (expressed in all dofs), such that

$$\mathbf{K} = \mathbf{K}_0 - \sum_{j=1, ne} \alpha_j \mathbf{K}^j = \sum_{j=1, ne} \mathbf{K}^j - \sum_{j=1, ne} \alpha_j \mathbf{K}^j,$$

where ne is number of all elements in the structure, $\mathbf{K}_0$ is reference model stiffness matrix and $\mathbf{K}^j$ is j th element expanded stiffness matrix expressed in all dofs corresponding to the reference model (i.e. size $n*n$). Values for damage factors should be in the range $0.0 \leq \alpha_j \leq 1.0$ to guarantee the physical meaning of the model. Clearly, during the iteration procedure there is a possibility that some damage factors α_j are temporarily less than zero, what implies a 'numerical increase' of the stiffness of the j th element. In damage identification procedures such elements are usually supposed to have suffered no damage and are subsequently excluded from the damage identification

algorithms. Similarly, zero or near zero value of the element damage factor α_j implies that the j th element has suffered no damage.

For a correctly assessed modified stiffness matrix, the trace function $f_i(\lambda_{id}) = f_i(\mathbf{K}, \mathbf{M}, \lambda_{id})$ tends to infinity for each of the available measured damaged eigenvalues λ_{id} , therefore an inverse of the trace function equals to zero.

$$\frac{1}{f_i(\mathbf{K}, \mathbf{M}, \lambda_{id})} \Rightarrow 0.$$

As trace functions depend on damage factors α_j , following matrix algebra [3], a Newton Raphson recursive iteration procedure can be established for the evaluation of successive iterates for the vector of element damage factors α , i.e.

$$\alpha^{n+1} = \alpha^n - \begin{bmatrix} \text{trace of } [\bar{\mathbf{K}}_1 \mathbf{K}^1 \bar{\mathbf{K}}_1 \mathbf{M}] & \dots & \dots & \text{trace of } [\bar{\mathbf{K}}_1 \mathbf{K}^{ne} \bar{\mathbf{K}}_1 \mathbf{M}] \\ \vdots & & \ddots & \vdots \\ \text{trace of } [\bar{\mathbf{K}}_{nf} \mathbf{K}^1 \bar{\mathbf{K}}_{nf} \mathbf{M}] & \dots & \dots & \text{trace of } [\bar{\mathbf{K}}_{nf} \mathbf{K}^{ne} \bar{\mathbf{K}}_{nf} \mathbf{M}] \end{bmatrix}^{-1} \times$$

$$\times \begin{Bmatrix} \text{trace of } [\bar{\mathbf{K}}_1 \mathbf{M}] \\ \vdots \\ \text{trace of } [\bar{\mathbf{K}}_{nf} \mathbf{M}] \end{Bmatrix}$$

where $\bar{\mathbf{K}}_i$ represents an abbreviated form for the inverse of the dynamic stiffness matrix

$$\bar{\mathbf{K}}_i = \left[\mathbf{K}_0 - \sum_{j=1, ne} \alpha_j \mathbf{K}^j - \lambda_i \mathbf{M} \right]^{-1}$$

A unique solution is obtained for the case when the number of the available equations nf (i.e. a number of measured frequencies) equals the number of unknowns ne (number of elements). When the number of elements to be considered in the iteration procedure is greater than the number of available equations, the system is underdetermined and may have several solutions. A solution can then be found by using the Pseudo-Inverse techniques (such as Moore-Penrose generalised inverse [5], [6]). In the case when the number of equations is greater than the number of unknowns, i.e. the system is overdetermined, a solution can be found by least square technique.

3 TRACE FUNCTION BASED DAMAGE IDENTIFICATION PROCEDURE

Starting from the nf known modified frequencies, the damage identification iterative procedure is divided into a series of initiation and iteration steps.

1. *Selection of elements to be included in the iteration process as possible damaged locations.*

Two criteria are used, based on the information from the reference model. Firstly a measure of participation of each element with respect to the strongest element of the structure (element with highest strain energy participation), and secondly a measure of participation of each element with respect to the average element strain energy for nf eigenfrequencies. The number of elements, which are finally selected to be included in the iteration process, is denoted by ner .

2. Ordering of modes by eigenfrequency closeness

The ratio between the suitably paired damaged and undamaged eigenfrequency is calculated, and modes are ordered according to their closeness to the undamaged eigenfrequencies. Pairing of modes is secured by the usual MAC criteria.

$$r_i = \frac{\omega_i^{damaged}}{\omega_i^{undamaged}}, i = 1, 2, \dots, nf$$

The trace function of the eigenvalue corresponding to the maximum ratio is chosen as the initial one. Initial damage factors for element stiffness matrices are set as $\alpha_j^1 = 1.0 - \max\{r_i\}$, $i = 1, 2, \dots, nf$; $j = 1, 2, \dots, ner$, whereas for the excluded elements the damage factors remain $\alpha_j^1 = 0$, $j = ner + 1, ne$.

Following steps 1 and 2 (initiation phase), a two stage iteration procedure, involving an ever increasing number of trace functions up to the number of available measured frequencies, has been implemented.

STAGE I - ITERATION

3. Crude approximation of damage factors

Iteration process begins for the eigenfrequency with the highest ratio ($\max\{r_i\}$), i.e. the starting trace function $f_1 = f(\lambda_k^d)$, where the index k denotes the eigenfrequency which is the closest to the corresponding undamaged eigenfrequency. A Newton Raphson process is conducted to achieve convergence for the damage vector α . Another measured damaged eigenvalue is added and the corresponding trace function is included and the Newton Raphson process is repeated. Stage I comprises a continuously increasing number of measured damaged eigenfrequencies and corresponding trace functions in every successive iteration. In every iteration the starting vector of damage factors is set to be the solution obtained at earlier iteration which was achieved for a smaller number of trace functions included. In Stage I it was found to be sufficient to adopt a very low value for tolerance ϵ_j as a convergence criteria.

STAGE II - ITERATION

4. Finer approximation of damage factors

Once the complete number of measured damaged eigenfrequencies is included into the Stage I iteration process, Stage II - iteration is used to iterate the vector of damage

factors to a higher tolerance ε_{II} ($\varepsilon_{II} \ll \varepsilon_I$), i.e. when $\Delta\alpha_j = \alpha_j^{n+1} - \alpha_j^n \leq \varepsilon_{II}$; $j = 1, ner$, the Stage II iteration is completed.

5. Exclusion of elements

If as a result the damage factors are found to be negative (i.e. element stiffness is formally increasing) or there is practically no damage detected as damage factors are smaller than some threshold value, such elements are excluded from the further iteration process as candidates for possible damage, and the corresponding damage factor is set to zero). Recalculation of the damage factors needs to be done whenever the new number of potential damaged elements ner is considered. The iteration process continues again at step 3, with the initial values set to $\alpha^1 = \alpha^{n+1}$, except for elements for which $\alpha_j^{n+1} \leq \text{threshold value}$.

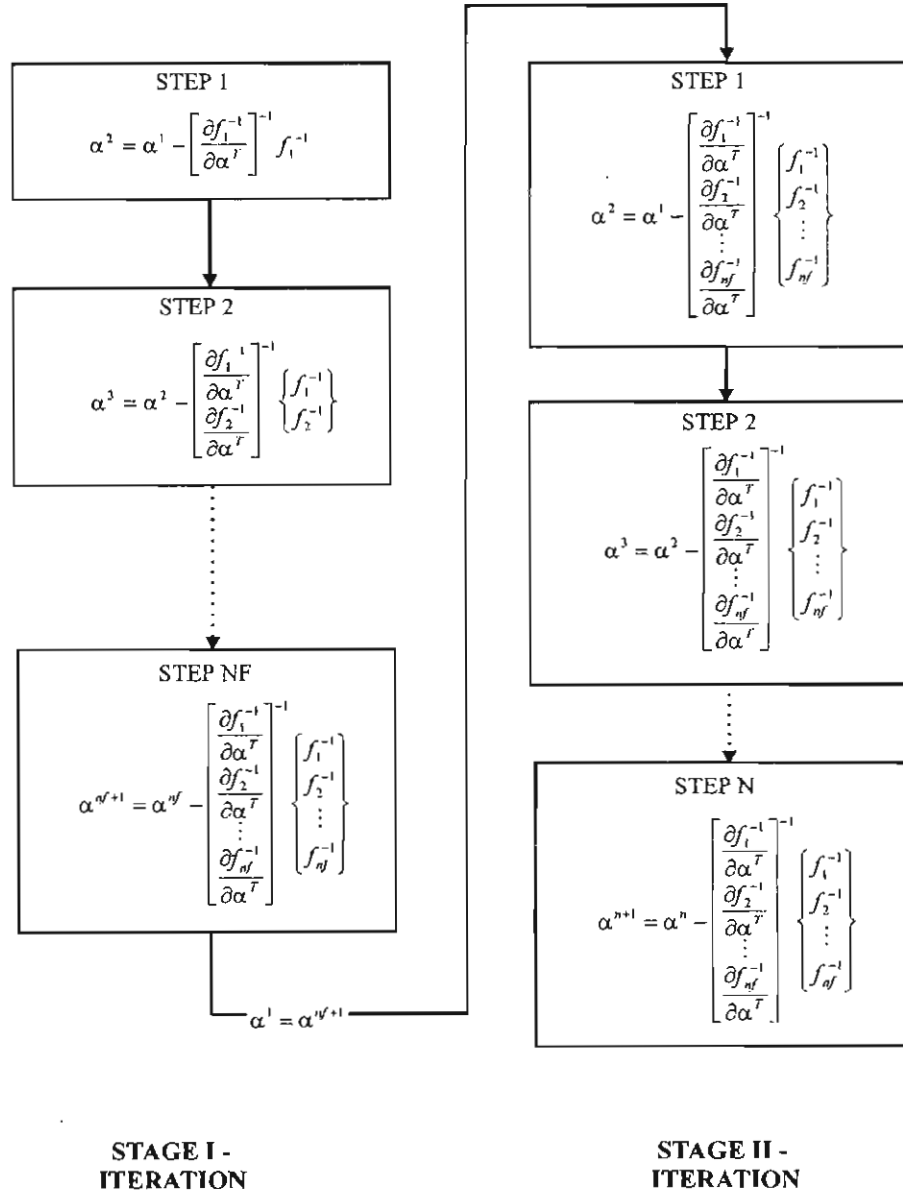


Fig 1: Stage I (ever increasing number of trace functions-up to nf -considered to a looser tolerance) and Stage II (all nf trace function considered to a tighter tolerance)

4 CHEN-GARBA 3 DOFS MODEL PROBLEM

Chen and Garba [7] presented a damage identification procedure, based on the known damaged eigenvalues and the corresponding complete eigenvectors. For a simple 3 dofs model problem, the damage assessment was completed successfully when at least two damaged eigenfrequencies and the corresponding eigenvectors were measured. The Chen-Garba model problem will be used here to demonstrate the trace function based damage identification method. Fig. 2 illustrates both the undamaged system and the damaged system, with the damage assumed to be caused by the absence of spring k_4 .

Initial (reference) system stiffness and mass matrices are given by

$$\mathbf{K}_0 = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 4 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

where the “expanded” element stiffness matrices (expressed in all dofs) are written as

$$\mathbf{K}^1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{K}^2 = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{K}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \quad \mathbf{K}^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There is no change in the mass matrix.

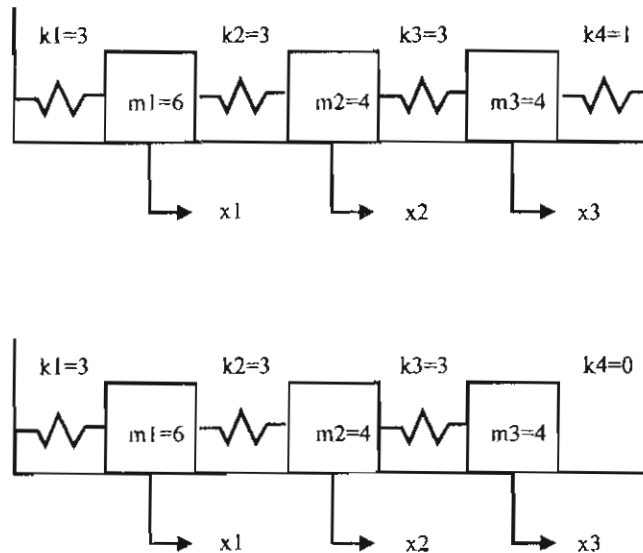


Fig 2: Undamaged and Damaged 3 dofs Chen-Garba model problem

The eigenvalues and eigenvectors of the undamaged system are presented in Table 1, whereas the element strain energy participation is given in Table 2. The eigenvalues of damaged system and the ratios between the damaged and the undamaged eigenvalues are included in Table 3.

Table 1: Eigenvalues and eigenvectors of the reference (undamaged) structure

Mode	1	2	3
Eigenvalue	0.25	1.00	2.25
Mode shape for x_1	2.0	1.0	2.0
Mode shape for x_2	3.0	0.0	-5.0
Mode shape for x_3	3.0	-1.0	3.0

Table 2: Strain energy distribution for undamaged structure

Mode	1	2	3	average
Element 1	3.33 %	50.00 %	30.00 %	27.78 %
Element 2	40.83 %	12.50 %	30.00 %	27.78 %
Element 3	53.33 %	0.00 %	30.00 %	27.78 %
Element 4	2.50 %	37.50 %	10.00 %	16.67 %

Table 3: Eigenvalues of the damaged system and the corresponding frequency ratios $\frac{\lambda_{id}}{\lambda_i}$

Mode	1	2	3
Eigenvalue	0.1406	0.909	2.200
$\frac{\lambda_{id}}{\lambda_i}$	0.562	0.909	0.978

Stage I and Stage II iteration process for damage factors based on the assumption that only two (*first and second*) damage eigenvalue are known is illustrated in Fig 3.

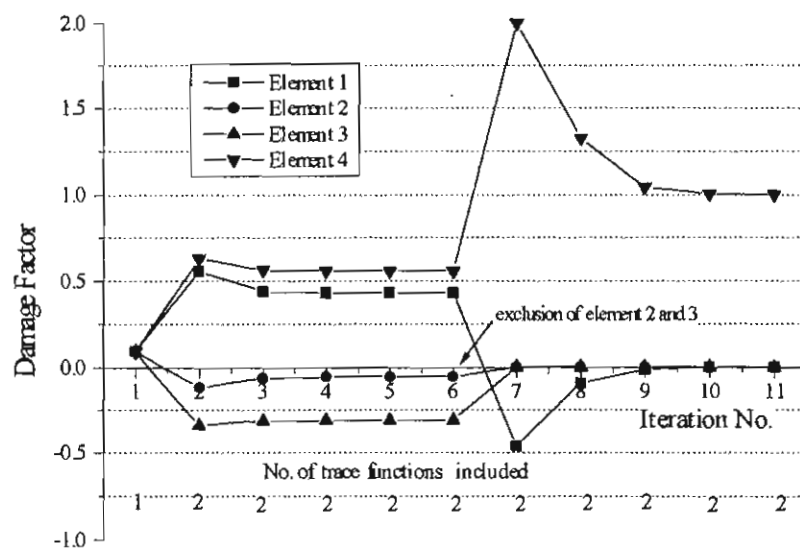


Fig 3: Stage I and Stage II iteration values for damage factors obtained with the 1st and 2nd damaged eigenvalues known

5 CONCLUSION

Experience with analysing larger model problems indicate that the convergence is usually achieved in only a few steps as the first derivatives of the inverse trace function are computed exactly at each step of the iteration. The number of elements selected as possible locations of damage can be significantly reduced by applying the exclusion criteria, which often leads to a unique solution of the damage identification problem. Knowledge of the eigenfrequencies and eigenvectors of the initial (reference) structure is required only for the initial selection of elements, which are considered as 'important' in the modal analysis. *Knowledge of eigenvectors of the damaged structure is not required. Eigenvectors do not have to be recalculated* in every step of the damage identification procedure. Matrices $\mathbf{K}^j$ which represent the contribution of elements to the global stiffness matrix are usually sparse, and as a result the computational time spent for procedures like matrix multiplication is highly reduced. Derivatives of the inverse trace function are also trace functions and only diagonal terms of the matrices have to be computed.

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Numerical Calculation of the Probability Distribution Function of the Critical Load in a Pressure Vessel

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ABSTRACT

The Cumulative Probability Distribution Function of the critical load for the elastic instability problem in a thin-walled pressure vessel with non perfectly spherical heads is numerically determined.

The critical load is an implicit function f of two Design Random Variables (x_1 , x_2) and is obtained from a Finite Element Method code.

The Cumulative Probability Distribution Function is calculated by means of an Advanced Mean Value Method.

This method uses a linear approximation of the function f near the mean value and the Rackwitz-Fiessler version of the First Order Reliability Method algorithm as Fast Probability Integration Method.

The results are compared with those given by a Monte Carlo simulation.

1. INTRODUCTION

Probabilistic structural analysis for the treatment of uncertainties in design, at least concerning First Level Methods [1], is becoming usual practice for civil engineers.

Every engineer is often concerned with problems that cannot be properly resolved in a deterministic way. In mechanical engineering the complicated problems in which machine elements are involved make still more popular the use safety factors. These factors, though, do not account properly of the probability distribution characterising loads, material parameters and geometry.

In many cases the first two statistical moments of a quantity can be obtained with elementary methods, but the characteristic function must be expressed in an explicit form.

An alternative procedure is to calculate the Cumulative Distribution Function (CDF) of the quantity of interest (typically the safety margin) in the whole range of the function or in some particular intervals. In the field of probabilistic design of structures the quantities of interest are often numerically evaluated by Finite Element Method (FEM) codes. Monte Carlo simulations are therefore impossible to perform, in the most cases, because of the large amount of computational time and memory requested. To determine the Probability Distribution Function (PDF) or the Cumulative Distribution Function of an implicit function f of Design Random Variables (DRV) approximated techniques like the Advanced Mean Value Method (AMV) [2] can be used.

In this paper AMV with linear approximation of $f(x_1, x_2)$ (AMVFO) is used.

The Cumulative Probability Distribution Function (CDF) is numerically calculated using the Rackwitz-Fiessler [5] version of the First Order Reliability Method (FORM)[3,4] algorithm in, valid for DRV with any kind of distribution. Only independent DRV are used since any case can be reduced to this by proper transformation. The results are compared with those given by a Monte Carlo simulation which took 10 minutes on a Intel Pentium® 200-based PC with 32 MB RAM for a simple case with less than 30 finite elements.

2. THE ADVANCED MEAN VALUE METHOD

This method is suitable when the physical quantity of interest is an implicit function of the design variables. A typical example is the calculation of the quantity by deterministic FEM codes.

In this case Monte Carlo techniques and iterative methods [2] are not practical because the need to run the FEM code at each iteration.

The implicit function is thus approximated by a linear expansion near the mean values of the design variables

$$f(X) \approx a_0 + \sum_{i=1}^n a_i (X_i - \mu_i)$$
$$a_i = \frac{\Delta f}{\Delta X_i}$$

The coefficients are numerically evaluated by arbitrary perturbations near the mean values. This way $n+1$ evaluations of the function are needed, where n is the number of

DRV. Actually, $n+2$ evaluations are performed, in order to improve the degree of approximation.

Then through the FORM algorithm the CDF and the design points in k (ad libitum) values of f are determined. These points form a rough approximation (MV) of the CDF of f .

Subsequently f is recalculated by means of the FEM code in each of the design points previously determined and a better approximation is obtained (AMV).

3. THE FIRST ORDER RELIABILITY METHOD

An implementation of a FORM algorithm in the Rackwitz-Fiessler version was performed. To determine the CDF the integral of the Joined PDF has to be evaluated over the domain defined by the implicit function

$$P(Z < z_0) = \text{CDF}(z_0) = \int_{Z < z_0} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

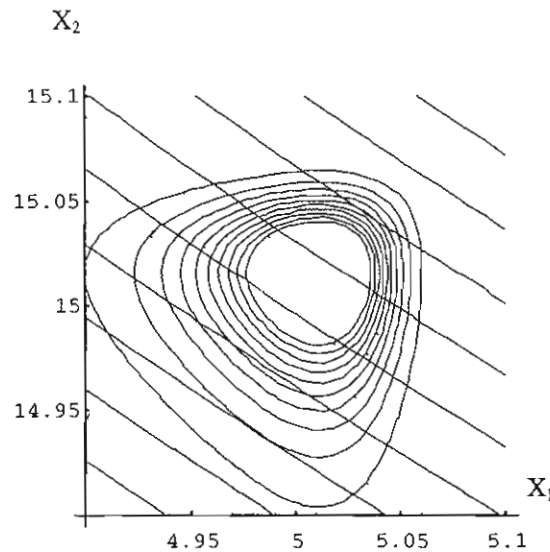


Fig. 1 Joined PDF of the design random variables and function f in the original Cartesian axes

By varying z_0 , it is possible to determine the whole CDF by points.

To evaluate the integral the random variables are reduced to Standard Normal Random Variables (SNRV),

$$x_i = \frac{X_i - \mu_i}{\sigma_i}$$

and the Reliability Index [6], defined as

$$\beta = \min\left(\sqrt{\sum x_i^2}\right)$$

is calculated. It represents the distance of the function graph from the origin.

If the function is not linear, it is approximated by the hyperplane tangent in the design point, which is determined by iterations.

It is possible to use non-Normal DRV by calculating the mean value and the standard deviation of the Normal DRV which have the same PDF and CDF of the non-Normal DRV in the design point. With these two quantities known, we can reduce again to Standard Normal DRV and evaluate the design point.

This way it is possible to find a β Normal-equivalent by iterations.

Once determined β , the CDF is simply $P = \Phi(-\beta)$ where ϕ is the CDF of the SNRV.

4. APPLICATION

The above illustrated methods were applied to the buckling problem in a thin-walled pressure vessel with non perfectly spherical heads. The profile of the head is obtained by revolving a Bezier curve defined by three poles. The critical load is the quantity we want to determine, and it is a function of the position of the poles. For simplicity, only the coordinates of only one pole C (Fig.2) are allowed to vary, which is equivalent to allowing the variation of the tangents in the points A and B.

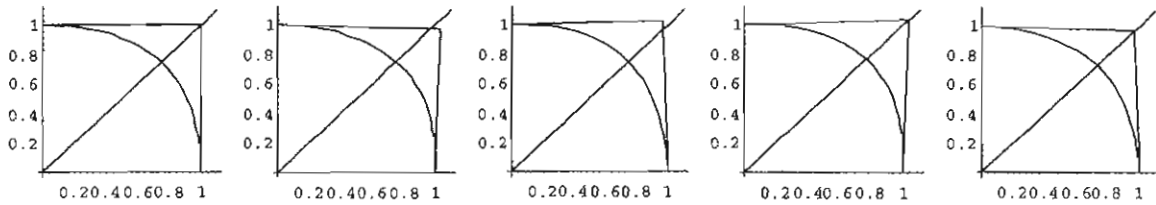


Fig. 2 Variation in the Vessel head.

This choice the two coordinates as design random variables requires the calculation of a new geometry, slightly different from the previous one, every time.

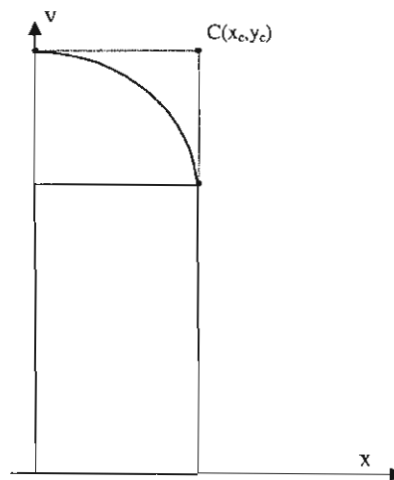


Fig. 3 Vessel geometry

4.1 The Buckling Problem

To determinate the value of the critical load, the displacements are assumed to be very small, until the sudden collapse of the vessel with great strain and stresses.

Under this assumption, the critical load can be predicted for structures that become unstable with negligible prebuckling displacements.

The problem is non-linear, because the displacement and the stress do not grow at the same rate of the load.

This is a case of geometric non-linearity; the hypothesis of little displacements and strains is no longer valid.

In this situation the equilibrium of the structure must be calculated considering the deformed body and the strain-displacement relationship must account for second order terms.

4.2 The FEM Codes

Two FEM codes [7] are used to determine the critical load; the first one calculates the statics of the vessel and the second one the critical load.

The codes are based upon axisymmetric conic shell elements and are written in FORTRAN.

They are managed by a main program that read the inputs, generates the mesh and calls them iteratively in order to perform a Monte Carlo simulation.

4.3 Results

Most of the graphs are plotted using *Mathematica*[®], which was also used for the random number generation in the Monte Carlo simulation.

A Weibull Distribution with mean value 5 m and standard deviation 0.034 m is chosen for x_p whereas a Weibull Distribution with mean value 15 m and standard deviation 0.034m is used for y_p . This corresponds to 2° of variation in the tangent. The same mean value (58.68 Mpa) and the same standard deviation (1.918 Mpa) are obtained both with the Monte Carlo simulation and the AMV.

Fig. 4 shows the PDF of f evaluated as relative frequencies in intervals, obtained by the Monte Carlo Simulation.

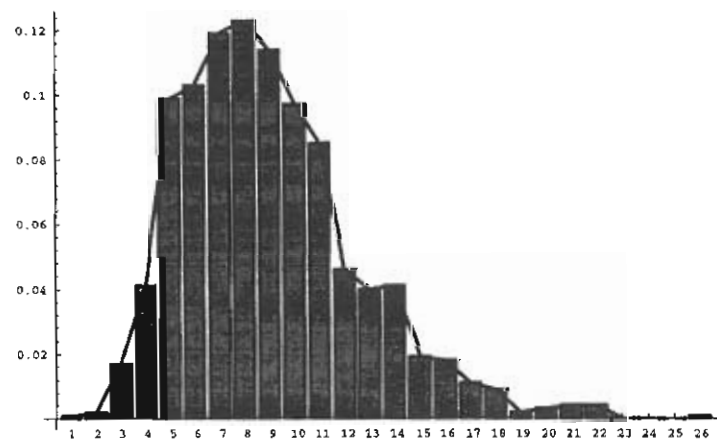


Fig. 4 PDF of f .

Fig. 5a compares the CDF obtained by:

- 1) the Monte Carlo simulation;
- 2) forcing the distribution of the critical load to be Normal;

3) forcing the distribution of the critical load to be Weibull, using as the parameters given by the simulation;

The curves obtained by means of the methods MV and AMV are shown in Fig.5b;

Fig. 5c represents the AMV curve and the Gaussian obtained with the simulation parameters. The AMV gives a better approximation than the Gaussian, whose parameters are hardly determined with precision.

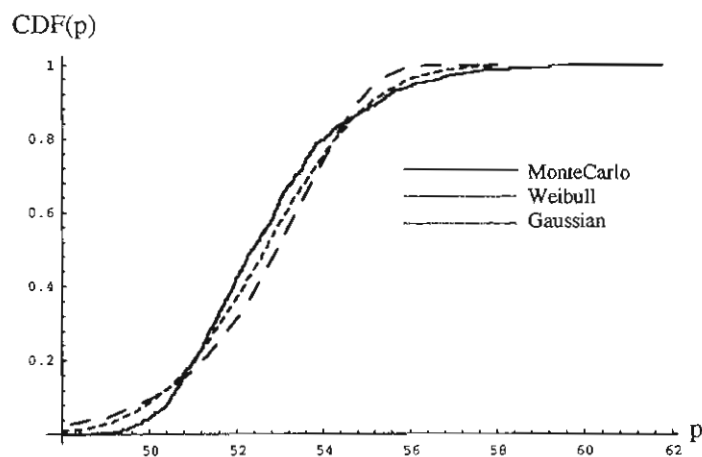


Fig. 5a

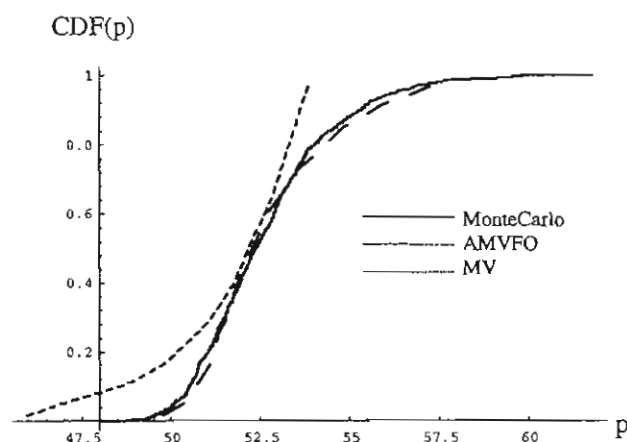


Fig. 5b

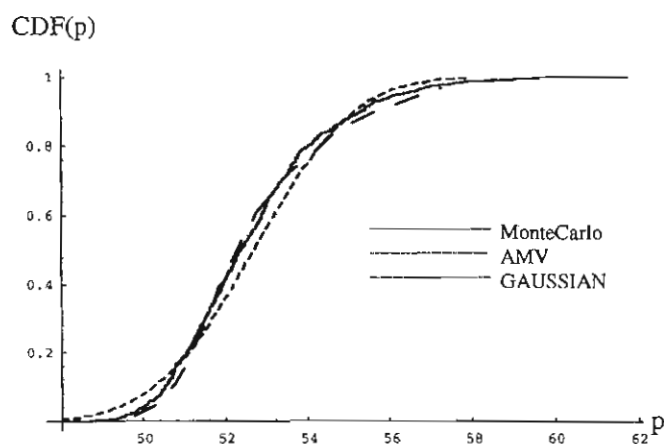


Fig. 5c

Finally, Fig. 6 shows the strain diagram corresponding to the calculated critical load

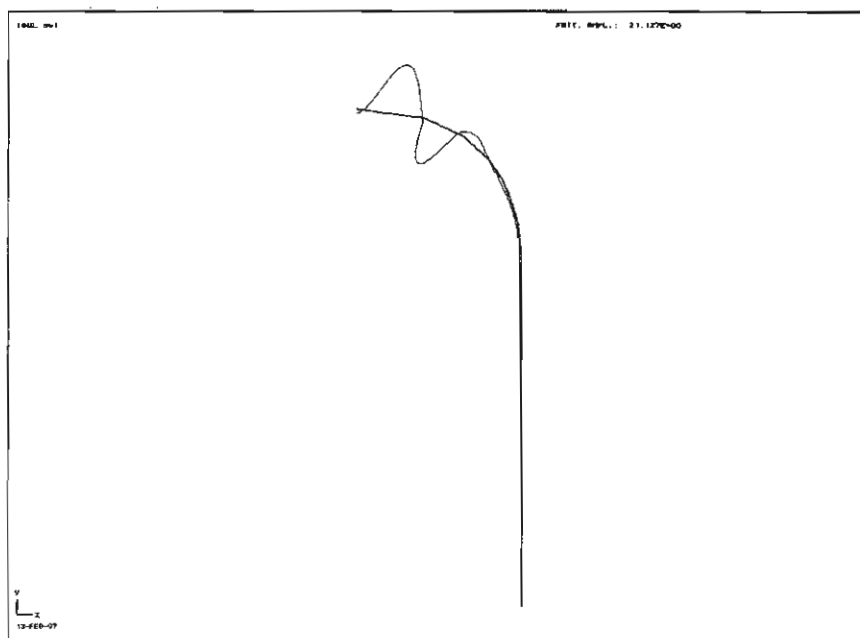


Fig. 6 The strain diagram for the vessel.

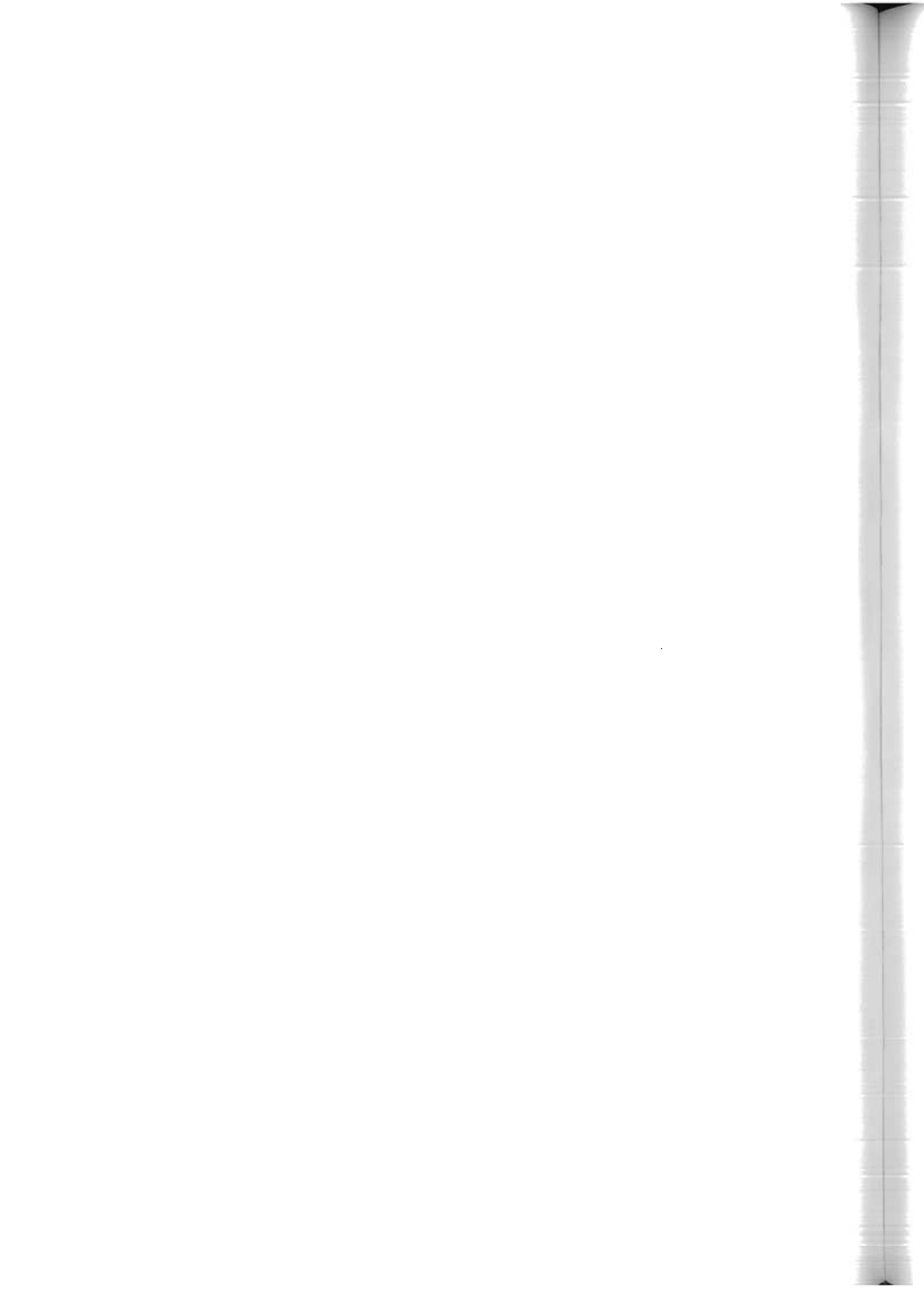
5. SUMMARY

The AMV method gives good approximations of the CDF of implicit functions. Knowledge of the CDF is very useful for the probabilistic design of structures and allows all the causes of uncertainties to be taken into account, instead of utilising crude safety factors.

The FEM implicit function needs to be evaluated $2n+1+k$ times, where n is the number of DRV, and k the number of points in which the CDF is calculated. In the case described in this paper $n=2$ and $k=15$ so we evaluated the function 20 times, as opposed to thousands of evaluations needed by the Monte Carlo simulation.

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Improved Constant Curvature Triangle Bending Element

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Introduction

An improved three-node Mindlin plate element with 1 point integration has been developed. The element is an extension of the CCT (Constant Curvature Triangle) originally derived by Řeřicha [1]. It will be shown that an addition of the simulated antisymmetric bending mode significantly improves the performance in most of cases. Because of its simplicity, this element can be effectively used in non-linear analysis.

Various results are presented to show the improved behavior of the developed ICCT element over similar plate elements.

Derivations

One of the most appealing features is that the strain fields are constant over the entire area of the element. Its derivation is based on the beam elements. Similarly to DKT and CST the CCT has nine degrees of freedom with vertical deflection w_i and two rotations φ_{xi} , φ_{yi} at each node. The functions w , φ_x , φ_y are independent, and consequently the shearing deformation of the cross section is allowed (Mindlin hypothesis).

The fundamental step is the construction of the basic function Fig.(1). Let us assume that the deflection w and both rotations φ_x , φ_y are approximated using linear interpolation in the form

$$w = \sum_{i=1,3} w_i L_i, \quad \varphi_x = \sum_{i=1,3} \varphi_{xi} L_i, \quad \varphi_y = \sum_{i=1,3} \varphi_{yi} L_i. \quad (1)$$

The rotation of the pseudo-normal at an arbitrary point along the edge 23 can be written as

$$\varphi_1 = \varphi(\xi) \cdot \mathbf{n}_1. \quad (2)$$

The variation of the function $\varphi_1 = \varphi_1(\xi)$ is shown in Fig. 1a. The nondimensional coordinate ξ varies from 1 to 0 between points 23. Let us now express the function φ_1 as

$$\varphi_1(\xi) = \frac{\varphi_{31} + \varphi_{21}}{2} + (\varphi_{31} - \varphi_{21})(\xi - 0,5). \quad (3)$$

The first term in eq. 3 corresponds to the skew-symmetric deformation of the element side with zero deflections and constant shear stress distribution over the element, Fig.1c. The second term then describes the symmetric deformations and constant shear stress, Fig.1d. The bending deformation $w_{23 \text{ sym}}$ can be obtained integrating

$$\frac{1}{l_1} \frac{dw_{sym}}{d\xi} \Big|_{23} = (\varphi_{31} - \varphi_{21})(\xi - 0.5), \quad (4)$$

resulting in

$$w_{sym} \Big|_{23} = -\frac{l_1}{2} (\varphi_{31} - \varphi_{21})(1 - \xi)\xi. \quad (5)$$

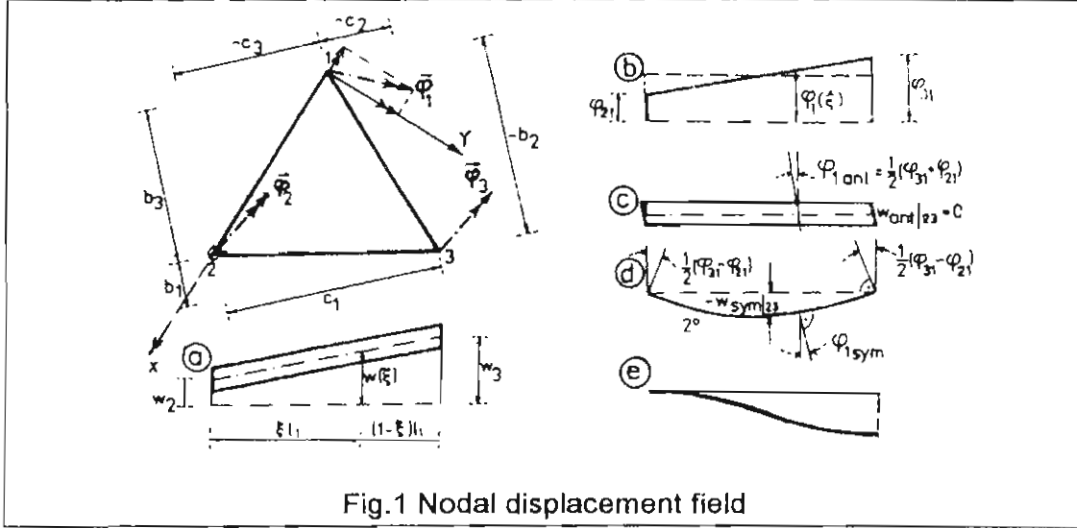


Fig.1 Nodal displacement field

Once the approximations w , w_{sym} , φ_x , φ_y , φ_{xy} are known, the shear distortions can be evaluated using the strain - displacement relationship given by

$$\begin{aligned} \gamma_{xz} &= \varphi_y + \frac{\partial}{\partial x} (w + w_{sym}) = \frac{\partial w}{\partial x} + (\varphi_y - \varphi_{y,sym}), \\ \gamma_{yz} &= -\varphi_x + \frac{\partial}{\partial y} (w + w_{sym}) = \frac{\partial w}{\partial y} - (\varphi_x - \varphi_{x,sym}). \end{aligned} \quad (6)$$

If the shear strains $(\varphi_x - \varphi_{x,sym})$ and $(\varphi_y - \varphi_{y,sym})$ are linear and computed with one-point quadrature rule, the actual linear variation of the shear strains is replaced by its average value. This can be simply achieved by evaluating the shear strains at point $L_1=L_2=L_3=1/3$.

The computation of the stiffness matrix in the case of the CCT element is thus reduced to matrix multiplication, as both κ , γ are element-wise constant vectors expressed as

$$\kappa = \begin{Bmatrix} \frac{\partial \varphi_y}{\partial x} \\ \frac{\partial \varphi_x}{\partial y} \\ \frac{\partial \varphi_y}{\partial y} - \frac{\partial \varphi_x}{\partial x} \end{Bmatrix}, \quad \gamma = \begin{Bmatrix} \frac{\partial w}{\partial x} + (\varphi_y - \varphi_{y,sym})_c \\ \frac{\partial w}{\partial y} - (\varphi_x - \varphi_{x,sym})_c \end{Bmatrix}. \quad (7)$$

Simulated Antisymmetric Bending Mode technique was developed by Martin in [2]. Simulated antisymmetric bending mode is the projection of cubic bending mode in the side direction with the area of the whole triangle. This is some what based on energy approach applied to simple beam bending

problem Fig. 1e. The concept of the simulated antisymmetric bending mode results in the change of shear stiffness so that

$$\begin{Bmatrix} q_{yz} \\ q_{zx} \end{Bmatrix} = \begin{bmatrix} c_{44} & c_{45} \\ c_{45} & c_{55} \end{bmatrix} \cdot \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}, \quad (8)$$

where q_{yz} , q_{zx} are shear forces and c are coefficients of shear stiffness matrix obtained solving the following questions.

$$\begin{aligned} (-\cos \alpha_{23})c_{55} + g_{23} &= (\sin^2 \alpha_{23})c_{44} + (\sin 2\alpha_{23})c_{45} \\ (-\cos \alpha_{31})c_{55} + g_{31} &= (\sin^2 \alpha_{31})c_{44} + (\sin 2\alpha_{31})c_{45} \end{aligned} \quad (9)$$

$$\begin{aligned} \text{where } c_{55} &= \frac{2G\kappa_{12}}{(1+2\kappa_{12})}, \quad g_{23} = \frac{2G\kappa_{23}}{(1+2\kappa_{23})}, \quad g_{31} = \frac{2G\kappa_{31}}{(1+2\kappa_{31})}, \\ \kappa_{ij} &= \frac{6(1+\nu)h}{5l_{ij}}. \end{aligned}$$

This modified shear stiffness results in artificially isotropic material with constants E, G .

Convergence tests

To illustrate the accuracy of the present element a number of examples are studied and compared with some existing results published in literature.

1. Simply supported and clamped thin a thick square plate under concentrated load

The classical square plates under concentrated loads applied at the center of the plate are considered to compare present element ICCT with the CCT [1], the classical DKT elements [4] and DST-BK elements [3] for thick plates. Solutions are presented for regular meshes (with orientations A and B) with $N=2,4,8$ elements per half side (Fig.2). Due to symmetry only one quarter of the plate is discretized.

The results for the central displacement w_c and for bending moment at mid-side of the clamped plate are stored in Table 1. Table 2. then shows the results for the central displacement w_c and for the bending moment in C of the Simply Supported plate in Tables 3 and 4.

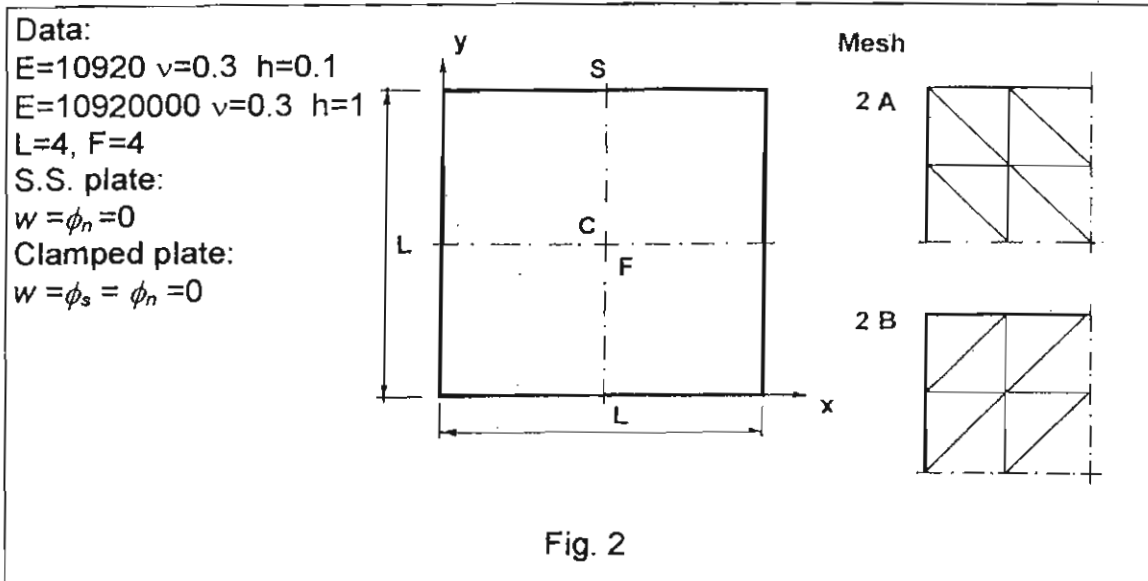


Table 1. Clamped plate $h=0.1\text{m}$

Exact solutions Timoshenko $w_c=0.3584$, $m_s=0.503$

mesh		ICCT		CCT		DST		DKT	
		A	B	A	B	A	B	A	B
2	w_c	0.364	0.48	0.102	0.115	0.375	0.352	0.375	0.407
	m_s	0.5	0.51	0.24	0.34	0.557	0.447	0.532	0.417
4	w_c	0.365	0.397	0.307	0.311	0.369	0.36	0.365	0.378
	m_s	0.486	0.5	0.425	0.5	0.518	0.49	0.516	0.476
8	w_c	0.366	0.375	0.351	0.354	0.367	0.364	0.361	0.365
	m_s	0.491	0.51	0.475	0.5	0.5	0.49	0.51	0.49

Table 2. Clamped plate $h=1\text{m}$

mesh		ICCT		CCT		DST		DKT	
		A	B	A	B	A	B	A	B
2	w_c	0.79	0.922	0.655	0.675	0.79	0.76	0.375	0.407
	m_s	0.36	0.4	0.25	0.26	0.43	0.392	0.532	0.417
4	w_c	0.93	0.967	0.884	0.892	0.925	0.914	0.365	0.378
	m_s	0.42	0.44	0.38	0.4	0.42	0.44	0.516	0.476
8	w_c	1.06	1.068	1.044	1.05	1.05	1.05	0.361	0.365
	m_s	0.46	0.45	0.5	0.5	0.43	0.47	0.51	0.49

Table 3. Simply supported plate $h=0.1\text{m}$

Exact solutions Timoshenko $w_c=0.7424$

mesh		ICCT		CCT		DST		DKT	
		A	B	A	B	A	B	A	B
2	w_c	0.713	0.889	0.492	0.556	0.745	0.749	0.748	0.82
4	w_c	0.74	0.785	0.69	0.705	0.746	0.7485	0.745	0.768
8	w_c	0.747	0.76	0.733	0.739	0.75	0.748	0.743	0.75

Table 4. Simply supported plate $h=1\text{m}$

mesh		ICCT		CCT		DST		DKT	
		A	B	A	B	A	B	A	B
2	w_c	1.145	1.333	1.0	1.09	1.162	1.163	0.748	0.82
4	w_c	1.294	1.34	1.25	1.27	1.301	1.295	0.745	0.768
8	w_c	1.426	1.441	1.413	1.42	1.43	1.426	0.743	0.75

2. Simply supported and clamped thin square plate under uniform load

To further assess the range of applicability of the present element the classical square plates under uniform loads applied at the center of the plate are analyzed and compared with the elements discussed in the previous example. Results for regular meshes (with orientations A and B) with the same degree of refinement as before are presented and can be found in Figs.4-7.

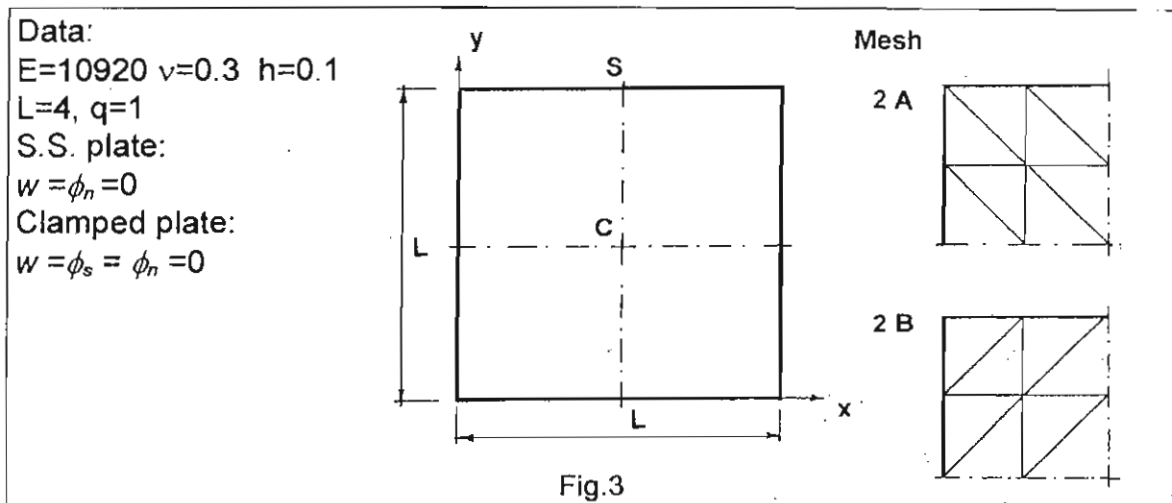


Fig. 4 Clamped plate deflection w_c ,
Exact $w_c=0,3256$

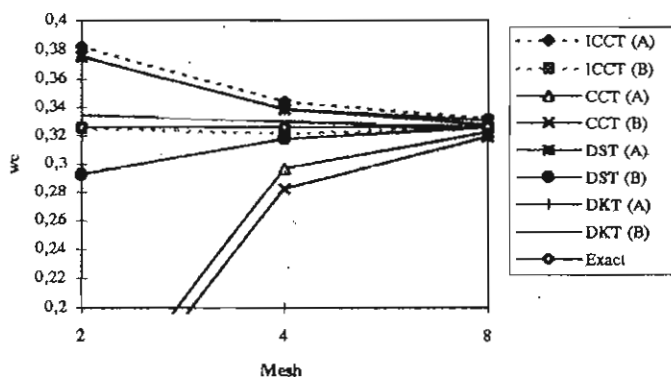


Fig. 5 Clamped plate moment m_s ,
Exact $m_s=0,821$

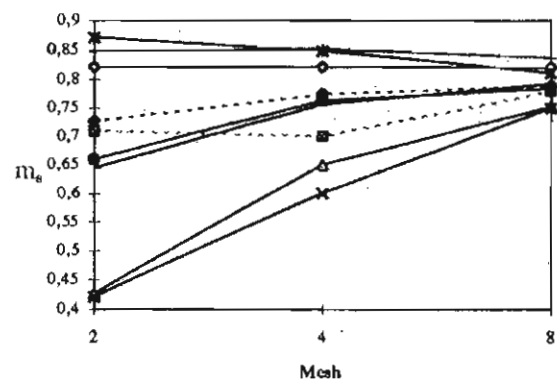


Fig. 6 Simply supported plate deflection w_c ,
Exact $w_c=1,0398$

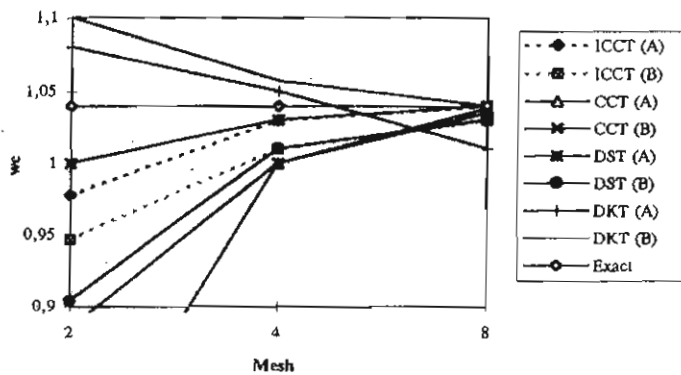
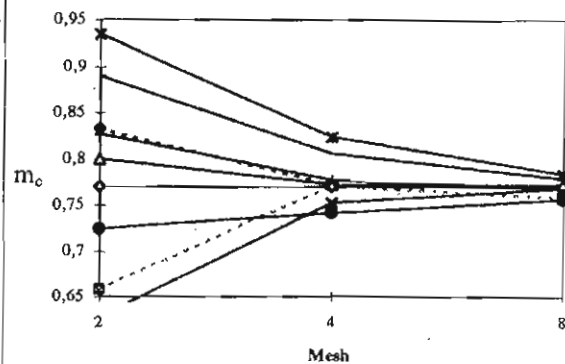


Fig. 7 Simply supported plate moment
 m_c ,
Exact $m_c=0,77$



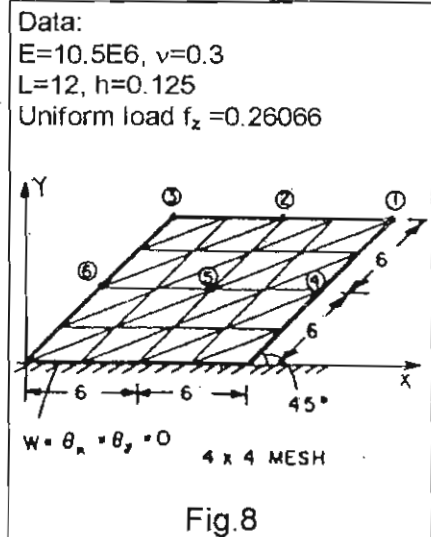
3. Analysis of a rhombic cantilever plate

This problem deals with the analysis of a rhombic cantilever plate subjected to a uniform load. The material properties and geometry are given in Fig.8 .

The results obtained with ICCT,CCT,DST and DKT elements are presented for comparison, see table 5.

Table 5. Deflections on node

node	mesh	ICCT	CCT	DST	DKT
1	4x4	0.291	0.177	0.253	0.28
	6x8	0.277	0.241	0.263	0.276
2	4x4	0.182	0.133	0.171	0.19
	6x8	0.182	0.167	0.178	0.186
3	4x4	0.103	0.098	0.105	0.112
	6x8	0.106	0.107	0.108	0.11
4	4x4	0.11	0.055	0.09	0.114
	6x8	0.107	0.086	0.099	0.11
5	4x4	0.049	0.039	0.047	0.055
	6x8	0.048	0.048	0.048	0.051
6	4x4	0.021	0.025	0.022	0.023
	6x8	0.019	0.021	0.02	0.02



4. Simply supported circular plate under uniform loading

The analytical solutions including the transverse shear effects are known for certain class of axisymmetric problems. Therefore to present the reliable performance of the ICCT element we present the numerical results for one quarter of the plate with symmetry conditions along the x,y axis using 6,24 and 96 elements Fig.9. Two aspect ratios are considered ($R/h = 50$, Table 6., $R/h = 2$, Table 7.).

Data: $E=1092E3$, $\nu=0.3$, Uniform loading: $f_z = 1$, Boundary condition : AB $w=0$

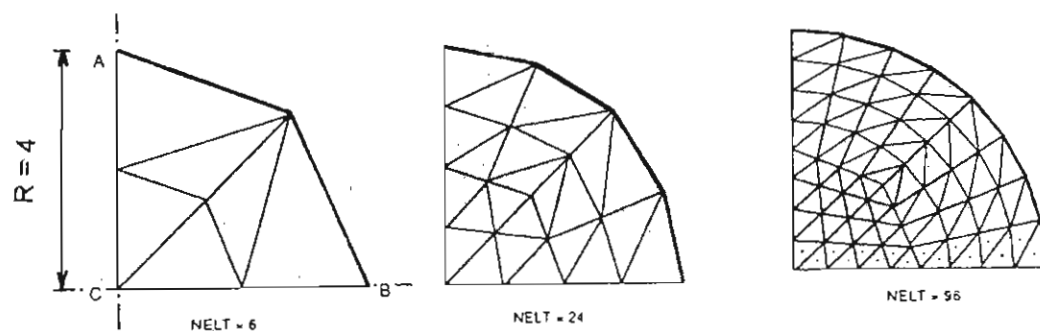


Fig. 9

Table 6. $R/h=50$ Solution: $w_c = 0.31865$, $m_{rc} = 3.3$

Nelt		ICCT	CCT	DST	DKT
6	w_c	0.302	0.305	0.299	0.344
24	w_c	0.316	0.315	0.314	0.326
96	w_c	0.317	0.318	0.317	0.32
96	m_{rc}	3.325	3.338	3.325	3.33

Table 7. $R/h=2$ Solution: $w_c = 3.262 \cdot 10^{-5}$, $M_{rc} = 3.3$

Nelt		ICCT	CCT	DST	DKT
6	$w_c \cdot 10^{-5}$	2.461	2.527	2.441	2.202
24	$w_c \cdot 10^{-5}$	2.583	2.587	2.571	2.085
96	$w_c \cdot 10^{-5}$	2.602	2.604	2.601	2.05
96	m_{rc}	3.325	3.338	3.325	3.33

Acknowledgements:

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Finite Volume Approach to Modelling of Plates

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Abstract

The Finite Volume modelling of bending of thin and thick plates is described in detail. The formulation employs the Mindlin plate theory, and preserves the efficiency and the economy of the original Finite Volume method for solid body stress analysis. Several validation cases are presented, which demonstrate the accuracy of the method and its applicability to a wide range of plate thicknesses.

1 Introduction

After the first publication in *Zavarivanje* [1], the Finite Volume method for structural analysis has been applied to a number of both linear and non-linear solid body deformation problems. A short review of those applications, as well as an outline of the method was presented at the *1st Congress of Croatian Society of Mechanics* [2]. Since then the method has been extended to a coupled solution of fluid-solid interaction problems [3, 4], and later furnished with a multigrid acceleration technique resulting in very impressive CPU times and memory requirements [5, 6].

Although the method is, in principle, applicable to solids of arbitrary shapes, it is not practical to apply it directly to solid bodies with a degenerated shape, like beams or plates. The reason is that, in case of plates, for example, due to relatively small plate thickness and in order to preserve a reasonable control volume (CV) shapes, a very large number of CVs are required resulting in unnecessary large memory and CPU requirements. To avoid that problem, an approximate set of equations is solved by employing the basic techniques of the FVM:

- discretisation that results in a system of linear algebraic equations with a very sparse, symmetric, diagonally dominant matrix, which is easy to solve by sparsity preserving iterative methods,
- segregated solution algorithm, which offers an extremely efficient memory management.

There are a number of 'plate theories' which approximate the behaviour of thin and/or thick plates under the transverse loads. Here, one of the most popular,

namely the Mindlin theory of plate deformation is used. A paper which also employs Mindlin theory and a FV discretisation has been published recently [7]. However, it does not use the segregated solution algorithm, which implies much larger memory requirements.

2 Governing equations

In case of an arbitrary shaped body of the volume V bounded by the surface S the equations governing equilibrium of forces and moments around an arbitrary point P may be expressed as:

$$\begin{aligned} \int_S \boldsymbol{\sigma} \cdot \mathbf{n} dS + \int_V \rho \mathbf{f}_b dV &= 0, \\ \int_S [(\mathbf{r} - \mathbf{r}_P) \times \boldsymbol{\sigma}] \cdot \mathbf{n} dS + \int_V \rho [(\mathbf{r} - \mathbf{r}_P) \times \mathbf{f}_b] dV &= 0, \end{aligned} \quad (1)$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{f}_b$ is the body force, ρ is the density and $\mathbf{r}$ is the position vector.

Consider a flat plate of arbitrary planeform and of constant thickness t , which is loaded by transverse loads only (Fig. 1). In that case Eqs. (1), valid for the plate as a whole as well as for any part of it of the volume V bounded by the surface $S = S_{xy} \cup S_z$ with the unit vector $\mathbf{n}$ pointing outwards, can be reduced to the following three equations:

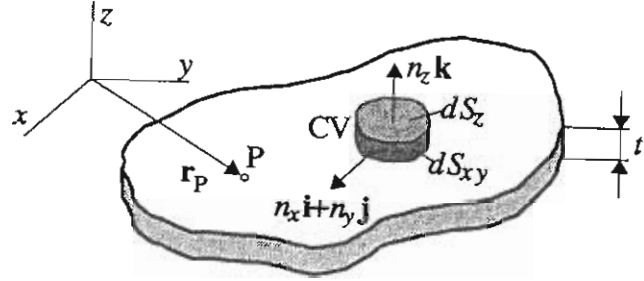


Fig. 1: Plate and a control volume

$$\begin{aligned} \int_l (Q_x n_x + Q_y n_y) dl + \int_{S_z} q dS &= 0, \\ \int_l [M_x n_x + M_{xy} n_y - (x - x_P) (Q_x n_x + Q_y n_y)] dl - \int_{S_z} (x - x_P) q dS &= 0, \\ \int_l [M_{xy} n_x + M_y n_y - (y - y_P) (Q_x n_x + Q_y n_y)] dl - \int_{S_z} (y - y_P) q dS &= 0, \end{aligned} \quad (2)$$

where the transverse shear forces per unit length Q_x and Q_y and the bending moments per unit length M_x , M_y and M_{xy} are given by:

$$\begin{aligned} Q_x &= \int_{-t/2}^{t/2} \sigma_{xz} dz, \quad Q_y = \int_{-t/2}^{t/2} \sigma_{yz} dz, \\ M_x &= \int_{-t/2}^{t/2} \sigma_{xx} z dz, \quad M_y = \int_{-t/2}^{t/2} \sigma_{yy} z dz, \\ M_{xy} &= \int_{-t/2}^{t/2} \sigma_{xy} z dz, \end{aligned} \quad (3)$$

and q includes (vertical) surface loads and body forces.

The plate theory [8] assumes that the transverse shear stresses σ_{xz} and σ_{yz} , associated with forces Q_x and Q_y vary quadratically with z , and that the normal stresses σ_{xx} and σ_{yy} , associated with bending moments M_x and M_y , and the shear

stress σ_{xy} , associated with twisting moment M_{xy} , vary linearly with z . Normal stress σ_{zz} is considered negligible. In case of homogeneous and linearly elastic plate material Eqs. (3) can be related to derivatives of the rotation components θ_x and θ_y , and the transverse displacement w by using the moment-curvature relations of Mindlin plate theory:

$$\begin{aligned} Q_x &= Gt \left(\theta_x - \frac{\partial w}{\partial x} \right), \quad Q_y = Gt \left(\theta_y - \frac{\partial w}{\partial y} \right), \\ M_x &= D \frac{\partial \theta_x}{\partial x} + \nu D \frac{\partial \theta_y}{\partial y}, \quad M_y = \nu D \frac{\partial \theta_x}{\partial x} + D \frac{\partial \theta_y}{\partial y}, \\ M_{xy} &= \frac{1 - \nu}{2} D \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right), \end{aligned} \quad (4)$$

where ν is the Poisson's ratio, G is the shear modulus and D , the flexural rigidity of the plate, is given by

$$D = \frac{Et^3}{12(1 - \nu^2)}, \quad (5)$$

with E being the Young's modulus.

Equations (2), when combined with Eqs. (4) represent a closed system of three equations with three unknown scalar functions w , θ_x and θ_y of two in-plane coordinates x and y , which can be written in the following generic form:

$$\int_l \Gamma_\phi \text{grad } \phi \cdot \mathbf{n} \, dl + \int_l \Gamma_\phi s_{\phi l} \, dl - \int_{S_z} s_{\phi S} \, dS = 0, \quad (6)$$

where ϕ stands for w , θ_x or θ_y and the respective coefficients Γ_ϕ and source terms $s_{\phi l}$ and $s_{\phi S}$ are given in Table 1.

Table 1: The meaning of Γ_ϕ , $s_{\phi l}$ and $s_{\phi S}$ in Eq. (6)

ϕ	Γ_ϕ	$s_{\phi l}$	$s_{\phi S}$
w	Gt	$-\boldsymbol{\theta} \cdot \mathbf{n}$	q
θ_x	D	$\nu \frac{\partial \theta_y}{\partial y} n_x + \frac{1 - \nu}{2} \frac{\partial \theta_y}{\partial x} n_y - \frac{1 + \nu}{2} \frac{\partial \theta_x}{\partial y} n_y + \frac{Gt}{D} (x - x_P) (\text{grad } w - \boldsymbol{\theta}) \cdot \mathbf{n}$	$(x - x_P) q$
θ_y	D	$\frac{1 - \nu}{2} \frac{\partial \theta_x}{\partial y} n_x + \nu \frac{\partial \theta_x}{\partial x} n_y - \frac{1 + \nu}{2} \frac{\partial \theta_y}{\partial x} n_x + \frac{Gt}{D} (y - y_P) (\text{grad } w - \boldsymbol{\theta}) \cdot \mathbf{n}$	$(y - y_P) q$

Equation (6) together with appropriate boundary conditions (clamped, free, simply supported) makes a mathematical model for analysing the bending deformation of thick and thin plates.

3 FV discretisation and segregated solution procedure

First, the 2D solution domain representing a plate $x - y$ cross-section is subdivided into a finite number of contiguous control volumes (CV) or cells of arbitrary polygonal shape (Fig. 2). The computational nodes are placed at the centres of

CVs, and boundary nodes required for the specification of boundary conditions at the centres of boundary cell-faces. Then Eq. (6) is applied to each CV. In doing that the only approximations involved are the assumption of linear variation of dependent variables over the CVs, and the midpoint rule used for calculation of all integrals. Both of these approximations are second-order accurate.

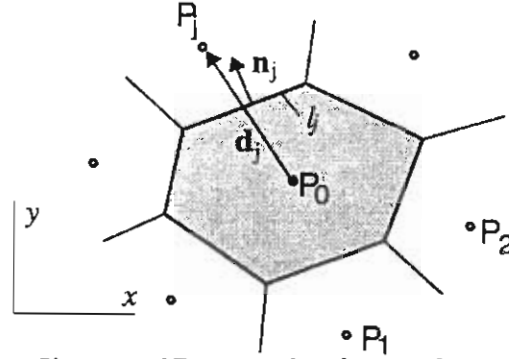


Fig. 2: 2D control volume of arbitrary polygonal shape

The first term in Eq. (6) is approximated in the following manner:

$$\int_l \Gamma_\phi \text{grad } \phi \cdot \mathbf{n} dl = \sum_{j=1}^n \int_{l_j} \Gamma_\phi \text{grad } \phi \cdot \mathbf{n} dl \approx \sum_{j=1}^n \Gamma_{\phi,j} (\text{grad } \phi)_j^* \cdot \mathbf{n}_j l_j, \quad (7)$$

where n is the number of faces of the CV, $\Gamma_{\phi,j}$ stands for the values of Γ_ϕ at the cell-face centre, $\mathbf{n}_j$ is the unit vector of the outer normal to the cell face j , l_j is the length of the cell face j , and the value of $(\text{grad } \phi)_j^*$ consists of the value of $\text{grad } \phi$ at the cell-face centre and a third order term added for stability reasons:

$$(\text{grad } \phi)_j^* = (\text{grad } \phi)_j + \left(\frac{\phi_{P_0} - \phi_{P_j}}{|\mathbf{d}_j|} - \frac{\overline{\text{grad } \phi} \cdot \mathbf{d}_j}{|\mathbf{d}_j|} \right) \mathbf{n}_j, \quad (8)$$

where the overbar denotes arithmetic average of values calculated at nodes P_0 and P_j .

The second term in Eq. (6) is approximated in a similar manner, e.g. for $\phi = w$:

$$-\int_l G t \boldsymbol{\theta} \cdot \mathbf{n} dl = -\sum_{j=1}^n \int_{l_j} G t \boldsymbol{\theta} \cdot \mathbf{n} dl \approx -\sum_{j=1}^n G t (\boldsymbol{\theta} \cdot \mathbf{n})_j l_j, \quad (9)$$

while the last term is calculated as:

$$\int_{S_z} s_{\phi S} dS \approx s_{\phi S, P} S_{z, P}, \quad (10)$$

where $s_{\phi S, P_0}$ is the value at the cell centre, and S_{z, P_0} is the cell area.

As a result of the above discretisation a system of $3 \times N$ linear algebraic equations with $3 \times N$ unknowns (values of w , θ_x and θ_y at the CV's centres) of the following form is obtained:

$$a_{\phi_0} \phi_{P_0} - \sum_{j=1}^n a_{\phi_j} \phi_{P_j} = b_\phi. \quad (11)$$

It should be said that only the first bracketed term in Eq. (8) is taken into account implicitly (forms coefficients: $a_{\phi_j} = \Gamma_{\phi,j} l_j / |\mathbf{d}_j| > 0$), while all the other terms form the right-hand side b_ϕ .

That system is solved by a segregated solution procedure, whereby the sets of equations for each dependent variable are temporarily decoupled by assuming that the right-hand side is known (calculated by using an initial guess or dependent

variables values from the previous iteration) resulting in a system of linear algebraic equations of the form

$$A_\phi \phi = \mathbf{b}_\phi \quad (\phi = w, \theta_x, \theta_y), \quad (12)$$

where A_ϕ is an $N \times N$ matrix, vector ϕ contains values of dependent variable ϕ at N nodal points, and $\mathbf{b}_\phi$ is the right-hand side vector. It is important to note that the matrix A_ϕ has the following desirable properties. It is sparse with the number of non-zero elements in each row equal to $n+1$, symmetric, positive definite and diagonally dominant ($a_{\phi_0} \geq \sum_{j=1}^n a_{\phi_j}$), which renders system (12) easily solvable by a number of iterative solvers which retain the sparsity of the matrix A_ϕ , resulting in very small computer memory requirements. Moreover, the segregated solution strategy enables reuse of the same storage for matrix A_ϕ and vector $\mathbf{b}_\phi$ for all dependent variables.

The system (12) is solved in turn for each dependent variable and the right-hand side vector $\mathbf{b}_\phi$ updated with the most recent dependent variables values. There is no need to solve Eq. (12) to a tight tolerance since its right-hand side is only approximated and reduction of the normalised sum of absolute residuals for one order of magnitude is usually sufficient. The procedure is stopped when a converged solution is obtained. In order to procure convergence, an under-relaxation is sometimes necessary. Also, positive contributions to the central coefficient a_{ϕ_0} from the source terms $s_{\phi l}$ and $s_{\phi s}$ and from the boundary conditions enhance both diagonal dominance and the implicitness of the system of equations, with beneficial effects on both linear equation solver and the overall convergence procedure.

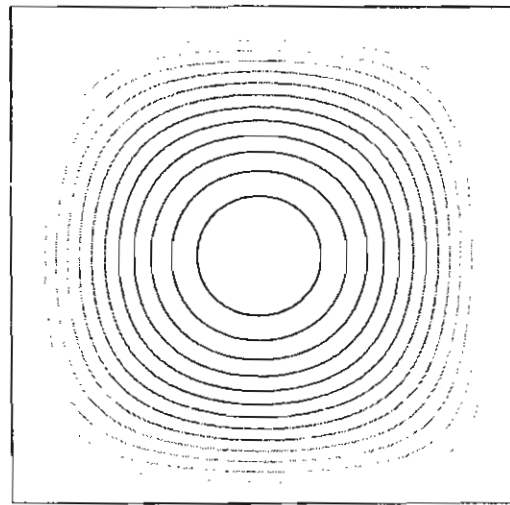
4 Validation cases

The method is verified against several classical plate solutions, one of which is described in detail below. Square plate, with sides $a = 2$ m, simply supported on all edges, is uniformly loaded by pressure p . The plate is made of a linear elastic material with $E = 210$ GPa and $\nu = 0.3$. The analytical solution for the deflection of the plate is given in [9]. The solution domain is discretised by five gradually refined, uniform FV meshes, ranging from 11×11 to 161×161 CVs.

The distributions of deflection w and θ_x and θ_y rotations for the plate of thickness $t = 0.2$ m, loaded by $p = 100$ MPa and calculated on 81×81 CV mesh are presented in Fig. 3. The comparison with the analytical solution for the deflection along the horizontal centre-line is presented in Fig. 4. A very good agreement is shown, demonstrating the accuracy of the FV method.

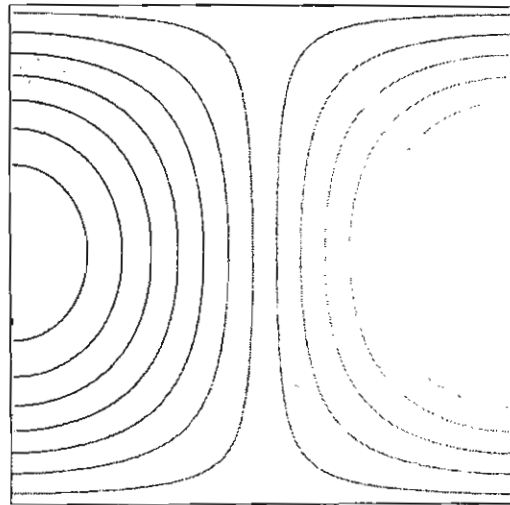
In Figure 5 the error in numerically predicted maximum deflection compared to the analytical solution is plotted against the CV size. It shows that the convergence towards the analytical solution is asymptotically of the second-order, as discussed in Section 3.

The Mindlin plate theory, used in this study, incorporates transverse shear deformation and should be valid for both thick and thin plates. In order to demonstrate that, the thickness of the plate is varied while the ratio $p/t^3 = 1/8 \cdot 10^{11}$ is kept constant so that deflections remain small compared to the thickness of the plate. Figure 6 shows the ratio between the FV 81×81 CV mesh results and the analytical maximum deflections obtained for $t/a = 0.1, 0.05, 0.01$ and 0.005 . This exercise demonstrates the 'locking free' behaviour of the FV method, which was also reported in [7].



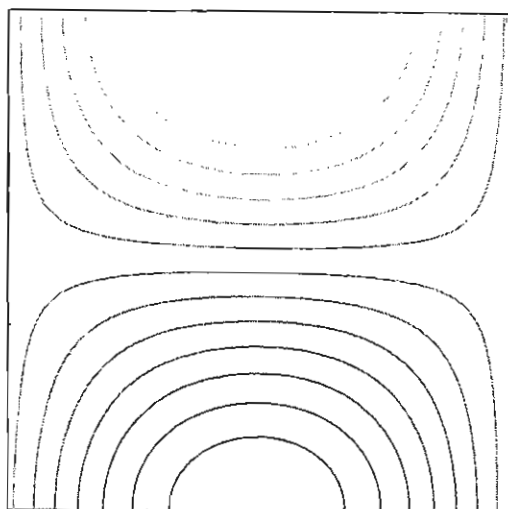
w (m)

-2.888E-03
 -5.754E-03
 -8.621E-03
 -1.149E-02
 -1.435E-02
 -1.722E-02
 -2.009E-02
 -2.295E-02
 -2.582E-02
 -2.869E-02
 -3.155E-02
 -3.442E-02
 -3.729E-02
 -4.015E-02



θ_x (rad)

5.794E-02
 4.903E-02
 4.011E-02
 3.120E-02
 2.229E-02
 1.337E-02
 4.457E-03
 -4.457E-03
 -1.337E-02
 -2.229E-02
 -3.120E-02
 -4.011E-02
 -4.903E-02
 -5.794E-02



θ_y (rad)

5.794E-02
 4.903E-02
 4.011E-02
 3.120E-02
 2.229E-02
 1.337E-02
 4.457E-03
 -4.457E-03
 -1.337E-02
 -2.229E-02
 -3.120E-02
 -4.011E-02
 -4.903E-02
 -5.794E-02

Fig. 3: Deflection w , and rotations θ_x and θ_y contours

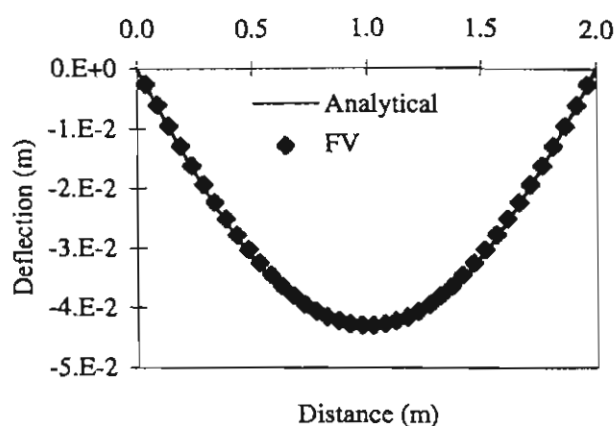


Fig. 4: FV vs. analytical results

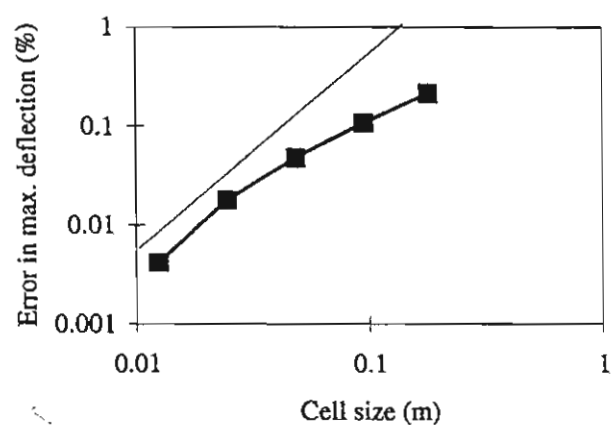


Fig. 5: Solution error as a function of the grid size (straight lines show ideal slope of a second-order scheme)

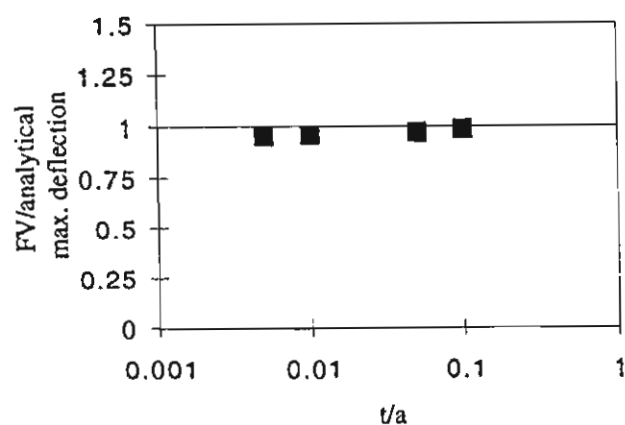


Fig. 6: Effect of the plate thickness on the FV results

5 Conclusions

A simple and efficient numerical procedure based on Mindlin plate theory and FV method is developed for the analysis of plates. The accuracy of the method was

demonstrated through the comparison between numerical and analytical results. The main features of the method are simplicity, since the method is based on basic conservation principles, low memory requirements due to the segregated solution algorithm, and second-order accuracy.

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A NEW NINE-NODE QUADRILATERAL ELEMENT FOR ANALYSIS OF THICK AND THIN PLATES

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Summary. A new nine-node, 27-d.o.f. quadrilateral plate bending element based on Mindlin-Reissner theory is proposed. The formulation of the element is derived on the basis of displacement variational principle using some simple criteria in order to eliminate undesirable shear locking effect. It is shown that the element is free of spurious modes, satisfies the patch test and its performance is good both for thick and thin plates. The shear strain-sampling points of element are located at optimal positions which is shown by using the concept of reduced minimization. This extends the previous work of the authors presented in Reference 16.

Keywords: Mindlin-Reissner plates, finite elements, shear locking, mixed interpolation approach, reduced minimization, matched/unmatched term.

1. Introduction

Several recent studies have exhibited the high accuracy attainable by isoparametric plate elements derived from Mindlin-Reissner thick plate theory, which requires C^0 continuity. By classical formulation the shear locking phenomenon can appear, which may lead to an incorrect solution for thin plates. To avoid this deficiency, various remedies based on selective and reduced integration have been proposed [1]. Since those approaches are not effective for very thin plates, other methods are used such as 'B approach' [2], 'mixed interpolation' [4, 6], 'substitute strain field' [3, 5], mixed formulation as well as new quadrilateral and triangular elements.

This paper represents an extension of the serendipity eight-node element CL8 given in Reference 12 on the nine-node Lagrangian element. The formulation of the element is derived from a displacement variational principle. To calculate the bending stiffness matrix, standard procedures are employed. However, the strain-nodal displacement matrix for the shear is derived in an inconsistent non-standard manner, as in the so-called 'assumed strain field method', but our approach differs from the above references. It was shown that the condition of vanishing shear strains for the thin plate limit imposes the condition that the coefficients defining the shear strains polynomial field must be a linear function of the nodal deflections and rotations. In this manner we formulate the element stiffness matrix in our approach by including the bending effects and transverse shear effects through different interpolations. Some simple criteria are used to derive the final interpolation of the transverse shear strains. The explanation is given how this approach avoids the shear locking by using reduced minimization [17, 18].

A series of numerical examples is presented to evaluate the performance of a new quadratic Mindlin-Reissner plate element. It has a proper rank, passes patch test and does not lock. The present results are compared with the corresponding theoretical and numerical predictions and fairly good convergence characteristics have been achieved for the very thin and very thick plates.

2. Finite element discretization

The basic expressions of Mindlin-Reissner plate theory [1] are as follows:

Displacement field

$$\mathbf{u} = [w, \boldsymbol{\theta}^T]^T, \quad \boldsymbol{\theta} = [\theta_x, \theta_y]^T, \quad (1)$$

where w is deflection and θ_x, θ_y two independent rotations of the plate.

Curvature field

$$\boldsymbol{\kappa} = [\kappa_x, \kappa_y, \kappa_{xy}]^T = [-\theta_{xx}, -\theta_{yy}, -(\theta_{xy} + \theta_{yx})]^T. \quad (2)$$

Shear strain field

$$\boldsymbol{\gamma} = [\gamma_{xz}, \gamma_{yz}]^T = [w_{,x} - \theta_x, w_{,y} - \theta_y]^T. \quad (3)$$

Constitutive relationships (only isotropic conditions presented)

Bending moment/curvature relationships

$$\mathbf{m} = [m_x, m_y, m_{xy}]^T = \mathbf{D}_b \boldsymbol{\kappa}. \quad (4)$$

Shear forces/shear strains relationships

$$\mathbf{s} = [Q_x, Q_y]^T = \mathbf{D}_s \boldsymbol{\gamma}, \quad (5)$$

$$\mathbf{D}_b = \frac{E t^3}{12 (1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}, \quad \mathbf{D}_s = G \alpha t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (6)$$

Equilibrium equation (principle of virtual work)

$$\iint_A (\delta \boldsymbol{\kappa}^T \mathbf{m} + \delta \boldsymbol{\gamma}^T \mathbf{s}) dA = \iint_A \delta w q dA. \quad (7)$$

In the above, E is the Young's modulus, G the shear modulus, ν the Poisson's ratio, t the plate thickness, α a parameter to account for non-uniform distribution of shear stresses across the section (usually taken as $\alpha = 5/6$), q the distributed pressure loading and A presents the area of the middle surface of the plate.

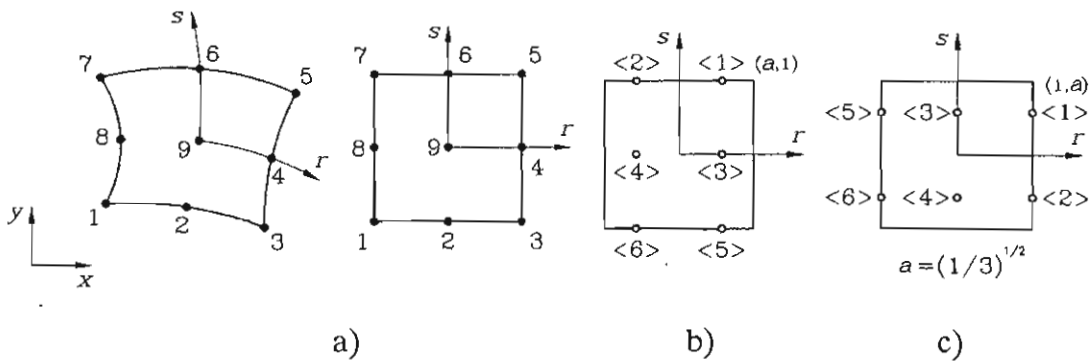


Figure 1. a) Isoparametric 9-noded plate element
b) Sampling positions for shear interpolation γ_{rz}
c) Sampling positions for shear interpolation γ_{sz}

Let us consider a nine-node quadrilateral plate element presented in Figure 1a. In order to derive discrete algorithms the Mindlin-Reissner displacement field is approximated by the quadratic Lagrangian interpolation functions $N_i(r, s)$ and the nodal point values $w_i, \theta_{xi}, \theta_{yi}$ in the following way

$$w = \sum_{i=1}^9 N_i(r, s) w_i, \quad \theta_x = \sum_{i=1}^9 N_i(r, s) \theta_{xi}, \quad \theta_y = \sum_{i=1}^9 N_i(r, s) \theta_{yi}, \quad (8)$$

where r and s are natural coordinates according to isoparametric techniques. After lengthy computations the element stiffness matrix may be derived in the form

$$\mathbf{K}^{(e)} = \mathbf{K}_b^{(e)} + \mathbf{K}_s^{(e)}, \quad (9)$$

where the first matrix presents bending stiffness

$$\mathbf{K}_b^{(e)} = \iint_{A^{(e)}} \mathbf{B}_b^T \mathbf{D}_b \mathbf{B}_b dA, \quad (10)$$

$\mathbf{K}_s^{(e)}$ denotes shear stiffness matrix

$$\mathbf{K}_s^{(e)} = \iint_{A^{(e)}} \mathbf{B}_s^T \mathbf{D}_s \mathbf{B}_s dA, \quad (11)$$

where $\mathbf{B}_b$ and $\mathbf{B}_s$ are the standard curvature-displacement matrix and the shear strain-displacement matrix respectively. Equations (8)-(11) define the C^0 classical nine-node isoparametric quadrilateral plate finite element. In the analysis of very thin plates, very stiff results may be obtained when the exact numerical integration for the computation element matrices is used. This phenomenon is termed shear locking effect. To eliminate this effect, some form of reduced or selective integration for the transverse shear strain-nodal displacement matrix $\mathbf{B}_s$ needs to be used.

To develop an efficient element, we are going to modify the matrix $\mathbf{B}_s$. The methodology proposed in Reference 12 for the serendipity four- and eight-node element is here applied and detailed in the nine-node Lagrangian element.

Mixed interpolation approach is based on the evolution of shear-displacement matrix $\mathbf{B}_s$ using the constrained shear strain fields. It starts with the expression of the transverse shear strain, which can be written by means of (8) and using natural coordinates (r, s) in the following form

$$\begin{aligned} \gamma_{rz} = w_{,r} - \theta_r = & \alpha_1(w_i, \theta_{ri}) + \alpha_2(w_i, \theta_{ri})r + \alpha_3(w_i, \theta_{ri})s + \alpha_4(\theta_{ri})r^2 + \alpha_5(w_i, \theta_{ri})rs + \\ & + \alpha_6(w_i, \theta_{ri})s^2 + \alpha_7(\theta_{ri})r^2s + \alpha_8(w_i, \theta_{ri})rs^2 + \alpha_9(\theta_{ri})r^2s^2. \end{aligned} \quad (12)$$

A similar expression for γ_{sz} may be obtained by simply interchanging r by s . The limiting Kirchhoff conditions ($\gamma_{rz} = \gamma_{sz} = 0$) imply $\alpha_1 = \alpha_2 = \dots = \alpha_9 = 0$. The conditions on the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6$ and α_8 impose a linear relationship between the nodal deflections and the average θ_r rotations (θ_r for γ_{rz}) on the element. This physically means that the average element rotation equals the element slope, which is clearly satisfied for thin-plate. On the other hand, $\alpha_4 = \alpha_7 = \alpha_9 = 0$ imposes a uniform or zero rotation on the element, which invariably leads to vanishing of the bending energy and to the undesirable locking effect [9]. In order to avoid the locking phenomenon the bending and the shear effect were included in the stiffness matrix using different interpolations. According to the classical isoparametric concept, the bending stiffness matrix is derived using equations (2) and (8).

The shear strains have to be evaluated only at points at which the spurious $\alpha_k(\theta_{ri})$ and $\alpha_k(\theta_{si})$ vanish. According to this approach, on the edges of the quadrilateral element in (r, s) natural coordinates are imposed as in [12, 16]. The polynomial of w must be one order higher than that assumed for the θ_r and θ_s . A classic quadratic Lagrangian interpolation for the rotations θ_r and θ_s is used, while the cubic interpolation for the transverse deflection is expressed as follows

$$w(r, s) = a_1 + a_2 r + a_3 s + a_4 r^2 + a_5 r s + a_6 s^2 + a_7 r^3 + a_8 r^2 s + a_9 r s^2 + a_{10} s^3 + a_{11} r^3 s + a_{12} r^2 s^2 + a_{13} r s^3 + a_{14} r^3 s^2 + a_{15} r^2 s^3. \quad (13)$$

Apart from the conventional degrees of freedom w_i , θ_{ri} , θ_{si} , new degrees of freedom representing the tangential derivative of w with respect to the natural coordinates r or s , such as $(w_{,r})_2$, $(w_{,s})_4$, $(w_{,r})_6$, $(w_{,s})_8$ in the mid-side nodes 2, 4, 6, 8 and $(w_{,r})_9$, $(w_{,s})_9$ in the center node 9 are introduced. In this way, undesirable degrees of freedom appear only in the constant and quadratic terms of the tangential transverse shear strains γ_{rx} and γ_{sx} along the edges $s = -1, 0, +1$ and $r = -1, 0, +1$ respectively. To eliminate this effect, the linear distribution of the tangential transverse shear strain along the edges is assumed, which leads to the following expressions

$$(w_{,r})_i = -\frac{1}{2}w_j + \frac{1}{2}w_k - \frac{1}{6}\theta_{ri} - \frac{1}{6}\theta_{rk} + \frac{1}{3}\theta_{ri}, \text{ for } \{(i,j,k)\} = \{(2,1,3), (6,5,7), (9,4,8)\} \quad (14a)$$

$$(w_{,s})_i = -\frac{1}{2}w_j + \frac{1}{2}w_k - \frac{1}{6}\theta_{sj} - \frac{1}{6}\theta_{sk} + \frac{1}{3}\theta_{si}, \text{ for } \{(i,j,k)\} = \{(4,3,5), (8,7,1), (9,6,2)\} \quad (14b)$$

Inserting equations (14a) and (14b) into (13), the sets of monomial terms $P(w_{,r})$ and $P(w_{,s})$ are determined

$$P(w_{,r}) = \{1, r, s, r^2, rs, s^2, r^2s, rs^2, s^3, r^2s^2, rs^3\}, \quad (15)$$

$$P(w_{,s}) = \{1, r, s, r^2, rs, s^2, r^3, r^2s, rs^2, r^3s, r^2s^2\}. \quad (16)$$

The rotations θ_r and θ_s are interpolated from the classic quadratic Lagrangian functions

$$P(\theta_r) = P(\theta_s) = \{1, r, s, r^2, rs, s^2, r^2s, rs^2, r^2s^2\}. \quad (17)$$

From (15), (17) and linearity of the tangential transverse shear strain along the edges, we can see that

$$\{\text{matched terms of } \gamma_{rx}\} = P(w_{,r}) \cap P(\theta_r) = \{1, r, s, rs, s^2, rs^2\} \quad (18)$$

$$\text{and } \{\text{unmatched terms of } \gamma_{rx}\} = P(w_{,r}) - P(\theta_r) = \{s^3, rs^3\}. \quad (19)$$

Likewise, from (16), (17) and linearity of the tangential transverse shear strain along the edges, it is obvious that

$$\{\text{matched terms of } \gamma_{sx}\} = P(w_{,s}) \cap P(\theta_s) = \{1, r, s, r^2, rs, r^2s\} \quad (20)$$

$$\text{and } \{\text{unmatched terms of } \gamma_{sx}\} = P(w_{,s}) - P(\theta_s) = \{r^3, r^3s\}. \quad (21)$$

Further, the shear strains are expressed as

$$\gamma_{rz}(r, s) = A_1 + A_2 r + A_3 s + A_4 rs + A_5 s^2 + A_6 r s^2 + A_7^* s^3 + A_8^* r s^3, \quad (22)$$

$$\gamma_{sz}(r, s) = B_1 + B_2 r + B_3 s + B_4 r^2 + B_5 r s + B_6 r^2 s + B_7^* r^3 + B_8^* r^3 s, \quad (23)$$

where the superscript * designates unmatched coefficients. From these equations it is evident that six points should be defined for determining the transverse shear strain components where the unmatched coefficients vanish. To obtain the linear variation of natural transverse shear strains in the tangential direction along the boundary, four points have to be located at the two edges of the element, while two inner points are introduced in order to ensure the quadratic variation in the normal directions. The shape functions $R_J(r, s)$ of the γ_{rz} for the points presented in Figure 1b are derived in the form

$$R_J(r, s) = 1/4 (1 + r/r_J) (1 + s/s_J) s s_J \quad \text{for } J = 1, 2, 5, 6, \quad (24)$$

$$R_J(r, s) = 1/2 (1 + r/r_J) (1 - s^2) \quad \text{for } J = 3, 4,$$

where J is the sampling point number. Interchanging r by s , the shape functions $S_J(r, s)$ of the γ_{sz} for the points presented in Figure 1c are described by analogous expression

$$S_J(r, s) = R_J(s, r). \quad (25)$$

The shear strain-sampling points of the element are located at optimal positions, which should be shown by using the concept of reduced minimization [17, 18]. The interpolation of γ_{rz} and γ_{sz} is obtained by

$$\gamma_{rz}(r, s) = \sum_{J=1}^{NS} R_J(r, s) \gamma_{rz}(r_J, s_J), \quad (26)$$

$$\gamma_{sz}(r, s) = \sum_{J=1}^{NS} S_J(r, s) \gamma_{sz}(r_J, s_J), \quad (27)$$

where NS is the number of strain-sampling points and $\gamma_{rz}(r_J, s_J)$, $\gamma_{sz}(r_J, s_J)$ are the shear strains computed from equations (22) and (23) at sampling point $(r = r_J, s = s_J)$. Under the penalty limit, the shear strains tend to vanish, which requires that all the coefficients in equations (22) and (23) go to zero. Now, let us consider the reduced minimization of shear strain energy (for the strain γ_{rz}): the error-moment equations for the reduced minimization of the shear strain energy are given as

$$\begin{aligned} \int_{-1}^{+1} \int_{-1}^{+1} \gamma_{rz} dr ds = 0, \quad \int_{-1}^{+1} \int_{-1}^{+1} \gamma_{rz} r dr ds = 0, \quad \int_{-1}^{+1} \int_{-1}^{+1} \gamma_{rz} s dr ds = 0, \\ \int_{-1}^{+1} \int_{-1}^{+1} \gamma_{rz} rs dr ds = 0, \quad \int_{-1}^{+1} \int_{-1}^{+1} \gamma_{rz} s^2 dr ds = 0, \quad \int_{-1}^{+1} \int_{-1}^{+1} \gamma_{rz} r s^2 dr ds = 0. \end{aligned} \quad (28)$$

These equations produce six consistent constraints as follows

$$\begin{aligned} 3A_1 + A_5 = 0, \quad 3A_2 + A_6 = 0, \quad A_3 + A_7^* = 0, \\ A_4 + A_8^* = 0, \quad 5A_1 + 3A_5 = 0, \quad 5A_2 + 3A_6 = 0. \end{aligned} \quad (29)$$

Substituting equation (29) into equation (22), we obtain the strain field resulted from reduced minimization such as $\gamma_{rz} = 0$. The reduced minimization for the shear strain γ_{sz} can be proved in the same manner. From the above, we see that γ_{rz} and γ_{sz} integrated by 3×3 Gaussian quadrature does not have shear locking.

Finally, expressions for the transverse shear strains (26) and (27) as a function of conventional degrees of freedom and only matched terms may be given in the following form

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_{rz}(r, s) \\ \gamma_{sz}(r, s) \end{bmatrix} = \mathbf{N}_{r-s} \bar{\boldsymbol{\gamma}}, \quad (30)$$

where $\mathbf{N}_{r-s}$ is the interpolation function matrix

$$\mathbf{N}_{r-s} = \begin{bmatrix} \mathbf{N}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_s \end{bmatrix}, \quad \text{where} \quad \begin{cases} \mathbf{N}_r = [R_1(r, s) \dots R_6(r, s)] \\ \mathbf{N}_s = [S_1(r, s) \dots S_6(r, s)] \end{cases} \quad (31)$$

and $\bar{\boldsymbol{\gamma}}$ is the shear strain matrix which can be written as a function of the element degrees of freedom

$$\bar{\boldsymbol{\gamma}} = \mathbf{A} \mathbf{v}, \quad \text{where} \quad \begin{cases} \bar{\boldsymbol{\gamma}}^T = [\gamma_{rz}(r_1, s_1) \dots \gamma_{rz}(r_6, s_6) \quad \gamma_{sz}(r_1, s_1) \dots \gamma_{sz}(r_6, s_6)] \\ \mathbf{v}^T = [w_i \quad \theta_i \quad \theta_{si}] \end{cases} \quad (32)$$

In this expression, the matrix $\mathbf{A}$ is deduced from the components of $\boldsymbol{\gamma}$ computed at the sampling point (r_j, s_j) . Finally, the shear-displacement matrix can be obtained as

$$\bar{\mathbf{B}}_i = \mathbf{J}^{-1} \mathbf{N}_{r-s} \mathbf{A}, \quad (33)$$

where $\mathbf{J}$ denotes Jacobian matrix and the shear stiffness matrix is now

$$\mathbf{K}_s^{(e)} = \iint_{A^{(e)}} \bar{\mathbf{B}}_i^T \mathbf{D}_s \bar{\mathbf{B}}_i dA. \quad (34)$$

By computing the stiffness matrices, full Gauss integration schemes (3×3) are used.

3. Element characteristics

To assess the performance characteristics of the present element several numerical examples were solved. A series of eigenvalue analyses of the stiffness matrices have shown that only three eigenvalues are always zero (corresponding to the rigid body modes of the element) for various element shapes and for very thick to very thin plates. The element always has a proper rank. The element also passes the patch tests [3, 4, 12] for bending, twisting and shear. In the locking test, a symmetric quadrant of a simply supported or clamped square plate ($E = 10.92$, $\nu = 0.3$, $L = 10$) under uniform load is idealized using 4 by 4 element mesh. Results in Figure 2 are presented in the following form:

$$\varepsilon_w(\%) = 100 \cdot (w - w_{\text{Kirchhoff}}) : w_{\text{Kirchhoff}}$$

where $\varepsilon_w(\%)$ is the percentage error. There is no locking behaviour and the central deflection becomes asymptotic to the Kirchhoff solution for thin-plate situations.

In the convergence test, one quarter of a uniformly loaded simply supported thick square plate is subdivided into 1, 2×2, 4×4 and 8×8 element meshes. The ratio between the side length L and the thickness t is 10. The structural properties of the plate are $E = 10.92$, $\nu = 0.3$, $L = 10$ and the intensity of the loading is $q = 1.0$. The results for central deflection and central bending moment are given in Table 1 in comparison with the elements which are derived in references [3,10,11].

The final example illustrated in Figure 3 has been performed for both thick and thin plates ($L/t = 10, 100$) with distorted 4×4 and 8×8 meshes (Reference 15). Results in Table 2 are presented in the following form:

$$\varepsilon_D(\%) = 100 \cdot (r_U - r_D) : r_U,$$

where $\varepsilon_D(\%)$ is the percentage error and r_U, r_D stand for both uniform and distorted mesh results.

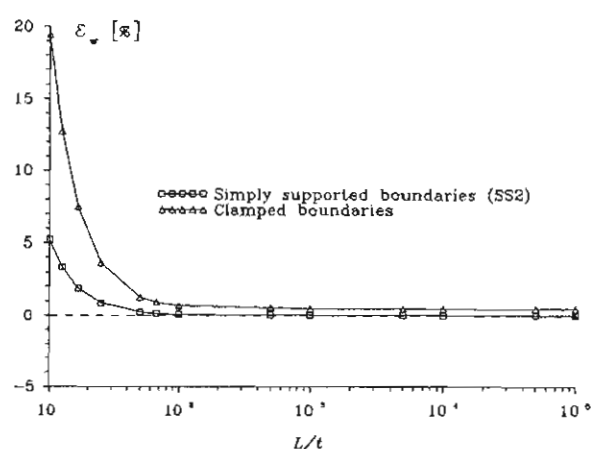
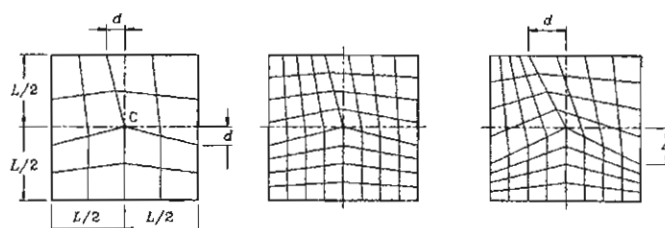


Fig. 2. Locking test

Table 1. Deflection and moment at the centre of uniformly loaded simply supported thick square plate ($L/t=10$)

Mesh	$(10^2 D w_c)/(q L^4)$				$(10 M_c)/(q L^2)$			
	Q4BL [11]	PQI [10]	QUAD9* [3]	Present element	Q4BL [11]	PQI [10]	QUAD9* [3]	Present element
1x1	-	0.48682	0.44137	0.440371	-	0.8115	0.58925	0.59186
2x2	0.426931	0.44709	0.42825	0.427945	0.471629	0.5591	0.49088	0.49306
4x4	0.427432	0.43225	0.42758	0.427323	0.477331	0.4979	0.47953	0.48166
8x8	0.427334	0.42852	-	0.427287	0.478497	0.4836	-	0.47952
Exact[15]	0.427284				0.478864			

The percentage errors for deflection and moments at the centre of uniformly loaded simply supported square plate show that present element is not sensitive to mesh distortions.



a) $d/L=0.125$, b) $d/L=0.125$, c) $d/L=0.25$

Fig. 3. Distorted 4×4 and 8×8 meshes

Table 2. Percentage error $\varepsilon_D(\%)$ due to distortion of 4×4 and 8×8 meshes of present elements (Figure 3). Uniformly loaded simply supported square plate

Mesh		4 x 4		8 x 8	
L/t	Quantity	$d/L=0.125$	$d/L=0.125$	$d/L=0.250$	
10	w_c	-0.01	-0.00	0.01	
	M_{xc}	-0.64	-0.18	-0.68	
	M_{yc}	-0.32	-0.12	-0.39	
100	w_c	1.73	0.09	0.97	
	M_{xc}	2.86	-0.04	0.92	
	M_{yc}	4.90	0.25	2.08	

4. Conclusion

The new nine-node and 27-d.o.f. quadrilateral element for the analysis of both thick and thin plates has been derived and implemented in a computer code. The element has been tested for a diverse set of problems, some of which are referred in the paper. It has been concluded that the element has the following characteristics: it passes the patch test for thick and thin plates, there are no extra zero-energy modes, there is no shear locking for thin plates, there is a very good convergence characteristic for thick and thin plates, it is relatively insensitive to geometric distortions, there are conventional engineering degrees of freedom at each node (i.e. one displacement and two rotations), it has curved edges and it is computationally efficient. This Lagrangian element gives further insight into the formulation of Reference 12. Furthermore, it has been shown that it is possible to explain optimal shear strain-sampling points by the reduced minimization theory. In this work, we have considered linear elastic problems only. Recently, we have also used new element to analyse nonlinear problems.

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Geometrically Exact Strain-invariant 3D Beam Finite Element

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Summary

Many recent finite element formulations for 3D beams have been based on the geometrically exact 3D beam theory. This theory satisfies the requirement of objectivity of the strain measures with respect to a rigid-body motion, but many popular finite element implementations often spoil this important property. In this paper we propose a new finite element implementation of the geometrically exact 3D beam theory, specifically designed to preserve the objectivity of adopted strain measures. In order to do so it becomes necessary to interpolate current local rotations in a manner similar to the co-rotational approach, but unlike many co-rotational approaches the proposed implementation preserves the geometric exactness of the theory. The new method is verified by running an illustrative numerical example.

1. Introduction

We talk about the geometrically exact beam theory when the relationships between the configuration and the strain measures are consistent with the virtual work principle and the equilibrium equations at a deformed state regardless of the magnitude of displacements, rotations and strains. Geometrically exact beam theory is generally referred to as "Reissner's beam theory", although a simplification in the parametrisation of the rotation matrix in space was made in [1] which facilitated the derivation of the rotational strain measures, but also spoiled the geometric exactness of the theory. A different approach in the geometrically exact beam theory was used by Simo [2] to propose a parametrisation of rotation matrix in space which furnished a full geometric exactness of the theory. The establishment of the relationships between the large overall 3D rotations and strains appears to be one of the biggest problems in the geometrically exact beam theory, mainly because the direction cosine matrices constitute a nonlinear manifold rather than a linear space.

The (nonadditive) update of 3D rotations has to be performed carefully in order to preserve the nature of the nonlinear manifold of 3D rotations, and in the implementation proposed by Simo and Vu-Quoc [3] the genuine iterative rotational changes were being interpolated along the length of the beam. In contrast, Cardona and G  radin [4] regarded the update of 3D rotations on the basis of last converged configuration as the most suitable choice and consequently chose to interpolate incremental rotational changes. Third interesting possibility is the update of 3D rotations on the basis of the initial configuration, which leads to the interpolation of total rotational (pseudo-)vectors [4,5]. In all cases were adopted rotational fields interpolated additively, despite the fact that 3D rotations cannot be additively composed. Possible problems due to such interpolation with particular emphasis on the objectivity of strain measures were analysed in [6], where a condition for strain measures to be objective in a finite element implementation was established and a new finite element proposed. These results were used in [7] to elaborate a finite element implementation of a new family of fully objective geometrically exact 3D beam elements. The main results of [7] are outlined here.

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2. Kinematics of 3D beams, strain measures and constitutive equations

The beam model analysed here is basically due to Reissner [1] with the parametrisation of rotations as given by Simo [2]. The line of centroids of cross-sections of the undeformed beam element is taken to be a straight line which coincides with the x axis of the orthonormal Cartesian frame (x, y, z) with $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ as the unit base vectors (Fig. 1). The cross-sections of the undeformed beam in the coordinate plane $x = \text{const.}$ are orthogonal to the line of centroids and their normals coincide with the base vector $\mathbf{g}_1$. The remaining two base vectors, $\mathbf{g}_2$ and $\mathbf{g}_3$, are directed along the principal axes of inertia of the cross-section. The shape of the cross-section is assumed to be arbitrary and constant along the axis of the beam. In the deformed state, the Bernoulli hypothesis of the cross-sections remaining planar after deformation and retaining their shape and area is assumed to hold, so the configuration of the beam at any coordinate x between its ends is completely defined by (i) the position vector $\mathbf{r}(x) = x\mathbf{g}_1 + \mathbf{u}(x)$ and (ii) the orthogonal transformation (called the rotation matrix) $\mathbf{A} \in SO(3)$ with well known properties $\det \mathbf{A} = 1$; $\mathbf{A}^{-1} = \mathbf{A}^t$. Here we proceed by parametrising the rotation matrix using the rotational (pseudo-)vector $\boldsymbol{\psi} = \psi^i \mathbf{g}_i$, whereby $\mathbf{A} = \exp \hat{\boldsymbol{\psi}}$, where $\hat{\boldsymbol{\psi}} = -e_k^{ij} \psi^k \mathbf{g}_i \mathbf{g}_j^t$ is the skew-symmetric matrix associated with the rotational pseudovector $\boldsymbol{\psi}$, e_k^{ij} is the permutation symbol and ψ^i ; $i = 1, 2, 3$ are the components of the rotational (pseudo-)vector $\boldsymbol{\psi}$ with respect to the spatially fixed frame $\mathbf{g} = [\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3]$. Throughout the paper, a superimposed hat will denote a skew-symmetric matrix of a corresponding vector and a product of two factors marked by the same subscript and superscript index will denote a sum of the products for all the possible values of the index. Note that the rotations are composed multiplicatively, i.e. the rotation matrices at configurations k and $k+1$ are related via a rotational pseudovector $\boldsymbol{\phi}$ which rotates the base vectors $\mathbf{G}_{i,k}$ into the base vectors $\mathbf{G}_{i,k+1}$ around the axis $\boldsymbol{\phi}/\phi$ for the angle ϕ through $\mathbf{A}_{k+1} = \exp \hat{\boldsymbol{\phi}} \mathbf{A}_k$.

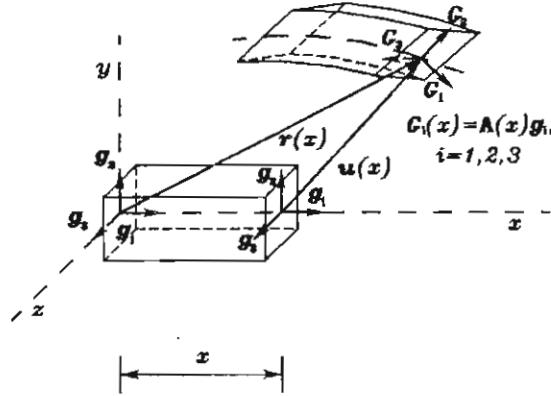


Figure 1. Reference coordinate system, body attached frame and deformed configuration of the beam

The translational and rotational strain measures $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are defined with respect to the body attached frame at x and are related to the configuration $(\mathbf{r}, \mathbf{A})$ through the equations given in Tables 1 and 2 of [3] as

$$\boldsymbol{\gamma} = \mathbf{A}^t \mathbf{r}' - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}; \quad \hat{\boldsymbol{\kappa}} = \mathbf{A}^t \mathbf{A}'. \quad (1)$$

The prime ($'$) in these equations denotes the derivative with respect to axial coordinate x . The constitutive law for elastic material is taken to be given by a linear relation between the stress and stress-couple resultants and the adopted strain measures as $\mathbf{N} = \mathbf{C}_N \boldsymbol{\gamma}$ and $\mathbf{M} = \mathbf{C}_M \boldsymbol{\kappa}$, where $\mathbf{C}_N = \text{diag}[EA \quad GA_2 \quad GA_3]$ and $\mathbf{C}_M = \text{diag}[GJ_t \quad EI_2 \quad EI_3]$ are constitutive matrices which define the relations between translational strains and cross-

sectional forces, and rotational strains and cross-sectional moments, respectively. Here, E and G denote elastic and shear moduli of material, A is the cross-sectional area, A_2 and A_3 are the shear areas in the directions of principal axes of inertia of the cross-section, J_t is the torsional inertial moment of the cross-section and J_2 and J_3 are the cross-sectional inertial moments about the principal axes of inertia of the cross-section.

3. Objectivity of strain measures in finite element implementation

The adopted strain measures (1) are objective in the sense that if a constant translation $\mathbf{r}_R$ and rotation $\mathbf{A}_R$ are superposed onto the current configuration $\mathbf{r}$, $\mathbf{A}$ so that as a result we have $\underline{\mathbf{r}} = \mathbf{A}_R(\mathbf{r}_R + \mathbf{r})$ and $\underline{\mathbf{A}} = \mathbf{A}_R \mathbf{A}$, it can be proved [6] that $\gamma(\mathbf{r}, \mathbf{A}) = \gamma(\underline{\mathbf{r}}, \underline{\mathbf{A}})$ and $\kappa(\mathbf{A}) = \kappa(\underline{\mathbf{A}})$.

However, once we introduce the finite element approximations (denoted by a superscript h), it can be shown [6] that the above is true only if we can demonstrate that

$$\underline{\mathbf{r}}^h = \mathbf{A}_R(\mathbf{r}_R + \mathbf{r}^h) ; \underline{\mathbf{A}}^h = \mathbf{A}_R \mathbf{A}^h \quad (2)$$

as a result of any constant $\mathbf{r}_R$ and $\mathbf{A}_R$. In order to apply (1), finite element formulations apply updatings of the form

$$\mathbf{A}^h = \exp \hat{\phi}^h \mathbf{A}_r,$$

where $\mathbf{A}_r$ depends on the particular implementation. For some implementations, $\mathbf{A}_r$ is the rotation matrix at the previous iteration [3], for others it is the rotation matrix at the end of the previous increment [4] and for others it is the Cartesian frame $\mathbf{g} = [\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3]$ [5]. Each use of $\mathbf{A}_r$ requires a different definition of ϕ but, in each case, the finite element interpolation is applied onto specific ϕ via

$$\phi(x) \doteq \phi^h(x) = I^i(x) \phi_i,$$

where ϕ_i are values taken by $\phi^h(x)$ at nodes of an N -noded finite element depicted in Fig. 2, $I^i(x)$ are standard interpolation polynomials and i runs from 1 to N . In each case it can be shown [6] that the formulation is non-objective in that it does not satisfy Eqn (2) except at specific nodal points or more generally in the limit as the number of nodal variables is increased towards infinity. In some cases [3,4], the formulations are also history dependent.

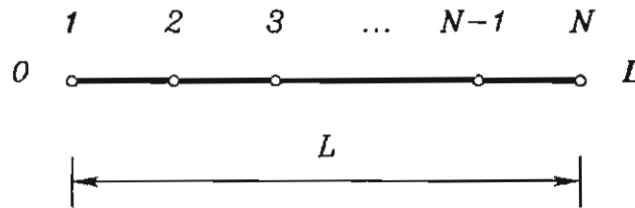


Figure 2. N-noded beam finite element

A fully strain invariant formulation can be defined by decomposing the current rotation matrix $\mathbf{A}(x)$ as

$$\mathbf{A}(x) = \mathbf{A}_I \exp \hat{\Theta}^I(x), \quad (3)$$

where $1 \leq I \leq N$ is taken arbitrarily and kept fixed throughout the analysis and $\mathbf{A}_I = \mathbf{A}(x_I)$ will be called the reference rotation matrix. An alternative choice for the reference rotation matrix is discussed in [6,7]. In Eqn (3), $\Theta^I(x)$ is the local rotation between the

reference rotation matrix Λ_I and the rotation matrix $\Lambda(x)$, the components of which are given in the coordinate system defined by the reference rotation matrix. In the strain invariant formulation, the finite element approximation is applied onto local rotation $\Theta^l(x)$ via

$$\Theta^l(x) \doteq \Theta^{lh}(x) = I^i(x) \Theta_i^l \quad (4)$$

where the local nodal rotations are extracted from the reference rotation matrix Λ_I and nodal rotation matrices via $\exp \hat{\Theta}_i^l = \Lambda_I^t \Lambda_i \rightarrow \Theta_i^l$; $i = 1, \dots, N$, by using e.g. Spurrier's algorithm [8] as given in [3]. The utilisation of a reference rotation matrix, which is rigidly connected to the element and the interpolation of current local rotations resembles the co-rotational technique [9], which in many cases preserves the objectivity of strain measures. However, in contrast to many co-rotational techniques, the proposed formulation is geometrically exact.

4. Equilibrium equations and FE discretisation

In order to numerically solve the problem of motion of 3D beam in space, we need to perform spatial and temporal discretisation of the space-time domain. Here we limit our attention only to the static problems, which do not require time discretisation. The variational finite element approach for solving the problem is based on the transformation of the differential equations of the problem into its weak (variational) form and the finite element approximation of the chosen test functions. We thus obtain $\langle \mathbf{f}_i^t, \boldsymbol{\varphi}_i^t \rangle \mathbf{g}^i = 0$, where $\mathbf{f}_i$ and $\boldsymbol{\varphi}_i$ are arbitrary nodal values of the adopted test functions $\mathbf{f}(x)$ and $\boldsymbol{\varphi}(x)$, which are interpolated using the standard interpolation polynomials $I^i(x)$ and $\mathbf{g}^i$ is the contribution of all the forces acting on the beam at node i . The arbitrariness of nodal values of the adopted test functions implies

$$\mathbf{g}^i \equiv \mathbf{q}_{int}^i - \mathbf{q}_{ext}^i = \mathbf{0}; \quad i = 1, \dots, N, \quad (5)$$

where the vectors of internal and external forces are derived in [3,7] as

$$\mathbf{q}_{int}^i = \int_0^L \begin{bmatrix} I^{i'} I & \mathbf{0} \\ -I^{i'} \hat{\mathbf{r}}' & I^{i'} I \end{bmatrix} \begin{Bmatrix} \Lambda N \\ \Lambda M \end{Bmatrix} dx, \quad \mathbf{q}_{ext}^i = \delta_1^i \begin{Bmatrix} F_0 \\ T_0 \end{Bmatrix} + \delta_N^i \begin{Bmatrix} F_L \\ T_L \end{Bmatrix}, \quad (6)$$

with δ_j^i being the Kronecker symbol and I a 3×3 unit matrix. For the sake of simplicity, we have only considered the concentrated forces and moments $\mathbf{F}$ and $\mathbf{T}$ at the ends of the element and neglected any distributed loads.

Eqn (5) represents the contribution to the algebraic nonlinear equilibrium equation of an isoparametric beam finite element at node i . This equation is iteratively solved using the Newton-Raphson method by developing it into a Taylor series around an arbitrary configuration and neglecting the higher order terms. In this way we obtain

$$\Delta \mathbf{g}^i = -\mathbf{g}^i; \quad i = 1, \dots, N, \quad (7)$$

where the right-hand side is defined by Eqns (5) and (6). By varying Eqn (6)₁ we obtain the left-hand side of Eqn (7) as

$$\Delta \mathbf{g}^i = \int_0^L \left(\begin{Bmatrix} \mathbf{0} \\ -I^{i'} \Delta \hat{\mathbf{r}}' \Lambda N \end{Bmatrix} + \begin{bmatrix} I^{i'} I & \mathbf{0} \\ -I^{i'} \hat{\mathbf{r}}' & I^{i'} I \end{bmatrix} \begin{Bmatrix} \Delta \Lambda N + \Lambda \Delta N \\ \Delta \Lambda M + \Lambda \Delta M \end{Bmatrix} \right) dx, \quad (8)$$

which may be written as $\Delta \mathbf{g}^i = \mathbf{K}^{ij} \Delta \mathbf{p}_j$; $i, j = 1, \dots, N$, where $\mathbf{K}^{ij}$ is the ij -contribution to the tangent stiffness matrix and the vector $\Delta \mathbf{p}_j$ consists of the infinitesimal changes

of displacements and rotations $\Delta \mathbf{u}_j$ and $\Delta \boldsymbol{\vartheta}_j$, where $\Delta \mathbf{u}_j = \Delta \mathbf{r}_j$; $j = 1, \dots, N$ and $\Delta \mathbf{A}_j = \Delta \hat{\boldsymbol{\vartheta}}_j \mathbf{A}_j$; $j = 1, \dots, N$.

The linearisation procedure leading to the tangent stiffness matrix is performed as described in [3] with one significant exception being that the infinitesimal change of the current rotation $\Delta \boldsymbol{\vartheta}$ is in the current approach not being interpolated using the standard interpolation polynomials $I^i(x)$. In contrast, it is crucial to the development of a strain-invariant finite element formulation that such an interpolation is applied onto current local rotations as shown in Eqn (4). The infinitesimal change of the current rotation $\Delta \boldsymbol{\vartheta}$, which occurs in $\Delta \mathbf{A}$, $\Delta \mathbf{N}$ and $\Delta \mathbf{M}$ in Eqn (8) can be related to the infinitesimal change of the current local rotation and the infinitesimal change of the reference rotation by varying Eqns (3) and (4). In this way we obtain [7]

$$\begin{aligned} \Delta \boldsymbol{\vartheta} &= \Delta \boldsymbol{\vartheta}_I + \mathbf{A}_I \mathbf{T}^{-1}(\boldsymbol{\Theta}^I) I^i \Delta \boldsymbol{\Theta}_i^I \\ \mathbf{T}(\boldsymbol{\Theta}^I) &= \frac{1}{\Theta^{I^2}} \boldsymbol{\Theta}^I \boldsymbol{\Theta}^{I^T} + \frac{\frac{\Theta^I}{2}}{\tan \frac{\Theta^I}{2}} \left(\mathbf{I} - \frac{1}{\Theta^{I^2}} \boldsymbol{\Theta}^I \boldsymbol{\Theta}^{I^T} \right) - \frac{1}{2} \hat{\boldsymbol{\Theta}}^I. \end{aligned} \quad (9)$$

Due to the orthogonality of the interpolation polynomials, inverted Eqn (9) at nodes reads

$$\Delta \boldsymbol{\Theta}_i^I = \mathbf{T}(\boldsymbol{\Theta}_i^I) \mathbf{A}_I^T (\Delta \boldsymbol{\vartheta}_i - \Delta \boldsymbol{\vartheta}_I),$$

so Eqn (9) can be re-expressed as

$$\Delta \boldsymbol{\vartheta}(x) = \tilde{\mathbf{I}}^i(x) \Delta \boldsymbol{\vartheta}_i, \quad (10)$$

with "generalised shape functions" $\tilde{\mathbf{I}}^i(x)$ being

$$\tilde{\mathbf{I}}^i(x) = \mathbf{A}_I \left\{ \delta_j^i \left[\mathbf{I} - \mathbf{T}^{-1}(\boldsymbol{\Theta}^I(x)) I^j(x) \mathbf{T}(\boldsymbol{\Theta}_j^I) \right] + \Delta_k^{ij} \mathbf{T}^{-1}(\boldsymbol{\Theta}^I(x)) I^k(x) \mathbf{T}(\boldsymbol{\Theta}_j^I) \right\} \mathbf{A}_I^T,$$

where Δ_k^{ij} may be named a "generalised Kronecker symbol" with $\Delta_k^{ij} = 1$ for $i = j = k = 1$ and $\Delta_k^{ij} = 0$ otherwise. By making use of the generalised interpolation (10) the ij -contribution to the tangent stiffness matrix can be derived by following approach presented in [3]. Note that the generalised shape functions $\tilde{\mathbf{I}}^i$; $i = 1, \dots, N$ satisfy the standard conditions $\tilde{\mathbf{I}}^i(x_j) = \delta_j^i \mathbf{I}$, $\sum_{i=1}^N \tilde{\mathbf{I}}^i(x) = \mathbf{I}$; $x \in [0, L]$ and $\sum_{i=1}^N \tilde{\mathbf{I}}^{i'}(x) = \mathbf{0}$; $x \in [0, L]$.

In the case of higher order elements with three or more nodes (see Fig. 2), the internal degrees of freedom may be condensed out of the linearised equilibrium (5) and resolved at the element level (see e.g. Cook et al. [10]). The assembly of the tangent stiffness matrix and the residual vector of the structure and the inclusion of kinematic boundary conditions are performed in the standard way (see e.g. Zienkiewicz and Taylor [11]). In this way we obtain following linearised equilibrium of the structure

$$\bar{\mathbf{K}} \Delta \bar{\mathbf{p}} = -\bar{\mathbf{g}}, \quad (11)$$

where $\bar{\mathbf{K}}$ is the tangent stiffness matrix and $\bar{\mathbf{g}}$ is the residual vector of the structure. This equation should be solved for the global unknowns $\Delta \bar{\mathbf{p}}$.

The update of the nodal quantities is performed in the standard manner [3]. The strain measures (1) are computed at the Gauss points by utilising the Gauss point rotation matrices $\mathbf{A}$ as obtained from Eqns (3) and (4). Such an update guarantees the objectivity of the adopted strain measures. More details on the theoretical aspects and finite element implementation are given in [7].

5. Numerical example

The described algorithm has been coded and mounted in the research version of the finite element system LUSAS [12] and will be referred to as Algorithm PG. The comparison will be made with respect to the research code, obtained by full utilisation of the theory proposed in [3] and referred to as Algorithm X. In both cases the nonsymmetric frontal solver will be utilised and the convergence tolerance for the sum of the squares of the iterative displacement components in $\Delta \bar{\mathbf{p}}$ in Eqn (11) as a percentage of the sum of the squares of the total displacement components and the sum of the squares of all the residual forces ($\bar{\mathbf{g}}$ in Eqn (11)) as a percentage of the sum of the squares of all external forces including reactions will be set up as 10^{-6} . Load step will be halved if the convergence has not been achieved within 30 iterations.

We analyse a rectangular elbow cantilever subject to a concentrated vertical force at its free end and a prescribed rotation at its fixed end (Fig. 3). The frame is composed of two equally long mutually perpendicular legs, the orientation of which is defined by the orthonormal element frames $\mathbf{g} = [\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3]$, where the unit base vector $\mathbf{g}_1$ is always directed along the centroidal axis and the base vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ are directed along the principal axes of inertia. The stiffness properties of legs are given in Fig. 3. Each leg is modelled by a single 3-noded isoparametric element, with quadratic interpolation for the displacement and iterative rotation fields in Algorithm X [3] or the displacement and current local rotation fields (Eqn (4)) in Algorithm PG.

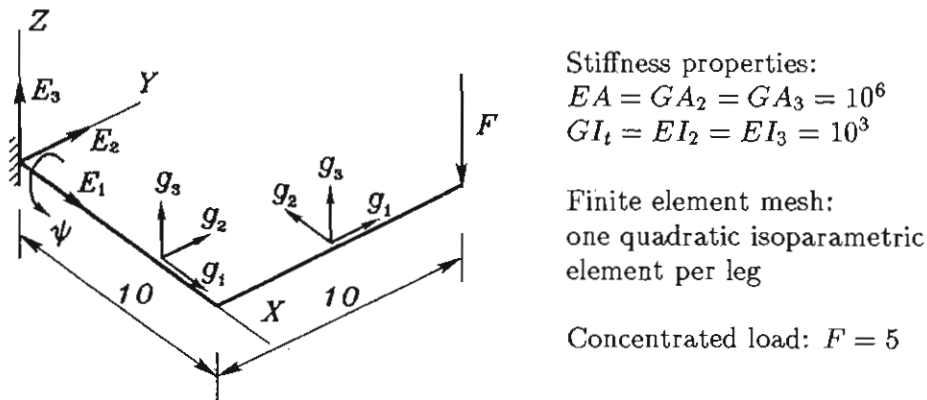


Figure 3. 3D cantilever frame subject to a fixed vertical force at the free end and a prescribed rotation around base vector $\mathbf{E}_1$ at the fixed end

The initial orientation of each element frame with respect to the orthonormal global frame $\mathbf{E} = [\mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3]$ is also shown in Fig. 3. The applied force F is of a fixed magnitude and constant vertical direction. The applied prescribed rotation ψ rotates the whole structure around the unit vector $\mathbf{E}_1$ of the global frame. This rotation does not change the direction of the applied load F .

This problem will be solved for displacements U , V and W along the coordinate axes X , Y , Z of the global frame $\mathbf{E}$ for different prescribed rotations ψ .

The example is first run for $\psi = 0$ and the free-end displacements obtained by Algorithm X are $U = 1.75069$, $V = -0.424614$ and $W = -6.76793$, while Algorithm PG gives $U = 1.74970$, $V = -0.424423$ and $W = -6.76754$. The difference in results obtained by the two algorithms is small and should be related to the difference in the two implementations.

These results should not change if the whole structure undergoes a rigid rotation of 2π . However, if the example is run for $\psi = 2\pi$, with ψ being applied in four equal increments and F kept constant throughout the incrementation, the results obtained by Algorithm X are $U = 1.74902$, $V = -0.434344$ and $W = -6.78935$, while Algorithm PG gives the same results as in the first run. The inability of Algorithm X to remain unaffected by a rigid

rotation is related to the nonobjectivity of the strain measures in the corresponding finite element implementation [6].

If the frame is left spinning as described above and the displacement components at the free end are monitored at the end of each whole turn, it can be observed that the error produced by Algorithm X is accumulating. In contrast, Algorithm PG, which was designed to guarantee the objectivity of strain measures, preserves the displacement components at the end of each whole turn. The results for both algorithms for first eighty-eight turns are collected and graphically presented in Fig. 4.

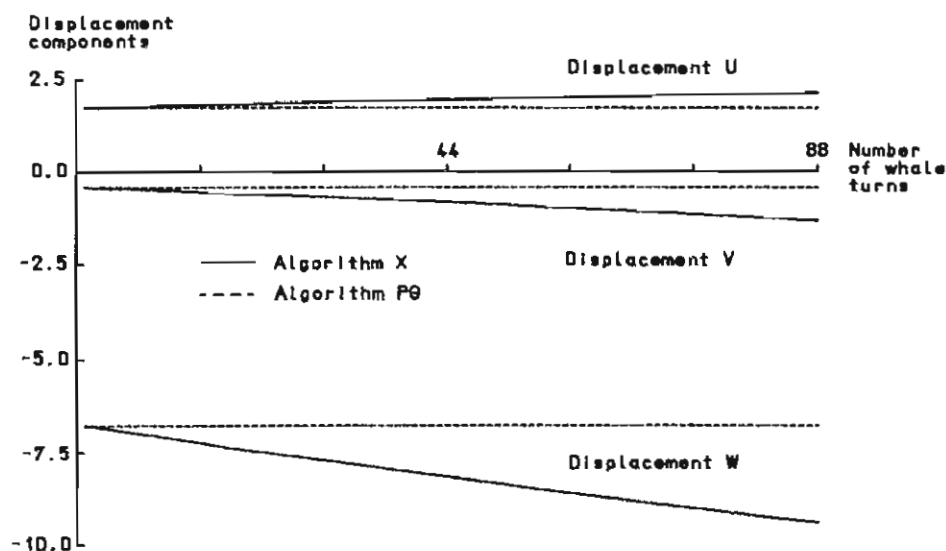


Figure 4. Displacement components at the free end of the cantilever at ends of each whole turn of the cantilever

It is interesting to note that Algorithm X not only fails to provide the objective strain measures, but it also seems to be less robust in coping with large increments once the error becomes to accumulate more considerably. On several occasions the solution procedure could not achieve equilibrium for given prescribed incremental rotation and the problem had to be overcome by halving the amount of the incremental rotation. In contrast, no reduction of the applied incremental rotation was ever needed for Algorithm PG. The applied incremental rotation against the increment number for both algorithms is shown in Fig. 5. Obviously, the total applied rotation for a fixed number of increments is eventually considerably greater in Algorithm PG than in Algorithm X. The problem was run by using Algorithm PG for the total amount of 2000 increments. The total amount of rotation applied at the fixed end of the cantilever at the end of analysis is thus $\psi = 1000\pi$ or 500 whole turns and the corresponding displacement components are the same as when $\psi = 0$.

6. Conclusions

In the presence of large 3D rotations the concept of objectivity of strain measures does not extend from the theory to the finite element formulation so naturally as it does otherwise. Consequently, the finite element implementation of a strain-invariant theory has to be specifically designed to inherit this important property. A new geometrically exact strain-invariant 3D beam finite element is proposed in the paper, which was used together with a popular but non-invariant formulation to run an illustrative numerical example. The theoretical findings are fully backed by the numerical results obtained.

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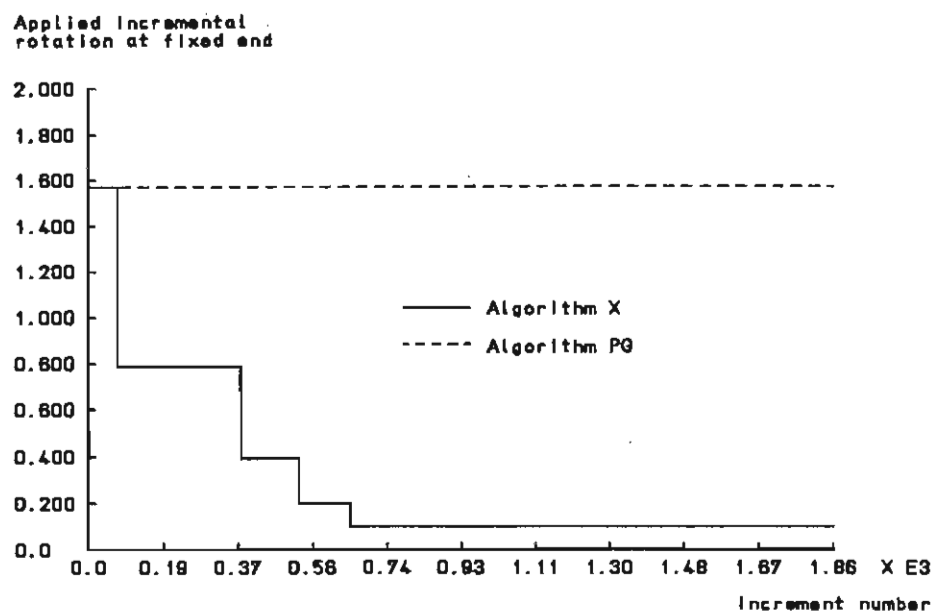


Figure 5. Amount of the applied incremental rotation at the fixed end of the cantilever against the increment number

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EXPERIMENTAL INVESTIGATIONS AND FINITE ELEMENT PROCEDURE OF THIN-WALLED LOCAL AND GLOBAL STABILITY PROBLEMS

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ABSTRACT

Thin-walled structures constitute very important class of structural components. In their design procedure, especially in the optimal one, an important considerations concern the accurate prediction of their limit stability state. This work deals with the phenomena of both local and global elastic buckling problems of thin-walled beam type structures with an opened cross-section. In the case of equal-leg angle as well as channel aluminium columns the influence of cross-sectional dimensions ratio on the above mentioned types of buckling form occurring has been investigated. Numerical procedure based on the finite element method has been used and obtained results have been compared to the experimentally obtained values.

1. INTRODUCTION

The most common type of metal beam structures used extensively in engineering applications are thin-walled beams, such as I-beams, channels, angles etc. In this, they are used both in their stand-alone forms an as stiffeners for shell or plate structures [1,2,3]. Thin-walled sections are appealing because they offer a high performances in terms of minimum weight for a given strength. At the same time, the structural behavior is very complex and the accurate prediction of their stability limit state or buckling strength is of fundamental importance in the design of such structures [4,5,6].

Thin-walled columns can be represented as an assemblage of rectangular plates. Depending on the relative cross-sectional dimensions they may fail by local buckling before they fail by global buckling [7,8]. This type of buckling form is well-known as sheet buckling, wrinkling, crimping etc. [9]. Local buckling may not cause immediate collapse of the column, but, because of the stress redistribution, it reduces the compressive stiffness and leads to the failure at loads less than global buckling load [10].

2. FINITE ELEMENT PROCEDURE

As it is well-known stability problems are characterized by the geometrical nonlinearity, i.e. there is a nonlinear strain-displacements relationship, so we have [11]:

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,j} u_{k,i}), \quad (1)$$

where Eq. 1 is Green-Lagrange strain tensor. In the vector form it can be rewritten as:

$$\varepsilon = \varepsilon_L + \varepsilon_N, \quad (2)$$

where ε_L is a linear part:

$$\varepsilon_L = \begin{Bmatrix} u_{,x} & v_{,y} & w_{,z} & u_{,y} + v_{,x} & v_{,z} + w_{,y} & w_{,x} + u_{,z} \end{Bmatrix} \quad (3)$$

and ε_N is a corresponding nonlinear one:

$$\varepsilon_N = \frac{1}{2} \mathbf{A} \Theta, \quad (4)$$

where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_x^T & 0 & 0 \\ 0 & \mathbf{a}_y^T & 0 \\ 0 & 0 & \mathbf{a}_z^T \\ \mathbf{a}_y^T & \mathbf{a}_x^T & 0 \\ 0 & \mathbf{a}_z^T & \mathbf{a}_y^T \\ \mathbf{a}_z^T & 0 & \mathbf{a}_x^T \end{bmatrix} \quad (5)$$

$$\Theta = [\mathbf{a}_x \quad \mathbf{a}_y \quad \mathbf{a}_z]^T \quad (6)$$

$$\mathbf{a}_i = [u_{,i} \quad v_{,i} \quad w_{,i}]^T, \quad i = x, y, z \quad (7)$$

Substituting Eq. (4) into (2), follows:

$$\varepsilon = \varepsilon_L + \frac{1}{2} \mathbf{A} \Theta \quad (8)$$

Because, in the finite element formulation displacements are [12]:

$$\mathbf{u} = \begin{Bmatrix} u & v & w \end{Bmatrix}^T = \mathbf{N} \mathbf{U} \quad (9)$$

where $\mathbf{N}$ is the shape functions matrix and $\mathbf{U}$ is the nodal displacements vector, from Eq. (6) we obtain:

$$\Theta = [\partial \mathbf{N} / \partial x \quad \partial \mathbf{N} / \partial y \quad \partial \mathbf{N} / \partial z]^T \mathbf{U} = \mathbf{G} \mathbf{U} \quad (10)$$

Also, as it is known from the linear theory that:

$$\varepsilon_L = \mathbf{B}_L \mathbf{U} \quad (11)$$

where $\mathbf{B}_L$ is the linear strain matrix, from Eq. (8) follows:

$$\varepsilon = \left(\mathbf{B}_L + \frac{1}{2} \mathbf{B}_N \right) \mathbf{U} \quad (12)$$

where:

$$\mathbf{B}_N(\mathbf{U}) = \mathbf{A} \mathbf{G} \quad (13)$$

On the basis of the principle of virtual works it is [13]:

$$\delta W = \delta U$$

or:

$$\delta \mathbf{U}^T \mathbf{F} = \int_V \delta \varepsilon^T \boldsymbol{\sigma} dV \quad (14)$$

with:

$$\delta \epsilon = (\mathbf{B}_L + \mathbf{B}_N) \delta \mathbf{U} = \bar{\mathbf{B}} \delta \mathbf{U} \quad (15)$$

where $\mathbf{F}$ is the nodal forces vector and σ is stress. Introducing Eq. (15) into (14), we have:

$$\delta \mathbf{U}^T \mathbf{F} = \int_V (\bar{\mathbf{B}} \delta \mathbf{U})^T \sigma dV = \delta \mathbf{U}^T \int_V \bar{\mathbf{B}}^T \sigma dV$$

or

$$\Psi(\mathbf{U}) = \int_V \bar{\mathbf{B}}^T \sigma dV - \mathbf{F} = 0 \quad (16)$$

Upon differentiation of Eq.(16) with respect to $\mathbf{U}$, we obtain:

$$d\Psi = \int_V d\bar{\mathbf{B}}^T \sigma dV + \int_V \bar{\mathbf{B}}^T d\sigma dV = \mathbf{K}_T d\mathbf{U} \quad (17)$$

As it is:

$$d\bar{\mathbf{B}} = d\mathbf{B}_N \quad (18)$$

and

$$\sigma = \mathbf{D}\epsilon \quad (19)$$

where $\mathbf{D}$ is the matrix of elastic constants [14], from Eq. (17) follows:

$$\mathbf{K}_T d\mathbf{U} = (\mathbf{K}_L + \mathbf{K}_\sigma + \mathbf{K}_R) d\mathbf{U}, \quad (20)$$

with:

$$\mathbf{K}_L = \int_V \mathbf{B}_L^T \mathbf{D} \mathbf{B}_L dV, \quad (21)$$

$$\mathbf{K}_\sigma d\mathbf{U} = \int_V d\mathbf{B}_N^T \sigma dV, \quad (22)$$

$$\mathbf{K}_R = \int_V (\mathbf{B}_L^T \mathbf{D} \mathbf{B}_N + \mathbf{B}_N^T \mathbf{D} \mathbf{B}_N + \mathbf{B}_N^T \mathbf{D} \mathbf{B}_L) dV, \quad (23)$$

where:

$\mathbf{K}_T$ - tangential stiffness matrix (total matrix)

$\mathbf{K}_L$ - linear stiffness matrix (small displacement matrix)

$\mathbf{K}_\sigma$ - geometric matrix (initial stress matrix)

$\mathbf{K}_R$ - initial displacement matrix (contains only terms that are linear and quadratic in $\mathbf{U}$) - stiffness due to large rotation.

Stability analysis is characterized by geometric nonlinearities which are manifested in problems involving large rotations and large deformation. The characteristics of these nonlinearities are follower forces due to large rotations, large strains due to large deformation etc. Large strain effects are pronounced in metal forming problems which could have strain exceeding 100%. In most structural applications moderately large strains 20-30% appear. In spite of necessity of total nonlinear stability analysis using large deformation matrix, in a large number of technical applications initial stability problems is considered. It means that stability problem was solved as eigenvalue matrix problem [15]:

$$d\Psi = (\mathbf{K}_L + \lambda \mathbf{K}_\sigma) d\mathbf{U} = 0, \quad (24)$$

although this type of approach is frequently used beyond its limits of applicability. In Eq. (24) factor λ is the eigenvalue or critical buckling load F_{cr} and the corresponding eigenvector represents the buckling form.

3. EXAMPLES

Both an equal-leg angle and a channel columns have been considered. For the same length l and boundary conditions (both clamped ends) as well as outer cross-sectional dimensions, but varying wall thickness t , it has been necessary to determine if the corresponding column would fail either by local or by global buckling. Profiles have been made of aluminium with experimentally determined modulus of elasticity $E = 65 \text{ GPa}$ and yield strength $\sigma_{0.2} = 210 \text{ MPa}$. The problems have been solved by the software MSC/NASTRAN V.68 using eight noded shell finite element CQUAD8. For the verification of obtained numerical results, the experimental verifications have been performed, too.

Example 1: Angle profile

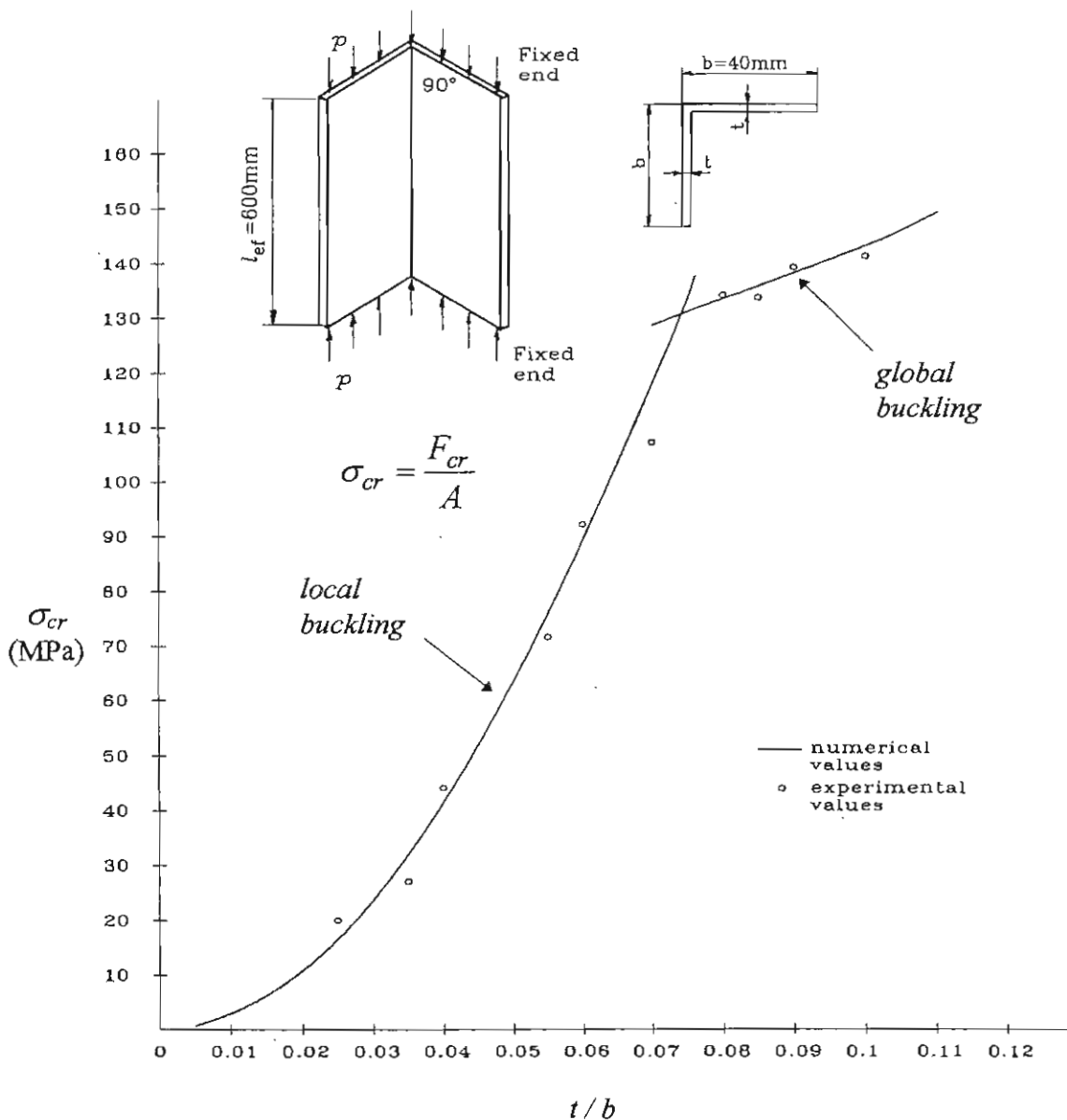
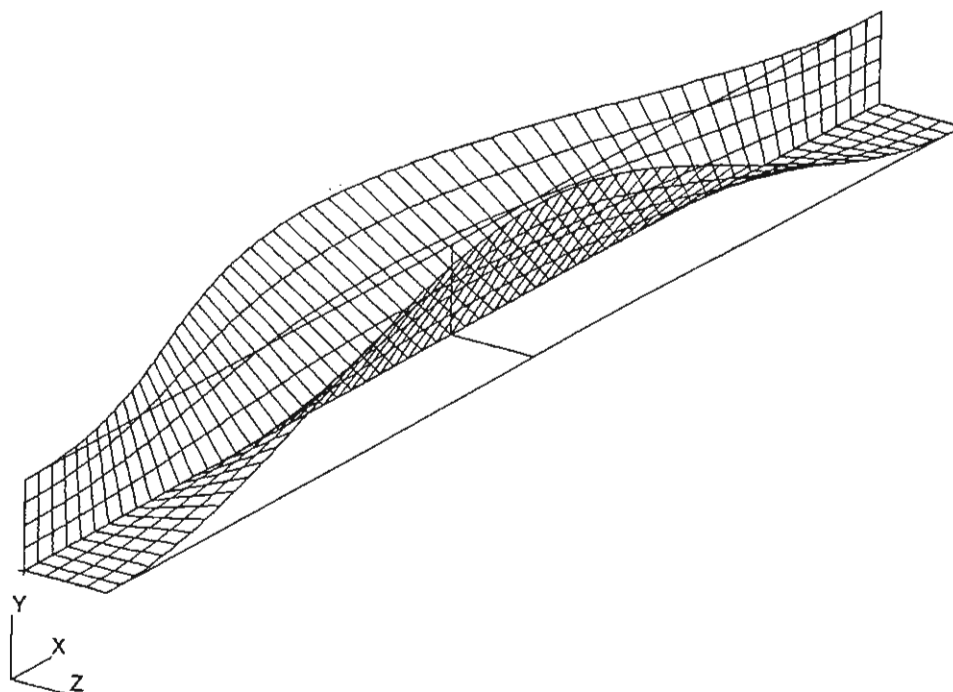
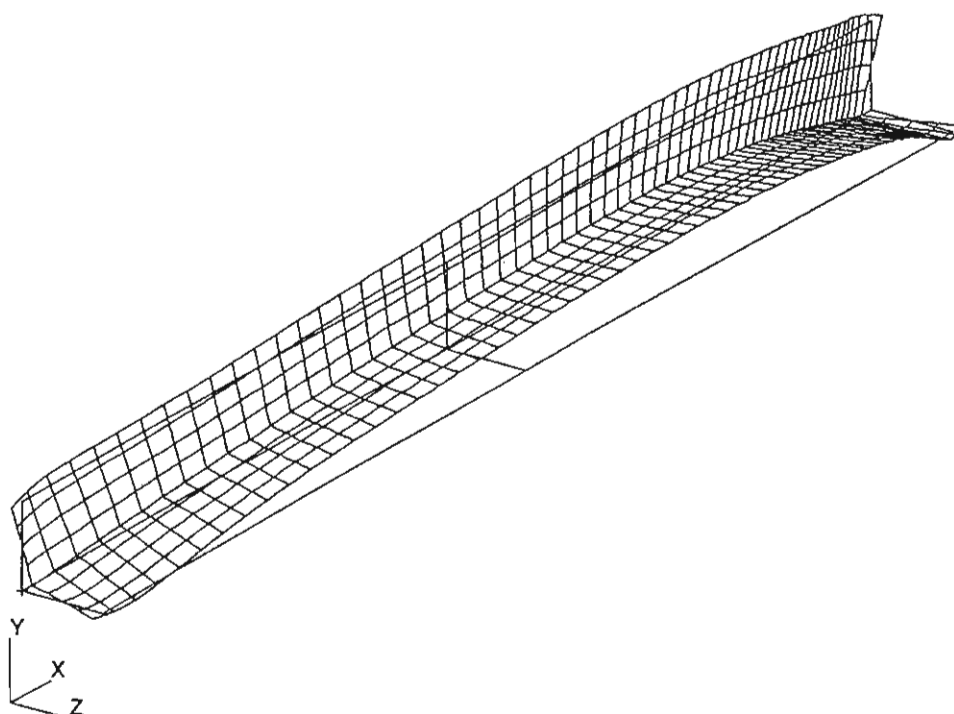


Figure 1: Buckling stress (σ_{cr}) vs. t/b for local buckling and global buckling for an angle profile of aluminium with $E = 65 \text{ GPa}$.

For any angle profile, if it has buckled by local buckling the lowest buckling mode has had eleven half-waves, Fig.2a. and if the profile has buckled by global buckling the corresponding mode has been of torsional-flexural type, Fig.2b.



a)



b)

Figure 2: b) first local buckling mode, b) first global buckling mode.

Example 2: Channel profile

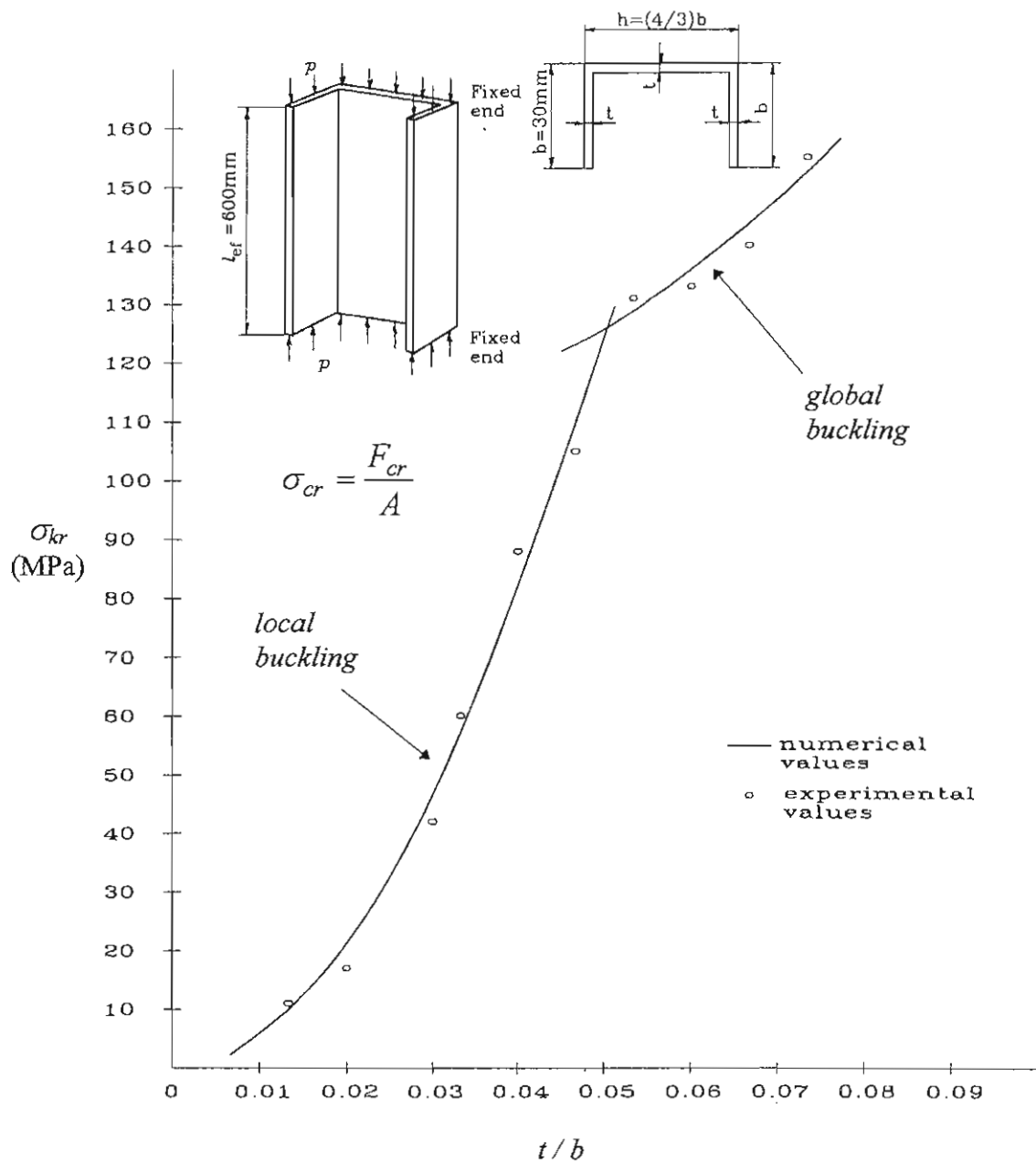
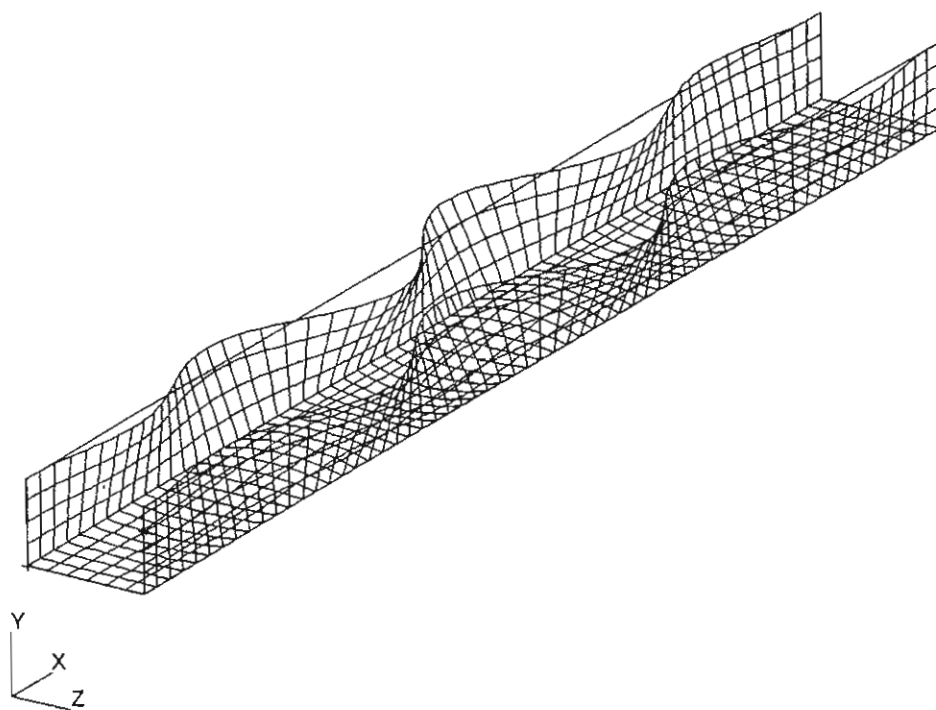
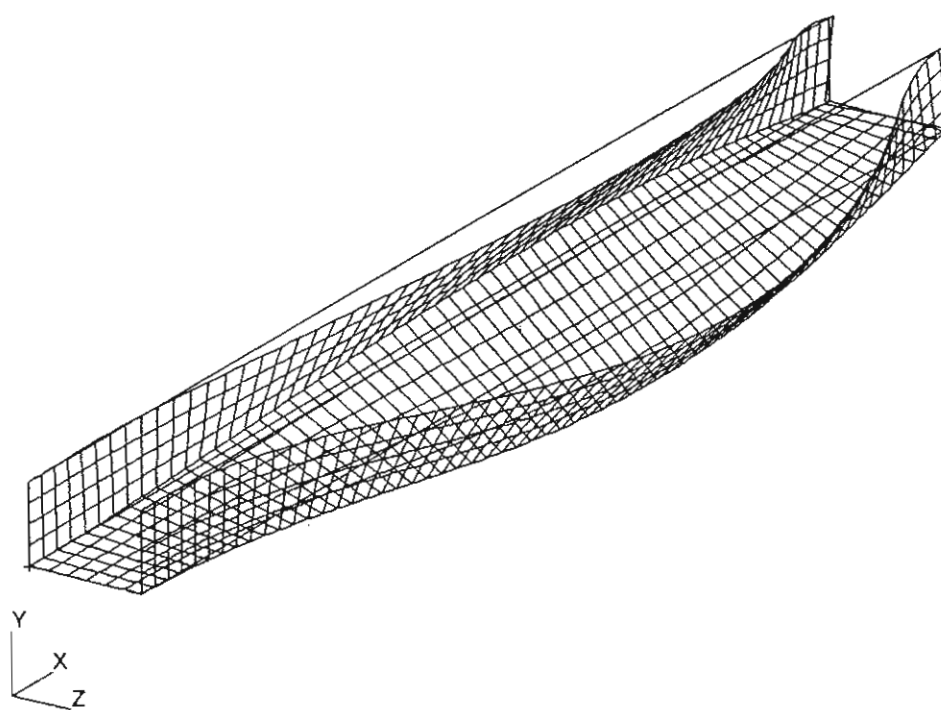


Figure 3. Buckling stress (σ_{cr}) vs. t/b for local buckling and global buckling for an angle profile of aluminium with $E = 65$ GPa.

For any channel profile, if it has buckled by local buckling the lowest buckling mode has had six half-waves, Fig.4a. Also, if the profile has buckled by global buckling the corresponding mode has been of torsional-flexural type, Fig .4b.



a)



b)

Figure 4: a) first local buckling mode, b) first global buckling mode.

5. CONCLUSION

Numerical approach presented, based on the finite element method, is very good tool in stability analysis. Using this procedure it is possible to decide if some thin-walled structure will fail by local buckling before it fails by global buckling. Finite element procedure provides modeling of complex structure independent on the number of structure members, their shapes as well as of boundary conditions. As one can see if t/b ratio is relatively small, the columns fail by local buckling before those buckle by global buckling. Experimentally obtained results are in good agreement with numerical values.

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NEW CLASS OF FINITE FUNCTIONS IN THE NUMERICAL ANALYSIS OF THE PROCESS ON THE CONTINUUM

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SUMMARY

This paper presents a new class of finite functions which possess the property of infinite smoothness and are referred to as R_{bf} - Rvachev's basis functions. R_{bf} functions retain the property of universality in a way similar to algebraic and trigonometric polynomials and, at the same time, they preserve the properties of a practical application of splines. The properties of these functions classify them between classical polynomials and spline functions; hence, they complete a set of elementary functions.

The application of R_{bf} functions as basis functions in approximate methods makes it possible to avoid various numerical difficulties frequently present in the previously applied basis functions and to overcome the main constraints of classical and spline functions. In the numerical procedure it is possible to use directly all degrees of derivations of R_{bf} basis functions as required by the differential equation and boundary conditions. The application of the strong formulation makes the numerical procedure much more efficient; at the same time, numerical integration can be avoided. The new class of finite basis functions makes it possible for all the fields derived from the approximate solution to be expressed by an infinite function in the considered areas. The universality property of R_{bf} functions enables the use of numerical solutions with a given accuracy, which makes R_{bf} functions more efficient than all previously known finite functions.

Exceptional approximation properties of the new class of R_{bf} basis functions are illustrated by the examples of a nonlinear analysis. By applying the R_{bf} functions a numerical model was developed which makes possible efficient determination of a nonlinear response of the structure with a high degree of accuracy and quality of the approximate solution. The advantages of R_{bf} functions become most prominent in the numerical analysis of nonlinear behavior of the structure since they ensure high accuracy and stability of the numerical procedure.

1. INTRODUCTION

The selection of the basis functions is of special importance for the numerical procedure and quality of the approximate solution. It is well-known that localized basis functions are most widely used in practice, particularly in the finite element method. In the FEM localized basis functions are selected most frequently from the respective class of polynomials. Therein, for practical reasons, the polynomials are selected according to the degree necessary to satisfy the minimum conditions on the finite element and ensure continuity between the elements. The selection of the basis functions of a lower degree of derivability entails the application of a "weak" formulation and, accordingly, the application of a numerical integration. Consequently, some fields derived from the solutions cannot be expressed by a continuous function in the considered area (e.g. stresses, bending moments, heat flow, etc.).

The evaluation of the characteristics of the basis functions used in numerical methods makes it necessary to seek new solutions and to expand the current selection of the basis functions.

This paper presents a new class of finite functions with a high degree of smoothness which are referred to as R_{bf} - Rvachev's basis functions. A linear combination of the displaced R_{bf} functions makes it possible to exactly present the algebraic polynomials, exponential and trigonometric functions. The possibility of accurate and simple calculation of the values of derivations of R_{bf} functions of high order enables the efficient application of a strong formulation. Special attention has been paid to the method of the collocation in a point considering a unique criterion for the selection of collocation points (generalized Taylor's series) which was not a property of the previously applied functions. By applying the recently introduced basis functions all fields derived from the approximate solution (which are usually more interesting than the solution function itself) can be expressed by a continuous function in the considered area with the same degree of accuracy.

2. CLASS OF R_{bf} FUNCTIONS

Rvachev's basis functions - R_{bf} are finite functions of class C^∞ . They are classified between classical polynomials and spline functions; however, their practical application as basis functions makes them closer to splines, so that the class of R_{bf} functions can be referred to as universal splines, i.e. splines of infinitely high degree. R_{bf} functions are determined as finite solutions of non-homogeneous differential functional equations of the following type:

$$L y(x) = \lambda \sum_{k=1}^M C_k y(ax - b_k) \quad (1)$$

where L is the linear differential operator with constant coefficients, λ is a scalar quantity different from zero, C_k are solution coefficients, a and b_k are function parameters. The solutions of equation (1) are C^∞ finite functions defined from $-\infty$ to $+\infty$, which have a compact support and can be written in the following form:

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} F(t) dt \quad (2)$$

where $F(t)$ is the Fourier transformation of the function $y(x)$. The solution presented in (2) satisfies the Paley-Wiener theorem in the following form:

$$\int_{-\infty}^{\infty} y(x) dx = 1 \quad (3)$$

By a linear combination of the displaced finite functions (2) it is possible to exactly present the polynomials, exponential and trigonometric functions [2]. Hence, R_{bf} functions are the elements of a linear vector space and they can, thus, be efficiently used as basis functions in numerical methods. The class of R_{bf} functions includes several types of functions where $up(x)$ and $Fup_n(x)$ are suitable for the development of algebraic polynomials, while $y_\omega(x)$ is suitable for the development of exponential functions and $y_{\omega,h}(x)$ for the development of trigonometric functions.

Finite function $up(x)$ from the class of R_{bf} functions with the support $\text{supp } up(x) = [-1, 1]$ is the solution of the differential functional equation (1) written in the following form:

$$up'(x) = 2 up(2x+1) - 2 up(2x-1) \quad (4)$$

Fourier transformation of the function $up(x)$ is:

$$\hat{up}(t) = \prod_{j=1}^{\infty} \frac{\sin t 2^{-j}}{t 2^{-j}} \quad (5)$$

Finite solution of equation (4), after satisfying the normed condition (3), is:

$$up(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \prod_{j=1}^{\infty} \frac{\sin t 2^{-j}}{t 2^{-j}} \quad (6)$$

Function $up(x)$ is graphically presented in Figure 1.a) and has the following properties:

$$\begin{aligned} up(0) &= 1 & ; & & up(x) &= up(-x) & ; \\ up(x) &\equiv 0 & \text{za } -1 \geq x \geq 1 & ; & up(x) &= 1 - up(x-1) & \text{za } 0 \leq x \leq 1 \end{aligned} \quad (7)$$

Expression (6) is not suitable for the calculation of the value of function $up(x)$. Reference [3] gives a numerically more appropriate expression:

$$up(x) = 1 - up(x-1) = 1 - \sum_{k=1}^{\infty} (-1)^{p_1 + \dots + p_k + 1} p_k \sum_{j=0}^k C_{jk} (x-0, p_1 \dots p_k)^j \quad (8)$$

where $(x-0, p_1 \dots p_k)$ represents the difference between the value of coordinate x and the presentation of its value with k bytes, $p_1 \dots p_k$ are the signs 0 or 1 of the binary development of the coordinate x value, and C_{jk} are the known coefficients obtained by (5). The accuracy of the computation of the function $up(x)$ value in an arbitrary point according to expression (8) depends upon the accuracy of the computer. However, exact function $up(x)$ values can be exactly computed in binary rational points of the support (characteristic points):

$$x_k = -1 + k \cdot 2^{-n}, \quad n \in \mathbb{N}, \quad 1 \leq k \leq 2^{n+1} \quad (9)$$

where k is the index of the characteristic point and n is the exponent which determines the interval between the points.

The function $up(x)$ derivations belong, as well as the $up(x)$ function itself, to the class of C^∞ finite functions. A general expression for the derivation of the m -th order is:

$$up^{(m)}(x) = 2^{C_{m+1}^2} \sum_{k=1}^{2^m} \delta_k up(2^m x + 2^m + 1 - 2k), \quad m \in \mathbb{N} \quad (10)$$

where C_{m+1}^2 is the binomial coefficient, and δ_k are the coefficients which determine the sign of each term:

$$\delta_{2k-1} = \delta_k, \quad \delta_{2k} = -\delta_k, \quad k \in \mathbb{N}, \quad \delta_1 = 1 \quad (11)$$

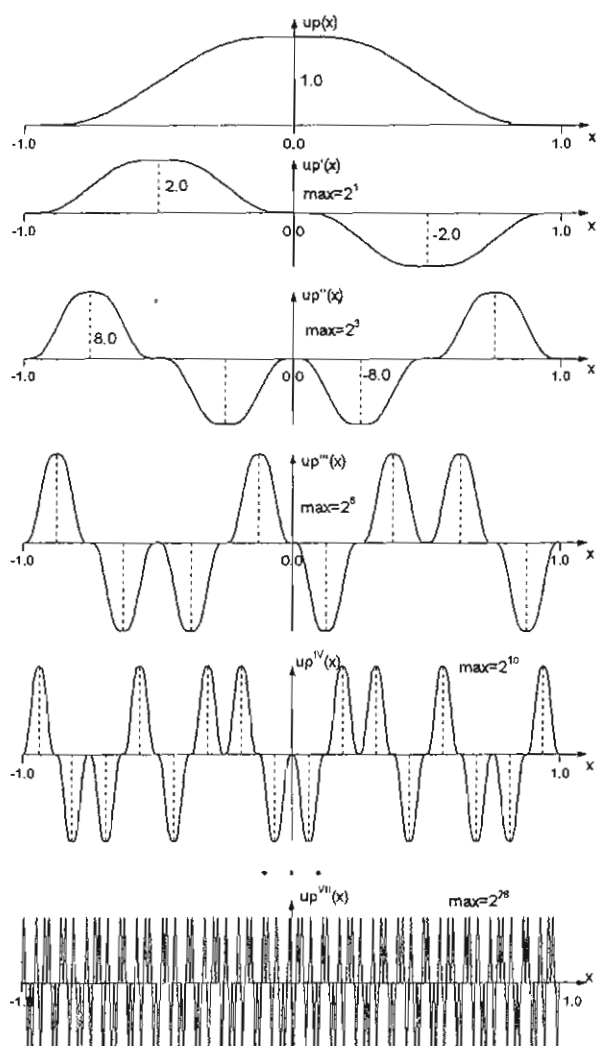


Figure 1. Function $up(x)$ and its derivations

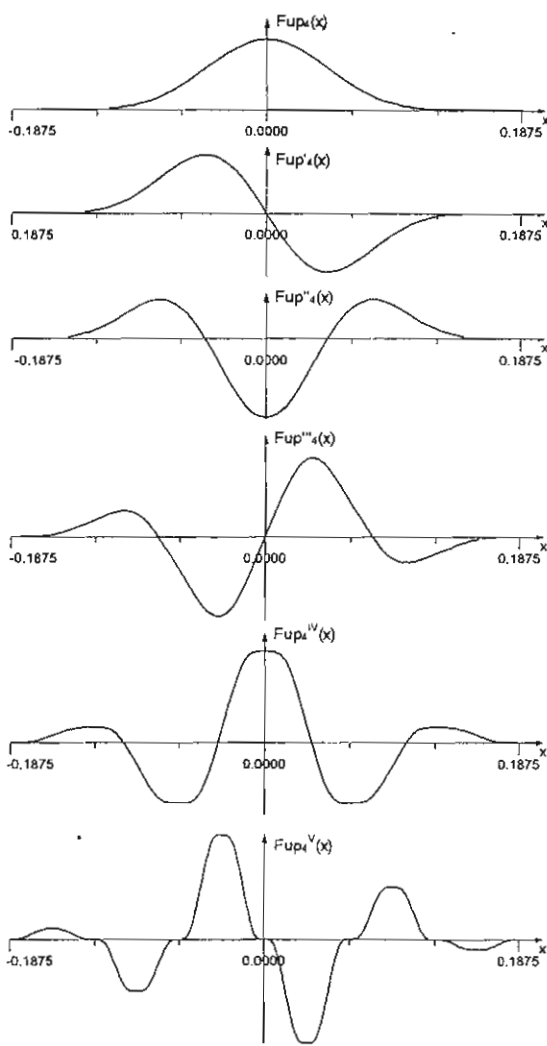


Figure 2. Derivations of $Fup_4(x)$ function of the 0,1,...,5 order

It is characteristic that the values of any derivation are computed as a linear combination of the $up(x)$ function value itself. The derivation of the $up(x)$ function of a high order, when $m \rightarrow \infty$, becomes a series with single members tending to Dirac's function (Figure 1.f).

A family of $Fup_n(x)$ functions was developed according to the $up(x)$ function. $Fup_n(x)$ functions and their derivations retain the properties of $up(x)$ functions, but they are more suitable for numerical analyses. They are also defined as finite solutions of equation (1) which therein satisfies condition (3); they have the form of expression (2). The support for the family of $Fup_n(x)$ functions is:

$$\text{supp } Fup_n(x) = [-(n+2) \cdot 2^{-n-1} ; (n+2) \cdot 2^{-n-1}] \quad (12)$$

where n denotes the degree of the algebraic polynomial which can be exactly developed along the basis which is formed by the displaced functions $Fup_n(x)$.

$Fup_n(x)$ function can also be expressed as a linear combination of displaced $up(x)$ functions:

$$Fup_n(x) = \sum_{k=0}^{\infty} C_k up\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right) \quad (13)$$

where coefficients C_k are computed and presented in tables in Ref. [3].

The derivations of the $Fup_n(x)$ function are also obtained by a linear combination of the derivations of displaced $up(x)$ functions. Figure 2. presents the $Fup_4(x)$ function and its first five derivations.

3. ELASTO-PLASTIC BENDING OF A STRAIGHT BEAM

The analysis of elasto-plastic bending of a beam with a rectangular cross-section was chosen to illustrate the approximate properties of R_{br} functions. Since the bending of the beam was described by a differential equation of the fourth order, the $Fup_4(x)$ function was selected from the class of R_{br} functions as the most suitable for this type of analysis.

3.1 Main assumptions

Elasto-plastic behavior is characterized by an initial elastic material response accompanied by plastic deformation when the bending moment M exceeds the yield moment $M_Y = \sigma_Y \cdot W_Y$ where σ_Y is the uniaxial yield stress and W_Y is the moment of resistance of the cross-section in the elastic state of the beam bending. After a further increase of the bending moment M beyond M_Y , the plastic zone expands towards the neutral axis. In the cross-sectional area there is an elastic core of height $h_e = 2\xi$ and a plastic zone (Figure 3).

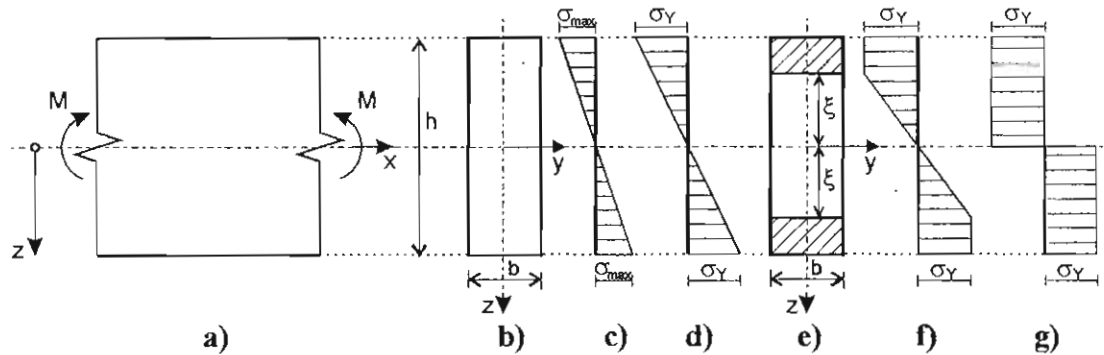


Figure 3. Propagation of the plastic zone towards the neutral beam axis

The value of moment M_{ep} , which corresponds to a partly plastified cross-section, is obtained according to expression:

$$M_{ep} = \sigma_Y \left(\frac{1}{\xi} \cdot I_{el} + S_{pl} \right) ; \quad I_{el} = 2 \int_0^{\xi} z^2 b \, dz \quad , \quad S_{pl} = 2 \int_{\xi}^{h/2} z b \, dz \quad (14)$$

where I_{el} is the moment of inertia of the elastic core of the cross section with regard to the axis y , whereas S_{pl} is the absolute value of the static moment of the cross-sectional plastic zone considering axis y .

The bending strength of the cross-section is completely exhausted when the elastic core disappears ($\xi=0$), and the cross-section becomes thoroughly plastified. The so-called plastic hinge is formed in the whole cross-section which takes over the constant moment which is equal to the limit bending moment M_{pl} . The highest moment the rectangular cross-section can take over, i.e. the moment of the cross-section plasticity, is $M_{pl} = \sigma_Y \cdot W_{pl} = \sigma_Y \cdot bh^2/4$.

3.2 Bending of a simply supported beam by a concentrated load in the middle

Bending of a beam with a rectangular cross-section loaded with a concentrated force F is considered, according to Figure 4.

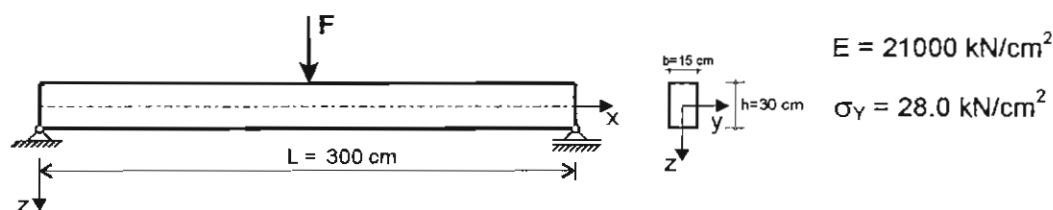


Figure 4.

In this case the plastic hinge forms at the center of the beam. The limit value of force F , where the elastic core completely disappears in the central cross-section $x = L/2$, i.e. when the central cross-section is completely plastified, for the given values E and σ_Y , is $F_{pl} = 1260$ kN. The maximum beam deflection which corresponds to the limit force F_{pl} is $\delta_{pl} = 14.815$ mm. At that moment, the beam turns into a mechanism with a plastic hinge where displacements increase incessantly without an increase in the force.

In order to evaluate the results obtained by a new class of finite R_{bf} functions, a computer program was developed, based on FEM, which uses three-node isoparametric finite elements and Lagrange polynomials of the 2nd degree as basis functions.

A method of strong formulation was used to obtain the numerical solution by the application of $Fup_4(x)$ functions as basis functions; therein, the number of basis functions along the length of the beam was changed instead of increasing the density of the mesh in FEM.

Figure 5 presents a comparison between the load-displacement relationships at the center of the beam which were obtained analytically, using FEM and by applying $Fup_4(x)$ functions. Figure 6 presents the convergence of numerical solutions, obtained by $Fup_4(x)$ functions and FEM, to an accurate solution.

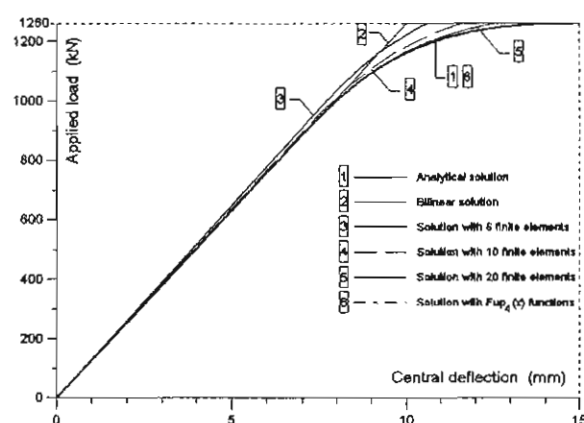


Fig. 5 Load-deflection relationships for a simple beam

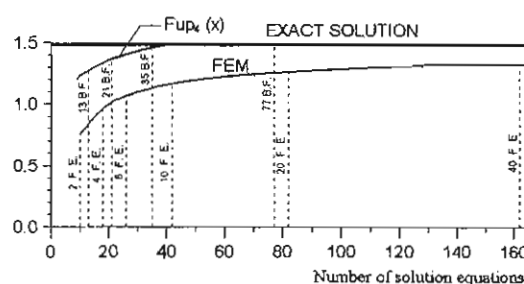


Fig. 6 Convergence of numerical solutions

The numerical solution obtained by $Fup_4(x)$ functions is almost completely in agreement with the analytical solution (Fig. 5). In addition, it is possible to obtain a more accurate solution, with approximately the same number of equations by applying $Fup_4(x)$ functions than with FEM (Figure 6).

3.3 Supported cantilever under uniformly distributed load

A beam clamped at one end, supported at the other end and loaded with a uniformly distributed load q , (as in Figure 7.a), is analyzed. The geometric characteristics of the beam and of the material properties are the same as in example 3.2 (Figure 4).

The bearing capacity of this statically undetermined structure will be completely exhausted only after the formation of two plastic hinges: on the left support and in the beam field. According to [4] the limit load, at which the mechanism is formed, is $q = 1.4571 \cdot q_0$, where q_0 is the limit load for a simple beam. A nonlinear numerical solution is determined by $Fup_4(x)$ functions with the incremental increase of load $q = f \cdot q_0$ until plastic failure, i.e. until load factor $f = 1.45$.

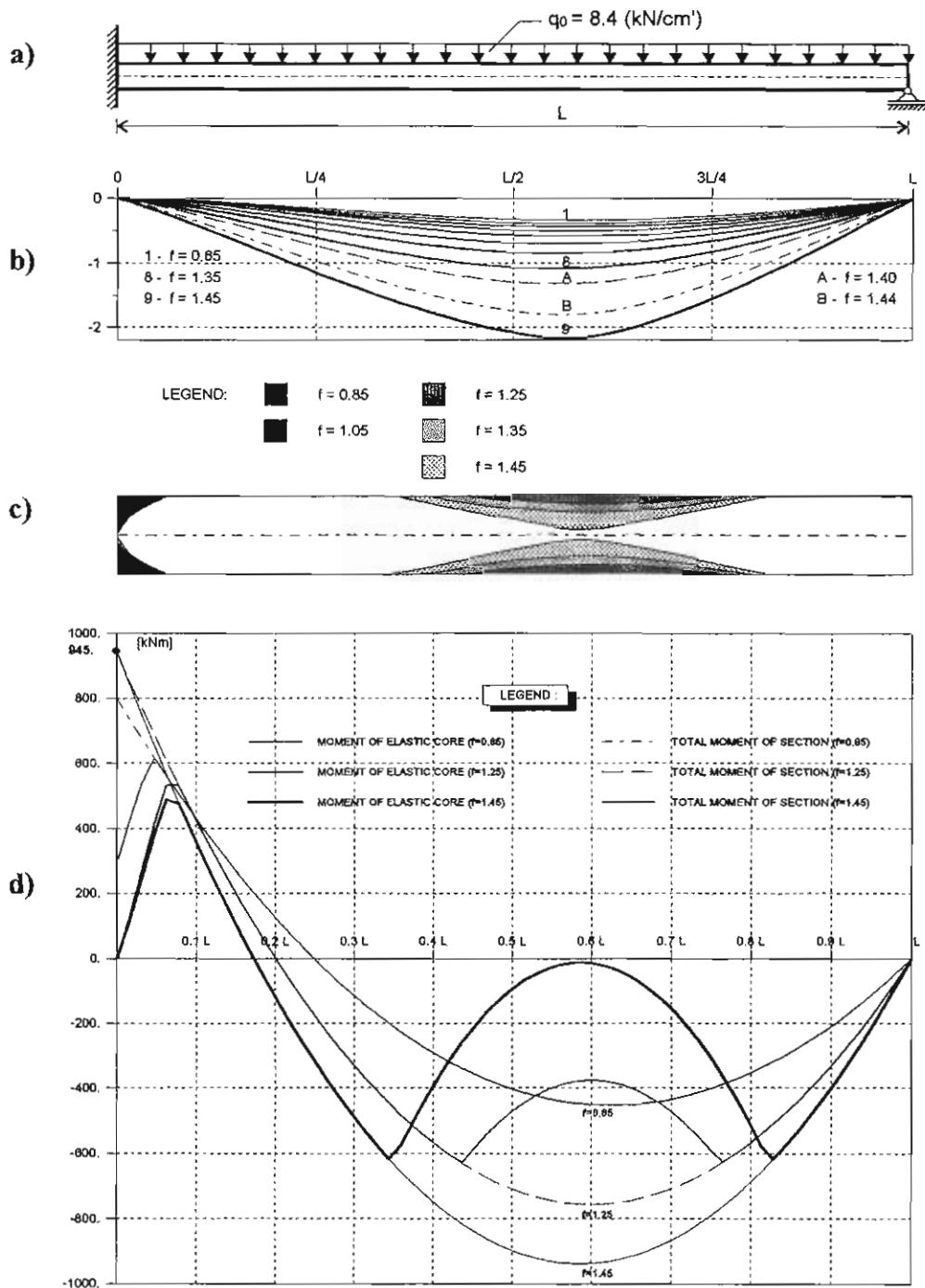


Figure 7.

Figure 7. presents the displacement diagrams for various load intensities (Figure 7 b), propagation of plastic zones until the occurrence of plastic deformations at the clamped end until failure, i.e. until the formation of the plastic hinge also in the field (Figure 7.c), and the diagrams of the bending moment with the separation of the elastic core moment and moment taken over by the plastified part of the cross-section (Figure 7.d). It should be noted that the previously mentioned program for FEM cannot monitor the behavior of the beam until it completely loses its bearing capacity but only until the load $q = 1.22 \cdot q_0$, i.e. until the load factor $f = 1.22$, when the numerical procedure is interrupted and the appearance of the mechanism is recorded.

The analysis of this example illustrates the exceptional approximate properties of R_{bf} functions. In other words, the numerical solution developed according to $Fup_4(x)$ basis functions completely follows the analytical solution until the factor $f = 1.45$, which corresponds to 99.5% of the value of the limit load factor $f_{ex} = 1.4571$. The numerical solution obtained by FEM, with an approximately five times greater number of conditional equations, (Figure 6) can be approximated only to the load factor $f = 1.22$ or 83.7% of the value of the limit factor $f_{ex} = 1.4571$. The great advantage of the numerical procedure using R_{bf} functions when compared with a procedure using FEM can be explained by the fact that the former enables continuous monitoring of all fields, whereas the latter follows the displacements in nodes while the stresses are monitored in the integration points of the finite elements.

4. CONCLUSION

The examples dealt with in the paper prove the exceptional approximation properties of R_{bf} functions considering the accuracy of the numerical solution. By applying a strong formulation the entire numerical procedure becomes more efficient and simpler. Furthermore, numerical integration is avoided so that all interesting fields derived from the solution function can be represented by a continuous function in the whole domain, while in FEM those fields are obtained in the discrete form only in Gaussian integration points. This advantage of R_{bf} functions is exceptionally important in nonlinear numerical analyses since it ensures accuracy and stability as well as a quicker convergence of the numerical procedure.

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Plastic Deformation Energy in the Analysis and Sensitivity of Strain Localization Problems

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ABSTRACT: The strain localization phenomena are considered with the use of the theory of viscoplasticity and finite element method. The aim of the study is to check if the strain localization is sensitive to variation of parameters specifying the physical model and initial and boundary conditions. Total and local plastic deformation energy as response functional for sensitivity analysis are discussed by the way of numerical examples. The paper demonstrates sensitivity of the form and intensity of strain localization to the variation of the above parameters.

1. INTRODUCTION

Structures made of materials with descending portion of σ - ϵ relation, when loaded sufficiently high, usually demonstrate high concentration of strains in small areas. This phenomenon is referred to as strain localization and indicates that damage of material has accumulated there. Therefore strain localization can be considered as a symptom of structural failure and plays important role in both theory of structural mechanics and in engineering practice.

This problem has focused much attention in the literature lately, where theoretical, numerical and experimental studies were reported and it remains still a challenging problem for future research. However, the sensitivity of strain localization to variations of parameters specifying the physical model and initial and boundary conditions has not yet been formulated and discussed in the literature. A study of sensitivity can provide better insight into the phenomena of strain localization for their better understanding and can deliver useful information for iterative improvement of a design of a structure.

2. MATHEMATICAL FORMULATION

The material under consideration exhibits strain softening as a result of temperature rise or/and evolution of porosity. Both of those effects for classical rate independent plastic strain formulation with negative stress-strain constitutive relation lead to ill-posed problems and in consequence to not unique results in numerical applications [Ref.1].

From theoretical point of view we observe the change of the type of partial differential equations (elliptic into hyperbolic in statics and conversely in dynamics). From numerical point of view we observe dramatic mesh sensitivity. The way to avoid this phenomena drives through the regularization. This role can be played for example by introducing the viscosity (inverse of the relaxation time for mechanical disturbances T). In evolution problem (dynamic formulation) the introduction of Perzyna's type viscoplasticity [Ref.2], naturally regularizes initial value problems. Then the governing equations do not change the type during the damage and the strain localization can be studied as a well posed problem. As a crucial effect the solution becomes unique, the results are stable and so called primary mesh dependency (PMD) is not observed any more [Ref.3]. The detailed rate type constitutive structure for elastic-viscoplastic damaged material was discussed in [Ref.4].

Let us restrict our consideration to elastic-viscoplastic associative model in the range of finite deformations. For the plastic part of the total deformation rate tensor $\mathbf{d}^p$ ($\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p$) we postulate the evolution equation with the elastic-viscoplastic model of the material (cf. Ref. 5) and the tensor function $\mathbf{P}$ is defined for associative plasticity

$$\mathbf{d}^p = \lambda \cdot \mathbf{P} \quad \lambda = \frac{1}{T_m} \langle \Phi(f - \kappa) \rangle \quad \mathbf{P} = \frac{\partial f}{\partial \boldsymbol{\tau}} \quad (1)$$

In the above T_m denotes the relaxation time for mechanical disturbances, κ is the isotropic work hardening/softening parameter, Φ , f denote the empirical overstress functions, $\langle \cdot \rangle$ defines the so called McCauly bracket and $\boldsymbol{\tau}$ is Kirchhoff stress tensor.

The different alternative hardening/softening forms of material function κ can be assumed including the effects of porosity nucleation and grows. We focus our attention only on that type of softening which appears as an effect of temperature rise what is well known phenomena observed in experiments. Following Perzyna's achievements we postulate the overstress viscoplastic function Φ in the form

$$\Phi(f - \kappa) = (f - \kappa)^m \quad \text{where } m = 1, 3, 5, \dots \quad (2)$$

We restrict our numerical tests to the initial boundary value problems for which the time scale covers only the fraction of a second and for this reason adiabatic inelastic flow process describes sufficiently close the physical phenomena which appear.

In this formulation an adiabatic flow process is defined by the evolution equations for velocity, mass, Kirchhoff stresses and temperature. The details of the formulation one can find in [Ref.5]. The boundary conditions for tractions and displacements are defined on the separate boundaries, assuming that the heat flux prescribed on the whole boundary is equal to zero and the initial conditions are given at time $t = 0$.

3. ANALYSIS AND SENSITIVITY OF THE PROBLEM

The problem formulated develops in time, so we deal with dynamic process.

The computations were performed using the environment of general purpose nonlinear finite element program ABAQUS [Ref.6]. For large scale computations it is well known, also proved and checked by the authors, that the explicit time integration of the dynamic system of governing equations is more efficient than implicit one, so we used this method in all cases under consideration.

The classical formulations of structural sensitivity analysis cannot be applied [Ref.7], because of time integration. The functional describing the structural response can be assumed as a functional at a prescribed time. In this case the functional would describe the terminal state of the structure. On the other hand the functional must provide information about the strain localization. Another approach to sensitivity functional is to use an integral over a prescribed period of time. In this case the functional would accumulate information and represent a history of the process.

Assume that the specimens are subjected to tension, with the controlled displacements imposed at one or two opposite sides with different velocities. Consider the terminal states when the total elongation of a specimen reaches prescribed value. We limit the formulation of the response functional to this terminal state [Ref.8].

The measure of strain localization can be assumed as a norm in L^p space. Let us introduce a local scalar measure v and consider a functional which is the global measure of strain localization in the form

$$v = \|v\| = \left[\int_V (v)^p dV \right]^{\frac{1}{p}} \quad (3)$$

where the integration is carried out over the whole volume of the specimen. When p tends to infinity this norm tends to maximum of the function. As the measure v will use plastic strains in the form of so called plastic equivalent strain PEEQ, local plastic deformation energy ELPD (dissipated in finite element) and density of that energy ELPDEN as well.

Energetic norm namely total plastic deformation energy for whole structure ALLPD (which strictly relates to above norm with $v=ELPD$ and $p=1$) is assumed, too.

$$\epsilon_{PEEQ} = \int \left(\frac{3}{2} \mathbf{d}^p : \mathbf{d}^p \right)^{\frac{1}{2}} dt \quad E_{PD} = \int (\boldsymbol{\sigma} : \mathbf{d}^p) dV \quad (4)$$

We carry out the sensitivity analysis with respect to the velocity of imposed displacements, distribution of the imposed displacements, boundary of the specimen and with respect to parameters of the physical model as the design variables.

4. NUMERICAL EXAMPLES

In the numerical examples presented in the paper we studied analysis and sensitivity of location and the width of the localized plastic zones for dynamic tension of a steel bar. The axisymmetric specimen under analysis is 19.05 mm long and has the radius of 3.175 mm. The other properties were assumed as follows: Young modulus $E = 200$ GPa, Poisson ratio $\nu = 0.3$, initial mass density $\rho_0 = 7850$ kg/m³, inelastic heat fraction 0.9, specific heat 460 J/kg °C, the yield stress $\sigma_0 = 1634$ MPa for initial temperature which changes (softens) nonlinearly to 1006 MPa for the rise of the temperature up to 610 °C. The viscosity $D = 1/T_m$ was assumed also as a function of temperature and varies between 40 1/s (for initial temperature) and 100 1/s for the temperature 200 °C. In the study there was also used the failure criteria which assumed that the element brakes down if the value of PEEQ reaches 100%.

Basically, two general cases of boundary and initial conditions were considered:

- (A) symmetric (double side) tension of the specimen which results in symmetric pattern of deformations,
- (B) asymmetric (single side) tension of the specimen with the opposite side fixed, which leads to non-symmetric deformation.

For both cases of boundary conditions a set of examples was computed with different initial velocities changing between $v_0 = 0.5$ m/s and $v_0 = 20$ m/s. The distributions of plastic equivalent strains PEEQ for 5 velocities (A05, A1, A5 A10 A20 m/s) are shown in Fig.1 and Fig.3 (B05, B1, B5, B10, B20).

In the processes under consideration for which the total elongation of the specimen was of order 25% the changes of temperature locally reached several hundreds degrees C.

The interesting response confirming the strong interaction of longitudinal waves is presented in Fig.2. In this figure the changes of forces which act on the edges versus total elongation of the specimen are presented. For slower processes ("quasi static") the post-critical behaviour is close one to the other while for the cases where the inertial effects are stronger the significant oscillations are observed.

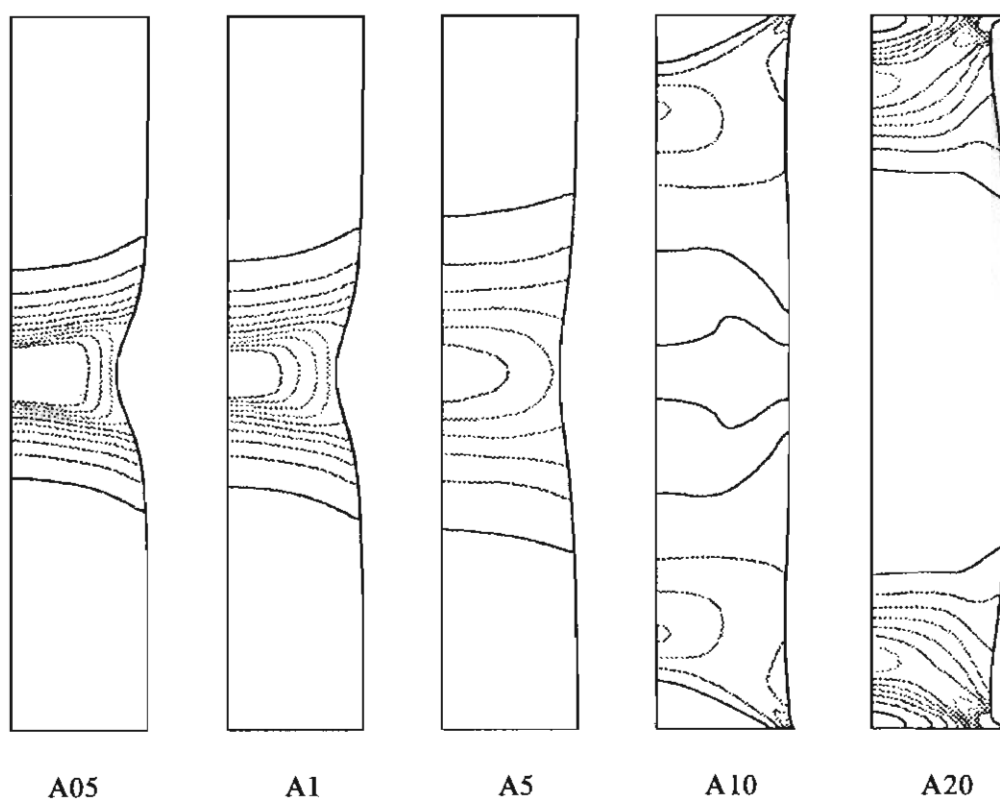


Figure 1. PEEQ strain contours (5-50%) for different double-side velocities.

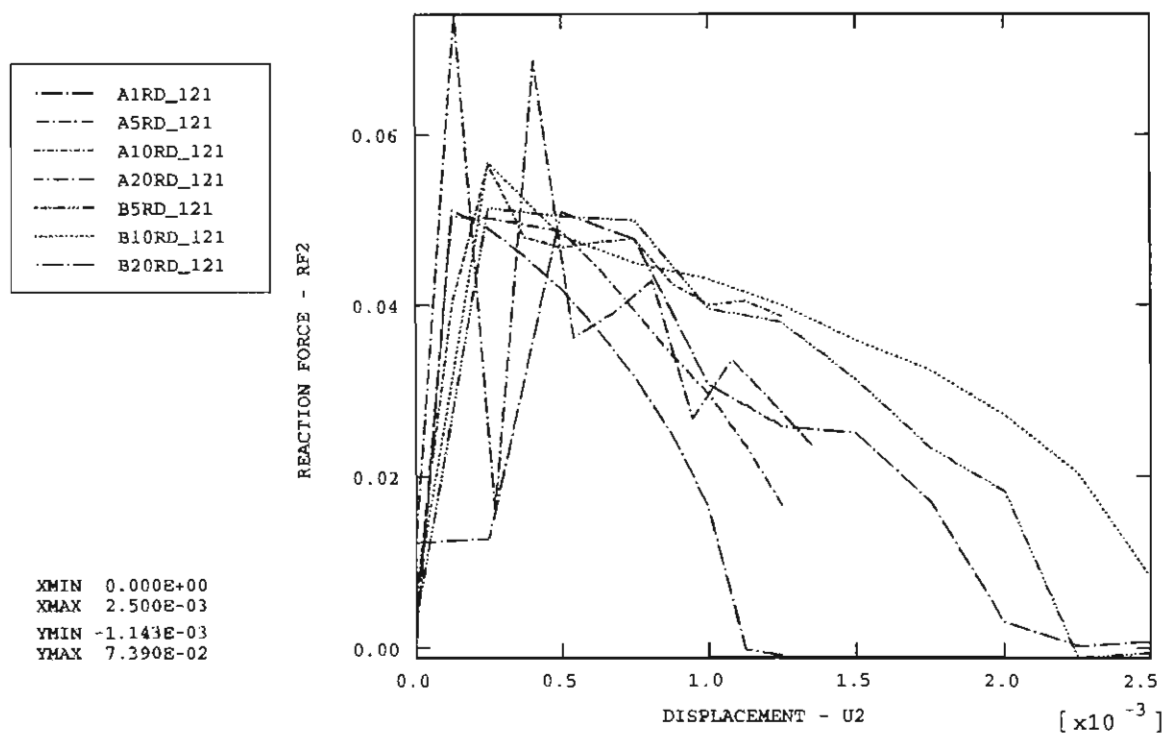


Figure 2. Reaction force versus displacement for different initial and boundary conditions.

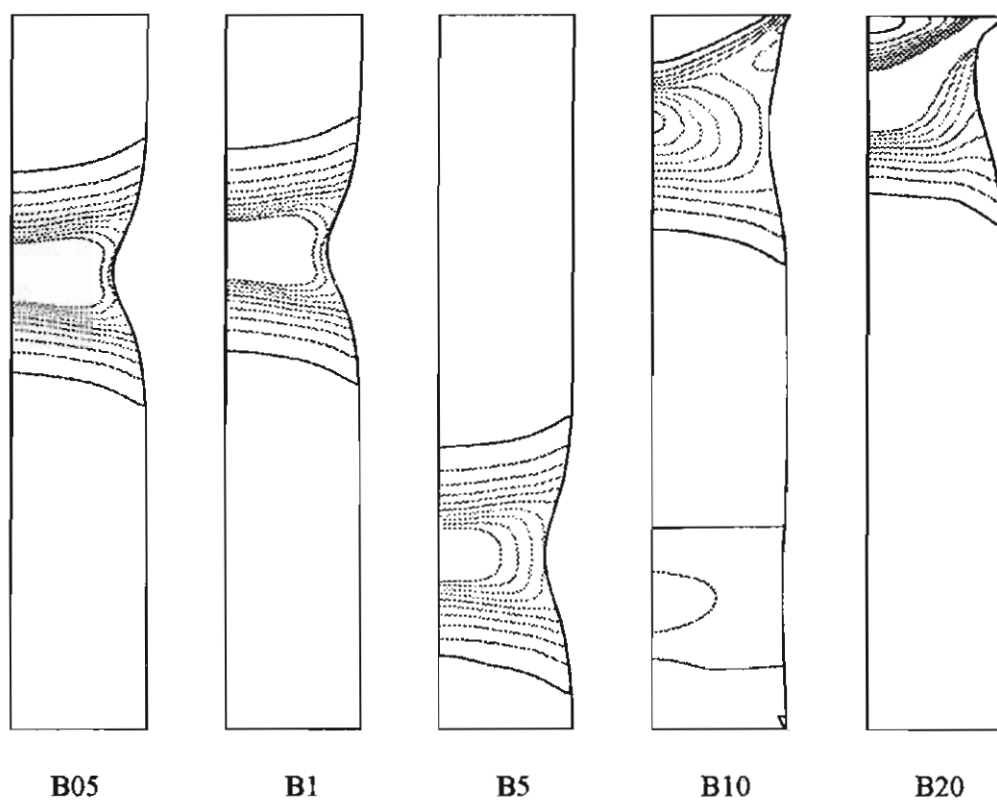


Figure 3. PEEQ stain contours (5-50%) for different single-side velocities.

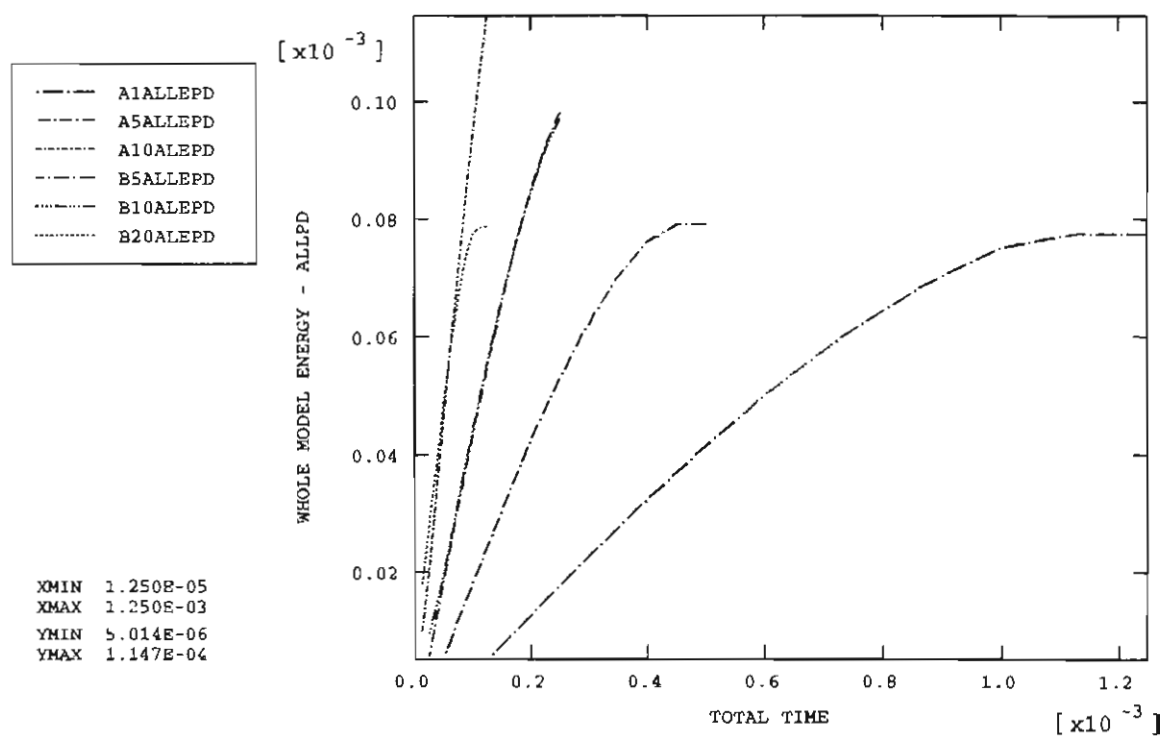


Figure 4. Total plastic dissipated energy for different initial and boundary conditions.

The symmetric conditions (double side) with different velocities also result in symmetric pattern of plastic deformation however for higher speed of deformation two zones of shear bands appear close to the loaded edges.

For non-symmetric (single side) loading the place of plastic strain localization varies depending on the assumed initial velocity that acts on the upper edge of the specimen. For quasi static (slow) cases zones of plastic concentrations tend to the middle of a specimen. For higher velocities this position changes and finally appears close to the loaded edge.

In Fig.4 we present the total plastic deformation energy dissipated during the same elongation of the specimen equal to $\Delta u = 2.5$ mm, for different initial and boundary conditions. One can find meaningful difference of dissipated energy amount during the processes with no strict relation to varied total times (for slower tension it takes longer time to obtain the same elongation). We would like to put your attention to curves A5 and B10 which almost quite coincide and A10 with B20 with the same relation: double side tension versus one side tension with doubled velocity and clamped opposite end.

Also at the local point of view - looking at the single element we have found interesting response. For the chosen problems: A1, A5, A10, B5 and set of points: three places within the width and seven along the specimen we control the same energy for finite element (see Fig.5 and Table 1). The column „position” points out the place within the specimen with the extremum of the measure v as the relative value to its adequate horizontal and vertical dimension. Now one can not only find difference of energy dissipation level but also recognize the places where dissipation lasts till the very end of observed processes and other ones with fast horizontal path on lower energy level.

Table 1. Values of functional $\|v\|$ for several problems *cases and parameters p*.

$\Delta u = 1.75$ mm	p=1	p=2	p=4	p=8	p=16	max= ∞	position
$v=PEEQ$							
A1	187.064	7.442	1.812	0.995	0.794	0.702	0.0*0.5
A5	190.217	5.859	1.120	0.580	0.419	0.332	0.0*0.5
A10	196.171	4.542	0.776	0.357	0.264	0.232	1.0*0.0
B5	186.191	7.700	1.893	1.039	0.826	0.726	0.0*0.23
$v=ELPDEN$	$[x 10^3]$						
A1	265.945	9.934	2.295	1.194	0.909	0.772	0.0*0.5
A5	301.794	9.473	1.851	0.893	0.643	0.508	0.0*0.5
A10	322.999	7.451	1.271	0.590	0.455	0.423	1.0*0.0
B5	269.555	10.507	2.464	1.288	0.981	0.832	0.0*0.23
$v=ELPD$	$[x 10^{-6}]$						
A1	69.099	2.879	0.712	0.380	0.286	0.231	1.0*0.5
A5	76.115	2.591	0.578	0.299	0.225	0.187	1.0*0.5
A10	82.283	2.116	0.404	0.244	0.221	0.208	1.0*0.0
B5	70.356	3.030	0.760	0.408	0.308	0.250	0.85*0.23

Beside of the results presented in this paper also the evident sensitivity of the width of zones of localized deformations to variations of constitutive parameters (viscosity) was observed for other types of the specimens (e.g. thin plates). Basically, for shorter relaxation times (higher viscosity) the shear bands were of the narrower width.

Next numerical examples will be demonstrated during the conference presentation.

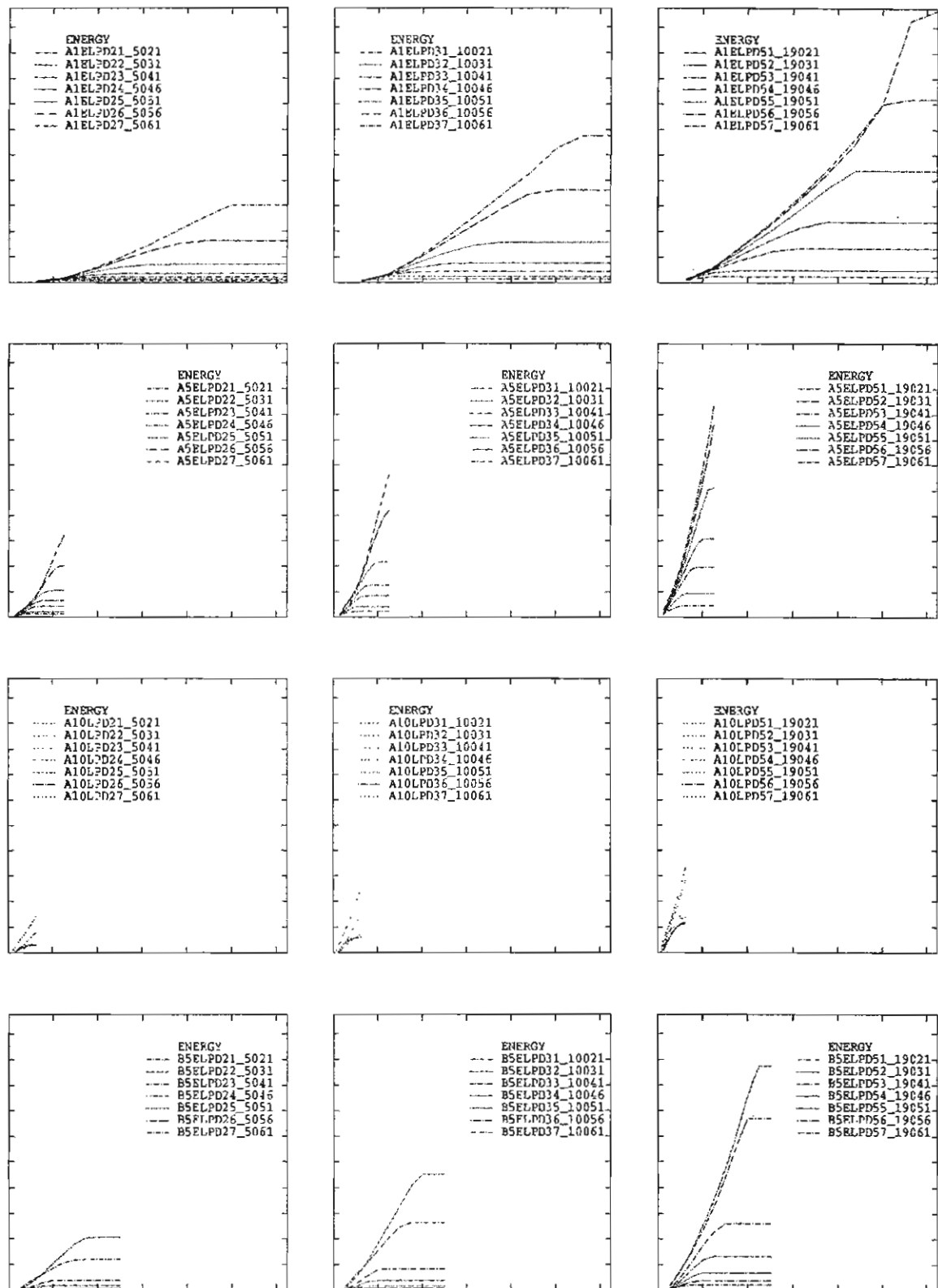


Figure 5. Local plastic dissipated energy for different initial and boundary conditions.

5. CONCLUSIONS

Sensitivity of strain localization to variation of parameters specifying boundary and initial conditions was studied in the paper by the way of numerical examples.

It was demonstrated that the position and form of the zone of localized strains and the total and the local intensity of plastic deformation energy are sensitive to variations of the rate of displacements (velocities) imposed at the boundary of the specimen.

These energy measures can play the role of response functionals in the future studies of sensitivity of strain localization.

From both, total and local energy histories, we hope for good promise for investigation of strain localization sensitivity analysis in the cases when we want to predict of localization zones, but also in reverse design when one wants to avoid such the physical phenomena.

The energy level itself, but also its derivative with respect to time, which describes power of plastic deformation energy arise to be the objective function for the optimization problem in the strain localization.

The study will be developed aiming at the theoretical and numerical verification of the sensitivity analysis to such a complex nonlinear problem.

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A COMPUTATIONAL ALGORITHM FOR PLANE STRESS ANALYSIS IN FINITE STRAIN ELASTOPLASTICITY

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Abstract. The main objective of this paper is to present the finite strain elasto-plastic return-mapping algorithm for plain stress analysis at the material point level, using the multiplicative decomposition of the deformation gradient. The algorithm was based on the use of consistent tangent elastoplastic modulus and closest point projection. The algorithm satisfies objectivity and incompressibility requirements for plastic flow. The spectral decomposition of the Kirchhoff stress tensor and the elastic left Cauchy-Green tensor has been used to obtain a return-mapping algorithm which is similar to return-mapping algorithms used in the infinitesimal theory.

1. Introduction

The problems like metal forming employing finite strain assumption, have been shown that additive strain decomposition on elastic and plastic part can no longer be used. The basic difference between finite strain problems and infinitesimal plasticity problems is that the difference between referential and current configuration for finite strain problems is not negligible any more. There are two basic approaches when solving finite strain problems. The first approach, based on a hypoelastic stress-strain description, employing Jaumann, Truesdell or Green-McInnis stress rate [1], shows disadvantages when describing cyclic loading [2,3]. Furthermore objectivity is not automatically satisfied, so this approach nowadays is not often used. The second approach, developed by Lee and Liu 1976 [1] and currently widely used, is based on the hyperelastic formulation and multiplicative decomposition of the deformation gradient. There are many different formulations of this approach, one of the most interesting is the formulation developed by Simo [4-7]. This formulation is based on the principle of maximum dissipation, hyperelastic material description, consistent linearisation and closest point projection. In this paper, based on work by Simo [4], the return-mapping algorithm is presented similar to infinitesimal strain algorithm described in [9]. This paper focuses on elastoplastic loading of a material point assuming the plane stress state. Plastic yielding is described by the von Mises yield criterion. The computation algorithm, based on consistent tangent elastoplastic modulus and closest point projection, automatically satisfies objectivity and incompressibility of plastic flow.

2. Return-mapping algorithm

By the discretisation of the variational formulation of boundary value problem, as shown in [9], and after the solution of the global finite element equation, nodal displacement increment is obtained. Nodal displacements are transformed into the displacement of a material point, using shape functions. Displacement increment $\Delta^i \mathbf{u}$ represents input variable in the predictor-corrector integration algorithm for plane stress analysis that is going to be presented here. Assuming that displacement ${}^{i-1} \mathbf{u}$ is known from previous iteration increment, then

$${}^i \mathbf{u} = {}^{i-1} \mathbf{u} + \Delta^i \mathbf{u}. \quad (1)$$

In this particular plane stress problem, displacement of the material point is described by the two vector components,

$${}^i \mathbf{u} = [{}^i u_1 \quad {}^i u_2]^T. \quad (2)$$

Position of a material point p in spatial (Eulerian) configuration Ω (Fig. 1) is defined by the vector

$$\mathbf{x} = [x_1 \ x_2]^T, \quad (3)$$

while the position of a material point P in material (Lagrangian) configuration Ω_0 is

$$\mathbf{X} = [X_1 \ X_2]^T. \quad (4)$$

Displacement represents the difference between this same point defined in these two configurations

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = [(x_1 - X_1) \ (x_2 - X_2)]^T. \quad (5)$$

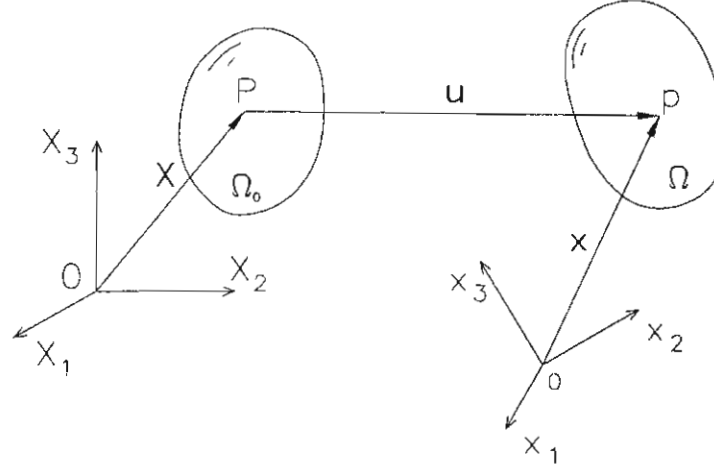


Figure 1. Material and spatial configuration

The deformation gradient is the main equation representing the relationship between the material (Lagrangian) and spatial (Eulerian) configuration

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix}. \quad (6)$$

The relative deformation gradient representing relationship between the current and updated current configuration, as shown in Figure 2, is defined by

$${}^i\mathbf{f} = \mathbf{I} + \nabla \mathbf{u}({}^{i-1}\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} & \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \\ \frac{\partial {}^i u_2}{\partial {}^{i-1} x_1} & \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} & \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \\ \frac{\partial {}^i u_2}{\partial {}^{i-1} x_1} & 1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \end{bmatrix}, \quad (7)$$

where $\nabla \mathbf{u}$ represents displacement gradient, i.e.,

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}. \quad (8)$$

Using (6), left Cauchy-Green tensor, for this particular plane stress problem, is defined as

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix}^T = \begin{bmatrix} \left(\left(\frac{\partial x_1}{\partial X_1} \right)^2 + \left(\frac{\partial x_1}{\partial X_2} \right)^2 \right) & \left(\frac{\partial x_1}{\partial X_1} \frac{\partial x_2}{\partial X_1} + \frac{\partial x_1}{\partial X_2} \frac{\partial x_2}{\partial X_2} \right) \\ \left(\frac{\partial x_1}{\partial X_1} \frac{\partial x_2}{\partial X_1} + \frac{\partial x_1}{\partial X_2} \frac{\partial x_2}{\partial X_2} \right) & \left(\left(\frac{\partial x_2}{\partial X_1} \right)^2 + \left(\frac{\partial x_2}{\partial X_2} \right)^2 \right) \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}. \quad (9)$$

Since additive decomposition of deformation does not hold for finite strain problems [1], the algorithm described herein is based on the multiplicative decomposition of the deformation gradient, defined as

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \quad (10)$$

where $\mathbf{F}^p$ represents the plastic part of a deformation gradient, and $\mathbf{F}^e$ represents the elastic part of a deformation gradient. Elastic left Cauchy-Green tensor is defined as a function of elastic deformation gradient

$$\mathbf{b}^e = \mathbf{F}^e \mathbf{F}^{eT} = \begin{bmatrix} b_{11}^e & b_{12}^e \\ b_{12}^e & b_{22}^e \end{bmatrix}. \quad (11)$$

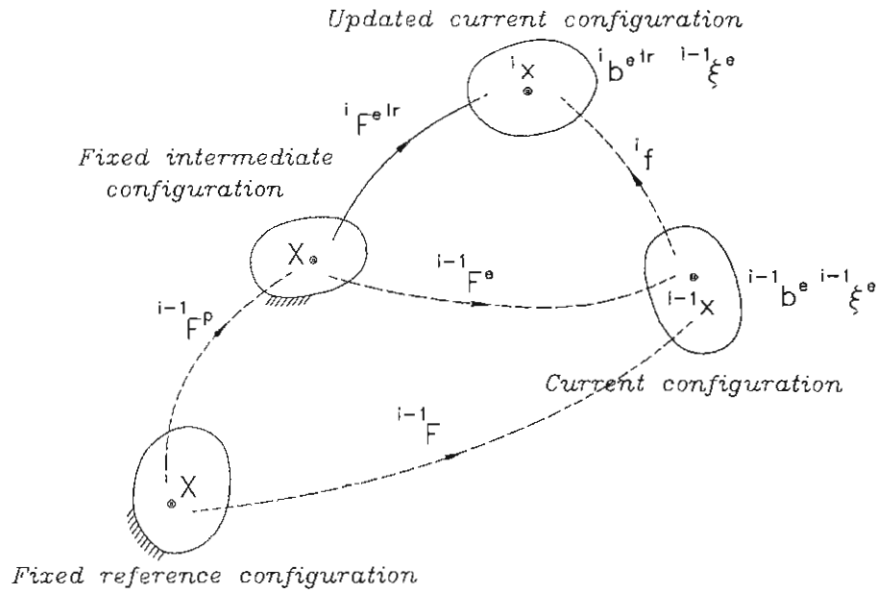


Figure 2. Predictor step (trial state)

The return-mapping algorithm, described herein, is based on the operator split of evolution equations and product formula algorithm [4-9]. The algorithm is conducted in two steps: (a) the elastic predictor (trial state) and (b) the plastic corrector. It is assumed that the elastic behavior of the material during the first, i.e., predictor (trial) step, the trial left elastic Cauchy-Green tensor, shown in Figure 2, is defined as

$$\begin{aligned} {}^i \mathbf{b}^{e tr} = {}^i \mathbf{f} \quad {}^{i-1} \mathbf{b}^e \quad {}^i \mathbf{f}^T &= \begin{bmatrix} 1 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} & \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \\ \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} & 1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \end{bmatrix} \cdot \begin{bmatrix} {}^{i-1} b_{11}^e & {}^{i-1} b_{12}^e \\ {}^{i-1} b_{12}^e & {}^{i-1} b_{22}^e \end{bmatrix} \cdot \begin{bmatrix} 1 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} & \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \\ \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} & 1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \end{bmatrix}^T, \\ {}^i \mathbf{b}^{e tr} &= \begin{bmatrix} {}^i b_{11}^{e tr} & {}^i b_{12}^{e tr} \\ {}^i b_{12}^{e tr} & {}^i b_{22}^{e tr} \end{bmatrix}, \end{aligned} \quad (12)$$

where components of matrix ${}^i \mathbf{b}^{e tr}$ are

$$\begin{aligned} {}^i b_{11}^{e tr} &= {}^{i-1} b_{11}^e \left(1 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} \right)^2 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \left[2 {}^{i-1} b_{12}^e \left(1 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} \right) + {}^{i-1} b_{22}^e \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \right], \\ {}^i b_{22}^{e tr} &= {}^{i-1} b_{11}^e \left(\frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \right)^2 + \left(1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \right) \left[2 {}^{i-1} b_{12}^e \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} + {}^{i-1} b_{22}^e \left(1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \right) \right], \end{aligned}$$

$${}^i b_{12}^{\epsilon^{ir}} = {}^{i-1} b_{11}^{\epsilon} \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \left(1 + \frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} \right) + {}^{i-1} b_{12}^{\epsilon} \left[\frac{\partial {}^i u_1}{\partial {}^{i-1} x_1} \left(1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \right) + \left(\frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \right)^2 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} + 1 \right] + {}^{i-1} b_{22}^{\epsilon} \frac{\partial {}^i u_1}{\partial {}^{i-1} x_2} \left(1 + \frac{\partial {}^i u_2}{\partial {}^{i-1} x_2} \right)$$

Spectral decomposition of ${}^i \mathbf{b}^{\epsilon^{ir}}$ leads to the following relationship

$${}^i \mathbf{b}^{\epsilon^{ir}} = \sum_{A=1}^2 \left({}^i \lambda_A^{\epsilon^{ir}} \right)^2 {}^i \mathbf{n}^{r(A)} \otimes {}^i \mathbf{n}^{r(A)}, \quad (13)$$

where $\left[\left({}^i \lambda_1^{\epsilon^{ir}} \right)^2 \quad \left({}^i \lambda_2^{\epsilon^{ir}} \right)^2 \right]^T$ is the principal vector while ${}^i \mathbf{n}^{\epsilon^{ir(l)}} = \left[{}^i n_1^{\epsilon^{ir(l)}} \quad {}^i n_2^{\epsilon^{ir(l)}} \right]^T$ and ${}^i \mathbf{n}^{\epsilon^{ir(2)}} = \left[{}^i n_1^{\epsilon^{ir(2)}} \quad {}^i n_2^{\epsilon^{ir(2)}} \right]^T$ are principal directions. The tensor product of two normals, for plane stress problem is

$${}^i \mathbf{n}^{\epsilon^{ir(A)}} \otimes {}^i \mathbf{n}^{\epsilon^{ir(A)}} = {}^i \mathbf{n}^{\epsilon^{ir(A)}} \left({}^i \mathbf{n}^{\epsilon^{ir(A)}} \right)^T = \begin{bmatrix} \left({}^i n_1^{\epsilon^{ir(A)}} \right)^2 & {}^i n_1^{\epsilon^{ir(A)}} {}^i n_2^{\epsilon^{ir(A)}} \\ {}^i n_1^{\epsilon^{ir(A)}} {}^i n_2^{\epsilon^{ir(A)}} & \left({}^i n_2^{\epsilon^{ir(A)}} \right)^2 \end{bmatrix}. \quad (14)$$

The objective of the presented work is to describe an algorithm which is similar to existing algorithms describing infinitesimal plasticity problems. For this reason deformation is described through the principal logarithmic stretches defined by

$${}^i \boldsymbol{\epsilon}^{\epsilon^{ir}} = \left[\log \left({}^i \lambda_1^{\epsilon^{ir}} \right) \quad \log \left({}^i \lambda_2^{\epsilon^{ir}} \right) \right]^T = \left[{}^i \epsilon_1^{\epsilon^{ir}} \quad {}^i \epsilon_2^{\epsilon^{ir}} \right]^T. \quad (15)$$

Trial Kirchhoff stress tensor can be also spectrally decomposed,

$${}^i \boldsymbol{\tau}^{\epsilon^{ir}} = \sum_{A=1}^2 {}^i \beta_A^{\epsilon^{ir}} {}^i \mathbf{n}^{r(A)} \otimes {}^i \mathbf{n}^{r(A)}, \quad (16)$$

where ${}^i \boldsymbol{\beta}^{\epsilon^{ir}}$ represents principal trial Kirchhoff stresses, while for isotropic materials, principal directions $\mathbf{n}^{r(A)}$ coincide. The relation between principal logarithmic stretches and principal trial Kirchhoff stresses is linear, i.e.,

$${}^i \boldsymbol{\beta}^{\epsilon^{ir}} = \mathbf{a} {}^i \boldsymbol{\epsilon}^{\epsilon^{ir}}, \quad (17)$$

where $\mathbf{a}$ is elastic modulus defined as

$$\mathbf{a} = K \mathbf{1} \otimes \mathbf{1} + 2\mu \left[\mathbf{I}_2 - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right], \quad (18)$$

where K is bulk modulus, μ is Lammé constant, $\mathbf{I}_2$ is rank-two identity tensor and $\mathbf{1}$ is unit tensor.

Hence, relations (17) and (18) becomes

$${}^i \boldsymbol{\beta}^{\epsilon^{ir}} = \left[{}^i \beta_1^{\epsilon^{ir}} \quad {}^i \beta_2^{\epsilon^{ir}} \right]^T, \quad (19)$$

where

$${}^i \beta_1^{\epsilon^{ir}} = K \left({}^i \epsilon_1^{\epsilon^{ir}} + {}^i \epsilon_2^{\epsilon^{ir}} \right) + \frac{4}{3} \mu {}^i \epsilon_1^{\epsilon^{ir}} - \frac{2}{3} \mu {}^i \epsilon_2^{\epsilon^{ir}}, \quad (20)$$

$${}^i \beta_2^{\epsilon^{ir}} = K \left({}^i \epsilon_1^{\epsilon^{ir}} + {}^i \epsilon_2^{\epsilon^{ir}} \right) - \frac{2}{3} \mu {}^i \epsilon_1^{\epsilon^{ir}} + \frac{4}{3} \mu {}^i \epsilon_2^{\epsilon^{ir}}. \quad (21)$$

Relation (17) is evaluated during the trial step. The deviatoric part of (19) is

$${}^i \bar{\boldsymbol{\beta}}^{\epsilon^{ir}} = {}^i \boldsymbol{\beta}^{\epsilon^{ir}} - \frac{1}{3} \left[{}^i \boldsymbol{\beta}^{\epsilon^{ir}} \cdot \mathbf{1} \right] \mathbf{1}, \quad (22)$$

$${}^i \bar{\boldsymbol{\beta}}^{\epsilon^{ir}} = \begin{bmatrix} {}^i \bar{\beta}_1^{\epsilon^{ir}} \\ {}^i \bar{\beta}_2^{\epsilon^{ir}} \end{bmatrix} = \begin{bmatrix} \frac{3}{9} K \left({}^i \epsilon_1^{\epsilon^{ir}} + {}^i \epsilon_2^{\epsilon^{ir}} \right) + \frac{2}{9} \mu \left(5 {}^i \epsilon_1^{\epsilon^{ir}} - 4 {}^i \epsilon_2^{\epsilon^{ir}} \right) \\ \frac{3}{9} K \left({}^i \epsilon_1^{\epsilon^{ir}} + {}^i \epsilon_2^{\epsilon^{ir}} \right) - \frac{2}{9} \mu \left(4 {}^i \epsilon_1^{\epsilon^{ir}} - 5 {}^i \epsilon_2^{\epsilon^{ir}} \right) \end{bmatrix}. \quad (23)$$

Norm 2 of the deviatoric principal trial Kirchhoff stresses, for plane stress problem is

$$\| {}^i \bar{\boldsymbol{\beta}}^{\epsilon^{ir}} \| = \sqrt{\left({}^i \bar{\beta}_1^{\epsilon^{ir}} \right)^2 + \left({}^i \bar{\beta}_2^{\epsilon^{ir}} \right)^2}. \quad (24)$$

Yield function is defined by the relation [4,9]

$$^i\hat{\phi}^{tr} = \| ^i\bar{\beta}^{tr} \| - \sqrt{\frac{2}{3}} \left[\sigma_0 + \hat{k} \left(^{i-1}\xi \right) \right], \quad (25)$$

where σ_0 represents yield stress and $\hat{k} \left(^{i-1}\xi \right)$ is the hardening function described in [9], while ξ is strain like internal variable [9]. If the yield function (25) is greater than zero, plastic yielding occurs and the corrector algorithm is activated. If the yield function is less than zero, stretches and deformation calculated during the trial step are accepted as the final solution during this load increment, i.e.,

$$^i\epsilon^e = ^i\epsilon^{e^{tr}}, \quad (26)$$

$$^i\beta = ^i\beta^{tr}. \quad (27)$$

After this, and before returning to the global algorithm, the Kirchhoff stress tensor and the left elastic Cauchy-Green tensor are determined using following relations

$$^i\tau = \sum_{A=1}^2 ^i\beta_A ^i\mathbf{n}^{tr(A)} \otimes ^i\mathbf{n}^{tr(A)} \quad (28)$$

$$^i\mathbf{b}^e = \sum_{A=1}^2 \exp \left[2 ^i\epsilon_A^e \right] ^i\mathbf{n}^{tr(A)} \otimes ^i\mathbf{n}^{tr(A)} \quad (29)$$

This represents a complete evaluation of the stress and strain state of a material point for displacement gradient $\Delta^i\mathbf{u}$ evaluated from the global algorithm.

If yielding occurs, i.e. yield function (25) is greater than zero, the corrector algorithm is activated. The principal logarithmic stretches evaluated in predictor step (15) and strain like internal variable $^{i-1}\xi$ evaluated from the previous corrector step, represent basic input variables. The principal deviatoric trial Kirchhoff stresses from predictor step are used to calculate normal to the yield surface

$$^i\bar{\mathbf{v}} = \frac{^i\bar{\beta}^{tr}}{\| ^i\bar{\beta}^{tr} \|} = \begin{bmatrix} ^i\bar{v}_1 \\ ^i\bar{v}_2 \end{bmatrix}. \quad (30)$$

This normal represents a path that is going to be used when returning (projecting) on the closest point of the yield surface. Since the result of the corrector algorithm is not an exact solution [9], the corrector algorithm represents an iteration process where $k = k + 1$ represents a counter of the iteration loop. An increment of the consistency parameter is evaluated at the end of every iteration loop. The increment was described in [9] as,

$$\delta \left(\Delta\gamma^{(k)} \right) = \frac{\partial \Delta\gamma^{(k)}}{\partial ^i\epsilon^{e^{tr}}} = \frac{1}{\hat{k}' \left(^{i-1}\xi + \sqrt{\frac{2}{3}} \Delta\gamma \right)} \begin{bmatrix} ^i\bar{v}_1 \\ ^i\bar{v}_2 \end{bmatrix} = \begin{bmatrix} \delta \left(\Delta\gamma^{(k)} \right)_1 \\ \delta \left(\Delta\gamma^{(k)} \right)_2 \end{bmatrix}, \quad (31)$$

which represents an enlargement of the yield surface during the plastic loading. For the first loop, the value of consistency parameter increment is taken to be zero,

$$\Delta\gamma^{(0)} = 0. \quad (32)$$

During every iteration, relations representing the current state of stress, hardening and yield function, are reevaluated according to the following relations:

$$^i\beta^{(k)} = ^i\beta^{tr} - 2\mu \Delta\gamma^{(k)} ^i\bar{\mathbf{v}}_{n+1} = \begin{bmatrix} ^i\beta_1^{tr} \\ ^i\beta_2^{tr} \end{bmatrix} - 2\mu \Delta\gamma^{(k)} \begin{bmatrix} ^i\bar{v}_1 \\ ^i\bar{v}_2 \end{bmatrix} = \begin{bmatrix} ^i\beta_1 \\ ^i\beta_2 \end{bmatrix}, \quad (33)$$

$$^i\xi^{(k)} = ^{i-1}\xi + \sqrt{\frac{2}{3}} \Delta\gamma^{(k)}, \quad (34)$$

$$^i\bar{\beta}^{(k)} = ^i\beta^{(k)} - \frac{1}{3} \left[^i\beta^{(k)} \cdot \mathbf{1} \right] \mathbf{1}, \quad (35)$$

$$^i\hat{\phi}^{(k)} = \| ^i\bar{\beta}^{(k)} \| - \sqrt{\frac{2}{3}} \left[\sigma_0 + \hat{k} \left(^i\xi \right) \right]. \quad (36)$$

Plastic residual, which defines the difference (in the radial direction) between the current logarithmic stretch state, obtained during the last iteration step and the current yield surface, is defined by the following relation

$$\mathbf{r}^{(k)} = -{}^i\boldsymbol{\varepsilon}^{e(k)} + {}^i\boldsymbol{\varepsilon}^{e,ir} - \Delta\gamma^{(k)} \nabla\hat{\phi}^{(k)} = -\begin{bmatrix} {}^i\varepsilon_1^{e(k)} \\ {}^i\varepsilon_2^{e(k)} \end{bmatrix} + \begin{bmatrix} {}^i\varepsilon_1^{e,ir} \\ {}^i\varepsilon_2^{e,ir} \end{bmatrix} - \Delta\gamma^{(k)} \begin{bmatrix} (\nabla\hat{\phi})_1^{(k)} \\ (\nabla\hat{\phi})_2^{(k)} \end{bmatrix}, \quad (37)$$

where $\nabla\hat{\phi}^{(k)}$ represents yield function gradient defined by [9]

$$\nabla\hat{\phi}^{(k)} = 2\mu\bar{\mathbf{v}}_{n+1} - \left(2\mu + \sqrt{\frac{2}{3}}\hat{k}'\left({}^{i-1}\xi + \sqrt{\frac{2}{3}}\Delta\gamma^{(k)}\right) \right) \delta(\Delta\gamma^{(k)}) = \begin{bmatrix} (\nabla\hat{\phi})_1^{(k)} \\ (\nabla\hat{\phi})_2^{(k)} \end{bmatrix}, \quad (38)$$

and $\hat{k}'\left({}^{i-1}\xi + \sqrt{\frac{2}{3}}\Delta\gamma^{(k)}\right)$ is the derivative of the hardening function [9]. If the material point is inside the yield surface $(\hat{\phi}^{(k)} \leq TOL)$, or near the yield surface $(\|\mathbf{r}^{(k)}\| \leq TOL)$, the corrector algorithm stops. If not, updated values of the principal logarithmic stretches, needed during the next iteration $k = k + 1$, are evaluated according to the following relationships

$${}^i\boldsymbol{\varepsilon}^{e(k+1)} = {}^i\boldsymbol{\varepsilon}^{e(k)} + ({}^i\mathbf{a})^{-1} \cdot \delta\boldsymbol{\beta}^{(k)} = \begin{bmatrix} {}^i\varepsilon_1^{e(k)} \\ {}^i\varepsilon_2^{e(k)} \end{bmatrix} + \begin{bmatrix} {}^i a_{11} & {}^i a_{12} \\ {}^i a_{12} & {}^i a_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \delta\beta_1^{(k)} \\ \delta\beta_2^{(k)} \end{bmatrix} = \begin{bmatrix} {}^i\varepsilon_1^{e(k+1)} \\ {}^i\varepsilon_2^{e(k+1)} \end{bmatrix}, \quad (39)$$

$${}^i\varepsilon_1^{e(k+1)} = \frac{{}^i a_{22} \delta\beta_1^{(k)} - {}^i a_{12} \delta\beta_2^{(k)}}{{}^i a_{11} a_{22} - ({}^i a_{12})^2} + {}^i\varepsilon_1^{e(k)}, \quad (40)$$

$${}^i\varepsilon_2^{e(k+1)} = \frac{{}^i a_{12} (\delta\beta_2^{(k)} - {}^i a_{22} \delta\beta_1^{(k)})}{{}^i a_{22} ({}^i a_{11} a_{22} - ({}^i a_{12})^2)} - \frac{\delta\beta_2^{(k)}}{{}^i a_{22}} + {}^i\varepsilon_2^{e(k)}, \quad (41)$$

where $\delta\boldsymbol{\beta}^{(k)}$ represents an increase of the principal Kirchhoff stresses,

$$\delta\boldsymbol{\beta}^{(k)} = {}^i\Xi^{(k)} [\mathbf{r}^{(k)} - \delta(\Delta\gamma^{(k)}) \nabla\hat{\phi}^{(k)}] = \begin{bmatrix} {}^i\Xi_{11}^{(k)} & {}^i\Xi_{12}^{(k)} \\ {}^i\Xi_{12}^{(k)} & {}^i\Xi_{22}^{(k)} \end{bmatrix} \cdot \begin{bmatrix} r_1^{(k)} \\ r_2^{(k)} \end{bmatrix} - \begin{bmatrix} \delta(\Delta\gamma^{(k)})_1 (\nabla\hat{\phi})_1^{(k)} \\ \delta(\Delta\gamma^{(k)})_2 (\nabla\hat{\phi})_2^{(k)} \end{bmatrix}, \quad (42)$$

$$\delta\boldsymbol{\beta}^{(k)} = \begin{bmatrix} {}^i\Xi_{11}^{(k)} (r_1^{(k)} - \delta(\Delta\gamma^{(k)})_1 (\nabla\hat{\phi})_1^{(k)}) + {}^i\Xi_{12}^{(k)} (r_2^{(k)} - \delta(\Delta\gamma^{(k)})_2 (\nabla\hat{\phi})_2^{(k)}) \\ {}^i\Xi_{12}^{(k)} (r_1^{(k)} - \delta(\Delta\gamma^{(k)})_1 (\nabla\hat{\phi})_1^{(k)}) + {}^i\Xi_{22}^{(k)} (r_2^{(k)} - \delta(\Delta\gamma^{(k)})_2 (\nabla\hat{\phi})_2^{(k)}) \end{bmatrix} = \begin{bmatrix} \delta\beta_1^{(k)} \\ \delta\beta_2^{(k)} \end{bmatrix}, \quad (43)$$

while ${}^i\Xi^{(k)}$ is a modified elastic modulus defined in principal directions, i.e.,

$${}^i\Xi^{(k)} = [({}^i\mathbf{a})^{-1} + \Delta\gamma^{(k)} \nabla^2\hat{\phi}^{(k)}]^{-1} = \begin{bmatrix} {}^i a_{11} & {}^i a_{12} \\ {}^i a_{12} & {}^i a_{22} \end{bmatrix}^{-1} + \Delta\gamma^{(k)} \begin{bmatrix} (\nabla^2\hat{\phi})_{11}^{(k)} & (\nabla^2\hat{\phi})_{12}^{(k)} \\ (\nabla^2\hat{\phi})_{12}^{(k)} & (\nabla^2\hat{\phi})_{22}^{(k)} \end{bmatrix}^{-1} = \begin{bmatrix} {}^i\Xi_{11}^{(k)} & {}^i\Xi_{12}^{(k)} \\ {}^i\Xi_{12}^{(k)} & {}^i\Xi_{22}^{(k)} \end{bmatrix}, \quad (44)$$

and $(\nabla^2\hat{\phi})$ is a second derivative of the yield function [9]. At the end of every iteration, the consistency parameter is reevaluated using (31), i.e.,

$$\Delta\gamma^{(k+1)} = \Delta\gamma^{(k)} + \delta(\Delta\gamma^{(k)}). \quad (45)$$

3. Consistent tangent elastoplastic modulus

In order to achieve quadratic convergence rate, consistent algorithmic elastoplastic tangent modulus is calculated at the end of the corrector algorithm. This modulus, obtained by the linearisation of relations described in [9], defines relationship between the principal Kirchhoff stresses and the principal logarithmic stretches as

$${}^i\mathbf{a}^{ep} = \frac{d {}^i\boldsymbol{\beta}}{d {}^i\boldsymbol{\varepsilon}^{e,ir}} = {}^i\Xi - \frac{{}^i\Xi \nabla\hat{\phi}({}^i\boldsymbol{\beta}) \otimes {}^i\Xi \nabla\hat{\phi}({}^i\boldsymbol{\beta})}{\nabla\hat{\phi}({}^i\boldsymbol{\beta}) \cdot {}^i\Xi \nabla\hat{\phi}({}^i\boldsymbol{\beta})}. \quad (46)$$

This modulus is identical to algorithmic modulus for infinitesimal strain described in [8]. For von Mises yield criterion, as shown in [9], (46) reduces to

$${}^i \mathbf{a}^{ep} = K \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 2\mu \left\{ {}^i s^{(k)} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) - {}^i \delta^{(k)} \begin{bmatrix} {}^i \bar{v}_1 \\ {}^i \bar{v}_2 \end{bmatrix} \cdot \begin{bmatrix} {}^i \bar{v}_1 \\ {}^i \bar{v}_2 \end{bmatrix}^T \right\}, \quad (47)$$

$${}^i \mathbf{a}^{ep} = \begin{bmatrix} K - \frac{2}{3}\mu \left(3 {}^i \delta^{(k)} ({}^i \bar{v}_1)^2 - 2 {}^i s^{(k)} \right) & K - \frac{2}{3}\mu \left(3 {}^i \delta^{(k)} {}^i \bar{v}_1 {}^i \bar{v}_2 + {}^i s^{(k)} \right) \\ K - \frac{2}{3}\mu \left(3 {}^i \delta^{(k)} {}^i \bar{v}_1 {}^i \bar{v}_2 + {}^i s^{(k)} \right) & K - \frac{2}{3}\mu \left(3 {}^i \delta^{(k)} ({}^i \bar{v}_2)^2 + 2 {}^i s^{(k)} \right) \end{bmatrix}, \quad (48)$$

$${}^i \mathbf{a}^{ep} = \begin{bmatrix} {}^i a_{11}^{ep(k)} & {}^i a_{12}^{ep(k)} \\ {}^i a_{12}^{ep(k)} & {}^i a_{22}^{ep(k)} \end{bmatrix}, \quad (49)$$

where ${}^i s^{(k)}$ and ${}^i \delta^{(k)}$ are defined as

$${}^i s = 1 - \frac{2\mu \Delta \gamma^{(k-1)}}{\| {}^i \bar{\mathbf{p}}^{tr(k-1)} \|} \quad (50)$$

$${}^i \delta = \frac{1}{1 + \frac{\hat{k}' \left({}^{i-1} \xi + \sqrt{\frac{2}{3}} \Delta \gamma^{(k-1)} \right)}{3\mu}} - \frac{2\mu \Delta \gamma^{(k-1)}}{\| {}^i \bar{\mathbf{p}}^{tr(k-1)} \|} \quad (51)$$

This elastoplastic algorithmic tangent modulus, calculated at the end of the corrector algorithm, is used to calculate spatial modulus, described in [4,9] when evaluating global stiffness matrix.

4. Concluding remarks

The algorithm described herein, based on [4,5] satisfies the objectivity and incompressibility requirements for plastic flow. Therefore, extra relations that would force the incompressibility requirement are not necessary. The use of consistent elastoplastic algorithmic tangent modulus enable a quadratic rate of convergence. This algorithm, defined in spatial configuration, is similar to infinitesimal algorithms described in [8]. There are also other formulations as [10], [11], similar to the one described herein. It is also worth noting that a formulation different from the one described herein [12], also reduces to a similar algorithm as the one described in this paper. This article focuses only on a material point. A complete explanation of variational formulation of boundary value problem, description of a four noded isoparametric finite element, etc., is provided in [9]. The relationships presented in this paper are to be implemented in finite element packages. Future work will be focused on enhancement of Elastic-linear plastic hardening model, as described in [9], and nonlinear hardening models, describing real material behavior. This algorithm constitutes a basis for further development of plate and shell elements.

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Influence of Boundary Conditions on Stress and Strain Results near Underground Rooms

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Abstract:

By excavation of opening in the rock, a stress concentration around the opening occurs. The aim of the paper is to determine the dimension of the Finite Element Method (FEM) model for the stress and strain analysis around an opening in rock massif. An elliptical opening is chosen, because the elliptical shape is better approximation to the real form of the underground rooms cross-section. The state of stresses and strains for an elliptical opening is analyzed, by theory of elasticity and by FEM analysis. Numerical calculation have been performed for both cases for given boundary conditions: with forces and separately with displacements. Comparison of the results calculated from the theory of elasticity and the results computed by the Finite Element Method is presented. The influence of the boundary condition given by forces and displacements is analyzed. Boundary conditions given by displacements are resulting with better approximation in the state of stress and strains.

1. Introduction

By the whole profile opening of underground room, applying for example the New Austrian Tunnel Method of excavation, the massif turns from the primary stress condition into the new secondary. Thereby the concentration of normal circular stresses occurs, while shearing and normal radial stresses disappear at the inner boundary. One of the important factor in underground room planning is stress analysis around underground room openings.

The calculation of stress and strain surrounding the underground room is based on the fact that the influence of the opening on certain distance disappears completely. The massif outside the influence zone is in the primary stress condition (state without opening).

In order to compute the stress and strain conditions in direct surrounding of the opening and the impact of the opening on the surroundings, there is necessary to know at least ten mechanical characteristics of the rock, which define mechanical elastic, plastic and viscous properties, anisotropy and other properties.

The characteristics can be established only, when particular location is reached by excavation. Unfortunately they are valid for only one limited massif area, so generally they have to be assumed on the experience basis. Besides, by bedding and crack system, the massif is actually a discontinue and not a continuum.

2. The theoretical plane strain analysis

Underground room at particular depth is observed as an opening in infinite plate. However, the first information about the occurrence around the opening can be determined for an elastic case. The plane strain analysis according to the theory of elasticity, by the Finite Element Method gives the first acceptable results. To compute the strains and stress around the opening, an area around the opening is to be considered. This area has to be large enough to get satisfactory results on both, inner and outer boundaries.

Theoretical solution is fined to determine dimensions of the area in which the opening existence provokes certain alterations in the stress condition related to the primary condition. The solution of stress condition for infinite plate with elliptical opening was established by Theodor Pöschl.

The opening size is given by half-axis a and b

$$a = c \cdot \operatorname{ch} \xi_0 \quad ; \quad b = c \cdot \operatorname{sh} \xi_0 \quad (1)$$

where c is linear eccentricity.

Uniform load p closes with bigger half-axis of ellipse a , angle $\pi/2+\alpha$ (Fig. 1).

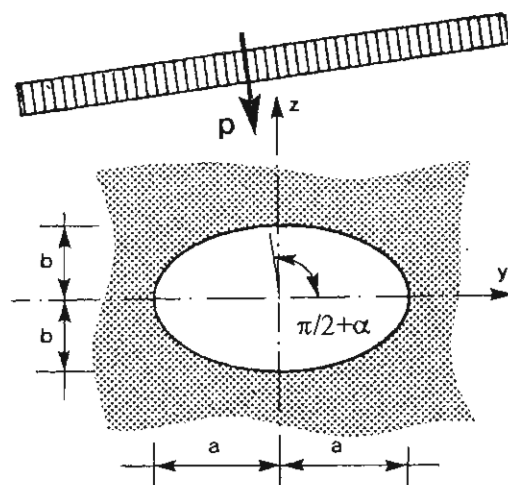
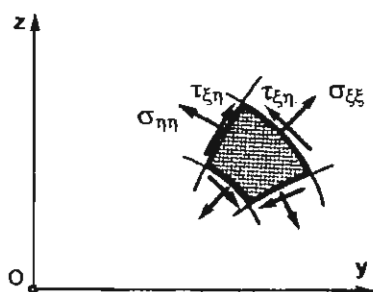


Fig. 1 Elliptical opening in mass under action of uniform load p

After Pöschl, the stress function is the function of curvilinear coordinates ξ and η . In form of

$$\Phi = \frac{p \cdot c^2}{8} \left\{ \text{sh} 2\xi - \cos 2\alpha \cdot e^{-2(\xi-\xi_0)} - 2(\text{ch} 2\xi + \cos 2\alpha) \cdot \xi + [\text{ch} 2(\xi - \xi_0) - 1] \cdot e^{2\xi_0} \cdot \cos 2(\eta - \alpha) \right\} \quad (2)$$



it satisfying Maxwell's differential equation:

$$\nabla^4 \Phi = 0 \quad (3)$$

and boundary conditions (Fig. 2) for:

$$\begin{aligned} \xi = \xi_0 & \quad \sigma_{\xi\xi} = 0 & \quad \sigma_{\xi\eta} = 0 \\ \xi = \infty & \quad \sigma_{yy} = p_h & \quad \sigma_{zz} = p_v \end{aligned} \quad (4)$$

Fig. 2 Boundary conditions

The stress components in curvilinear coordinates are given by expression:

$$\begin{aligned} \sigma_{\xi\xi} = & \frac{2}{c^2(\text{ch} 2\xi - \cos 2\eta)} \frac{pc^2}{2} e^{2\xi_0} \left[1 - \text{ch} 2(\xi - \xi_0) \right] \cos 2(\eta - \alpha) + \frac{4}{c^3 \sqrt{2(\text{ch} 2\xi - \cos 2\eta)^3}} \frac{c\sqrt{2}}{2} \frac{pc^2}{4} \\ & \frac{\text{sh} 2\xi}{\sqrt{\text{ch} 2\xi - \cos 2\eta}} \cdot \left\{ \text{ch} 2\xi + \cos 2\alpha \cdot e^{-2(\xi-\xi_0)} - \cos 2\alpha - \text{ch} 2\xi + \text{sh} 2(\xi - \xi_0) \cdot e^{2\xi_0} \cos 2(\eta - \alpha) \right\} - \\ & \frac{4}{c^3 \sqrt{2(\text{ch} 2\xi - \cos 2\eta)^3}} \frac{c\sqrt{2}}{2} \cdot \frac{pc^2}{4} \frac{\sin 2\xi}{\sqrt{\text{ch} 2\xi - \cos 2\eta}} \cdot e^{2\xi_0} \left[1 - \text{ch} 2(\xi - \xi_0) \right] \cdot \sin 2(\eta - \alpha) \end{aligned} \quad (5)$$

$$\begin{aligned} \sigma_{\eta\eta} = & \frac{2}{c^2(\text{ch} 2\xi - \cos 2\eta)} \frac{pc^2}{2} \cdot e^{2\xi_0} \cdot \left\{ \text{sh} 2\xi + \cos 2\alpha \cdot e^{-2(\xi-\xi_0)} + \text{ch} 2(\xi - \xi_0) \cdot e^{2\xi_0} \cos 2(\eta - \alpha) \right\} - \frac{4}{c^3 \sqrt{2(\text{ch} 2\xi - \cos 2\eta)^3}} \cdot \\ & \frac{c\sqrt{2}}{2} \frac{pc^2}{4} \frac{\text{sh} 2\xi}{\sqrt{\text{ch} 2\xi - \cos 2\eta}} \cdot \left\{ \text{ch} 2\xi + \cos 2\alpha \cdot e^{-2(\xi-\xi_0)} - \text{ch} 2\xi_0 - \cos 2\xi\alpha + \text{sh} 2(\xi - \xi_0) \cdot e^{2\xi_0} \cos 2(\eta - \alpha) \right\} + \\ & + \frac{4}{c^3 \sqrt{2(\text{ch} 2\xi - \cos 2\eta)^3}} \frac{c\sqrt{2}}{2} \cdot \frac{\sin 2\eta}{\sqrt{\text{ch} 2\xi - \cos 2\eta}} \frac{pc^2}{4} \cdot e^{2\xi_0} \left[1 - \text{ch} 2(\xi - \xi_0) \right] \cdot \sin 2(\eta - \alpha) \end{aligned} \quad (6)$$

$$\tau_{\xi\eta} = \frac{2}{c^2(\operatorname{ch}2\xi - \cos2\eta)} \frac{pc^2}{2} e^{2\xi_0} \sin2(\eta - \alpha) \cdot \operatorname{sh}2(\xi - \xi_0) + \frac{4}{c^3\sqrt{2(\operatorname{ch}2\xi - \cos2\eta)^3}} \frac{c\sqrt{2}}{2} \frac{pc^2}{4} \frac{\sin2\eta}{\sqrt{\operatorname{ch}2\xi - \cos2\eta}} \cdot \left\{ \operatorname{ch}2\xi + \cos2\alpha \cdot e^{-2(\xi-\xi_0)} - \operatorname{ch}2\xi_0 + \cos2\alpha + \operatorname{sh}2(\xi - \xi_0) \cdot e^{2\xi_0} \cos2(\eta - \alpha) \right\} + \frac{4}{c^3\sqrt{2(\operatorname{ch}2\xi - \cos2\eta)^3}} \frac{c\sqrt{2}}{2} \cdot \frac{pc^2}{4} \frac{\operatorname{sh}2\xi}{\sqrt{\operatorname{ch}2\xi - \cos2\eta}} \cdot e^{2\xi_0} [1 - \operatorname{ch}2(\xi - \xi_0)] \cdot \sin2(\eta - \alpha) \quad (7)$$

For the calculation of secondary stress condition the computer program is made. Program calculates stresses in particular points on the basis of input data (half-axis of the ellipse a , b , load p and angle α , or p_h/p_v). The calculation is performed for different ratio of ellipse half-axis $b/a=1/2$; $2/3$; $1/1$ and $3/2$ and ratio of horizontal and vertical load $p_h/p_v=0$; 0.25 ; 0.50 ; 0.75 and 1.00 .

To discuss possible failure conditions in different planes, the sketch of stress trajectory can be used, presented by the program. Let's assume a model composed of blocks circumscribed just with these trajectories (Fig. 3 and 4). The blocks circumscribed with trajectories in the area, where both major stresses are compressible, exist in conditions similar to those testing the samples in three-axial apparatus. During failure, a sliding occurs on a plane, they normal close an angle $45^\circ - \varphi/2$ with direction σ_{min} . In the roof where a major stress is tensile, radial cracks will occur. Tracing the lines with $45^\circ - \varphi/2$ in respect to σ_{max} direction the probable sliding planes can be constructed.

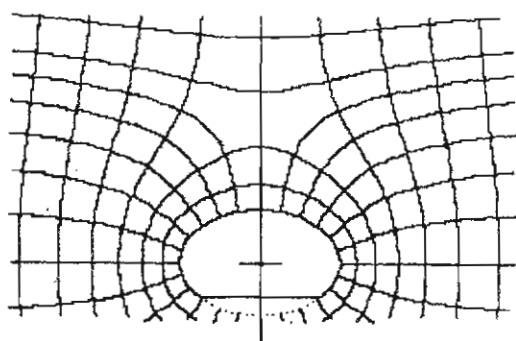


Fig. 3 Trajectories of stress for elliptical opening with half-axis ratio $b/a = 2/3$ and load ratio $p_h/p_v = 0.50$

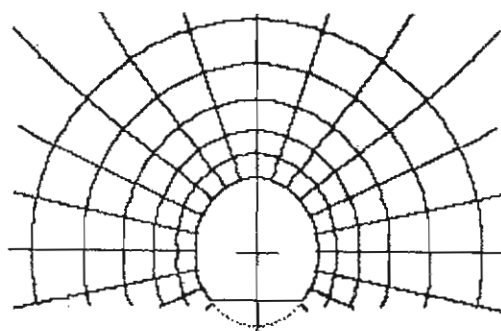


Fig. 4 Trajectories of stress for elliptical opening with half-axis ratio $b/a = 3/2$ and load ratio $p_h/p_v = 1.00$

Table 1. Values of stresses σ_{yy} and σ_{zz} along the axis' y and z for elliptical opening with half-axis ratio $b/a = 2/3$ and vertical load $p_v = 1.00$

y/a	$p_h = 0.00$		$p_h = 0.50$		$p_h = 1.00$		z/b	$p_h = 0.00$		$p_h = 0.50$		$p_h = 1.00$	
	σ_{zz}	σ_{yy}	σ_{zz}	σ_{yy}	σ_{zz}	σ_{yy}		σ_{zz}	σ_{yy}	σ_{zz}	σ_{yy}	σ_{zz}	σ_{yy}
1.00	3.999	0.000	3.499	0.000	2.999	0.000	0.67	0.000	-1.00	0.000	0.166	0.000	1.333
1.41	1.556	0.504	1.547	0.659	1.538	0.814	1.20	0.175	-0.07	0.300	0.599	0.424	1.274
2.00	1.184	0.281	1.200	0.610	1.216	0.938	1.86	0.520	0.047	0.608	0.603	0.695	1.159
2.83	1.075	0.144	1.087	0.559	1.098	0.974	2.73	0.744	0.043	0.794	0.564	0.844	1.085
4.00	1.034	0.072	1.040	0.530	1.047	0.988	3.93	0.868	0.029	0.894	0.535	0.921	1.043
5.66	1.016	0.036	1.019	0.515	1.023	0.994	5.61	0.933	0.014	0.946	0.518	0.960	1.022
8.00	1.008	0.018	1.009	0.507	1.011	0.997	7.96	0.966	0.007	0.973	0.509	0.980	1.011
11.31	1.003	0.009	1.004	0.503	1.005	0.998	11.29	0.983	0.003	0.861	0.504	0.990	1.005
16.00	1.001	0.004	1.002	0.501	1.002	0.999	15.98	0.991	0.001	0.993	0.502	0.995	1.002

The values of stress σ_{yy} and σ_{zz} along the axis's y and z for elliptical opening with the half-axis ratio $b/a=2/3$ and loading ratio $p_H/p_V=0.5, 0.0, 1.00$ are given in Table 1. It is obvious, that the differences in stress along the axis y are almost two times smaller than along the axis z . Similar results are obtained also for other relations of the half-axis b/a and loads p_H/p_V .

Table 2. Differences in stresses related to primary stress along y and z axis

$\Delta \sigma$	10.0 %	5.0 %	3.0 %	2.5 %	2.0 %	1.0 %	0.5 %
y/a	2.66	3.62	4.63	5.22	5.61	7.81	11.05
z/b	4.05	5.80	7.64	8.53	9.47	13.60	20.12

Certain area around the opening must be included in the FEM model. Better results can be expected with a larger area, but this would require a considerably bigger number of nodal points and elements, as well as of the scope of numerical calculation itself. The expected error in stress could estimate from the Table 2, and the score of considered area chosen. Much greater errors are possible in load estimation as well as by the assuming mechanical characteristics.

3. Finite element method analysis

The choice of numbers, dimensions and forms of elements is based on the previously comparisons and an area of six times half-axis a and almost nine times half-axis b is chosen. Intersection of confocal ellipses $\xi = \text{constant}$ and confocal hyperbolas $\eta = \text{constant}$ are chosen as nodal points, defining the elements. The network of quadrilateral elements with side ratio 1,41 is accepted. Element dimensions are increased with constant step toward outer boundary. This does not cause a slighter approximation, since the state of stress on the outer contour boundary approximates the homogenous one, with smaller gradients. The outer contour has form of an ellipse - confocal with an opening and very approximately corresponding to the circle. For an elliptical opening with the ratio of half-axis $b/a=2/3$ the network with 90 nodal points and 72 quadrilateral elements is accepted (Fig. 5). The symmetry of displacement and load is provided by restraining along y and z axis.

The boundary conditions on the outer contour are given in two ways:

- the forces at the nodal points (FEM-f)
- the displacements at the nodal points (FEM-d).

The forces for nodal points of the outer contour are determined by trapezoidal rule.

For the boundary conditions given by displacements it was necessary to assign new 18 nodal points and 18 new boundary elements. The displacements of contour points are calculated by integration of strain components.

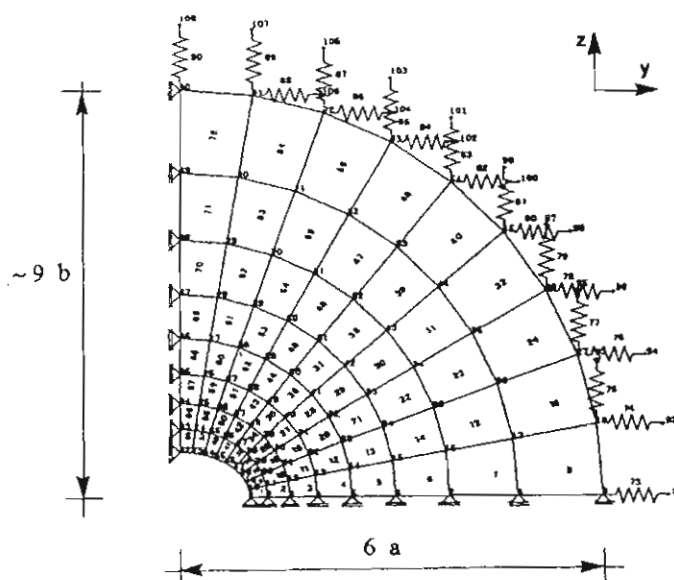


Fig. 5 Network of elements for using the Finite elements method

4. Comparison of the results

Stresses in centroids of the outer contour elements for the ratio half-axis $b/a=2/3$ and load $p_H/p_V=0.5$ are compared with theoretical results in Table 3. Comparison of stresses in centroids of elements along the opening see in Table 4. The differences in stress between the theoretical and the numerical solution are presented in percentages of basic vertical load $p_V=1,0$. The table illustrates the mean values of stress differences $\Delta_{average}$ (%) as well.

Table 3. Comparison of stresses σ_{yy} and σ_{zz} in centroids of outer contour elements

Element	Stress σ_{yy}					Stress σ_{zz}				
	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)
8	0.515	0.507	0.8	0.517	0.2	1.022	0.999	2.3	1.027	0.5
16	0.507	0.503	0.4	0.508	0.1	1.025	1.008	1.7	1.028	0.3
24	0.495	0.499	0.4	0.494	0.1	1.029	1.021	0.8	1.028	0.1
32	0.484	0.500	1.6	0.481	0.3	1.028	1.032	0.4	1.022	0.6
40	0.480	0.509	2.9	0.472	0.8	1.019	1.038	1.9	1.009	0.1
48	0.484	0.527	4.3	0.471	1.3	1.001	1.035	3.4	0.900	1.3
56	0.495	0.550	5.5	0.477	1.8	0.979	1.023	4.4	0.963	1.6
64	0.509	0.573	6.4	0.486	2.3	0.957	1.009	5.2	0.940	1.7
72	0.518	0.588	7.0	0.493	2.5	0.943	1.000	5.7	0.926	1.7
		$\Delta_{average}$	3.3	$\Delta_{average}$	1.0		$\Delta_{average}$	2.9	$\Delta_{average}$	0.9

Table 4. Comparison of stresses σ_{yy} and σ_{zz} in centroids of elements along the opening

Element	Stress σ_{yy}					Stress σ_{zz}				
	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)
1	0.500	0.448	5.2	0.444	5.6	2.292	2.747	45.5	2.315	2.3
9	0.416	0.394	2.2	0.371	4.5	2.148	2.321	17.3	2.138	1.0
17	0.295	0.296	0.1	0.285	1.0	1.890	1.999	10.9	1.855	4.5
25	0.186	0.202	1.6	0.202	1.6	1.563	1.617	5.4	1.519	4.4
33	0.110	0.126	1.6	0.134	2.4	1.230	1.235	0.5	1.183	4.7
41	0.068	0.083	1.5	0.096	2.8	0.933	0.907	2.6	0.895	3.8
49	0.059	0.067	0.8	0.080	2.1	0.696	0.649	4.7	0.669	2.7
57	0.054	0.071	1.7	0.080	2.6	0.527	0.465	6.2	0.511	1.6
65	0.063	0.085	2.2	0.087	2.4	0.437	0.363	7.4	0.427	1.0
		$\Delta_{average}$	1.9	$\Delta_{average}$	2.8		$\Delta_{average}$	11.2	$\Delta_{average}$	2.9

The most expressive is concentration of stress along the axis y , which is evident from Table 5. and diagram in Figure 6.

Theoretical stress σ_{zz} is presented in Figure 6a and theoretical stress σ_{yy} in Figure 6c.

The differences in stress $\Delta\sigma_{zz}$ of numerical estimations for the case of assumption of the boundary conditions expressed in forces (FEM-f) or displacements (FEM-d) Figure 6b, and the stress differences $\Delta\sigma_{yy}$ for both numerical estimations Figure 6d.

Comparison of stresses σ_{zz} and σ_{yy} in centroids of elements along the axis z in midpoints of elements' sides and differences according to FEM-f and FEM-d are given in Table 6. and in Figure 7.

Noticeable is the comparison of the nodal points displacements of the outer contour. FEM-d reproduces the displacements which are given as boundary conditions in advance. Considerable differences from theoretical solution are obtained in case of the boundary conditions given by forces in outer contour points (FEM-f). In Table 7. deviations are expressed in percentages of theoretical values.

Table 5. Comparison of stresses σ_{xx} and σ_{yy} along axis y in midpoints of elements' sides

Element	Stress σ_{xx}					Stress σ_{yy}				
	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)
1	2.310	2.726	41.6	2.305	0.5	0.496	0.400	9.6	0.401	9.5
2	1.546	1.602	5.6	1.486	6.0	0.654	0.665	1.1	0.631	2.3
3	1.276	1.319	4.3	1.230	4.6	0.631	0.636	0.5	0.616	1.5
4	1.159	1.186	2.7	1.113	4.6	0.586	0.587	0.1	0.579	0.7
5	1.090	1.115	2.5	1.056	3.4	0.562	0.552	1.0	0.552	1.0
6	1.054	1.069	1.5	1.029	2.5	0.540	0.528	1.2	0.533	0.7
7	1.033	1.029	0.4	1.016	1.7	0.526	0.510	1.6	0.520	0.6
8	1.021	0.988	3.3	1.016	0.5	0.516	0.500	1.6	0.512	0.4
		Δ_{average}	7.7	Δ_{average}	3.0		Δ_{average}	2.1	Δ_{average}	2.1

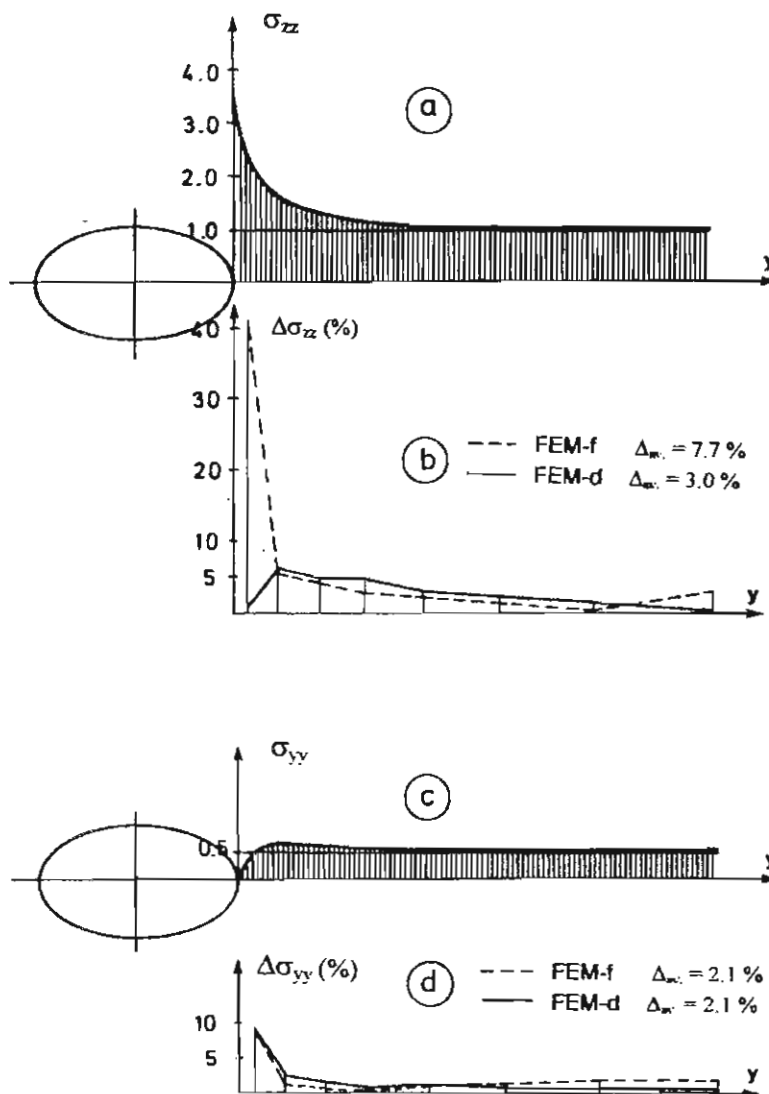


Fig. 6 Diagram of stresses σ_{xx} and σ_{yy} along axis y and differences according to FEM-f and FEM-d

Table 6. Comparison of stresses σ_{zz} and σ_{yy} along axis z in midpoints of elements' sides

Element	Stress σ_{zz}					Stress σ_{yy}				
	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)	Theory of elasticity	FEM-f	Δ (%)	FEM-d	Δ (%)
65	0.431	0.367	6.4	0.429	0.2	0.064	0.088	2.4	0.089	2.5
66	0.603	0.605	0.2	0.618	1.5	0.294	0.276	1.8	0.270	2.4
67	0.621	0.683	6.2	0.668	4.7	0.506	0.450	5.6	0.439	6.7
68	0.677	0.768	9.1	0.723	4.6	0.589	0.516	7.3	0.508	8.1
69	0.786	0.863	7.7	0.804	1.8	0.566	0.524	4.2	0.511	5.5
70	0.860	0.933	7.3	0.865	0.5	0.545	0.524	2.1	0.497	4.8
71	0.910	0.982	7.2	0.908	0.2	0.530	0.531	0.1	0.480	5.0
72	0.942	1.017	7.5	0.937	0.5	0.520	0.552	3.2	0.461	5.9
		Δ_{average}	6.5	Δ_{average}	1.8		Δ_{average}	3.3	Δ_{average}	5.1

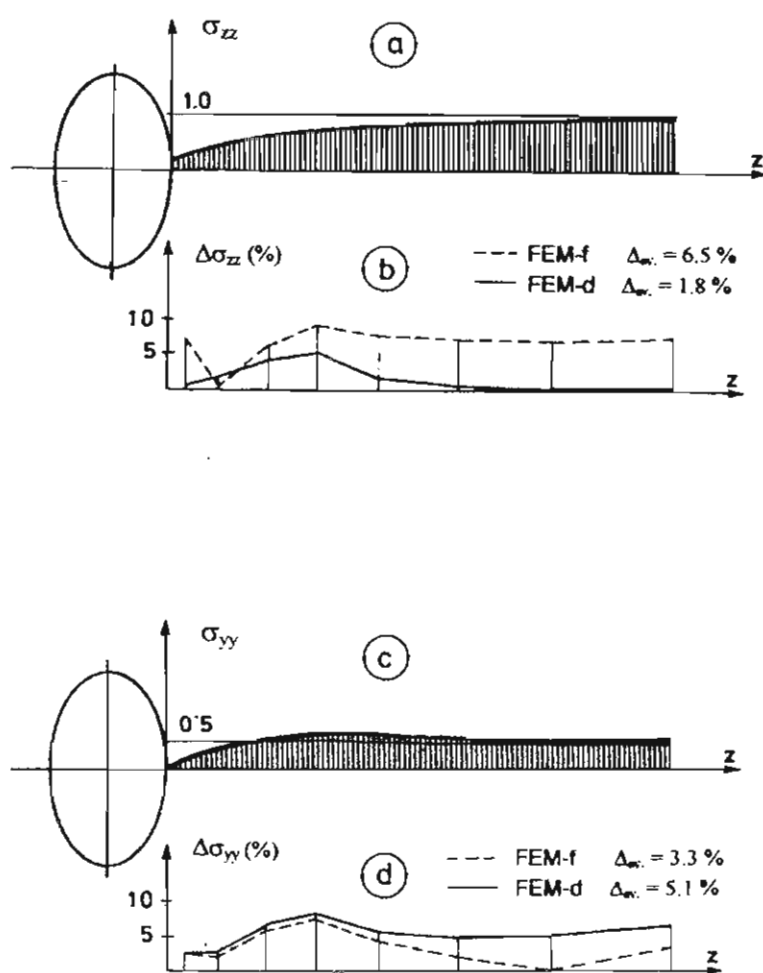


Fig. 7 Diagram of stresses σ_{zz} and σ_{yy} along axis z and differences according to FEM-f and FEM-d

Table 7. Comparison of displacements v and w of outer contour nodal points

Node	Displacement v in y direction			Displacement w in z direction		
	Theory of elasticity	FEM-f	Δ (%)	Theory of elasticity	FEM-f	Δ (%)
9	-0.001439	-0.001284	10.8	0.000000	0.000000	0.0
18	-0.001418	-0.001286	9.3	-0.000893	-0.000848	5.0
27	-0.001353	-0.001287	4.9	-0.001759	-0.001691	3.7
36	-0.001247	-0.001271	1.9	-0.002571	-0.002522	1.9
45	-0.001103	-0.001216	10.2	-0.003306	-0.003319	0.4
54	-0.000926	-0.001102	19.0	-0.003940	-0.004054	2.9
63	-0.000720	-0.000917	27.4	-0.004454	-0.004687	5.2
72	-0.000493	-0.000660	33.9	-0.004833	-0.005177	7.1
81	-0.000250	-0.000346	38.4	-0.005065	-0.005489	8.4
90	0.000000	0.000000	0.0	-0.005143	-0.005970	16.1
		$\Delta_{average}$	17.3		$\Delta_{average}$	5.6

5. Conclusion

Comparison of the results according to the theory of elasticity with the results of numerical analysis indicates, that boundary conditions given by displacements (FEM-d) results with more accurate stresses both on the outer contour and around the opening. The error in stresses is 4.3% smaller. The greatest differences occur on the places of the greatest stress concentration, when the boundary conditions are given by forces (FEM-f).

The error in displacements on the outer contour in giving boundary conditions given by forces (FEM-f) are in average 11.4% theoretical value.

Boundary conditions given by displacements are resulting with better approximation in the state of stress and strains.

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SOME ASPECTS OF THE NONLINEAR FINITE ELEMENT ANALYSIS OF ELASTOMERS

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ABSTRACT

The finite element methods are well established in many branches of engineering and are now routinely used in the solution procedure of many large scale industrial problems. Computational procedures, which are almost exclusively based on the finite element method, are now increasingly employed in the modelling of the elastomers due to (in addition to improving computational power) remarkable recent advances in the numerical treatment of large strain elastic and elasto-plastic deformation phenomena in the presence of frictional contact. Therefore, this paper discusses in short, some computational aspects of the nonlinear finite element analysis of the elastomers, namely: material behaviour, automatic determination of the material parameters from test data, damage and failure, dynamics and vibration, automated contact analysis techniques, and automated solution strategies. Although the main focus of the paper is on the elastomers (or rubber-like materials) and foams, many of these concepts are also applicable to the finite element analysis of glass, plastics, and biomaterials. At the end, areas which still require further research and development are mentioned. The paper is concluded with some questions which should be asked when selecting a code for nonlinear finite element analysis of elastomers or rubber-like materials.

1. INTRODUCTION

The finite element methods are well established in many branches of engineering and are now routinely used in the solution procedure of many large scale industrial problems [1]. This situation has been brought about by the increased maturity of the numerical analysis methods, particularly for the nonlinear problems, and dramatic developments in the computer performance. The performance / cost ratio of the computers continues to improve and this situation is set to continue, and even accelerate, with the emergence of the workstation based parallel processing hardware.

Computational procedures, which are almost exclusively based on the finite element method, are now increasingly employed in the modelling of the elastomers due to (in addition to improving computational power) remarkable recent advances in the numerical treatment of large strain elastic and elasto-plastic deformation phenomena in the presence of frictional contact.

In the computational circles, effort has been directed at the formulation of algorithms for integration of the constitutive equations relying on operator split methodology and in particular the concept of consistent linearisation has been introduced to provide quadratically convergent solution procedures. By employing logarithmic stretches as strain measures a particularly simple model for large inelastic deformations at finite strains is recovered.

Considerable advances in the numerical modelling have also been made in recent years in the solution of contact-friction phenomena which arise in most technology operations. Recognition of the complete analogy which exists between contact-friction behaviour and the classical theory of elasto-plasticity has ensured that operator split algorithms and consistent linearisation procedures developed for the latter case translate directly to contact-friction models to provide robust and rapidly convergent numerical solutions.

Therefore, this paper discusses, in short according to Ref. [2], some computational aspects of the nonlinear finite element analysis of the elastomers. Although the main focus of the paper is on the elastomers (or rubber-like materials) and foams, many of these concepts are also applicable to the finite element analysis of glass, plastics, and biomaterials, not only in the rubber and tire industries, but also to those involved in the following: glass, plastics, ceramics, and solid propellant industries, biomechanics and the medical/dental professions (implantable surgery devices, prostheses, orthopedics, orthodontics, dental implants, artificial limbs, artificial organs, wheelchairs and beds, monitoring equipment), highway safety and flight safety (seat belt design, impact analysis, seat and padding design, passenger protection), packaging industry and sports and consumer industries (helmet design, shoe design, athletic protection gear, sports equipment safety), etc.

Elastomers are used extensively in many industries because of their wide availability and low cost. They are also used because of their excellent damping and energy absorption characteristics, flexibility, resiliency, long service life, ability to seal against moisture, heat, and pressure, non-toxic properties, moldability and, variable stiffness.

It is a very unique material. During processing and shaping, it behaves mostly like a highly viscous fluid. After its polymer chains have been crosslinked by vulcanization or by curing, it can undergo large reversible elastic deformations. Unless damage occurs, it will return to its original shape after removal of the load.

Proper analysis of elastomer and/or rubber components requires special material modelling and nonlinear finite element analysis tools that are quite different than those used for the metallic parts. The unique properties of elastomer/rubber are such that:

- 1) It can undergo **large deformations** under load, sustaining strains of up to 500% in engineering applications.
- 2) Its load-extension behaviour is **markedly nonlinear**.
- 3) Because it is viscoelastic, it exhibits significant damping properties. Its behaviour is **time- and temperature-dependent**, making it similar to glass and plastics in this respect.
- 4) It is **nearly incompressible**. This means its volume does not change appreciably with stress. It cannot be compressed significantly under hydrostatic load.

Therefore, the reliable, efficient and robust nonlinear finite element analysis program should possess adequately formulated elements, material and friction models, and automated contact analysis procedures to handle elastomers and/or rubber like materials.

2. SOME COMPUTATIONAL ASPECTS OF THE FINITE ELEMENT ANALYSIS OF ELASTOMERS

As mentioned earlier, this paper discusses in short some computational aspects of the nonlinear finite element analysis of the elastomers, namely: material behaviour, automatic determination of the material parameters from test data, damage and failure, dynamics and vibration, automated contact analysis techniques, and automated solution strategies. More detailed descriptions of these topics can be found in Ref. [2] from where the next sections are extracted and condensed.

2.1 Material behaviour

This section should discuss the issues central to the description of material modelling of elastomers and other rubber-like materials starting with the concepts of nonlinearity and the stress-strain descriptions suitable for nonlinear analysis. It should also include the features of time-independent (strain energy density functions, incompressible behaviour) and time-dependent (linear and nonlinear viscoelasticity, temperature effects) material behaviour, anisotropy, hysteresis, and stability. Due to the lack of space it is omitted, but can be found in short in Ref. [2] or in some well known classic book on material behaviour.

2.2 Automatic determination of material parameters from test data

Successful modelling and design of rubber components hinges on the selection of an appropriate strain energy function, and accurate determination of the material constants in the function. The computer programme MARC has the capability to evaluate the material constants for nonlinear elastic and viscoelastic materials via a data fitting programme, CURVE.

For time-independent nonlinear elasticity, the fitting procedure may be carried out for polynomial representations of incompressible materials, the generalized Ogden model for slightly compressible materials, and Foam model for compressible materials. Six different types of experiments are supported, namely, uniaxial tension, uniaxial compression, equibiaxial, planar or pure shear, simple shear, and volumetric tests. The

significance of non-equivalent multiple tests for material modelling cannot be overemphasized. In general, a combination of uniaxial tension/compression and simple shear is required at the very least. Data from equilibrium tension or planar shear may also be needed depending on the deformation modes of structures. Volumetric data must be included for materials undergoing large compressible deformations, e.g., foams. Also, CURVE allows a combined input of more than one test to obtain the appropriate material constants.

The phenomenon of negative material constants and therefore physically not meaningful is a numerical serendipity and not a fundamental material behaviour. In this case, an optimization process can be invoked, based on sequential linear programming, in order to obtain non-negative material constants.

Automated facilities are available to help the user determine these material parameters from test data. The curve-fitting programme is interactive and consists of four steps: (1) data entry - where the user inputs experimental data; (2) evaluation - where the programme mathematically fits the data; (3) plotting/display - where the user sees graphical verification of the results; (4) write - where the programme automatically creates a data set and the necessary coefficients for the strain energy density function of choice.

For the generalized Ogden as well the Foam model, the material constants follow from a set of nonlinear equations and CURVE fits the data based on the Downhill-Simpson algorithm.

The data representing a time-dependent or viscoelastic response of materials can be approximated by a Prony series, based on relaxation or creep test. For a linear viscoelastic material, either the shear and bulk moduli, or the Young's modulus and Poisson's ratio may be expressed in terms of Prony series. For large strain viscoelasticity, the elastic strain energy or the stress is expressed in terms of Prony series.

2.3 Damage and failure

The most important and perhaps the most difficult aspect of design analysis is failure prediction. The failure in rubber can occur because of flaws introduced during the manufacturing processes (e.g., compound mixing, extrusion, molding, or vulcanization, etc.) or fatigue caused by service loads and/or material degradation due to environmental/mechanical/thermal conditions. Along these lines, some very promising models were proposed [2].

In a study of the fracture of bonded rubber blocks under compression, Gent et al. [3], found that: (1) under static compression, two modes of fracture are possible - circumferential tearing at or near the bonded edges, and splitting open of the free surface; (2) under cyclic compression, the most likely fracture mode of the rubber is by crack propagation, breaking away the bulged volume.

For cord-reinforced composites, besides damage and fracture of the rubber matrix, the critical modes of failure are ply separation, debonding between layers of dissimilar materials, fiber pull-out due to lack of adhesion, and microbulking of cords. Besides mechanical loading, thermal and viscoelastic effects play a critical role in failure of cord-rubber composites. Frictional heating at cord-rubber interface and internal heat build-up due to hysteresis in rubber cause the temperature of the material to rise. Due to low thermal conductivity of rubber, the temperatures can rise to a very high value,

causing adhesion failures and microcracking in the rubber matrix. Unfortunately, no good models exist currently to simulate the above failures.

2.4 Dynamics and vibration

A widespread use of rubber is for shock/vibration isolation and noise suppression in transportation vehicles, machinery, and buildings. These common rubber components include: snubbers, load bearing pads, engine mounts, bearings, bushings, air springs, bumpers, and so forth. Recent seismic isolation applications have seen increased usage of laminated rubber bearings for the foundation designed of buildings, highway and bridge structures (especially in the United States and Japan). These applications take advantage of well-known characteristics of rubber: energy absorption and damping, flexibility, resilience, long service life, and moldability.

A dynamic analysis is required whenever inertial effects are important, e.g., high speed rolling of tires or sudden loss of contact in a snap-through buckling analysis. When inertial effects are unimportant, such as for engine mounts and building bearings, performing a dynamic analysis is unnecessary. When the viscous effects are important for such cases, a quasi-static analysis is performed to obtain the overall deformation which is followed by a harmonic analysis to obtain frequencies and mode shapes.

Damping

The nature of damping is complex and is still poorly understood. Four common damping models include: (1) *Proportional* or *Rayleigh damping* - assumes that damping may be decomposed as a linear combination of the stiffness and mass matrices; (2) *Coulomb damping* or *dry friction* - comes from the motion of a body on a dry surface (e.g., on the areas of support); (3) *Viscous damping* - occurs when a viscous fluid hinders the motion of the body. The damping forces are proportional to velocity in the equation of motion; (4) *Joint damping* - results from internal friction within the material or at connections between elements of a structural system.

Because of the viscoelastic behaviour of rubber, damping is dependent on frequency of the excitation. The presence of damping forces progressively reduces the amplitude of vibration, and ultimately stops the motion when all energy initially stored in the system is dissipated.

Modal extraction

A popular and efficient modal (eigenvalue) extraction method in the finite element analysis codes is the Lanczos method. This method has been proven to be faster and more accurate than the basic subspace iteration method. For the case of proportional damping, real modes give useful information (the natural frequencies). In the case of nonproportional damping, complex modes result. Natural frequencies are dependent upon pre-stress and material properties; both of these would require nonlinear analysis.

Time vs. frequency domain analysis

In viscoelastic problems, both time and frequency domains are used. In time domain analysis, experimental data is required over the time domain of interest and a Prony series is usually used to represent the data. In frequency domain analysis, Laplace transform techniques and harmonic excitation are commonly used. The storage modulus and loss modulus are dependent upon frequency (and amplitude for filled rubbers), and one needs to be aware of the in-phase and out-of-phase concepts. In linear viscoelastic problems with harmonic loading, the behaviour can be characterized in the frequency

domain in terms of the storage and loss moduli. This subject is very complex: as the frequency increases, the state of the rubber changes from an elastomer to a glass, with the maximum in the loss modulus signaling the transition to the glassy state. In unfilled rubbers, the storage and loss moduli are dependent on the frequency, but the former is largely independent on the strain amplitude. In filled rubbers, the storage modulus depends significantly on the strain amplitude.

Direct time integration methods

In transient nonlinear dynamics, both implicit and explicit direct integration methods are available for solving the equations of motion. The choice of whether to use an implicit (Newmark- β , Wilson- θ , Hilber-Hughes-Taylor, and Houbolt) or explicit (Central difference) method is very subtle and depends on: the nature of the dynamic problem and the material; the type of finite elements making up the model; and the magnitude of the speed of sound in the material.

Implicit methods - In this method, the nonlinear matrix equations of motion are solved at each time step to advance the solution. Treatment of boundary nonlinearities must occur within a time step. Large time steps may be used in implicit, dynamic analysis. The most popular is the Newmark- β method, which is unconditionally stable; however, it does not preserve the total angular momentum of the system. The Houbolt method should be avoided because it is known to exhibit artificial damping problems. As for the use of dynamic methods in viscoelastic analysis, no additional damping should be introduced in the Newmark- β or Hilber-Hughes-Taylor methods because viscoelastic effects are already included in the material properties.

Explicit methods - In this method, the solution is advanced without forming a stiffness matrix, which makes the coding much simpler, reduces storage requirements, and improves computational efficiency. Explicit methods are conditionally stable for undamped linear systems. For a given time step, an explicit operator requires fewer computations per time step than an implicit one. Explicit methods possess some disadvantages, and it is important for users to bear in mind that a definite stability limit exists, which means that sometimes extremely small time steps may be required - resulting in higher computer costs. In nearly incompressible problems, the speed of sound in the material approaches infinity, and hence an extremely small time step is required. A common solution to overcome these numerical difficulties using explicit methods is to conjure up the mass matrix - which is very often assumed to be diagonal. (This "diagonalized mass matrix" assumption is invalid in many cases!) Finally, if Lagrange multipliers are included in the analysis, special formulations are required because they do not have any associated mass.

2.5 Automated contact analysis techniques

Rubber products and contact problems always seem to involve contact versus no-contact conditions - for instance, rubber gaskets and the contact of a car tire with the road.

Contact, by nature, is a nonlinear boundary value problem. During contact, both forces transmitted across the surface and the area of contact change. Since rubber is flexible, this change in the contact area is both significant and difficult to model using earlier methodologies (such as gap elements). The contact stress is transmitted in the normal direction, and if the friction is present, shear stress is also transmitted. Mathematically, the contact problem occurs as a constrained optimization problem where frictional contact conditions occur as inequalities.

In the computer programme MARC, *automatic* means that the user interaction is not required in treating multibody contact and friction. The user needs only to identify *bodies* and their *boundary surfaces* which are potential candidates for contact during the analysis.

Friction is very complex phenomenon, including: surface imperfections, stick-slip motions, material softening due to heat in the contact area, time- and rate- dependence of the coefficient of static friction, and the oscillatory and unstable nature of sliding [4, 5]. The proper simulation of friction is extremely important for success in rubber contact analysis, but will not be discussed in details here. One promising computational model can be found in Ref. [6].

2.6 Automated solution strategies

A good programme should offer a well-balanced combination of sophisticated analysis code, integrated seamlessly with an easy-to-use Graphical User Interface. The core of a typical design process encompasses three phases: preprocessing of data, solution, and postprocessing.

In the *preprocessing phase*, besides the data required in a typical linear analysis, a user must specify certain nonlinear analysis controls (analysis procedures, contact control parameters, convergence controls, etc.) and additional material properties required for a nonlinear analysis.

In the *solution phase*, the key difference between nonlinear and linear finite element analysis is that the solver performs the analysis in load *increments*. Within each increment, the code seeks a solution by *iteration* until equilibrium is achieved. A modern code automatically increases or decreases the step size in order to achieve a *converged* solution using a *minimum* number of increments. The objective of a successful nonlinear analysis is therefore to obtain an *accurate, converged solution* at the least cost.

Optimal *postprocessing phase* should consider as much as possible graphics capabilities including a wide array of geometry modelling features and a variety of mesh generation capability, both in 2-D and 3-D.

3. CURRENT TRENDS AND FUTURE RESEARCH

Nonlinear finite element analysis of elastomers has come a long way in the past twenty five years. Previous difficulties in the 1970-1985 period with handling complex contact boundary conditions have now been solved; recently, significant progress also has been made in 2-D and 3-D automated adaptive meshing, and these automated procedures are now being used in the design/analysis of rubber components.

Areas which still require further research and development include: (a) adaptive meshing and rezoning for nonlinear finite element analysis (especially for 3-D problems); (b) coupling of design optimization methods with nonlinear finite element analysis; (c) methods for dealing with crack or void initiation and propagation in elastomers; (d) improved modelling of friction effects; (e) material instabilities - e.g., surface folds and wrinkling; (f) viscoelastic effects in filled rubbers; (g) improved plastics and other polymer models (to model large elastic as well inelastic deformations); (h) coupled processes involving interaction between mechanical, chemical, thermal, and electrical phenomena.

4. CONCLUSIONS

This paper presents in short information and findings of the MARC Analysis Research Corporation Report on the nonlinear finite element analysis of elastomers [2].

One should have in mind that the finite element analysis of elastomeric or viscoelastic structures, especially nonlinear one, is a non-trivial task. Some of the following questions should be asked when selecting a code for rubber finite element analysis:

- Does the finite element code contain the proper material models? Which is the proper material model?
- Are there suitable finite elements for incompressible analysis?
- Does the code have modern automated contact analysis capabilities?
- Does the code offer the best choice of elements, material models, solution algorithms, and convergence criteria for your situation?
- Does the code developer possess examples and verification problems similar to your situation?

After the code has been selected, the user should bear in mind that there are other additional considerations which help to ensure success, like: mesh refinement, specification of the incremental load schedule, tolerance selection in the convergence criterion used, and so forth. These "tricks of the trade" come with experience, and very often they mean the difference between success and failure.

As can be seen, the modelling of real world rubber parts is often complicated by a lack of good material data, boundary conditions, and knowledge of the actual field service conditions. Finally, professional engineering judgment must be applied to interpret the numerical simulation results.

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Numerical Study on Interface Modelling in Metal-Matrix Composite Materials

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Abstract: Estimation of the interphase (the layer between matrix and reinforcement) properties in metal-matrix composites (MMC) is proposed in this paper. Composites with continuous unidirectional SiC fibres and titanium matrix have been considered. The mechanical properties of the interphase are crucial for the behaviour of the composite. To investigate processes performed, different models of the fibre/matrix interaction have been studied. The main attention has been paid to the fracture model with friction which successfully described the evolution of crack growth for the pull-out test. As the residual thermal stresses are critical for the MMCs properties, a simulation of the technological process of the composite creation has been performed. Some preliminary results of phase transition are shown in the presentation.

1. Introductory remarks

In the recent years composite materials become more and more popular and find the application in different kinds of industry. The general definition of this group of materials says, that composites consist of at least two components (phases): the matrix and the reinforcement. This definition includes also engineering and natural materials like for example reinforced concrete, several kinds of wood or specially prepared materials. In the production of the composite, the matrix and the reinforcement are mixed together, and it is important to ensure a proper connection between separated phases. A physical-chemical, thermodynamic and kinetic agreement must be fulfilled. The properties of composite material can be changed by controlling the phenomena that influence the process of fabrication.

There are three basic types of composite materials:

- dispersion-strengthened (the particle diameter 0.01-0.1 μm),
- particle-reinforced (the particle diameter 0.1-1.0 μm),
- fibre (whisker)-reinforced (the fibre length 0.1-250 μm).

The reinforcement should be uniformly distributed in the matrix. However, three types of composite materials can be distinguished concerning the matrix material:

- ceramic,
- polymeric,
- metallic.

In our consideration, restriction is made to the metal-matrix composites (MMC) with fibre reinforcement.

2. Selected properties and applications

The fibre volume fraction of the considered composites ranges from several up to 70% and its degree of anisotropy depends on the fibres orientation. Basically, the fibres have to carry loads, while the matrix transfers and distributes the load to the fibres. Assuming high interface efficiency, the mechanical properties of the composite depend more on the properties of the fibre than those of the matrix.

MMCs have several advantages that are very important for their use as structural materials, [6]. These advantages include a combination of the following properties: high strength, high elastic modulus, high toughness and impact properties, low sensitivity to temperature changes or thermal shock, high surface durability and low sensitivity to surface flaws, high electrical and thermal conductivity, high vacuum environment resistance, low density.

Due to high cost and production difficulties the using of MMCs is restricted to cost-effective applications, where the strength-stiffness savings outweigh the increased cost.

Composites are considered to be more reliable high-temperature material system than any other type of materials. The application of MMCs as various components of engines in space vehicles is primal. They are also predicted to be applicable as turbine blades, sealing elements in turbines and compressors, electrical conductors, cutting tools or sliding parts.

3. Mechanical problems

Some mechanical problems that should be solved when thinking of proper mechanical description are discussed in this chapter. The difficulties arise from: physical nonlinearity of materials (plasticity, ductility), roughness interaction between the fibre and the matrix, strain localisation, fracture, crack propagation, friction and large deformation. When purely looking at the mechanical properties of the interface one can conclude that it differs from both the matrix and the fibre, however it is not the separate phase.

The shear strength of the fibre/matrix interface is a key property when investigating the micromechanical behaviour of composites, since it is a measure of integrity of the interface. Accurate evaluation of this parameter has until now been very difficult although there has been keen interest for many years.

The reason for the difficulties in estimating the mechanical properties of composites arises also from the fact that the properties are created on micromechanical level in complicated chemical processes. In this situation the restriction to pure mechanical field can lead to inadequate description of the processes performed in MMCs.

There are many experimental tests developed for investigating the composites behaviour, specially for determining properties of the interfacial region (or interphase). Gray [1] lists sixteen different tests of varying complexity and concludes that of these only six, involving pulling the fibre from the matrix, come close to a satisfactory test method. Among them, there is pull-out test which has been one of the most commonly used tests for ceramic matrix composites, now successfully adopted for other composite materials.

In our paper, only thermo-mechanical formulation of MMCs behaviour is assumed. The phase transition allowing for evaluating the amount of residual stresses is conducted.

4. Pull-out test

The single-fibre pull-out test is employed experimentally to model the failure of fibre-reinforced composite materials. In this test the fibre embedded in the matrix is pulled out of the matrix, Fig. 1. The response of the fibre/matrix system to axial loading depends on properties of the fibre, the matrix and the interface.

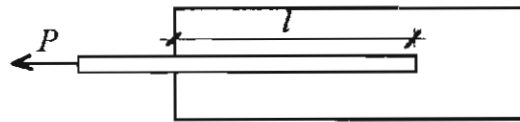


Figure 1. Scheme of pull-out test

If the fibre is bonded at the interface and if the load on the fibre required for interfacial crack growth is less than the load for fibre failure, then the fibre debonding will occur. Fibre debonding will continue until the fibre is completely debonded or until the load required for fibre failure is reached. Interface properties are typically deduced from the relationship between the peak loads and the embedded lengths and from the applied load versus embedded length relationship during pullout.

Theoretical analyses dealing with the stress states and the debond propagation in various loading geometries are essential to understand how and to what extent the interface properties influence the mechanical/structural performance and fracture behaviour of the composites. Usually, the evaluation of micromechanical test methods is based on a strength criterion: if the applied load on an embedded single fibre reaches a critical value, the interface fails. If the maximum applied load is known and the embedded fibre surface, a strength value may be calculated. It can be shown with finite element method that the stress distribution along the loaded fibre is generally very inhomogeneous. Therefore, the computation of an 'interfacial shear strength' based on the assumption of a homogeneous shear stress in the interface is only a rough estimation.

It should be clear that the shear and radial stresses around the fibre strongly depend on the fibre diameter, the embedded length of the fibre, the stiffness ratio between the fibre and the matrix and a fibre volume fraction. There is also a great influence of thermal residual stresses, caused by the mismatch between expansion coefficients of the fibre and the matrix and Poisson's effect. A significant effort should be put into proper characterization of the interface properties as they influence composite damage and rupture properties.

Most micromechanic theories are developed basing on a shear-lag theory. The basic assumption of this theory is that all the tensile loads are carried by the fibre and all the shear loads are carried by the matrix, although in practice the matrix surrounding the fibre also sees some tensile stresses.

The problems and formulations listed above, and in particular the study on numerical modelling of the interface behaviour in MMCs are in focus of our interest.

5. Interface

The interface governs the mode and extent of load transfer between the fibre and the matrix. It also affects the damage initiation and the way of cracks propagation. The properties of the interface are strongly influenced by physical-chemical treatment in the preparation process and by residual stresses due to the process of technology of MMCs. The effect of the thermal residual stresses on the stress state of the interface is crucial for the global mechanical properties of the composite.

The stress transfer across the fibre/matrix interface and the interface debond process are phenomena of vital importance in fibre composite technology, particularly in the neighbourhood of a fibre break or matrix crack. It is obvious that attractive mechanical performance of MMCs depends on well-adjusted bond strength and sliding resistance of the fibre/matrix interface. The bond strength is usually described by the interface toughness. It is proved that interface debonding, which is necessary for non-catastrophic

failure, depends on low values of the interface toughness. When debonding occurs, composite failure is dictated by frictional sliding at the fibre/matrix interface during progressive debonding and pulling the fibre out of the matrix.

The micromechanics representations of the microdetails of fibre-reinforced composites have generally assumed the existence of two phases, namely the fibre and the matrix. However in reality, an additional phase may exist between the fibre and the matrix. This phase, commonly known as the interphase, is a junction that results when the matrix bonds with the fibre surface or the fibre sizing. Its incorporation into the micromechanical analyses of composite systems is critical to the understanding of composite behaviour. The properties of the region depend directly upon the chemical, mechanical and thermodynamical nature of the bonding process between the matrix and the fibre material. As a direct result of this dependence, the interphase often has spatially nonuniform properties, i.e., the properties vary from point to point along the thickness of the interphase. In [2] the third phase was introduced. Here the interphasial coefficient of thermal expansion has been chosen to remain constant and Young's modulus varies across the thickness of the interphase of T300/Ni/Al 6061 composite. Parametric study was conducted to demonstrate how the interphase thickness, the fibre volume fraction and the functional form of interphase Young's modulus variation affect the local thermal stresses. Large variations were observed.

The influence of roughness interaction between fibre and the matrix during pull-out is calculated in [3]. The composite of SiC fibre (SCS-6) and aluminosilicate glass matrix ($\text{SiC}_2\text{-Al}_2\text{O}_3\text{-B}_2\text{O}_3$) was investigated. The authors proved that pull-out phenomenon cannot be explained based on an assumption of perfect cylinders. They assumed frictional theory of tribological fundamentals which fits the experimental data very well. The proposed 'rough surface model' enables direct calculation of the interfacial clamping stress. Here the separation of fibre and matrix during fibre sliding equals the roughness amplitude. The solution of ideal axisymmetric elastic separation of cylindrical bodies was applied to quantify that model. It should be noted that the total separation of the fibre and the matrix due to roughness interaction is lower than the roughness amplitude because of statistical distribution of the profile asperities and the large deformation of the asperities in contact. As the result of numerical calculations, for certain data, the debonding plateau and the coefficient of friction were evaluated. The problem of rough interaction between two bodies with aspect of asperities was carefully discussed by Mróz, [4].

The thorough discussion on the interphase properties is crucial when thinking of numerical modelling. In particular it helps to choose the proper model of interface behaviour (the type of interaction of between matrix and reinforcement) and to accept its parameters. The numerical experiments show that qualitatively consistent results can be obtained for different models of the interface behaviour (like spring type interaction, plasticity contact with friction and fracture), [5]. The main difficulty is to predict the parameters of the models which results from the technology of the composite creation. What can be now proposed is the estimation of residual stresses in MMCs and in particular in the interface as a result of phase transition which is a critical step in the process of composite creation. Some preliminary results will be discussed.

6. Numerical models

The single-fibre model of a cylindrical shape, containing the fibre embedded in the matrix is assumed. It is represented by 2D symmetric net of 8-node plane strain continuous elements with reduced integration. The total number of elements is 302 with 30 and 10 elements in longitudinal and lateral (or radial) directions, respectively and two elements

more from both sides of the fibre. Such a mesh ensures sufficient resolution and thus accuracy of the results at a reasonable time needed for computation. This is a rough estimation for the need of the analyses considering interaction between the fibre and the matrix. It is assumed that large deformation is possible and the pulling process is static under isothermal conditions. The fibre is subjected to a stepwise increment of the displacement at the free end; the total displacement $u=5\mu\text{m}$ (it makes 1/12 of the fibre length). As the simplification, the fibre and the matrix are assumed to be perfectly elastic and isotropic with elastic constants $E_f=393\text{ GPa}$, $\nu_f=0.25$ and $E_m=113.7\text{ GPa}$, $\nu_m=0.3$, respectively, where E is Young's modulus and ν is Poisson's coefficient. The geometry and the boundary conditions are shown in Fig. 2. The boundary conditions are imposed such that the upper edge is constrained, so there is no vertical (or radial) displacement, and the lower edge is fixed. The fibre radius a and the matrix radius b are related to the fibre volume fraction $V_f=a^2/b^2$. Here $a=2\mu\text{m}$, $V_f=0.01$ (such a low volume fraction allows to neglect interactions between neighbouring fibres), so b is equal $20\mu\text{m}$. The length of the embedded fibre is $l=60\mu\text{m}$.

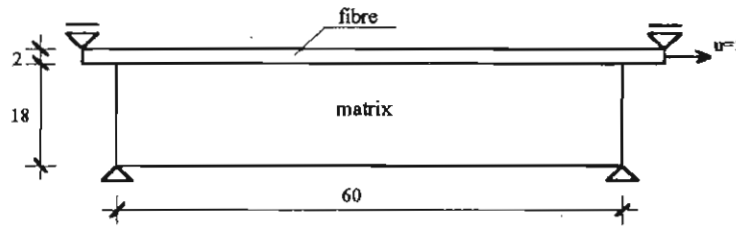


Figure 2. Geometry and boundary conditions assumed in the simulation

Some different models of the interface with additional or modified data, comparing with those given above, have been prepared. These models were described in [5]. Three of them have been chosen for further investigation. First model enables to introduce Coulomb's friction at the contact surfaces of the fibre and the matrix. The coefficient of friction is assumed to have the value of 0.2. The second one introduces spring elements at the fibre/matrix interface. The behaviour of the specimen with springs of different characteristics was analysed. Generally, the horizontal springs are non-linear and the vertical ones are linear. The usage of that kind of elements allows to observe the growth of a shear force at the interface and the relative displacement of the fibre and the matrix. The last model allows for implementing a failure criterion. Initially bonded surfaces of the fibre and the matrix may debond after fulfilling the critical stress criterion which is calculated as follows:

$$f = \sqrt{\left(\frac{\sigma_n}{\sigma_f}\right)^2 + \left(\frac{\tau_1}{\tau_1^f}\right)^2 + \left(\frac{\tau_2}{\tau_2^f}\right)^2}, \quad \sigma_n = \max(\sigma, 0)$$

where σ is the normal component of stress carried across the interface at the distance specified, τ_1 and τ_2 are the shear stress components in the interface, and $\sigma_f, \tau_1^f, \tau_2^f$ are normal and shear failure stresses which must be specified by the user. The crack-tip node debonds whenever the fracture criterion reaches the value of 1.0.

The debonding is associated with a crack propagation along the fibre surface. This phenomenon was observed for two kinds of boundary conditions. First was described above. The other one enables the bottom edge of the matrix to move in the direction of applied load, except one point at the bottom right corner of the model.

The last model seems the most appropriate for modelling the effects performed at the interface during the pull-out test. Here, a decrease of the force needed for continuation of the debonding is achieved during the debond process. This model enables the debonded surfaces to be in contact again, with frictional slip, and to carry some stresses. Figure 3 shows details of the deformed shapes of that model for the case without and with the residual stresses, respectively. The evaluation of relative horizontal and vertical displacements of the fibre and the matrix for the case without residual stresses are shown in Fig. 4.

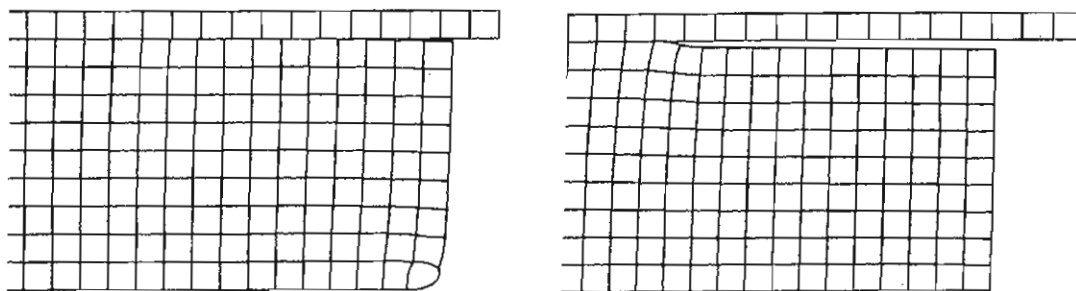
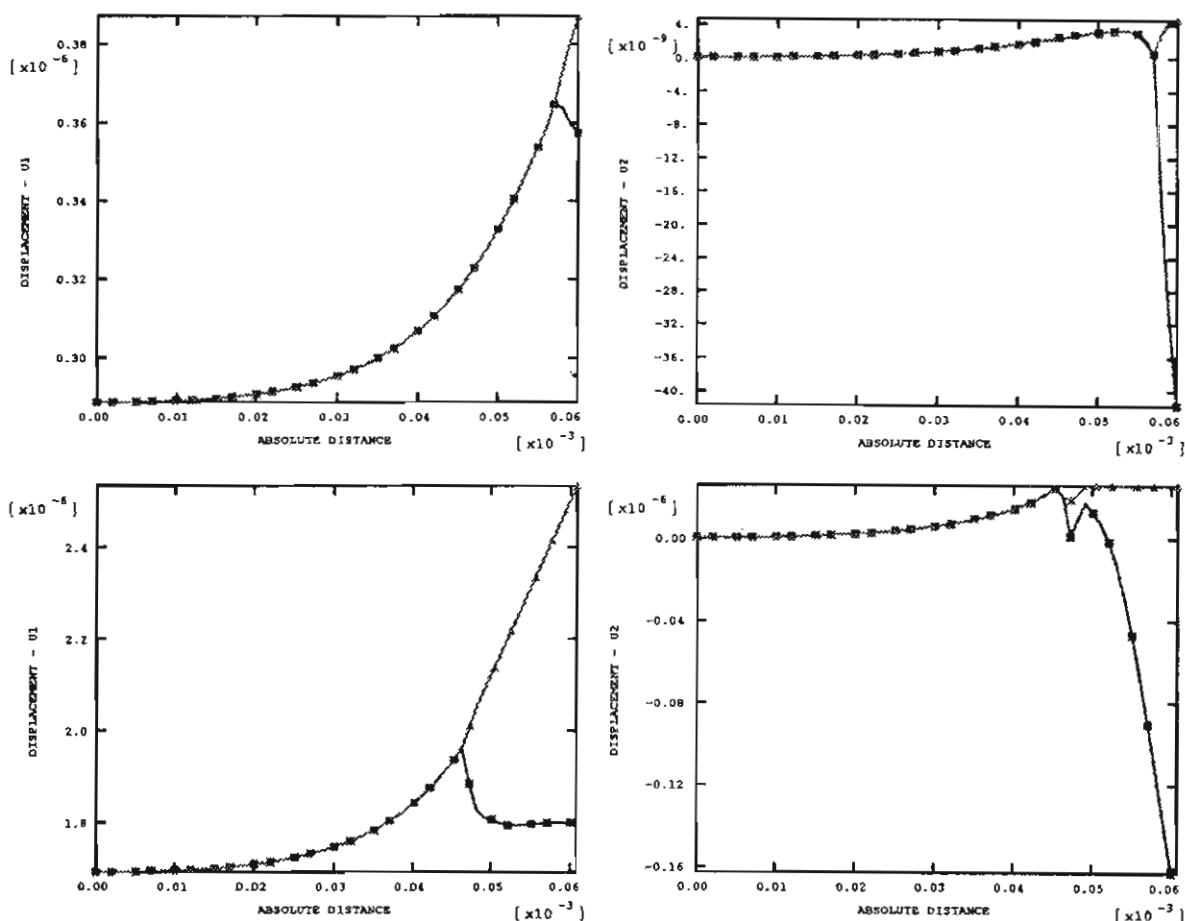


Figure 3. Detail of deformed shape for the case without and with residual stresses (fibre $\triangle-\triangle$ and matrix $\square-\square$)

It can be seen that for certain deformations a part of the crack closes, so the contact with friction may be reasonably included in computations. In our case, the coefficient of friction has the value of 0.2, but as the materials are assumed linear-elastic and there is no residual stresses we can conclude that the friction does not play a significant role.



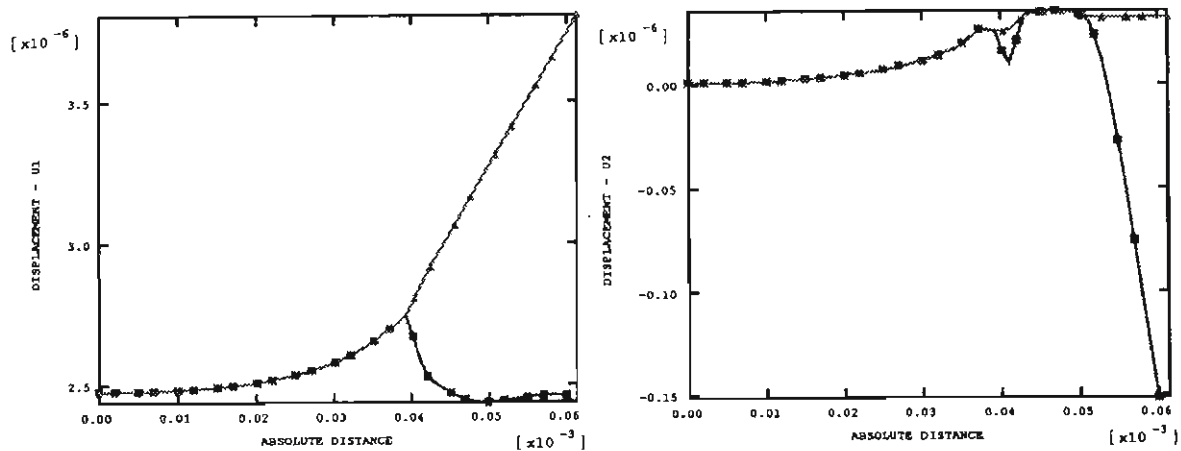


Figure 4. The evaluation of relative horizontal (left column) and vertical (right column) displacements of the fibre and the matrix

Another model enables to estimate the amount of thermal residual stresses due to phase transition of the composite materials. Here cooling from 1900°C to 20°C was set. Some specific data taken in calculation are as : for SiC - specific heat 1190 J/kg·K, density 6000 kg/m³, conductivity 25 W/m·K, expansion coefficient 3.9907e-6 1/°C and for titanium - specific heat 540 J/kg·K, conductivity 19 W/m·K, expansion 9.44e-6 1/°C, density 4530 kg/m³.

In Fig. 5 the distribution of residual stresses after the phase transition is shown (expected stress concentration regions are visible).

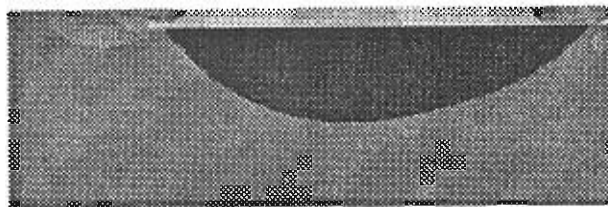


Figure 5. Distribution of stresses after the phase transition

7. Conclusions

The pull-out test model with single fibre has been studied for evaluation of mechanical properties of fibre/matrix interface. The model allowing for implementing of fracture criterion was analysed. The nucleation and propagation of a crack was observed. Also the problem of fibre pullout with frictional sliding at fibre/matrix interface was studied; Coulomb friction low was introduced. It has been demonstrated that introducing the friction for the case with critical stress criterion is insignificant in our calculations. The thermoelastic analysis was assumed and applied to the prediction of the residual stresses. In all cases the materials were assumed elastic.

Acknowledgement

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Analaysis of Time-dependent Response of Composite Structures

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Summary

The paper deals with the procedure of numerical computational analysis of time-dependent response and redistribution of internal forces of a composite plane beam structure due to the rheology of material. Principles and the theoretical bases of the developed computational method and the elaborated software are presented. Concrete creep is taken into account according to the linear theory of creep. The influence of concrete ageing and shrinkage is, similarly to the influence of concrete creep, considered by the adequate constitutive law of material.

1. Introduction

In order to make an adequate computational simulation of the behaviour of composite structures within time, it is necessary to consider also the influence of the rheology of material as well as the influence of cracks on the time-dependent response of the structure. Here, it is very important that the starting conditions of the analysis and the actual environmental conditions of the structure match as much as possible the natural conditions in which the real structure is situated.

Due to the rheology of material some extreme stresses appearing in the structure during the construction can decrease significantly after a certain time, but their influence on the strains or displacements of the structure often remains to a large extent the same. For this reason also the influence of gradual construction and the building technology have to be considered in the numerical simulation of the time dependent stress-strain relationship of structures.

The time-dependent response and the rheological redistribution of internal forces of composite structures could be considered with a numerical nonlinear analysis. In the presented computational method also the geometrical and physical nonlinearity of the structure as well as the rheology of material are taken into account.

2. Response of the Structure to the External Load

The time dependent response of the composite structure is simulated on a computer by using the finite element method. The geometrical nonlinearity of the structure is considered with adequate kinematic equations of the structure [1]. Physical nonlinearity and the rheology of material are taken into account using the adequate constitutive laws of materials. The influence of cracks on the behaviour of composite structures is taken into account by way of the constitutive equations of cross-sections [2]. The analysis of the displacements and internal forces of the structure are carried out on the basis of the adopted Bernoulli-Navier hypothesis of a linear distribution of strains across the cross-section.

3. Constitutive Laws of Materials

The time-dependent behaviour of the concrete according to the linear theory of concrete creep can be modelled in the accordance with the well known constitutive law of concrete in its integral form:

$$\varepsilon(t, t_1) = \sigma(t_1)\phi(t, t_1) + \int_{t_1}^t \dot{\sigma}(\tau)\phi(t, \tau)d\tau + \varepsilon_k(t), \quad (1)$$

where, t is the time of analysing the structure, t_1 the time or the age of the concrete at the moment of the first loading of the structure and τ the current variable of time. The concrete creep and ageing are taken into account by the function of creep $\phi(t, \tau)$. $\varepsilon_k(t)$ is the strain due to shrinkage of the concrete in time t . Assuming that the stresses within the individual time intervals are changing linearly, the constitutive law of concrete (1), with the numerical integration according to the trapezoidal rule, can be transformed into the algebraic form (2).

$$\sigma(t) = \frac{\varepsilon(t, t_1)}{A_{t,t}} - \frac{1}{A_{t,t}} \left[\sum_{j=1}^n B_{j,t} \sigma(t_j) + \varepsilon_k(t) \right] \quad (2)$$

Coefficients $A_{t,t}$ and $B_{j,t}$ are defined by making use of the creep function $\phi(t, t_j)$ with the equations:

$$\phi(t, t_j) = \frac{1}{E_c(t_j)} + \frac{\phi(t, t_j)}{E_{c28}}; \quad A_{t,t} = 0.50[\phi(t, t_n) + \phi(t, t)]; \quad B_{j,t} = 0.50[\phi(t, t_{j-1}) - \phi(t, t_{j+1})] \quad (3)$$

were, t_j is the beginning of an individual time interval, so that t_1 is the beginning of the first and t_n the beginning of the last or the end of the last but one time interval considered [3]. According to the Bernoulli-Navier hypothesis the strains can be expressed by the elongation $e_0(t, t_1)$ and the curvature ($1/\rho = \beta_{,x}$) of the system axis of the finite element [2,7].

$$\varepsilon(t, t_1) = e_0(t, t_1) + z\beta_{,x}(t, t_1) \quad (4)$$

Considering the Bernoulli hypothesis (4) and the constitutive law (2) the concrete stress in time t at the point with co-ordinate z can be expressed with the following expression:

$$\sigma_c(t, z) = \frac{1}{A_{t,t}} \left[e_0(t, t_1) + z\beta_{,x}(t, t_1) - \sum_{j=1}^n B_{j,t} \sigma(t_j) + \varepsilon_k(t) \right] \quad (5)$$

For the reinforcing steel and profile steel of composed structures, a bilinear stress-strain diagram is taken into account. Following Goldberg's proposal, the modelling of the nonlinear stress-strain relationship of prestressing steel is considered [2].

4. The Internal Forces of the Cross-section

The constitutive equations of the cross-section as the relationships between the internal forces, the elongation and the curvature of the element axis are obtained by the integration of the stresses through the whole composite cross-section consisting of concrete, reinforcing steel, prestressing steel and profile steel [4].

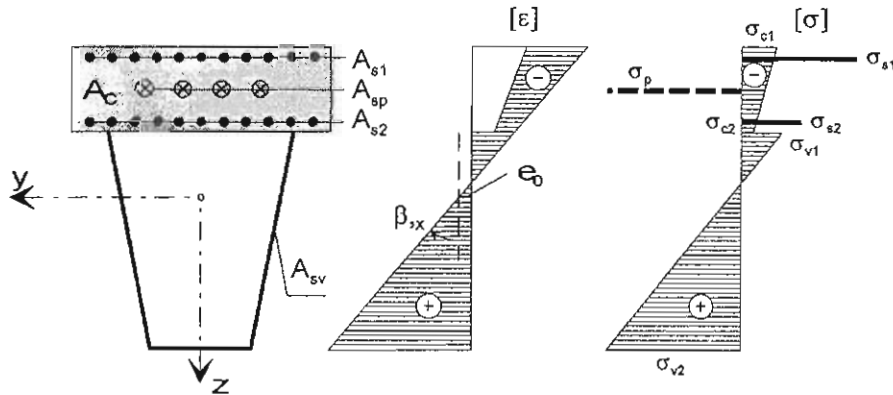


Fig. 1: Strain and stress distribution through the cross-section

Internal forces as indicated by the equations (6) are considered to be the partial contributions of concrete, reinforcing steel, prestressing steel, and profile steel, respectively.

$$\begin{aligned} N_x(t) = \sum_{i=1}^4 \int_{A_i} \sigma(t, z) dA = & \int_{A_c} \sigma_c(E_c; \varphi; e_0; \beta_x; z; t) dA + \int_{A_s} \sigma_s(e_0; \beta_x; z; f_{sy}; \varepsilon_{sy}) dA \\ & + \int_{A_{sp}} \sigma_s(f_{sp}; f_{sps}; \varepsilon_{sp}; \varepsilon_{sps}; e_0; \beta_x; z; \frac{\partial \sigma_{sp}}{\partial \varepsilon_{sp}}; t) dA + \int_{A_{sv}} \sigma_{sv}(f_{sv}; \varepsilon_{sv}; e_0; \beta_x; z) dA \end{aligned} \quad (6)$$

$$\begin{aligned} M_y(t) = \sum_{i=1}^4 \int_{A_i} z \cdot \sigma(t, z) dA = & \int_{A_c} z \cdot \sigma_c(E_c; \varphi; e_0; \beta_x; z; t) dA + \int_{A_s} z \cdot \sigma_s(e_0; \beta_x; z; f_{sy}; \varepsilon_{sy}) dA \\ & + \int_{A_{sp}} z \cdot \sigma_s(f_{sp}; f_{sps}; \varepsilon_{sp}; \varepsilon_{sps}; e_0; \beta_x; z; \frac{\partial \sigma_{sp}}{\partial \varepsilon_{sp}}; t) dA + \int_{A_{sv}} z \cdot \sigma_{sv}(f_{sv}; \varepsilon_{sv}; e_0; \beta_x; z) dA \end{aligned}$$

The increments of internal forces of cross-sections $\Delta N(\Delta t_i)$ and $\Delta M_y(\Delta t_i)$ which occur in the time interval Δt_i due to the rheology of material or due to the change of external load can be expressed by means of the matrix of tangential stiffnesses of the cross-section in the previous time interval and by the increments of elongation Δe_0 and curvature $\Delta \beta_{,x}$ of the system axis of the element.

$$\begin{Bmatrix} \Delta N_x \\ \Delta M_y \end{Bmatrix}_j = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{i-1} \cdot \begin{Bmatrix} \Delta e_0 \\ \Delta \beta_{,x} \end{Bmatrix}_i \quad (7)$$

The elements of the stiffness matrix in the previous interval (i-1) are determined by the Equations (8), where j is the number of the basic constitutive materials of the cross-section (concrete, reinforcing steel, prestressing steel, profile steel).

$$K_{11} = \sum_{j=1}^4 \int_{A_j} \frac{d\sigma_j}{d\varepsilon} dA_j; \quad K_{12} = K_{21} = \sum_{j=1}^4 \int_{A_j} z \cdot \frac{d\sigma_j}{d\varepsilon} dA_j; \quad K_{22} = \sum_{j=1}^4 \int_{A_j} z^2 \cdot \frac{d\sigma_j}{d\varepsilon} dA_j \quad (8)$$

5. Kinematic and Equilibrium Equations of the Structure

The finite element method is used for the purpose of computational modelling of the response of the structure within arbitrary time, by taking into account the rheology of material. Along with adequate kinematic equations of the finite element also the geometrical nonlinearity of the structure is considered. The equilibrium of the finite element is defined by the stationary value of the potential energy of the system. Deriving from the kinematic and equilibrium equations of the finite element, the equations of the whole structural system are formed according to the geometry of the structure [5]. Nonlinear equations of structure are iteratively solved step by step with the Newton-Raphson method:

$$[\bar{K}]_s \cdot \{\Delta U\}_{s+1} = \{F\} - \{\bar{S}_r\}_s \quad (9)$$

Here, s is the iteration step, $\{F\}$ the vector of the given node load, $\{\bar{S}_r\}_s$ the vector of condensed boundary forces in the s-th iteration, $\{\Delta U\}_{s+1}$ the correction of the vector of nodal displacements in the iteration s+1, and $[\bar{K}]_s$ the tangential stiffness matrix of the structure in the s-th iteration. In the presented computational method the beam finite element P_4 was used. The curvature of the system axis of that element was approximated by means of the polynomial of the fourth degree. The numerical integration along the finite element axis is carried out by the Lobatto integration formula.

6. Computer Program

The influence of the gradual construction of structures on their stress-strain relationship could be considered through the changing of structural system during the analysis by adding or taking away whole elements or individual subsections of cross-sections. Time

changing of the structural system and the formulation of adequate equations of the structure is managed by a special program module. The influence of the rheology on the time response of the composed structure is considered by means of adequate constitutive laws of materials, or constitutive equations of composed cross-sections.

7. Computational Example

A composite bridge presented in Fig. 2 with spans of 40.0 + 60.0 + 40.0 m was analysed with the prepared software for the prediction of the behaviour of structures. The analysis takes into account concrete ageing, shrinkage and creep, relaxation of the prestressed steel and the influence of gradual construction. A part of the obtained results is presented on figures 3 to 6.

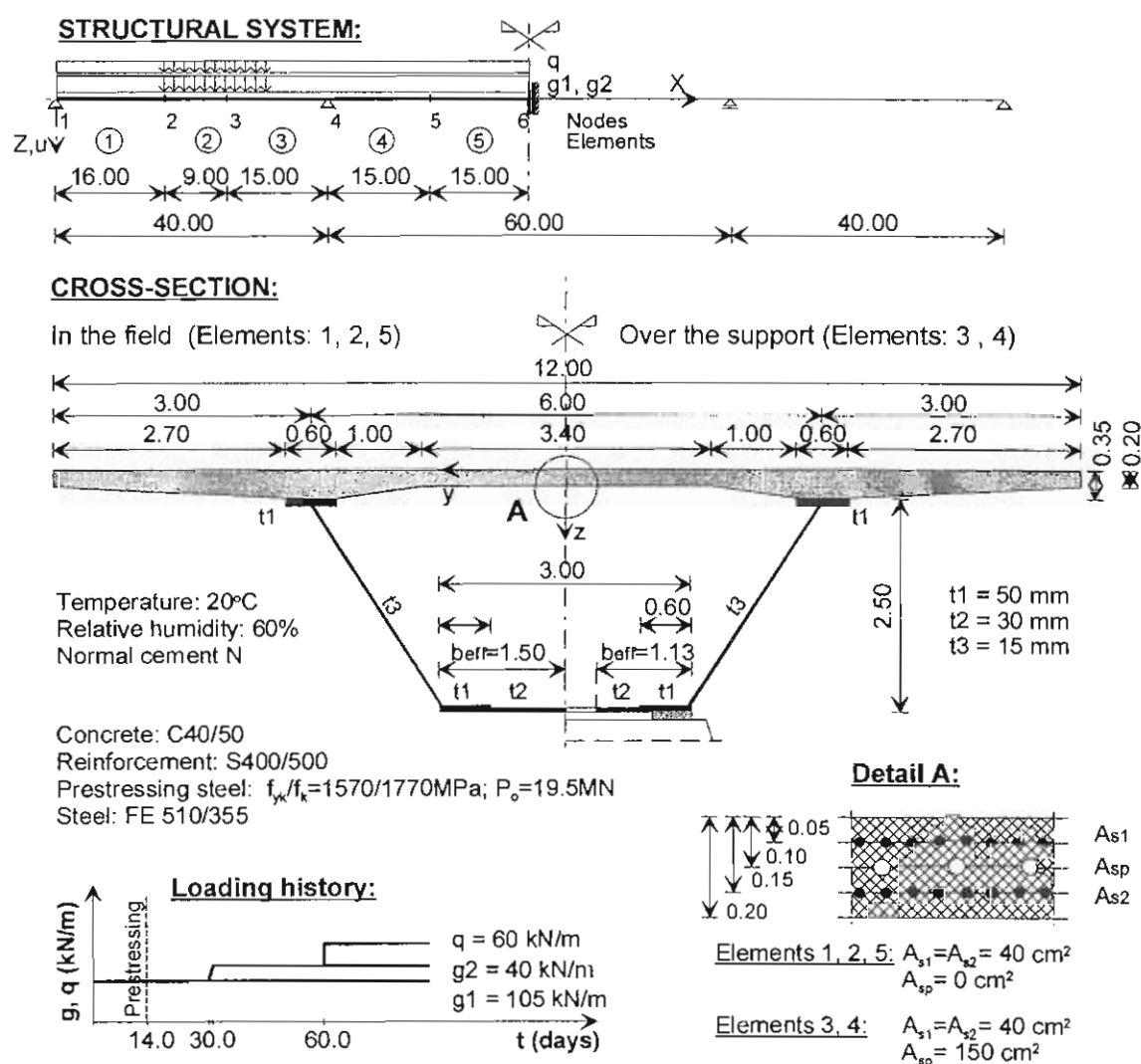


Fig. 2: Computational example - composite box girder

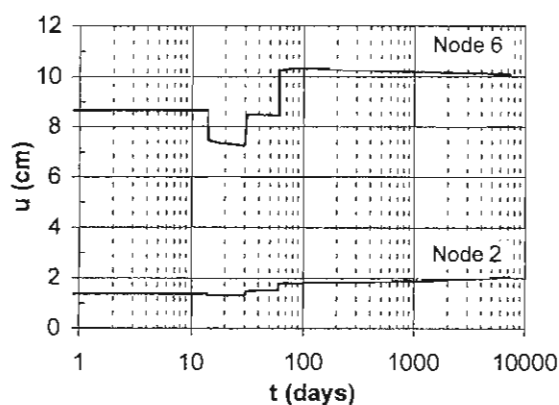


Fig. 3: Deflections time-history for the nodes 2 and 6

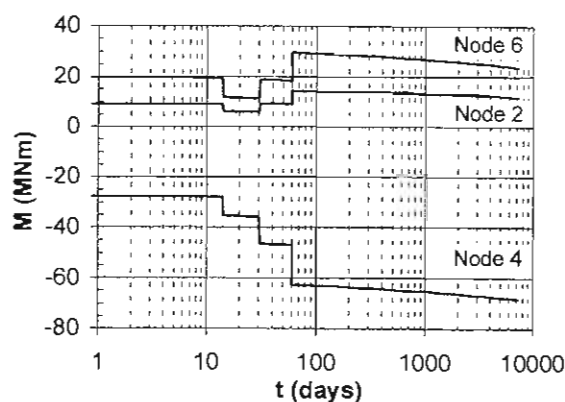


Fig. 4: Bending moment time-history for the nodes 2, 4 and 6

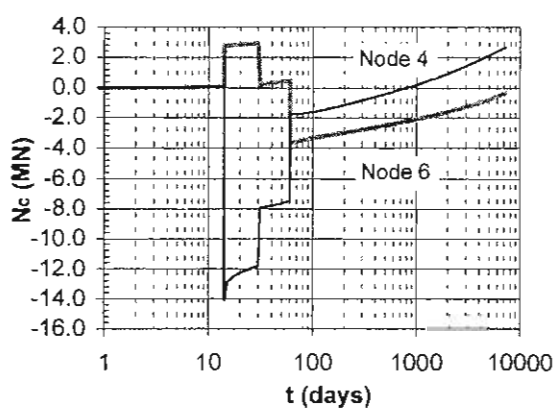


Fig. 5: Force in the concrete time-history for the nodes 4 and 6

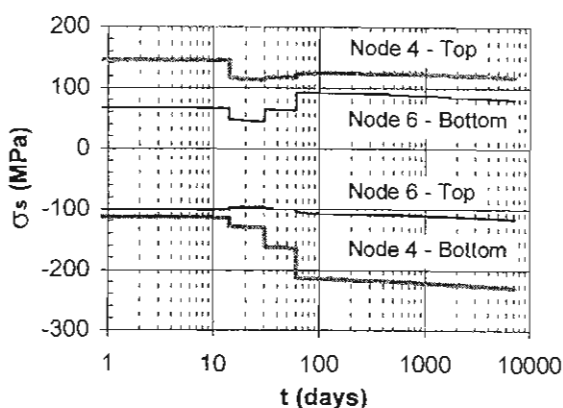


Fig. 6: Profile steel stress time-history for the nodes 4 and 6

8. Conclusion

Using the developed computational method and the elaborated software the time dependent response of composite structures can be simulated. In this way also the influence of gradual construction and arbitrary history of the loading can be considered. If concrete creep is considered according to the non-linear theory and other material non-linearities are additionally included [6], the computational method and the software can be also used for the analysis of the response of the structure in the state of ultimate bearing capacity.

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AN ANALYSIS OF DAMAGED LAMINATES WITH MATRIX CRACKING UNDER IN-PLANE LOADS

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ABSTRACT

Mori - Tanaka's averaging scheme has been employed on the evaluation of the compliance and stiffness of the laminated composites. The overall in-plane properties of the laminate are calculated from those of the laminae. Lamina stresses are related to the stress averages in the constituents by means of the stress-concentration factors defined using Mori-Tanaka's approach. Lamina stress concentration factors have been defined providing the relation between stress tensor in the laminae and laminate. Drucker-Prager failure criterion has been applied on the both constituents thus enabling the identification of different failure modes of which the most important are slit matrix cracks. The stiffness loss due to the appearance of slit crack damage has been taken into account using self-consistent approach based on the Eshelby's model of equivalent inclusions.

1. INTRODUCTION

The process of designing with multilayered fibrous composites includes both the structure and the material. Failure modes and damage state in general are particularly difficult to predict and numerically evaluate. It has been noticed that multilayered composites under in-plane loading show several damage modes, among which are the most important matrix cracking in the form of slit cracks and fiber breaking in the form of the penny-shaped cracks [1]. The geometry, density and distribution pattern of these has the crucial influence on the mechanical properties of the damaged solid [1], [2], [3]. In this paper, only longitudinal slit matrix cracks have been evaluated. There are several approaches in the calculation of the effective stiffness and compliance of the damaged solid [3], [4]. Self-consistent model with the introduction of damage parameter has been employed in the paper. This method takes into account the approximation that cracked solid could be substituted with the equivalent medium without cracks, but of changed material properties [2]. In the stiffness calculation of undamaged solid, Mori-Tanaka's principle is chosen due to its general applicability in the micromechanics of composites [4]. The establishment of failure criteria for fiber and matrix is to be performed separately, thus predicting the failure mode of both constituents [5]. In the case of the brittle matrix, failure appearance results in the resizing of the failure surface.

2. PRELIMINARIES

The notation used in the paper is standard in the theory of composite materials. Anisotropic linear elastic solid has been considered whose stiffness L and compliance M are relating stress σ and strain ε through the relations

$$\varepsilon = M \sigma, \quad \sigma = L \varepsilon. \quad (1)$$

Standard engineering 6×6 notation has been used in description of M and L . Matrix has been considered as brittle isotropic material. Fiber has been considered as a transversely

isotropic material with main material axis A denoting the fiber direction, and axes T and T' defining the plane of the transversal isotropy. Lower indices f and m denote fiber and matrix, respectively, while upper index i denotes i -th layer in the laminate. Stress and strain tensor in main material axes system of a transversely isotropic fiber are written as 6×1 vectors

$$\begin{aligned}\bar{\varepsilon}_f &= [\varepsilon_{TT} \ \varepsilon_{TT'} \ \varepsilon_{AA} \ \gamma_{TA} \ \gamma_{TA'} \ \gamma_{TT'}]^T, \\ \bar{\sigma}_f &= [\sigma_{TT} \ \sigma_{TT'} \ \sigma_{AA} \ \sigma_{TA} \ \sigma_{TA'} \ \sigma_{TT'}]^T,\end{aligned}\quad (2)$$

with fiber compliance written through

$$\bar{M}_f = \begin{bmatrix} 1/E_T & -\nu_T/E_T & -\nu_A/E_A & 0 & 0 & 0 \\ & 1/E_T & -\nu_A/E_A & 0 & 0 & 0 \\ & & 1/E_A & 0 & 0 & 0 \\ & & & 1/G_A & 0 & 0 \\ & Sym. & & & 1/G_A & 0 \\ & & & & & 1/G_T \end{bmatrix}, \quad (3)$$

$$\text{with } \nu_A = \nu_{AT}, \quad \frac{\nu_{AT}}{E_A} = \frac{\nu_{TA}}{E_T}, \quad G_A = G_{AT}, \quad G_T = G_{TT} = \frac{E_T}{2(1+\nu_T)}, \quad (4)$$

$$\text{giving } \bar{\varepsilon}_f = \bar{M}_f \bar{\sigma}_f, \quad \bar{\sigma}_f = \bar{L}_f \bar{\varepsilon}_f. \quad (5)$$

Stiffness and compliance of a matrix are easily defined using relations for Young's modulus, shear modulus and Poisson's ratio for isotropic material.

3. THE APPLICATION OF THE MORI - TANAKA'S SCHEME TO THE FIBROUS COMPOSITES

The aim of this part of the paper is to define proper expressions for the effective stiffness and compliance of the layered fiber composite. Approach formulated by Benveniste [4] which combines the Eshelby's equivalent inclusion principle with the Mori - Tanaka's concept of the average stress in the matrix has been used. Mori - Tanaka's concept has the advantage that direct evaluation of the effective properties is obtained by means of stiffness and compliance of the phases knowing their volume fractions. The analysis is limited to the two phase composite with aligned particles (inclusions). The fiber has been considered as a cylindrical inclusion embedded in the matrix. Perfect bonding is assumed between the two phases, with the inclusions of the similar topology. The relation between average stress tensor in the inclusion and the average stress tensor in the matrix is defined with Wu's tensor T_w^i [4]

$$T_w^i = \left[I + S (L_m^i)^{-1} (L_f^i - L_m^i) \right]^{-1}. \quad (6)$$

The effective compliance of the i -th lamina is thus defined using expressions

$$M^i = (M_f^i - M_m^i) \left\{ I + c_m^i L_m^i [(I - S)(M_f^i - M_m^i)] \right\}^{-1}, \quad (7)$$

$$W^i = L_f^i \left[I + S (L_m^i)^{-1} (L_f^i - L_m^i) \right]^{-1} M_m^i, \quad (8)$$

with Eshelby's tensor S defined by the expressions given by Gavazzi and Lagoudas [6]. In the previous expressions I is the unity tensor and value c_r^i , $r = f, m$ is the volume fraction of the phase r . Eshelby's tensor is calculated using surface integral parametrized on the surface of the unit sphere [6]

$$S_{ijkl} = \frac{1}{8\pi} L_{mnkl} \int_{-1}^{+1} d\zeta_3 \int_0^{2\pi} \{G_{imjn}(\bar{\xi}) + G_{jmin}(\bar{\xi})\} d\omega, \quad (9)$$

$$G_{ijkl}(\bar{\xi}) = \bar{\xi}_k \bar{\xi}_l N_{ij}(\bar{\xi}) / D(\bar{\xi}),$$

$$\bar{\xi}_i = \zeta_i / a_i, \quad \zeta_1 = \sqrt{1 - \zeta_3^2} \cos \omega, \quad \zeta_2 = \sqrt{1 - \zeta_3^2} \sin \omega,$$

$$D(\bar{\xi}) = \varepsilon_{mnl} K_{m1} K_{n2} K_{l3},$$

$$N_{ij}(\bar{\xi}) = \frac{1}{2} \varepsilon_{ikl} \varepsilon_{jmn} K_{km} K_{ln},$$

$$K_{ik} = L_{ijkl} \bar{\xi}_j \bar{\xi}_l$$

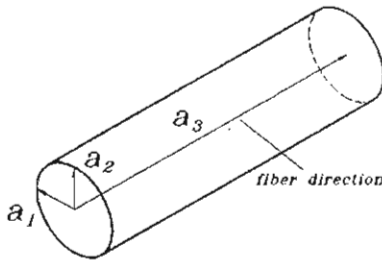


Fig. 1. Fiber geometry

In the previous expressions L_{ijkl} are components of the stiffness tensor L and ε_{klm} is the permutation tensor. Values $a_{1,2,3}$ define the aspect ratio of the fiber, with value a_3 in the longitudinal direction of the fiber, and values a_1 and a_2 defining circular cross section of the fiber (Fig.1.) Numerical evaluation of the Eshelby's tensor S has been performed using ratios $a_1 / a_2 = 1$ and $a_1 / a_3 = 10^{-40}$ (numerically corresponding to $a_3 \rightarrow \infty$) to simulate cylindrical inclusion. Gauss integration of the expression (9) is performed by the following formula

$$S_{ijkl} = \frac{1}{8\pi} \sum_{p=1}^M \sum_{q=1}^N L_{mnkl} \{G_{imjn}(\omega_q, \zeta_{3p}) + G_{jmin}(\omega_q, \zeta_{3p})\} w_{pq}, \quad (10)$$

with $M=2$ and $N=16$ number of Gauss points used for integration over ζ_3 and ω , respectively, assuring accuracy of up to the six significant digits [6].

4. STRESS DISTRIBUTION FACTORS FOR LAMINAE AND PHASES

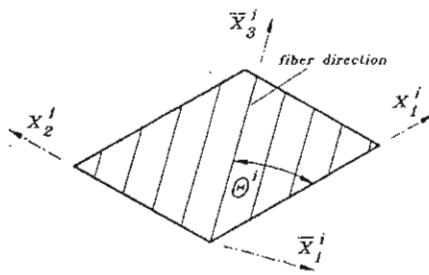


Fig.2. Coordinate systems

The overall in-plane properties of the laminate are calculated from those of the laminae, which are derived from those of the fiber and matrix of each lamina. Therefore, two coordinate systems have to be defined. At each lamina local coordinate system $\bar{x}_j^i, j=1,2,3$ is introduced, with axis $\bar{x}_3^i$ corresponding to the fiber direction. Global coordinate system $x_j^i, j=1,2,3$ is defined using angle θ^i between local axis $\bar{x}_3^i$ and global axis x_1^i , (Fig. 2.). Stress tensor for the i -th layer in the local coordinate system of the lamina is written as

$$\bar{\sigma}^i = [\bar{\sigma}_{11}^i \quad \bar{\sigma}_{33}^i \quad \bar{\sigma}_{13}^i]^T \quad (11)$$

and as

$$\sigma^i = [\sigma_{11}^i \quad \sigma_{22}^i \quad \sigma_{12}^i]^T \quad (12)$$

in the global coordinate system. The transformation relations between the two stress and corresponding strain tensors are defined through

$$\bar{\sigma}^i = T^i \sigma^i, \quad \bar{\varepsilon}^i = N^i \varepsilon^i. \quad (13)$$

The transformation matrix is defined using expression

$$T^i = \begin{bmatrix} \sin^2 \theta^i & \cos^2 \theta^i & -\sin 2\theta^i \\ \cos^2 \theta^i & \sin^2 \theta^i & \sin 2\theta^i \\ \frac{1}{2} \sin 2\theta^i & -\frac{1}{2} \sin 2\theta^i & -\cos 2\theta^i \end{bmatrix} \quad (14)$$

with T^i and N^i matrices related as

$$(T^i)^T = (N^i)^{-1}. \quad (15)$$

In-plane local compliance matrix of the lamina is derived by extraction of those elements of the 6×6 matrix $\bar{M}^i$ which correspond to the in-plane stress state

$$\bar{M}^i = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{13} & \bar{M}_{15} \\ & \bar{M}_{33} & \bar{M}_{35} \\ Sym. & & \bar{M}_{55} \end{bmatrix}. \quad (16)$$

The transformation from local lamina coordinate system to global coordinate system is performed through the expressions

$$M^i = (T^i)^T \bar{M}^i T^i, \quad L^i = (N^i)^T \bar{L}^i N^i. \quad (17)$$

Lamina stresses $\bar{\sigma}^i$ are related to the stress averages in the phases (constituents) $\bar{\sigma}^i_r$ through the expressions

$$\bar{\sigma}^i_r = B^i_r \bar{\sigma}^i, \quad r = f, m, \quad (18)$$

where B^i_r are stress-concentration factors defined using Mori - Tanaka's approach [4]

$$B^i_m = \frac{I - c^i_f W^i (c^i_m I + c^i_f W^i)^{-1}}{c^i_m}, \quad (19)$$

$$B^i_f = \frac{1}{c^i_f} (I - c^i_m B^i_m), \quad (20)$$

with W^i defined in expression (8).

If the laminate consists of n laminae, with thickness of each lamina h^i , the volume fraction of particular lamina in the laminate is defined as

$$c^i = \frac{h^i}{\sum_{i=1}^n h^i}. \quad (21)$$

The overall in-plane properties of the laminate are derived from those of the laminae applying the basic assumption that laminae are perfectly bonded

$$\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_n.$$

Lamina stress concentration factor H^i gives the relation between global stress σ^i in particular i -th lamina and global overall stress σ in the laminate [7].

$$\sigma^i = H^i \sigma \quad (22)$$

and is defined with

$$H^i = L^i (L)^{-1}. \quad (23)$$

Here L is the stiffness of the laminate. Lamina stress concentrations factors have to satisfy the relation

$$c^1 H^1 + c^2 H^2 + \dots + c^n H^n = I \quad (24)$$

leading to the expression that relates the stiffness and compliance of individual laminae with those of the laminate

$$L = c^1 L^1 + c^2 L^2 + \dots + c^n L^n, \quad (25)$$

$$M = M^1 H^1 = M^2 H^2 = \dots = M^n H^n. \quad (26)$$

Finally, the relation between global overall applied stress σ and local stresses in matrix and fiber $\bar{\sigma}_r$, are obtained using transformations

$$\bar{\sigma}_r = B_r^i \bar{\sigma}^i, \quad r = f, m, \quad (28)$$

$$\bar{\sigma}_f = B_f^i T^i H^i \sigma, \quad (29)$$

$$\bar{\sigma}_m = B_m^i T^i H^i \sigma.$$

5. THE APPLICATION OF FAILURE CRITERIA ON THE COMPOSITE PHASES

In contrast to the most composite failure criteria that are macroscopic and try to predict the general failure of the laminate, the criterion presented here is dealing with the local failure surface defined for both phases - matrix and fiber [5]. Thus, it is possible to take into account different modes of failure, where the most interesting for the future research are longitudinal cracks in the matrix, spreading parallel to the fibers. Drucker-Prager failure criterion has been used, defined in the lamina local system as

$$f(\bar{\sigma}) = \bar{\alpha} \bar{I}_1 + \sqrt{\bar{J}_2} - \bar{k} = 0, \quad (29)$$

$$\bar{I}_1 = \bar{\sigma}_{11} + \bar{\sigma}_{22} + \bar{\sigma}_{33},$$

$$\bar{J}_2 = \frac{1}{6} \left[(\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + (\bar{\sigma}_{22} - \bar{\sigma}_{33})^2 + (\bar{\sigma}_{33} - \bar{\sigma}_{11})^2 \right] + \bar{\sigma}_{12}^2 + \bar{\sigma}_{23}^2 + \bar{\sigma}_{31}^2,$$

$$\bar{\alpha} = -\frac{1}{\sqrt{3}} \frac{f_t^i - f_c^i}{f_t^i + f_c^i},$$

$$\bar{k} = \alpha f_t^i + \frac{1}{\sqrt{3}} f_c^i,$$

where f_t^i and f_c^i are tensile and compressive strength, respectively. Here $\bar{\sigma}_{ij}$ depicts components of stress tensor in the local lamina system for both phases. Failure criterion applied on the matrix and fiber of the i -th lamina written in terms of the overall applied stress σ states

$$f_i(\bar{\sigma}_m^i) = f_i(B_m^i T^i H^i \sigma) = 0, \quad (30)$$

$$f_i(\bar{\sigma}_f^i) = f_i(B_f^i T^i H^i \sigma) = 0.$$

6. EFFECTIVE PROPERTIES OF A FIBROUS COMPOSITE WITH SLIT LONGITUDINAL MATRIX CRACKS

Damage mode analyzed in the paper appears in the form of longitudinal matrix cracks parallel to the fiber direction. The theory has been established for the cracks of the generally ellipsoidal geometry which are homogeneously distributed and spread over the thickness of the lamina. In this paper the cracks are considered to have a form of a slit, i.e. one of the dimensions of the crack tends towards zero. Degradation of mechanical properties is calculated using self-consistent model of Dvorak, Laws and Hejazi based

on the Eshelby's principle of equivalent inclusions [1], [2], [3]. The loss of the stiffness due to the appearance of damage is described by means of crack density parameter for longitudinal slit cracks, β , and Hill's Λ interaction tensor of the fourth order which depends on the geometry of the cracks. Crack density parameter β measures the average distance between cracks in the lamina of the thickness $2a$, implying that when $\beta = 1$, the average distance between cracks equals $2a$. The effective compliance of the homogeneous solid with cracks is calculated by the expression

$$\mathbf{M} = \mathbf{M}^0 + \frac{1}{4} \pi \beta \Lambda \quad (31)$$

with $\mathbf{M}^0$ as the compliance of the uncracked solid. In this case it is the compliance of the lamina under investigation. The appearance of the cracks influences only the following elements of the compliance tensor $\mathbf{M}$

$$M_{ii} = M_{ii}^0 + \frac{1}{4} \pi \beta \Lambda_{ii} \quad i = 2, 4, 6 \quad (32)$$

Corresponding non-zero values of Λ interaction tensor are

$$\begin{aligned} \Lambda_{22} &= \frac{M_{22} M_{33} - M_{23}^2}{M_{33}} \left(\sqrt{\alpha_1} + \sqrt{\alpha_2} \right), \\ \Lambda_{44} &= \sqrt{M_{44} M_{55}}, \\ \Lambda_{66} &= \frac{\sqrt{M_{22} M_{33} - M_{23}^2} \sqrt{M_{11} M_{33} - M_{13}^2}}{M_{33}} \left(\sqrt{\alpha_1} + \sqrt{\alpha_2} \right), \end{aligned} \quad (33)$$

with α_1 and α_2 written as the solutions of the equation

$$\left(M_{22} M_{33} - M_{23}^2 \right) \alpha^2 - \left\{ M_{33} M_{66} + 2 \left(M_{12} M_{33} + M_{12} M_{23} \right) \right\} \alpha + M_{11} M_{33} - M_{13}^2 = 0 \quad (34)$$

Compliance coefficients changed due to the appearance of the cracks are given by

$$\begin{aligned} M_{22} &= M_{22}^0 + \frac{1}{4} \pi \beta \frac{M_{22} M_{33} - M_{23}^2}{M_{33}} \left(\sqrt{\alpha_1} + \sqrt{\alpha_2} \right), \\ M_{44} &= M_{44}^0 + \frac{1}{4} \pi \beta \sqrt{M_{44} M_{55}}, \\ M_{66} &= M_{66}^0 + \frac{1}{4} \pi \beta \frac{\sqrt{M_{22} M_{33} - M_{23}^2} \sqrt{M_{11} M_{33} - M_{13}^2}}{M_{33}} \left(\sqrt{\alpha_1} + \sqrt{\alpha_2} \right). \end{aligned} \quad (35)$$

M_{44} is evaluated directly through the following expression

$$M_{44} = M_{44}^0 + \frac{1}{8} \pi \beta \left(\frac{1}{4} \pi \beta M_{55} + \sqrt{\left(\frac{1}{4} \pi \beta \right)^2 M_{55}^2 + 4 M_{44}^0 M_{55}} \right). \quad (36)$$

The calculation of the M_{22} and M_{66} values is carried out solving the following equations

$$\alpha_1 \alpha_2 = \frac{M_{11} M_{33} - M_{13}^2}{M_{22} M_{33} - M_{23}^2}, \quad (37)$$

$$\alpha_1 + \alpha_2 = \frac{M_{33} M_{66} + 2 \left(M_{12} M_{33} - M_{13} M_{23} \right)}{M_{22} M_{33} - M_{23}^2}. \quad (38)$$

This system has been solved using Newton-Raphson method giving the compliance of the cracked solid in the local lamina system. The calculation of the effective compliance

of the lamina with cracks in the global coordinate system is performed by the previously defined transformation (17).

7. EXAMPLES

Example 1

Drucker-Prager criterion has been applied on the E-glass/epoxy composite. Tensile and compressive strength for the fiber are 3.45 and 1.73 GPa and for the matrix 58.5 and 80.0 MPa, respectively [5]. The case of the in-plane applied loading has been considered, therefore the values with the index 2 in the expression (29) are equal to zero. Figure 3. shows the failure surface for both constituents.

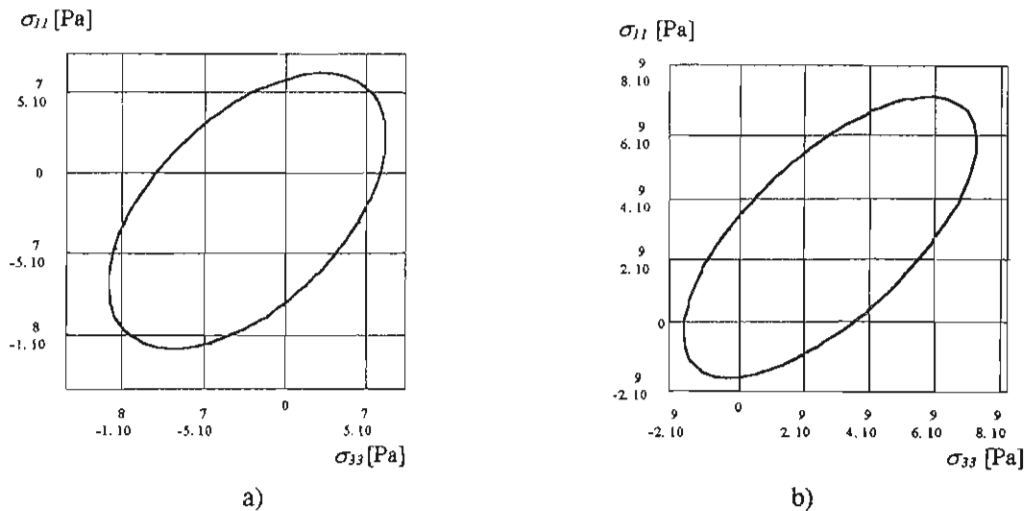


Fig. 3. Failure surface for a) matrix , b) fiber

Example 2

For the T300 graphite/epoxy composite, the degradation of mechanical properties has been calculated expressed through the increase of the compliances M_{44} and M_{66} as they are written in (35). The graphite fiber has the following material properties using notation as in expression (3): $E_A = 227.7$ GPa, $\nu_A = 0.41$, $E_T = 15.5$ GPa, $G_A = 23.2$ GPa, $G_T = 5.4$ GPa. The material properties of the isotropic epoxy matrix are defined with $E_m = 3.46$ GPa, $\nu_m = 0.35$, $G_m = 1.28$ GPa [2]. Crack density parameter β takes the value in the range from 0.0 to 1.0, and three fiber volume fractions have been analyzed: 0.2, 0.4 and 0.6 (Figure 4). It has been noticed that compliances change almost linearly with the increase of crack density parameter. The largest increase in the compliance is for the smallest fiber volume fraction.

8. CONCLUSION

The results submitted in the paper represent one stage in the research of mechanical behavior of layered composites. The presented computational approach shows the procedure for efficient evaluation of stiffness degradation due to the appearance of longitudinal slit matrix cracks. Coupled with the application of failure criteria on matrix and fiber separately, this is the basis for the application of this work on the problems of delamination of the multilayered composites in the case of the low velocity impact loading [8]. It is well known that matrix cracks in the lower layer are the indicator of the

possible appearance of the delaminations. Future work will focus onto application of the model described in the paper on the transient dynamic analysis of impact loading, in order to more accurately describe the delamination process.

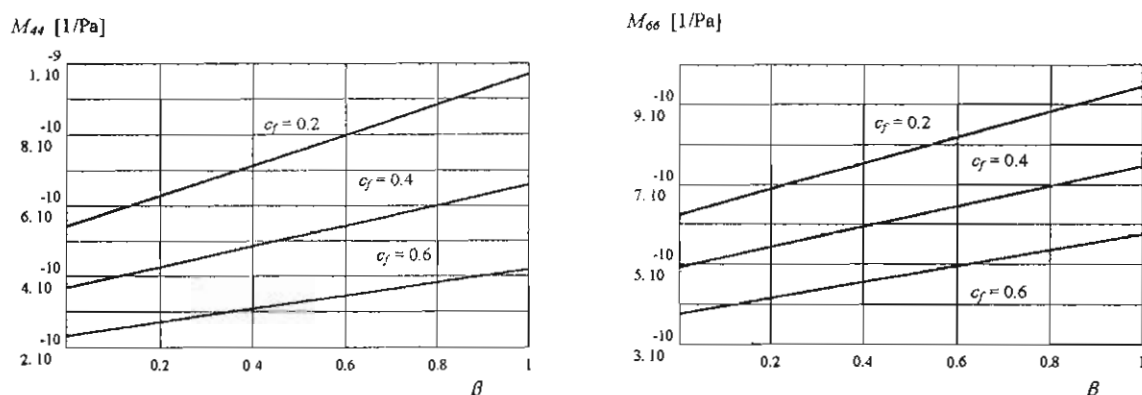


Fig. 4. Compliance degradation induced by matrix cracks

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MODELING OF COMPOSITE LAP JOINTS

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Abstract

A multilayered plate element is derived to study the problem of composite lap joints. Mathematical formulation of the element matrices originates from the refined plate theory developed by Mau. The theory relies on Mindlin's kinematic assumptions with independent approximation of in-plane displacements of each layer. The continuity of displacement field at layer interfaces is attained by imposing a continuity conditions as a set of constraints. These constraint conditions are introduced into modified variational principle through Lagrange multipliers representing the interlaminar shear stresses. Such an accurate estimation of the interlaminar shear stresses then serves to a reliable prediction of the tensile failure of surface plies of adhesively bonded composite plates.

1 Introduction

From the analysis point of view the conventional joint can be characterized as an interfacial problem on structural level. Indeed, most of the works devoted to the design of adhesively bonded joints have been concerned with a prediction of internal adhesive stresses. A number of different material models have been developed to account for inelastic deformation of adhesive to accurately estimate the shear stress transmitted between two or more adherents. Variety of treatments, such as variation of thickness of adhesive bonds, have been proposed to increase strength of the adhesive and to prevent adhesive failure.

In the present work we focused on a rather different failure mode frequently observed in adhesively bonded composite laminates, particularly, the tensile failure of the surface plies in highly stressed areas in the immediate vicinity of the edges of bonded members. The emphasis, therefore, is on accurate prediction of the distribution of laminate tensile stress in composite adherent. In the present paper this goal is achieved with help of a multilayered plate element derived in §2. Results are presented for $[0]_N$ Graphite-Epoxy laminate subjected to a shear loading.

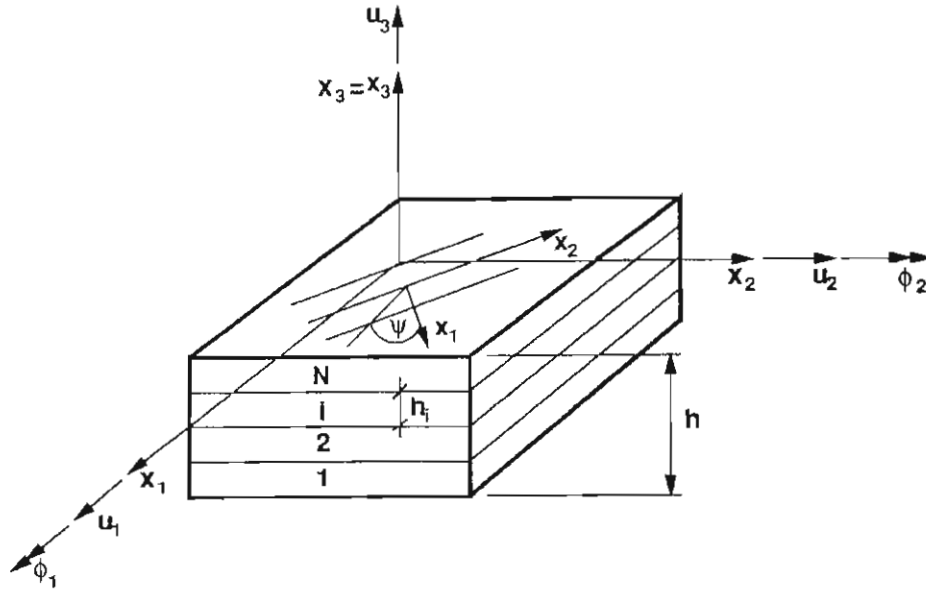


Figure 1: Representative volume element

2 Element Formulation

Consider the representative volume V of Fig. 1. The composite plate consists of N orthotropic layers with arbitrary laminate configuration. The Cartesian coordinate system is chosen such that the middle surface of the plate occupies domain S_m in X_1X_2 plane, and the X_3 axis is normal to this plane. The thickness of each layer is $h_i = c_i h$, where h represents the uniform thickness of the plate and the volume fraction, c_i , satisfies

$$\sum_{i=1}^N c_i = 1.$$

Perfect bonding is assumed between the layers; each is made of a homogeneous matrix bonded to fibers aligned in a certain direction x_1^i that contains the angle ψ_i with the X_1 -axis and serves to define a local cartesian coordinate system $\mathbf{x}_i$ with $x_3^i \equiv X_3$. The elastic symmetry of the plies is at least orthotropic; material symmetry planes coincide with the planes of the $\mathbf{x}_i$ system. The bold lower-case Greek letters $\boldsymbol{\epsilon}, \boldsymbol{\sigma}$ denote the strain and stress vectors in local coordinate system, whereas the bold upper-case italic letters $\boldsymbol{E}, \boldsymbol{S}$ represent corresponding quantities in overall $\mathbf{X}$ coordinate system. The subscript m denotes the in-plane variables as oppose to subscript s , which represents the out-of plane variables, and $(\)_{,j}$ is used to denote partial differentiation with respect to X_j .

2.1 Kinematics and constitutive equations

To obtain system of governing equations we assume that the displacement field within each ply is derived based on Reissner-Bolle-Mindlin's kinematic assumptions:

- No stretching of the material line in thickness direction.
- Normal material line remains straight after deformation but not necessary normal to the deformed plane of the ply.

Following above assumptions we write the displacement field within each layer as

$$\begin{aligned} u_1^i(x_1, x_2, x_3) &= U_1^i(x_1, x_2) + x_3^i \phi_2^i(x_1, x_2) \\ u_2^i(x_1, x_2, x_3) &= U_2^i(x_1, x_2) - x_3^i \phi_1^i(x_1, x_2) \\ u_3^i(x_1, x_2, x_3) &= U_3^i(x_1, x_2) \end{aligned} \quad i = 1, 2, \dots, N. \quad (1)$$

where x_3^i is measured from the middle surface of the i^{th} ply. To ensure displacement continuity at the interfaces we impose following constraint equations

$$\begin{aligned} g_1^i &= U_1^{i+1} - U_1^i - \frac{1}{2} [h^{i+1} \phi_2^{i+1} + h^i \phi_2^i] \\ g_2^i &= U_2^{i+1} - U_2^i + \frac{1}{2} [h^{i+1} \phi_1^{i+1} + h^i \phi_1^i] \end{aligned} \quad i = 1, 2, \dots, N-1. \quad (2)$$

The in-plane strains $\epsilon_m^i = [\epsilon_{11}^i, \epsilon_{22}^i, \epsilon_{12}^i]^T$ are related to the displacement field by

$$\begin{aligned} \epsilon_m^i &= (\epsilon_m^0)^i + x_3^i \mathbf{S} \kappa^i \\ (\epsilon_m^0)^i &= [U_{1,1}^i, U_{2,2}^i, U_{1,2}^i + U_{2,1}^i]^T \\ \kappa^i &= [\phi_{1,2}^i, \phi_{2,1}^i, \phi_{1,1}^i - \phi_{2,2}^i]^T, \end{aligned} \quad (3)$$

where orthogonal matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (4)$$

The out-of plane strains $\epsilon_s^i = [\epsilon_{23}^i, \epsilon_{13}^i]^T$ are written in the form

$$\epsilon_s^i = [U_{3,2}^i - \phi_1^i, U_{3,1}^i + \phi_2^i]^T. \quad (5)$$

The macroscopic response of each ply is that of a homogeneous layer with certain effective properties. In the local coordinates, x_k^i ($k = 1, 2, 3$), we write the ply constitutive relations for the ply (i) as

$$\sigma_m^i = \mathbf{L}_m^i (\epsilon_m^i - \mu_m^i), \quad \sigma_s^i = \mathbf{L}_s^i (\epsilon_s^i - \mu_s^i), \quad (6)$$

where $\mathbf{L}_m^i$ is the (3x3) in-plane stiffness matrix of the ply; $\mathbf{L}_s^i$ is the (2x2) matrix which contains the out-of plane shear moduli. The in-plane stress vector σ_m^i and out-of plane shear stress vector σ_s^i , written in contracted notation are

$$\sigma_m^i = [\sigma_{11}^i, \sigma_{22}^i, \sigma_{12}^i]^T, \quad \sigma_s^i = [\sigma_{23}^i, \sigma_{13}^i]^T. \quad (7)$$

The μ_m^i and μ_s^i denote overall eigenstrains (stress free strains) in the plies. In the present formulation we admit only thermal effects in the form

$$\mu_m^i = \mathbf{m}^i \Delta \theta^i, \quad \mu_s^i = \mathbf{0}. \quad (8)$$

2.2 Element matrices

The familiar Navier solution has very limited range of applicability. In most practical applications more complex numerical techniques such as finite element or boundary element methods are required. In this section, the refined laminated plate theory will be used to derive an accurate 4-node plate element for the finite element analysis of moderately thick laminated plates.

Following the formulation introduced by Mau [2] an element stiffness matrix of the layered plate element can be found from the modified total potential energy functional Π_m

$$\Pi_m(\mathbf{u}, \boldsymbol{\beta}) = U_{int} + \int_{S_m} \sum_{i=1}^{N-1} (\boldsymbol{\beta}^i)^T \mathbf{g}^i dS - U_{ext}, \quad (9)$$

where the term U_{ext} represents the work done by an externally applied load and the first term in (9) is the strain energy of the system. In the absence of inelastic effects the U_{int} reads

$$U_{int} = \frac{1}{2} \int_{S_m} \sum_{i=1}^N \left[(\mathbf{E}_m^0 - \mathbf{T}_F \Delta \boldsymbol{\eta})^T \mathbf{A} (\mathbf{E}_m^0 - \mathbf{T}_F \Delta \boldsymbol{\eta}) + (\mathbf{S} \boldsymbol{\kappa} - \mathbf{T}_D \Delta \boldsymbol{\eta})^T \mathbf{D} (\mathbf{S} \boldsymbol{\kappa} - \mathbf{T}_D \Delta \boldsymbol{\eta}) + (\mathbf{E}_s)^T \mathbf{C} \mathbf{E}_s \right] dS, \quad (10)$$

where the $\mathbf{A}$, $\mathbf{D}$, $\mathbf{C}$ are the (3x3) transformed material matrices; $\mathbf{T}_F \Delta \boldsymbol{\eta}$, $\mathbf{T}_D \Delta \boldsymbol{\eta}$ represent the thermal effects.

In the spirit of finite element method the components of the strain vector $\mathbf{E}^i$ are interpolated in terms of nodal parameters $\mathbf{d}^i$

$$(\mathbf{E}_m^0)^i = B_u \mathbf{d}_u^i \quad \boldsymbol{\kappa}^i = B_\phi \mathbf{d}_\phi^i \quad \mathbf{E}_s^i = B_{s1} \mathbf{d}_\phi^i + B_{s2} \mathbf{d}_w^i \quad (11)$$

where the matrices B_u , B_ϕ , B_{s1} , and B_{s2} are

$$B_u^a = \frac{1}{|J|} \begin{bmatrix} N_{a,\xi} & 0 \\ 0 & N_{a,\eta} \\ N_{a,\eta} & N_{a,\xi} \end{bmatrix}, \quad B_\phi^a = \frac{1}{|J|} \begin{bmatrix} N_{a,\eta} & 0 \\ 0 & N_{a,\xi} \\ N_{a,\xi} & -N_{a,\eta} \end{bmatrix} \quad (12)$$

$$B_{s1}^a = \frac{1}{|J|} \begin{bmatrix} 0 & N_a \\ N_a & 0 \end{bmatrix}, \quad B_{s2}^a = \frac{1}{|J|} \begin{bmatrix} N_{a,\xi} \\ N_{a,\eta} \end{bmatrix}, \quad a = 1, 2, \dots, 4,$$

where Jacobian $|J|$ is introduced to account for coordinate transformation. Similar to (11) the constraint conditions (2) can be written in terms of interpolation functions N and nodal parameters $\mathbf{d}_g^i$ as

$$\mathbf{g}^i = \mathbf{G}^i \mathbf{M} \mathbf{d}_g^i, \quad (13)$$

where $\mathbf{M}$ is the matrix of interpolation functions defined as

$$\mathbf{M} = \begin{bmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & N \end{bmatrix}, \quad N = [N_1, N_2, \dots, N_{n_{el}}]. \quad (14)$$

The matrix $\mathbf{G}^i = [G_1, G_2, G_3, G_4]^i$ is given by

$$\begin{aligned}
G_1^i &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, & G_2^i &= \begin{bmatrix} 0 & -\frac{h_i}{2} \\ \frac{h_i}{2} & 0 \end{bmatrix} \\
G_3^i &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & G_4^i &= \begin{bmatrix} 0 & -\frac{h_{i+1}}{2} \\ \frac{h_{i+1}}{2} & 0 \end{bmatrix},
\end{aligned} \tag{15}$$

with h_i being the thickness of lamina (i); the vector of nodal parameters $\mathbf{d}_g^i$ contains the components of nodal rotations and in-plane displacements associated with the i^{th} and $i+1^{th}$ plies

$$(\mathbf{d}_g^i)^T = \langle (\mathbf{d}_u^i)^T, (\mathbf{d}_\phi^i)^T, (\mathbf{d}_u^{i+1})^T, (\mathbf{d}_\phi^{i+1})^T \rangle. \tag{16}$$

Substituting from (11) and (13) into (10) provides the expression for Π_m in terms of nodal parameters $\mathbf{d}$ and Lagrange multipliers β , in the form

$$\begin{aligned}
\Pi_m(\mathbf{d}, \beta) &= \sum_{n_{el}} \left\{ \sum_{i=1}^N (\mathbf{d}^i)^T \mathbf{K}^i \mathbf{d}^i dS + \int_{S_m} \sum_{i=1}^{N-1} (\beta^i)^T \mathbf{G}^i \mathbf{M} \mathbf{d}_g^i dS \right. \\
&\quad - \int_{S_m} \sum_{i=1}^N \left[\mathbf{d}_u^T \mathbf{B}_u^T \mathbf{A} \mathbf{T}_F \Delta \boldsymbol{\eta} + \mathbf{d}_\phi^T \mathbf{B}_\phi^T \mathbf{S}^T \mathbf{D} \mathbf{T}_D \Delta \boldsymbol{\eta} \right]^i dS \\
&\quad - \int_{S_m} \left[\mathbf{d}_u^T \mathbf{N}^T \mathbf{p}^- - \frac{t}{2} \mathbf{d}_\phi^T \mathbf{N}^T \mathbf{p}^- \right]_1 dS \\
&\quad - \int_{S_m} \left[\mathbf{d}_u^T \mathbf{N}^T \mathbf{p}^+ - \frac{t}{2} \mathbf{d}_\phi^T \mathbf{N}^T \mathbf{p}^+ \right]_N dS - \int_{S_m} \mathbf{d}_w^T \mathbf{N}^T \mathbf{p}_3 dS \\
&\quad - \int_{\partial S_m} \sum_{i=1}^N \left[\mathbf{d}_{un}^T \mathbf{N}^T \hat{\mathbf{F}}_n + \mathbf{d}_{us}^T \mathbf{N}^T \hat{\mathbf{F}}_{ns} \right. \\
&\quad \left. + \mathbf{d}_w^T \mathbf{N}^T \hat{\mathbf{Q}}_n + \mathbf{d}_{\phi n}^T \mathbf{N}^T \hat{\mathbf{M}}_n + \mathbf{d}_{\phi s}^T \mathbf{N}^T \hat{\mathbf{M}}_{ns} \right]^i \Big\}, \tag{17}
\end{aligned}$$

where n_{el} is the total number of elements. The stationary conditions with respect to nodal parameters $\mathbf{d}$ and Lagrange multipliers β , i.e. $\partial \Pi_m / (\partial \mathbf{d}, \partial \beta) = 0$, yield the system of equilibrium and constraint conditions

$$\begin{aligned}
\mathbf{K} \mathbf{d} + \mathbf{Q}^T \beta &= \mathbf{R} \\
\mathbf{Q} \mathbf{d} &= \mathbf{0},
\end{aligned} \tag{18}$$

where $\mathbf{K}$ is the global stiffness matrix after assembly and after imposing the essential boundary conditions. Matrix $\mathbf{Q}$ represents the approximation of constraint conditions (2) and vector $\mathbf{d}$ is the global vector of free nodal parameters. Finally, the vector $\mathbf{R}$ is the vector of nodal forces due to externally applied mechanical and thermal loading.

Before proceeding with examples and numerical results it is appropriate to point out a few remarks regarding the performance of the plate element considered in the present analysis. First, the transverse shear stresses, represented here by Lagrange multipliers, are assumed to be constant over the element to avoid phenomenon called shear locking. In addition, the selective reduced integration is applied in the present formulation. Similar results can be derived based on assumed strain formulation [Hughes et. al. (1981), [3] Fish & Guttal (1995) [4]].

3 Examples

We opene our exposition to above problem by examining a $[0]_N$ Graphite-Epoxy laminate under uniform shear applied to the top and bottom surfaces over a certain distance d representing here the length of the overlap, see Fig. 2.

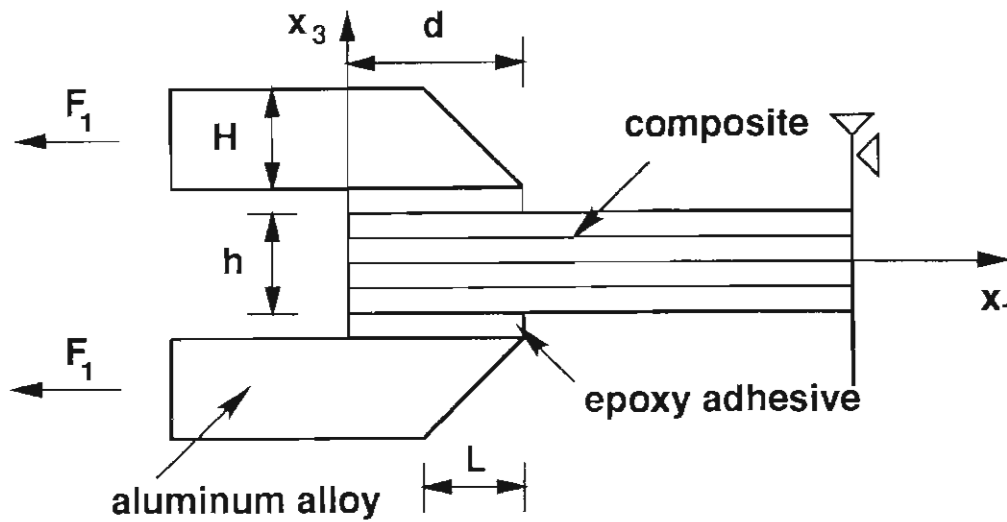


Figure 2: Geometry and loading conditions of typical lap-joint

The effect of the variation of assumed length of the overlap on tensile stress S_{11} is illustrated in Fig. 3. It is evident that the peak stresses detected in the top and bottom layers decrease rather rapidly to attain constant magnitude equal to $2d/h$. Since the uniform adhesive shear stress would generally occur only in case of very stiff adherents, we considered as our next application more realistic lap joint configuration shown in Fig. 4. Although the stress distribution in the bonded region is quite different from that of Fig. 3, the trend in stress degradation observed in latter example is similar. In addition, Fig. 4 indicates that the maximum tensile stresses, causing the laminate failure, further increase. This can be contributed to the fact that the true adhesive shear stress is highly nonuniform. Due to complexity of the problem the distribution of adhesive shear stresses was obtained from simplified 2D analysis. Both composite and steel adherents, Fig.2, were discretized into standard 2D plane strain elements. The adhesive interlayer was modeled using simple 4-node interface element capable of transferring shear stress only. All phases were assumed to be elastic with material properties listed in Tables 1.

Table 1: Mechanical properties of Graphite/Epoxy composite, Titanium adherent and Epoxy adhesive

Material	E_L (GPa)	E_T (GPa)	G_L (GPa)	G_T (GPa)	ν_{LT}	ν_{TT}
composite	172.4	6.9	3.5	1.4	0.25	0.25
adherent	72.	72.	27.7	22.7	0.3	0.3
adhesive		2.46	2.46			

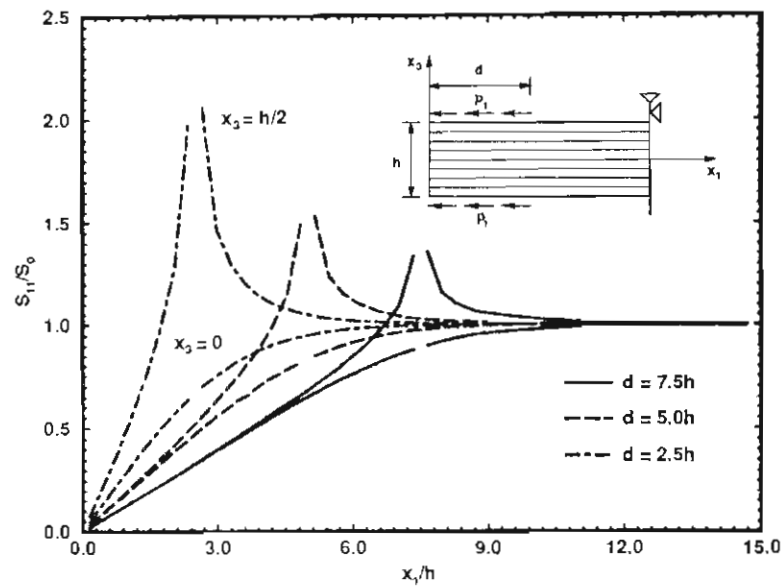


Figure 3: Distribution of tensile stress S_{11} of $[0]_n$ Graphite / Epoxy laminated plate in shear; $S_0 = 2F/h$, $p = 2F/d$

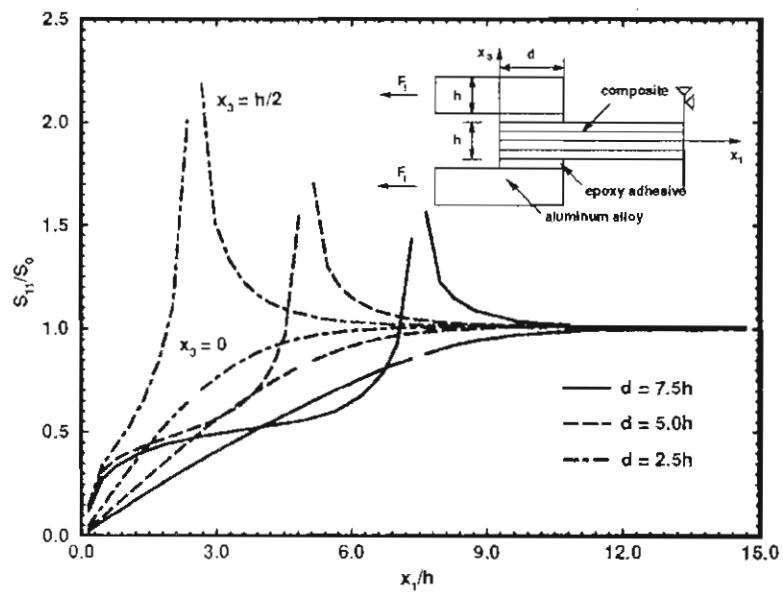


Figure 4: Distribution of tensile stress S_{11} of $[0]_n$ Graphite / Epoxy laminated plate in shear; $S_0 = 2F/h$, $p = 2F/d$

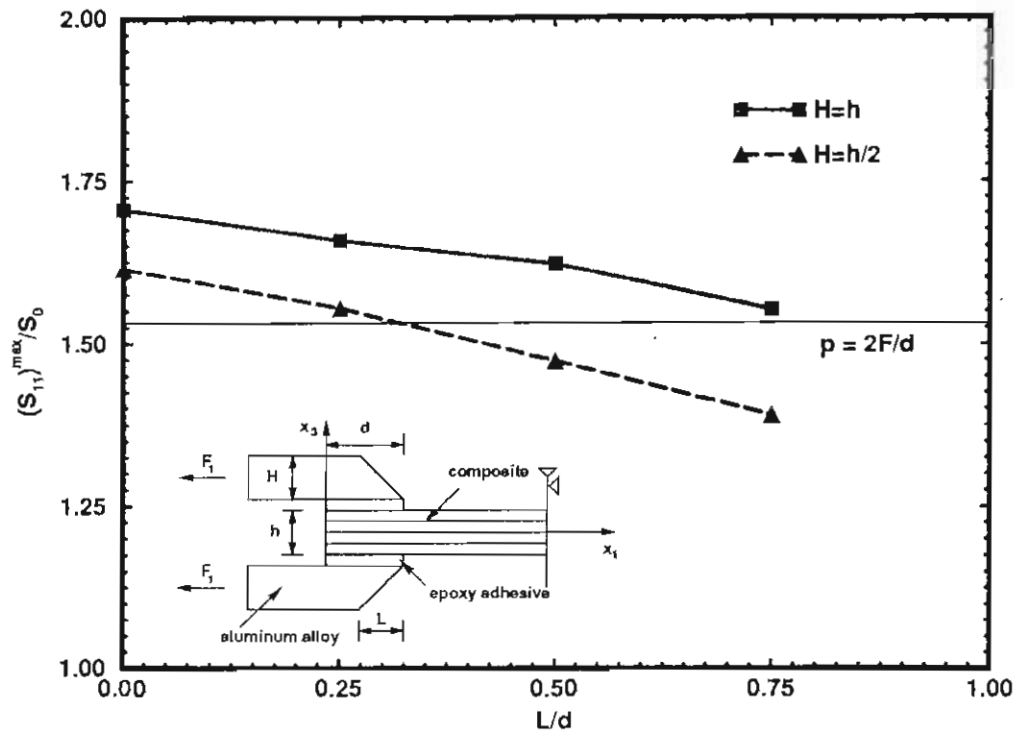


Figure 5: Maximum tensile stress S_{11} of $[0]_n$ Graphite/Epoxy laminated plate in shear evaluated at $x_3 = h/2$, $x_1 = d$.

It can be shown that the shear stress concentration near the edges can be reduced by cutting off the edge of the steel adherent as illustrated in Fig. 2. We should point out that the adhesive shear stresses tend to decrease more for smaller thickness of the steel adherent.

To illustrate this effect, the adhesive shear stresses derived for various ratios of L/d were then applied to composite laminate as a surface tractions. Results are plotted in terms of maximum tensile stress S_{11} in Fig. 5 as a function of L/d ratio. Once again the maximum tensile stress obtained assuming uniform adhesive shear stress is included for illustration.

Acknowledgments

Financial support was provided by the GACR 103/97/1255.

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DETERMINISTIC OPTIMIZATION OF LAMINATED PLATES

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Abstract

A refined laminated plate theory, originally devised to improve modeling of bending problems in laminates, is applied to specific class of optimisation problems of laminated plates. In particular, internal strains or eigenstrains developed during manufacturing processes are used to minimize magnitudes of tensile or compressive stresses in individual plies. Several examples are presented to support theoretical foundations.

1 Introduction

A number of different optimization problems have been formulated in order to improve design and the overall performance of the composite structures. Both the deterministic and the statistical approaches have been used in finding the optimum weight, strength and stiffness of laminated plates, shells or composite cylinders. Mostly the lamination parameters and ply thickness have been used as design parameters [Fukunaga & Vanderplaats (1991) [2], Miki et. al. (1993) [3]].

Recently, Dvorak & Prochazka (1995) [4], Sejnoha & Prochazka (1995) [6], and Naprstek et. al. (1996) [5] have introduced an eigenstrain as a new control variable in optimizing the distribution of hoop stresses through the thickness of composite cylinder loaded by external pressure. This approach opens a way to a new class of optimization problems. For example, eigenstrains as control parameters can be effectively used in estimating the bounds of instantaneous overall stiffness or compliance of composite materials undergoing inelastic deformation.

In the present study, the eigenstrains are used in conjunction with the refined plate theory to minimize the sum of squares of in-plane stresses of composite laminates. Although from laminated plates point of view this problem does not offer any practical applications, the mathematical formulation is quite general and can be applied to any other and perhaps more realistic problems.

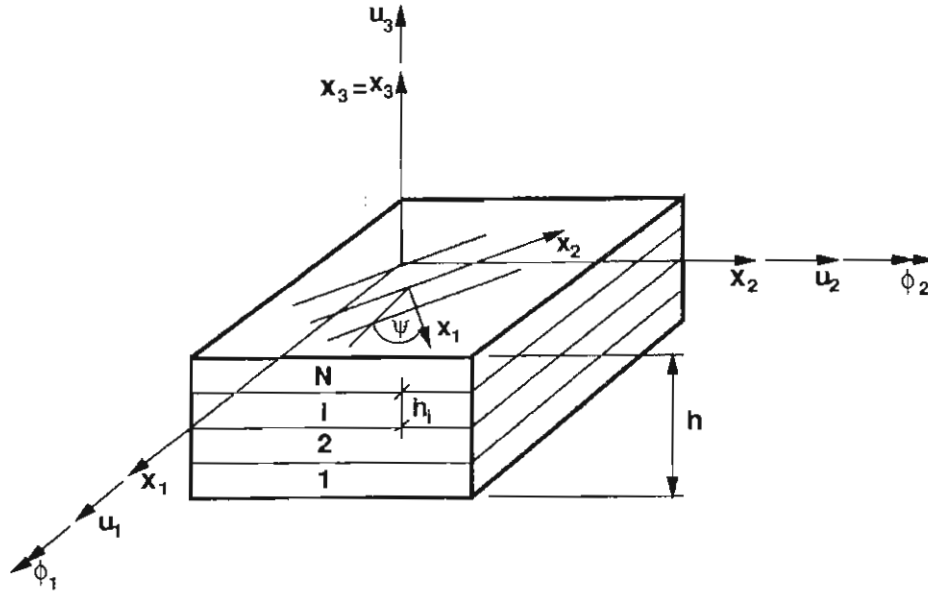


Figure 1: Representative volume element

2 Formulation of the problem

Consider a representative volume V of Fig. 1 with S_m^i being the midplane of the i^{th} ply. The laminate consists of N orthotropic layers with no restriction on laminate configuration. Definition of local and overall coordinates is that of [1].

The objective of the present study is to minimize the sum of squares of in-plane stresses by introducing the uniform ply eigenstrains as control parameters. Hence, using the subscript m to denote the in-plane variables we define the objective functional ϕ in the form

$$\phi = \frac{1}{2} \sum_{i=1}^N \int_{V_i} (S_m^i)^T S_m^i dV, \quad (1)$$

where

$$S_m^i = L_m^i [I_3, x_3^i I_3] \left\{ \begin{matrix} (E_m^0)^i \\ S\kappa^i \end{matrix} - \mu_m^i \right\}. \quad (2)$$

In eq. (2) L_m^i is the (3×3) in-plane stiffness matrix of the ply (i) given in overall coordinate system, I_3 is the (3×3) identity matrix; $(E_m^0)^i$ and κ^i are the uniform in-plane strain and curvature vectors, respectively, and μ_m^i is the unknown eigenstrain, yet to be determined. Integrating (1) over the thicknesses of individual plies we obtain

$$\phi = \frac{1}{2} \sum_{i=1}^N \int_{S_m^i} \varphi^i [E_m^0(\mu), \kappa(\mu), \mu] dS. \quad (3)$$

The stationary condition $\delta\phi = 0$, which yields the system of algebraic equations for yet unknown eigenstrains μ_m^i , is attained by performing a variation of (3). Note that in order to obtain a non-trivial solution, the number of independent eigenstrains M must be less or equal to $N-1$, where N is the number of plies. Following the rules

of differentiation of (3) with respect to μ_m^i , the stationary solution reads

$$\delta\phi = \sum_{i=1}^N \int_{S_m^i} \sum_{\alpha=1}^M \delta(\mu_m^\alpha)^T \sum_{j=1}^N \frac{\partial(\mu_m^i)^T}{\partial\mu_m^\alpha} \left\langle \frac{\partial(\mu_m^i)^T}{\partial\mu_m^j} \frac{\partial\varphi^{iT}}{\partial\mu_m^i} + \frac{\partial(E_m^0)^{iT}}{\partial\mu_m^j} \frac{\partial\varphi^{iT}}{\partial(E_m^0)^i} + \frac{\partial\kappa^{iT}}{\partial\mu_m^j} \frac{\partial\varphi^{iT}}{\partial\kappa^i} \right\rangle dS. \quad (4)$$

Since $\frac{\partial(\mu_m^i)^T}{\partial\mu_m^i} = \mathbf{I}$ and denoting $\frac{\partial(\mu_m^j)^T}{\partial\mu_m^\alpha}$ as $Z_{j\alpha}^T$, eq. (4) can be recast as

$$\delta\phi = \sum_{i=1}^N \int_{S_m^i} \sum_{\alpha=1}^M \delta(\mu_m^\alpha)^T \left\langle -Z_{i\alpha}^T h^i (\mathbf{L}_m^i)^T \mathbf{L}_m^i \left\{ (E_m^0)^i - \mu_m^i \right\} + \sum_{j=1}^N Z_{j\alpha}^T \left[\frac{\partial(E_m^0)^{iT}}{\partial\mu_m^j} h^i (\mathbf{L}_m^i)^T \mathbf{L}_m^i \left\{ (E_m^0)^i - \mu_m^i \right\} + \frac{\partial\kappa^{iT}}{\partial\mu_m^j} \frac{h^{i3}}{12} \mathbf{S}^T (\mathbf{L}_m^i)^T \mathbf{L}_m^i \mathbf{S} \kappa^i \right] \right\rangle dS. \quad (5)$$

Eq. (5) requires an evaluation of partial derivatives $\frac{\partial(E_m^0)^{iT}}{\partial\mu_m^j}$ and $\frac{\partial\kappa^{iT}}{\partial\mu_m^j}$. For this purpose assume that the solution of $(E_m^0)^i$ and κ^i can be found in the form

$$\begin{aligned} (E_m^0)^i &= (\hat{E}_m^0)^i + \sum_{j=1}^N \mathbf{D}_{ij}^\epsilon \mu_m^j \\ \kappa^i &= \hat{\kappa}^i + \sum_{j=1}^N \mathbf{D}_{ij}^\kappa \mu_m^j, \end{aligned} \quad (6)$$

where $(\hat{E}_m^0)^i$ and $\hat{\kappa}^i$ are derived as a solution of the original problem with $\mu_m^i = 0$, and $\mathbf{D}_{ij}^\epsilon$ and $\mathbf{D}_{ij}^\kappa$ represent certain influence functions. From (6) readily follows that

$$(\mathbf{D}_{ij}^\epsilon)^T = \frac{\partial(E_m^0)^{iT}}{\partial\mu_m^j}, \quad (\mathbf{D}_{ij}^\kappa)^T = \frac{\partial\kappa^{iT}}{\partial\mu_m^j}. \quad (7)$$

As we already pointed out there exist only $M \leq N-1$ independent eigenstrain vectors μ_m^α . It is evident that the solution of (5) will depend on the selection of sub-space of eigenstrains μ_m^α , $\alpha = 1, 2, \dots, M$. Reminding that $\frac{\partial(\mu_m^j)^T}{\partial\mu_m^\alpha} = Z_{j\alpha}^T$, we can express every eigenstrain vector μ_m^j as a linear combination of the independent eigenstrains μ_m^β

$$\mu_m^j = \sum_{\beta=1}^M Z_{j\beta} \mu_m^\beta. \quad (8)$$

Since variations $\delta(\mu_m^\alpha)^T$ are mutually independent, the substitution from (6)-(8) into (5) represents a system of linear algebraic equations to be solved for unknown

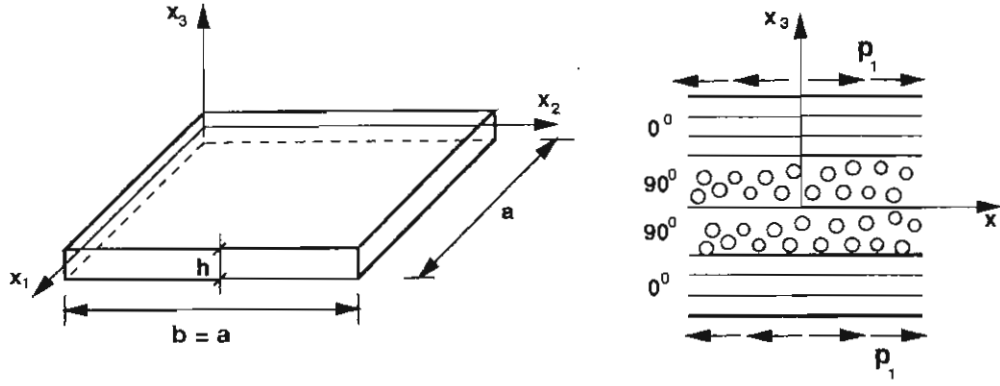


Figure 2: Geometry and loading conditions

eigenstrains μ_m^α , $\alpha = 1, 2, \dots, M$. Note that the influence functions D_{ij}^ϵ and D_{ij}^κ defined in (7) are solved *a priori*, such that the ply (j) is loaded, in turn, by each of the three component of μ_m^j equal to unity. The components of the in-plane strain and curvature vectors evaluated in ply (i) are then used to extract desired ($N \times N \times 3$) influence functions D_{ij}^ϵ and D_{ij}^κ . Finally, as we already mentioned the in-plane strains and curvatures $(\hat{E}_m^0)^i$ and $\hat{\kappa}^i$, respectively, are found as a solution of the original problem under condition $\mu_m^i = 0$ for $i = 1, 2, \dots, N$.

3 Examples

As an application we consider a simple example of $[0/90]_s$ and $[90/0]_s$ laminates loaded by surface tractions p_1 , Fig. 2. For this type of laminates the in-plane forces, moments and uniform in-plane eigenstrains of the i^{th} ply can be expressed in terms of Fourier series as

$$\begin{aligned} (\hat{E}_m^0)^i(x_1, x_2) &= \sum_{mn} \Phi_{mn} (\hat{E}_m^0)_{mn}^i \\ \hat{\kappa}^i(x_1, x_2) &= \sum_{mn} \Phi_{mn} \hat{\kappa}_{mn}^i \\ \mu_m^i(x_1, x_2) &= \sum_{mn} \Phi_{mn} \Upsilon_{mn}^i, \end{aligned} \quad (9)$$

where the (3x3) matrix Φ_{mn} is given by

$$\Phi_{mn} = \begin{bmatrix} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b} & 0 & 0 \\ 0 & \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b} & 0 \\ 0 & 0 & \cos \frac{m\pi x_1}{a} \cos \frac{n\pi x_2}{b} \end{bmatrix},$$

with a and b being the in-plane dimensions of the laminated plate. Substituting from (9) into (5) then provides the desired results.

In the present analysis, the effect of free eigenstrain parameters on laminate response was studied by selecting several different sets of eigenstrains μ_m^α , $\alpha = 1, 2, \dots, M$. For both laminate configurations the results are listed in tables of Figures 4 and 5. Among all cases considered herein the absolute minimum of (9) was

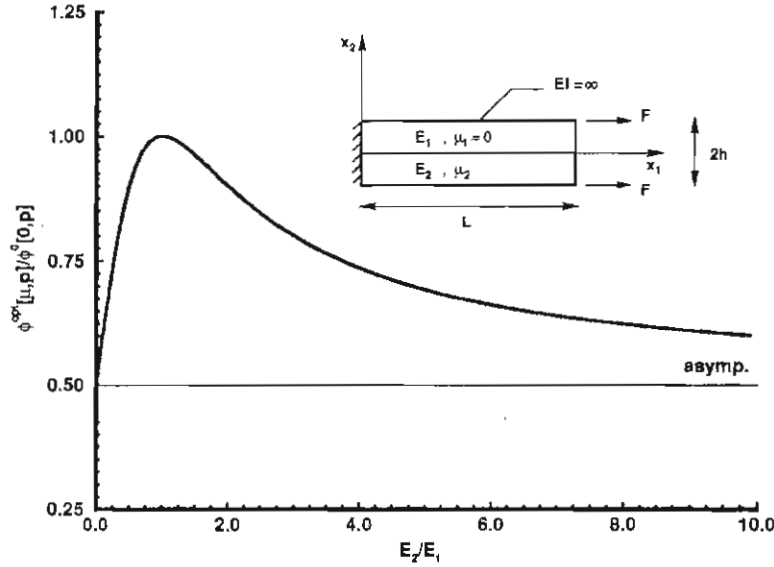


Figure 3: Optimization of two-layered beam structure

attained when the μ_{m11} and μ_{m22} components of the in-plane eigenstrain vector μ_m were applied in 90° plies. Distribution of normalized in-plane force F_{11} for both $[0/90]_s$ and $[90/0]_s$ lay-ups, respectively, is shown in Figures 4 and 5 as well. By examining individual results it is evident that the effect of free eigenstrain parameters is more pronounced when applied in rather compliant 90° plies. This particular result can be demonstrated in a very simple example.

Consider a stiff two layer composite beam, Fig. 3, loaded by externally applied forces. To comply with the requirement that the number of free eigenstrain parameters $M \leq N-1$, we consider only a single eigenstrain μ_2 applied in the bottom layer. For this type of structure the objective functional is simply written as

$$\phi = \phi(\epsilon(\mu), \mu, F) = \int_V \left(\frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} \right) dV. \quad (10)$$

Then the stationary condition $d\phi/d\mu_2 = 0$ provides

$$\frac{d\phi}{d\mu_2} = \frac{\partial \phi}{\partial \mu_2} + \frac{\partial \phi}{\partial \epsilon} \frac{\partial \epsilon}{\partial \mu_2} = 0 \implies \mu_2 = \frac{F}{E_2 h} \left(1 - \frac{E_2}{E_1} \right). \quad (11)$$

Results plotted in Fig. 3 indicate that the ratio $\phi^{opt}(\mu, p)/\phi^0(0, p)$, $p = F/h$ converges to its minimum value more rapidly when the ratio E_2/E_1 decreases, thus supporting the results derived for laminates.

Acknowledgments

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case/ply	0°	90°	90°	0°	ϕ^{opt}	ϕ^{opt}/ϕ^0
1	μ_{11}	0	0	μ_{11}	0.567	0.889
2	μ_{22}	0	0	μ_{22}	0.626	0.945
3	μ_{11}, μ_{22}	0	0	μ_{11}, μ_{22}	0.559	0.846
4	0	μ_{11}	μ_{11}	0	0.451	0.682
5	0	μ_{22}	μ_{22}	0	0.662	1.001
6	0	μ_{11}, μ_{22}	μ_{11}, μ_{22}	0	0.441	0.667

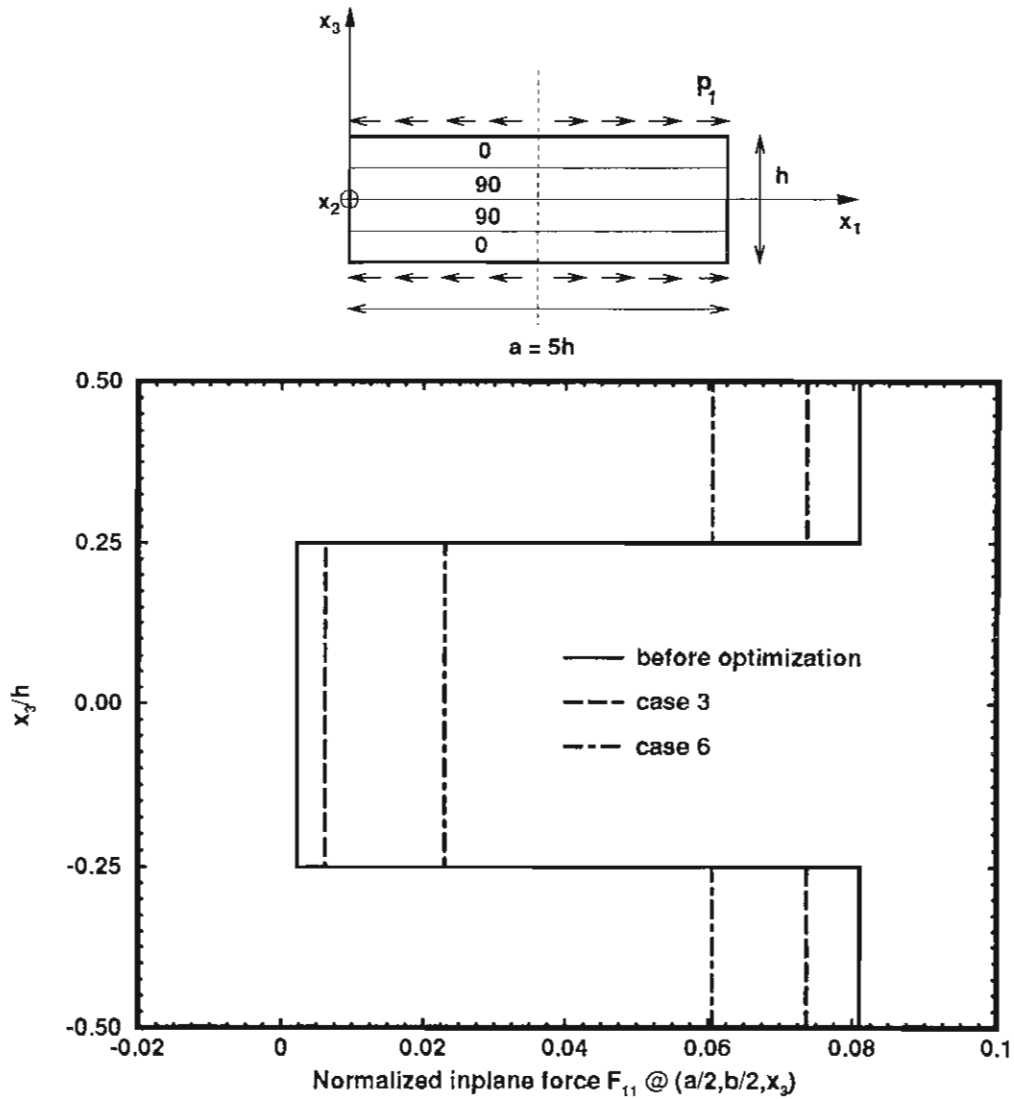


Figure 4: Optimization of $[0/90]$, Graphite/Epoxy laminate; Distribution of normalized in-plane force F_{11} .

case/ply	90°	0°	0°	90°	ϕ^{opt}	ϕ^{opt}/ϕ^0
1	μ_{11}	0	0	μ_{11}	0.397	0.734
2	μ_{22}	0	0	μ_{22}	0.543	1.001
3	μ_{11}, μ_{22}	0	0	μ_{11}, μ_{22}	0.392	0.725
4	0	μ_{11}	μ_{11}	0	0.469	0.867
5	0	μ_{22}	μ_{22}	0	0.507	0.947
6	0	μ_{11}, μ_{22}	μ_{11}, μ_{22}	0	0.454	0.839

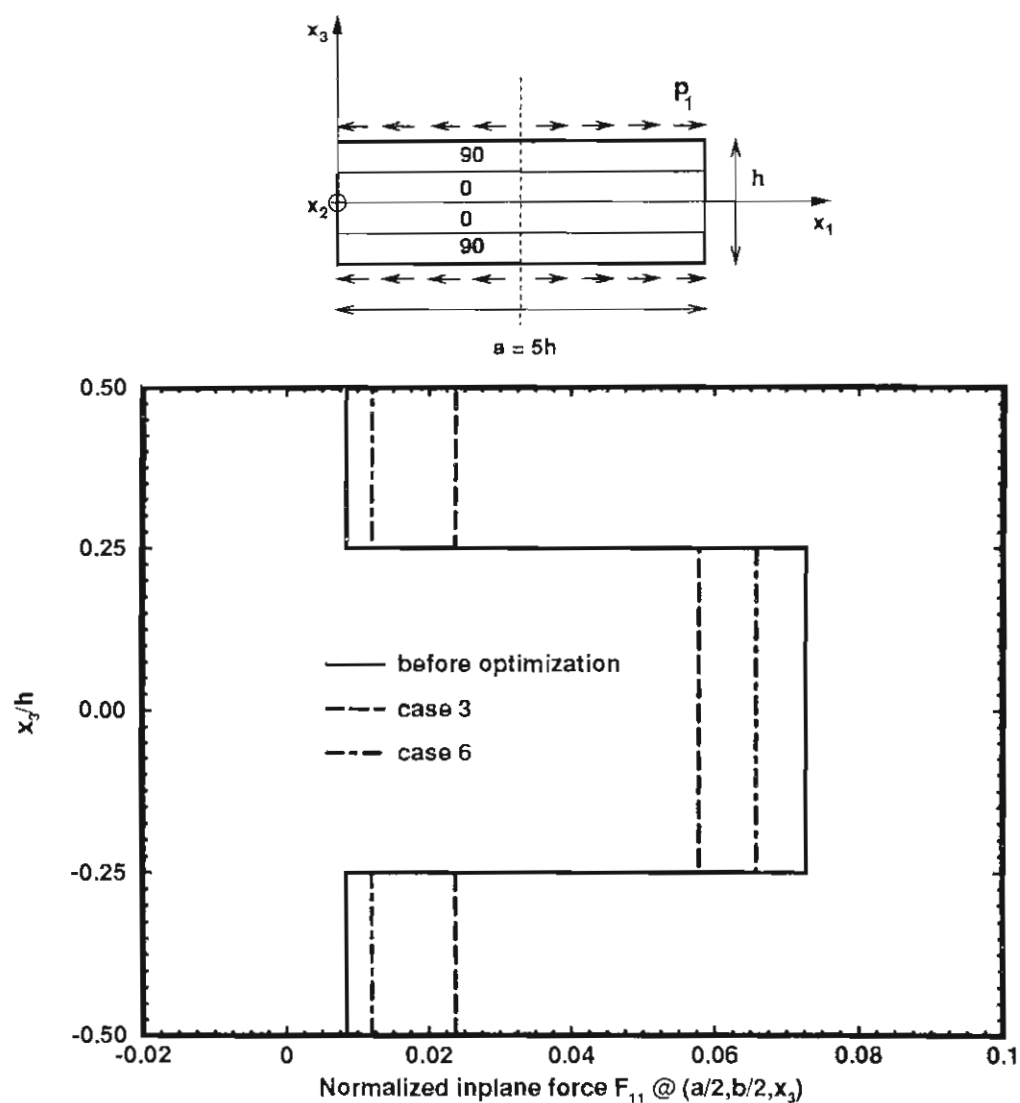


Figure 5: Optimization of $[90/0]_s$ Graphite/Epoxy laminate; Distribution of normalized in-plane force F_{11} .

The Numerical Analysis of EHD-Lubricated Contact Problems

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Abstract

The influence of EHD-lubrication on stress field in contact area of rolling and sliding mechanical elements is presented in this paper. Computational analysis is performed using the equivalent model of two contacting cylinders that have the same radii as are the curvature radii of contacting mechanical elements at any point of a real contact. The equivalent cylinders are subjected to normal and tangential tractions that are determined with the Hertz contact theory with consideration of additional external contact forces that arise from the EHD-lubrication conditions. The finite element method is used for numerical computations of the stress fields for different lubrication conditions.

The results from such computations provide the basis for the influence determination of typical lubrication parameters, like viscosity of lubricant and mean surface velocity of contacting cylinders, on the stress field in contact area.

1. INTRODUCTION

The theory of contact mechanics is concerned with description of stresses and deformations, which arise when the surfaces of two solid bodies are brought into contact. In non-conforming contact, which is usually appearing in gears, bearings, wheels, etc., the contact area is generally small comparing with the dimensions of the bodies themselves. The stresses are concentrated in the region close to the contact zone and are not significantly influenced by the body geometry [1, 2, 3]. To prevent any fatal surface damage occurring on contact surfaces the maximum contact stresses should be lower than the local yield stress [4]. The contact conditions between two mechanical elements are complex and many parameters influence the distribution and extent of contact stresses that are appearing in the contact area. In this paper only the influence of EHD-lubrication is studied. The EHD-lubrication is a type of lubrication where the elastic deformations of the lubricated surfaces resulting from a fluid-film surface interaction become significant. This type of lubrication is usually associated with highly stressed machine elements such as bearings and gears.

2. HERTZ CONTACT CONDITIONS

According to Hertz contact theory the distribution of contact pressure in the contact area can be analytically determined with [4]

$$p(x) = \frac{2F_N}{\pi b^2} \sqrt{b^2 - x^2}, \quad (1)$$

in which F_N is the normal force per unit length of equivalent cylinders and b is the half-width of the contact area, which is given as

$$b = \sqrt{\frac{8F_N r^*}{\pi E^*}}, \quad (2)$$

where r^* is the equivalent radius and E^* is the equivalent Young's modulus defined as

$$E^* = \frac{2E_1 E_2}{E_2(1 - \nu_1^2) + E_1(1 - \nu_2^2)} \quad (3)$$

$$r^* = \frac{r_1 r_2}{r_1 + r_2}. \quad (4)$$

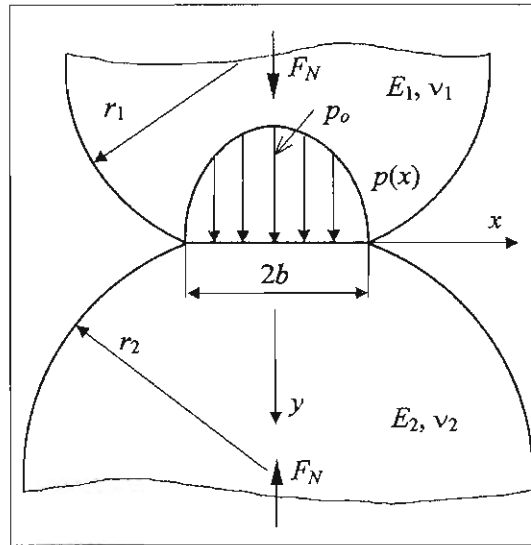


Figure 1. Equivalent model of two contacting cylinders

Utilising the Coulomb friction law the tangential loading $q(x)$ due to sliding of contacting surfaces can be easily determined from the normal loading $p(x)$ and coefficient of friction μ with

$$q(x) = \mu \cdot p(x). \quad (5)$$

For the purpose of easier consideration of contact parameters in numerical computations it is very useful to replace usually complex geometry of contacting mechanical elements with an equivalent model of two cylinders (see Figure 1). The equivalent cylinders have the same curvature radii as the contacting mechanical elements in any chosen point of

the real contact. This enables use of the Hertz contact theory for determination of contact conditions as it is presented above. Such approach enables usage of the general finite element codes for simulation of rolling and sliding contact conditions between two contact bodies, without the need for implementation of special algorithms that are needed for solving the contact problems. The influence of EHD-lubrication can also be considered with the additional external contact loading.

3. INFLUENCE OF EHD-LUBRICATION CONDITIONS

Under EHD-lubrication conditions the contacting bodies are separated in a contact zone with a fluid-film, which transmits the contact stresses by its pressure. The transmitting pressure develops in the entry region by hydrodynamic action and is accompanied by a very large increase in viscosity. The film thickness at the outlet region, *i.e.* at the end of converging zone is limited by the necessity of maintaining a finite pressure. This condition virtually determines the film thickness in terms of the mean surface speed u , roller radii r and the viscous properties of the lubricant (see Fig. 2). The highly viscous fluid passes through the parallel zone until the pressure and viscosity collapse at the exit, which dictates thinning of the film. The inlet and exit regions are effectively independent; they meet at the end of the parallel zone with a discontinuity in a contact strain field, which is associated with a sharp peak in pressure.

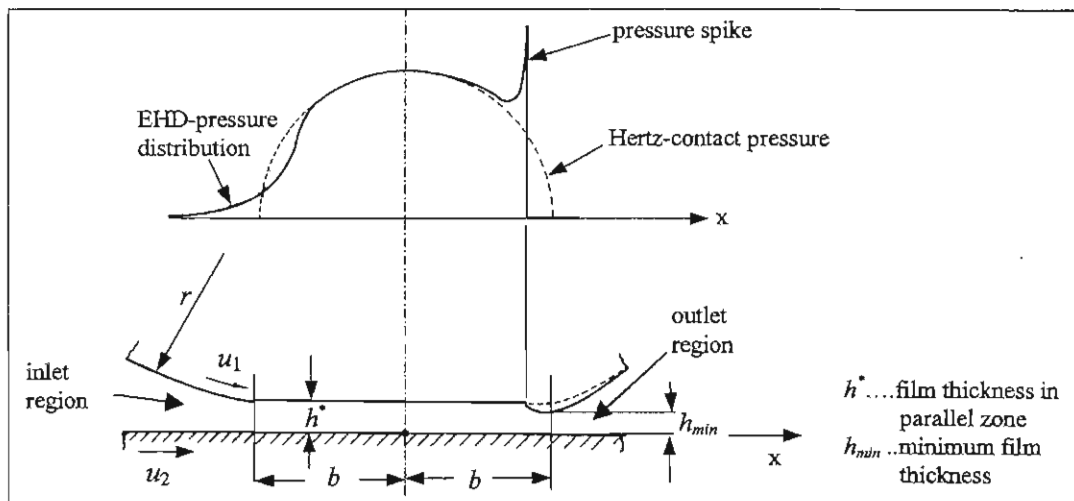


Figure 2. The conditions within an EHD-lubricated line contact [4]

The presence of a pressure spike in the outlet region of the EHD-lubricated contact produces large shear stresses that are localized very close to the surface. The operating parameters normally used in EHD-lubrication line contact analysis are [5, 6, 7, 8]

- Dimensionless load parameter:
$$W = \frac{F_N}{E^* r^*} \quad (6)$$

- Dimensionless speed parameter:
$$U = \frac{\eta_o u}{E^* r^*} \quad ; \quad \eta_o = \rho \cdot v \quad (7)$$

- Dimensionless material parameter:
$$G = \alpha E^* \quad (8)$$

where F_N is the load per unit width, η_o is the dynamic viscosity at the atmospheric pressure, $u = (u_1 + u_2)/2$ is the mean surface velocity and u_1 and u_2 are respective surface

velocities, ρ is the lubricant density, ν is the kinematic viscosity and α is the pressure-viscosity coefficient.

In the present paper it is assumed that W and G are constant while the dimensionless speed parameter U varies with relation to different u and ν . The following conditions are assumed: fully flooded contact conditions, no side leakage of the lubrication fluid, isothermal behavior, smooth surfaces and Newtonian fluid. Figure 3 shows the effect of dimensionless speed parameter U on the pressure profile throughout the contact area. The lubrication fluid inlet is at the left and the outlet at the right, with the center of the contact occurring at $x=0$. From Figure 3, it can be observed that with the decrease of U the dimensionless pressure spike also decreases and its location moves towards the outlet.

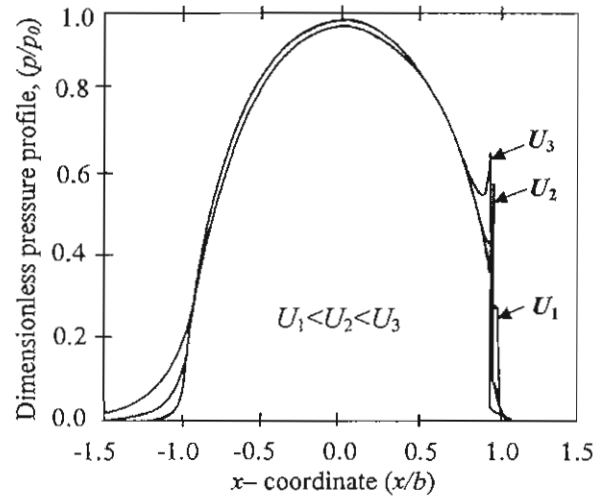


Figure 3. Dimensionless pressure profiles for different U [5]

Hamrock proposed the following equations for the dimensionless spike amplitude Y and dimensionless spike location X (see Fig. 4) [9]

$$Y = 0.267 W^{-0.375} U^{0.174} G^{0.219} \quad (9)$$

$$X = 1 - 2.469 W^{-0.941} U^{0.206} G^{-0.848} \quad (10)$$

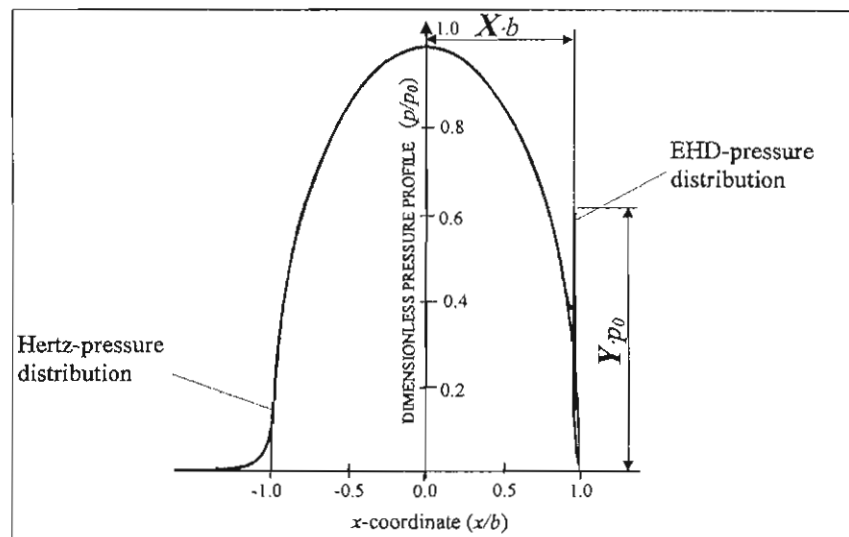


Figure 4: Pressure spike amplitude Y and its location X

4. NUMERICAL EXAMPLE

For the numerical calculations an equivalent model of two cylinders with radii $r_1=r_2=20$ mm, Young's modulus $E=2.06 \cdot 10^5$ MPa, Poisson's ratio $\nu=0.3$ and maximum contact pressure $p_o=1550$ MPa has been used. Taking into account the recommended value for the coefficient of friction in contact problems with good lubrication $\mu=0.04$ [10], the distribution of Misses equivalent stress σ_{EM} in the contact area has been determined with the finite element model shown in Figure 5. The influence of the dimensionless speed parameter U on the stress field in the contact area is studied for different values of mean velocity and kinematic viscosity (see Table 1 and Figures 6 and 7), where the value $\rho=9 \cdot 10^{-7}$ kg/m³ is used for lubricant density. Two cases are considered:

- case 1: variable mean velocity and fixed viscosity,
- case 2: fixed mean velocity and variable viscosity.

Table 1: The loading parameters (see Figure 4).

LOAD CASE	VARIABLE PARAMETER	Dimensionless spike location X	Dimensionless spike amplitude Y
CASE 1 $\nu=220 \text{ mm}^2/\text{s}$	$u=1 \text{ m/s}$	0.9610	0.6156
	$u=5 \text{ m/s}$	0.9462	0.8146
	$u=10 \text{ m/s}$	0.9379	0.9190
	$u=20 \text{ m/s}$	0.9284	1.0368
	$u=30 \text{ m/s}$	0.9215	1.1126
	$u=50 \text{ m/s}$	0.9135	1.2160
CASE 2 $u=10 \text{ m/s}$	$\nu=10 \text{ mm}^2/\text{s}$	0.9665	0.5456
	$\nu=50 \text{ mm}^2/\text{s}$	0.9664	0.7219
	$\nu=100 \text{ mm}^2/\text{s}$	0.9462	0.8145
	$\nu=200 \text{ mm}^2/\text{s}$	0.9379	0.9189
	$\nu=400 \text{ mm}^2/\text{s}$	0.9284	1.0366
	$\nu=600 \text{ mm}^2/\text{s}$	0.9222	1.1124

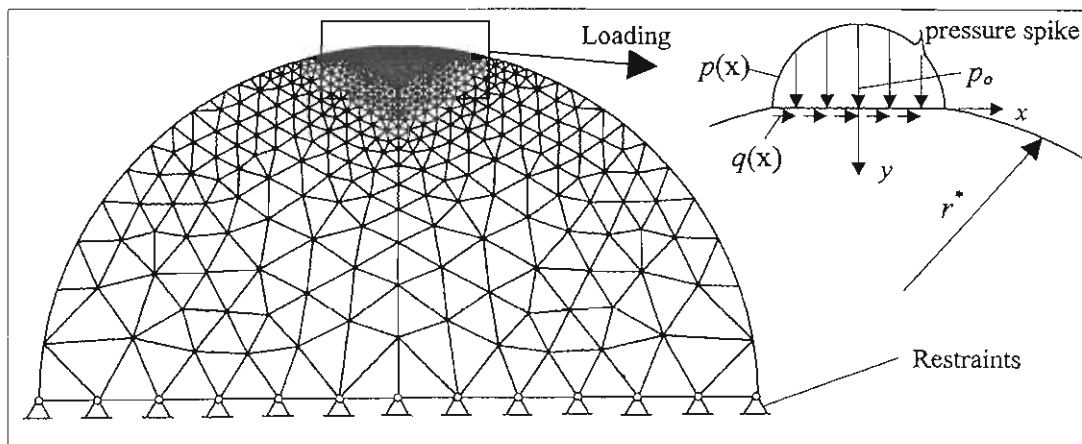


Figure 5: Numerical model for determination of σ_{EM}

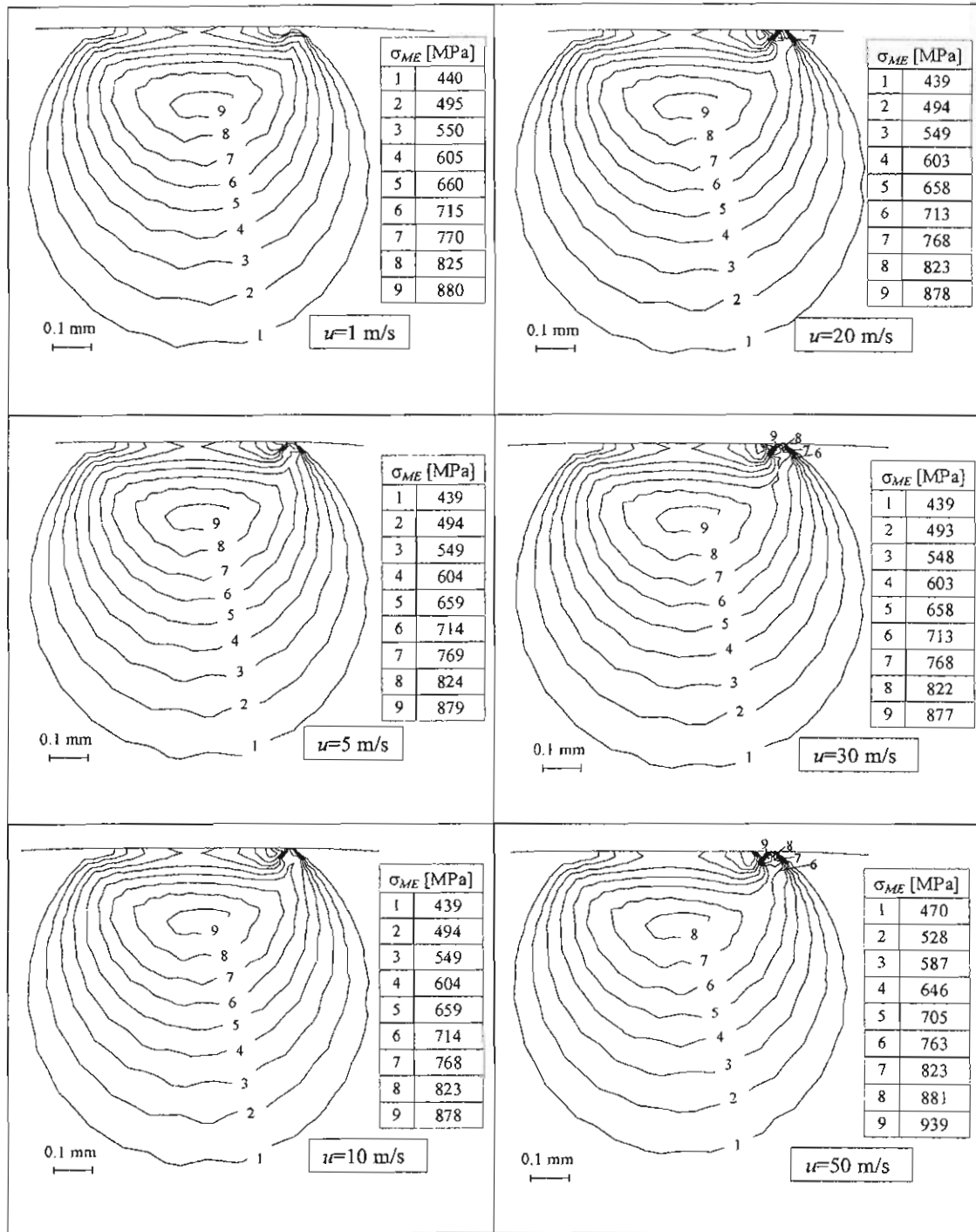


Figure 6: Distribution of von Misses equivalent stress σ_{EM} in the contact area of equivalent cylinders for kinematic viscosity $\nu = 220 \text{ mm}^2/\text{s}$

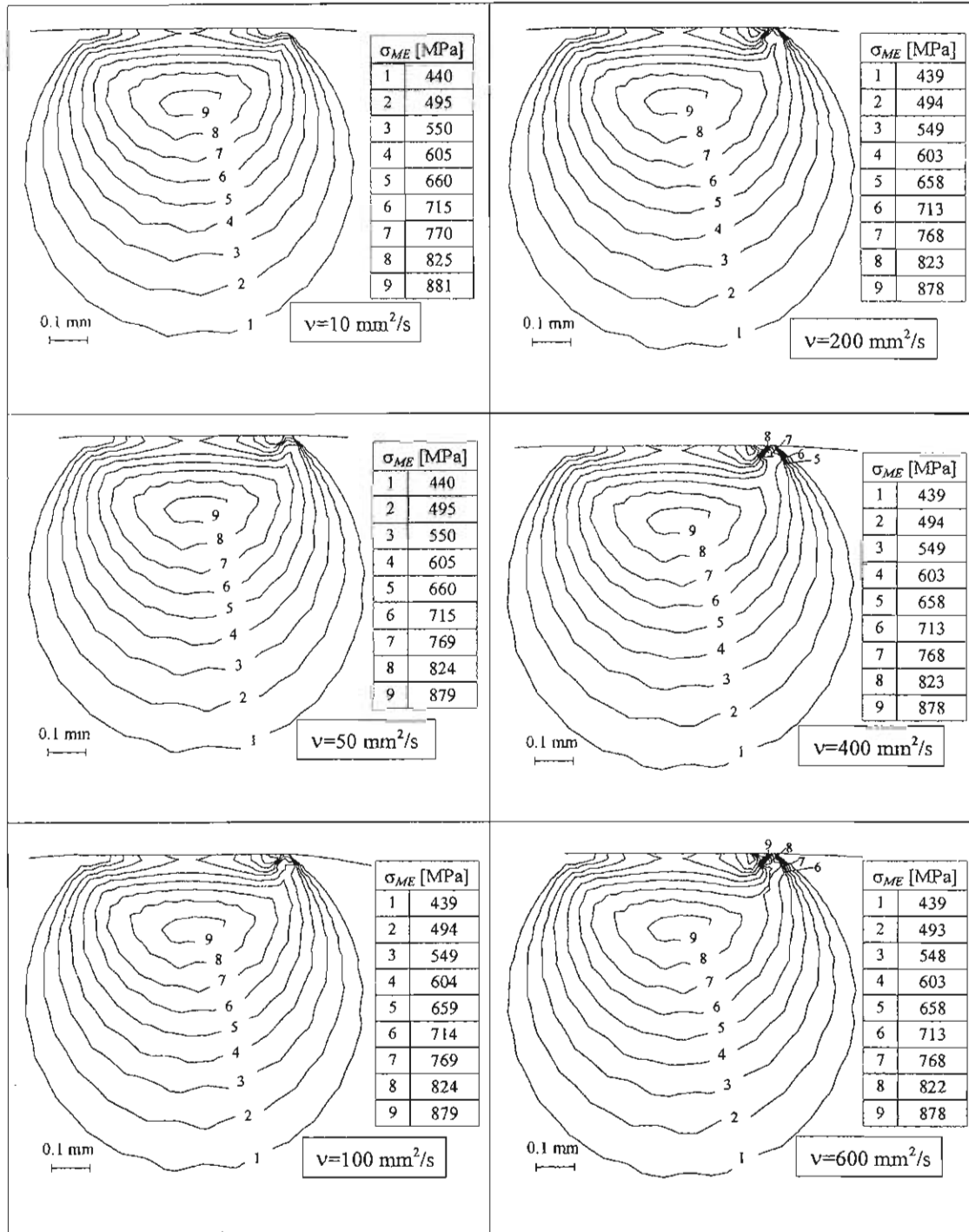


Figure 7: Distribution of von Mises equivalent stress σ_{EM} in the contact area of equivalent cylinders for mean velocity $u = 10$ m/s

The results in Figures 6 and 7 show that the distribution of equivalent stress σ_{EM} in contact area significantly depends on the mean velocity and kinematic viscosity of the lubricant. For small values of mean velocity ($u < 10$ m/s) its influence on the equivalent stress is very small and can be neglected. With increase of the mean velocity the maximum equivalent stress also increases and its maximum $(\sigma_{EM})_{max}$ moves closer to the contact surface. At $u > 30$ m/s the maximum equivalent stress $(\sigma_{EM})_{max}$ reaches the contact surface. The same applies to the influence of the kinematic viscosity. For values

of $v < 200 \text{ mm}^2/\text{s}$ the influence on the stress field can be neglected, while for $v > 400 \text{ mm}^2/\text{s}$ the maximum equivalent stress moves very close to the surface.

5. CONCLUSIONS

The influence of EHD-lubrication on the stress field in the contact area of mechanical elements like gears and bearings is presented in the paper. An equivalent contact model of two contacting cylinders is used for computational analyses. The equivalent cylinders are subjected to normal and tangential Hertzian loading conditions with the addition of extra contact loading due to EHD-lubrication.

The finite element method is used for computational analyses of the resulting stress field. The influence of EHD-lubrication on the stress field in contact area is studied for different values of mean velocity of contacting bodies and viscosity of lubricant, where the stress field in the contact area is represented with the von Mises equivalent stress. The results of numerical analyses show that the distribution of the equivalent stress in the contact area is significantly dependent on mean velocity and kinematic viscosity. With increase of both parameters the maximum equivalent stress approaches the contact surface, which can ultimately lead to surface cracking.

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NUMERICAL ESTIMATION OF FATIGUE LIFE TO CRACK INITIATION IN METAL STRUCTURES

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Abstract

This paper deals with numerical aspects of estimation of fatigue life in metal structures, on the basis of calculated strains and stresses, as well as their amplitude and mean values, due to cyclic loading of the structure. The material behaviour can be described by the four, so called, fatigue properties. Computer program LifeNISA is developed as a module which estimates fatigue life, on the basis of previously performed linear or materially non-linear static calculations. An example of its implementation is also given.

1. Introduction

Fatigue cracks are the most important cause of collapse of metal structures, with serious consequences, such as failures and fractures. In the design stage, or even in some latter stages, it is important to determine structural fatigue behaviour. One of the aspects of this behaviour is fatigue life of the structure or its part until the initiation of a fatigue crack. Fatigue life can be determined (estimated) numerically, on the basis of previously calculated strains and stresses, by means of a linear or materially non-linear calculation procedure [1].

Stresses and strains are calculated by means of NISA II static module [2]. Numerical estimation of fatigue life is performed by means of the computer program LifeNISA, developed by the author in [3] and based on the theory and presumptions described hereafter. Stresses and strains obtained by means of non-linear analysis by NISA are transferred to this program as input data, as well as material fatigue properties. Program LifeNISA calculates estimated fatigue life to the initiation of a new crack in the nodes of the model, expressed as a number of loading cycles $N(x,y,z)$. Fatigue life is determined by means of Manson-Coffin equation [4], or Smith-Watson-Topper's parameter [5], on the basis of normal stress in the direction perpendicular to the direction of expected crack propagation and the relevant longitudinal strain (or equivalent strain ϵ_{HMH} , as described in [1]).

2. Presumptions

Real structure or its part is represented by a numerical finite element model and exposed to a cyclic external loading. In any of the model points, during any loading cycle, strains and stresses are time dependent functions $\epsilon = f_1(t)$ and $\sigma = f_2(t)$. Fatigue life calculations are based upon the following presumptions:

- Displacements and strains are small enough, so that a theory of small displacements and strains may be applied
- Material model follows non-linear Ramberg-Osgood law

In any point of the model, at selected time step, on the basis of the known loading dependence on time and the known distribution of initial stresses and strains, the

displacements, components of tensor of elastic, plastic and total strain, as well as stress tensor components are calculated. Equivalent stress, equivalent strain and effective plastic strain are determined as well. These quantities play a significant role in fatigue life estimations.

Implementing material fatigue properties, determined experimentally on special testing specimens without notches, on the basis of calculated stresses and strains in a selected model point it is possible to determine estimated fatigue life to initiation of a new crack in that point, expressed by means of the number of loading cycles.

3. Mean and amplitude strains and stresses

Minimal and maximal stress for every loading cycle, $\sigma_{\min}$ and $\sigma_{\max}$ play the most significant role during these calculations, as they are used to determine amplitude stress and mean stress.

Amplitude stress for every particular loading cycle is defined as:

$$\sigma_a = \frac{\sigma_{\max} - \sigma_{\min}}{2} \quad (1)$$

Mean stress, for each particular loading cycle is defined as:

$$\sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2} \quad (2)$$

Amplitude strain is determined in an analogous way, from the minimal and maximal strain, during a loading cycle:

$$\varepsilon_a = \frac{\varepsilon_{\max} - \varepsilon_{\min}}{2} \quad (3)$$

Influence of the mean strain is neglected, as it is practically irrelevant for the estimation of fatigue life. It is not discussed here which components of strain and stress tensor are to be selected in above equations, as a basis for fatigue life calculations.

4. Fatigue life for neglectable mean stress

Fatigue life estimation for neglectable mean stress is dependent only on material fatigue properties, amplitude strain and amplitude stress. At every level of external loading in a model part there exist elastic strains only, while in the remaining part of the model a combination of elastic and plastic strains appears. The former part of the model is equivalent to a high cyclic regime, in which the strain amplitude is related to the fatigue life by the following equation [6]:

$$\varepsilon_a = \frac{\sigma_a}{E} = \frac{\sigma_f}{E} \cdot (2N_f)^{b_f} \quad (4)$$

In the latter part of the specimen, which is elastoplastically deformed (in the region of low cycle regime) the following equation is valid [6]:

$$\varepsilon_a = \varepsilon_a^{(e)} + \varepsilon_a^{(p)} = \frac{\sigma_f}{E} \cdot (2N_f)^{b_f} + \varepsilon_f \cdot (2N_f)^{c_f} \quad (5)$$

Fatigue properties of material σ_f , b_f , ε_f i c_f are determined by experiments with controlled deformation of specimens without notches. Both of the above relations are valid subject to the presumption that the influence of mean stress can be neglected:

$$\sigma_m = 0 \quad (6)$$

The relationship $\epsilon_a = f(N)$ is usually presented in a logarithmic chart (log-log). An example of such a presentation is given in Figure 1:

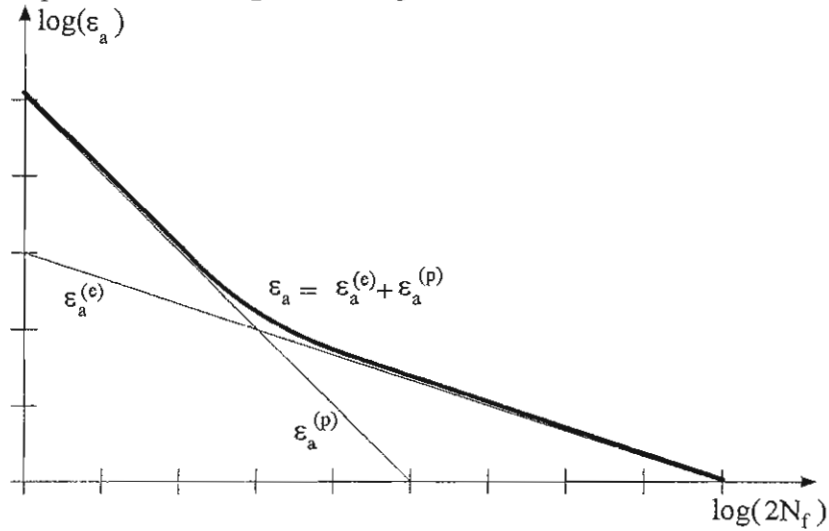


Figure 1: Relationship between fatigue life and amplitude strain

Instead of the number of loading reversals (number of loading change sign points) N_f , the number N of loading cycles is usually used:

$$N = 2 \cdot N_f \quad (7)$$

Calculation of stresses and strains, in the case of cyclic loading, is to be carried out taking into account the cyclic material properties, not the static ones. The reason is that the material of a structural part, at the beginning of its exploitation, possesses mechanical properties obtained by usual static mechanical tests, so called, static properties. During exploitation, under the influence of cyclic loads, after a certain number of loading cycles, material properties change gradually into its cyclic properties, which are to be used in calculations.

One of the stresses (amplitude or mean) used in fatigue life calculations may be substituted by the well-known stress ratio R_σ , an important cyclic loading property:

$$R_\sigma = \frac{\sigma_m - \sigma_a}{\sigma_m + \sigma_a} \quad (8)$$

5. Influence of mean stress

Experimental results show significant discrepancies of the estimated fatigue life value from the quantity calculated in accordance with the above described procedure, in the cases when the mean stress influence cannot be neglected. Influence of the mean stress can be taken into account in several ways. Manson-Coffin equation [4], or the damage parameter, called Smith-Watson-Topper's parameter P_{swt} may be implemented, in the classical [5] or modified form [7].

6. Manson-Coffin equation

Manson-Coffin equation is given in the following form [4]:

$$\epsilon_a = \frac{(\sigma_f' - \sigma_m)}{E} \cdot N^{b_f} + \epsilon_f' \cdot N^{c_f} \quad (9)$$

where the correction value is assigned to the fatigue strength coefficient and is usually used in the whole range of strains. However, implementation of Manson-Coffin equation, compared to the experimental results, does not give so good results for a high cycle regime, as it does for a low cycle regime. Regardless of that, in accordance with the present state-of-the-art it is the most precise (and the most convenient) to include the mean stress influence into the fatigue life calculation expression, through the relevant change of the fatigue strength coefficient. For the high cyclic regime (elastic strains), the following expression is recommended:

$$E \cdot \varepsilon_a = \frac{(\sigma'_f)^2}{\sigma_{\max}} \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_{\max}} \cdot N^{b_f+c_f} \quad (10)$$

For the low cycle regime (elastoplastic strains) Manson-Coffin equation is still valid in the original form, from which it follows:

$$E \cdot \varepsilon_a = \left(\sigma'_f - \frac{1+R_\sigma}{2} \sigma_{\max} \right) \cdot N^{b_f} + E \cdot \varepsilon'_f \cdot N^{c_f} \quad (11)$$

7. Smith-Watson-Topper's parameter

Damage parameter, called Smith-Watson-Topper's parameter of stresses, P_{SWT} is defined by the following relation [5]:

$$P_{\text{SWT}} = \sqrt{\sigma_{\max} \cdot E \cdot \varepsilon_a} \quad (12)$$

which can be transformed into the following form:

$$P_{\text{SWT}} = \sqrt{\sigma'_f \cdot N^{b_f} \cdot (\sigma'_f \cdot N^{b_f} + E \cdot \varepsilon'_f \cdot N^{c_f})} \quad (13)$$

Relationship of the estimated fatigue life and the damage parameter P_{SWT} in implicit form, can be shown in one of the following ways, dependent on the character of strains and the mean stress level:

a) Parameter P_{SWT} , for elastic strains and neglectable mean stress, $\sigma_m = 0$:

$$P_{\text{SWT}} = \sigma'_f \cdot N^{b_f} \quad (14)$$

b) Parameter P_{SWT} , for elastic strains and mean stress, $\sigma_m \neq 0$:

$$P_{\text{SWT}} = \sqrt{(\sigma'_f)^2 \cdot N^{2b_f} + \sigma'_f \cdot E \cdot \varepsilon'_f \cdot N^{b_f+c_f}} \quad (15)$$

c) Parameter P_{SWT} , for elastoplastic strains and neglectable mean stress, $\sigma_m = 0$:

$$P_{\text{SWT}} = \sqrt{\sigma_{\max} \cdot (\sigma'_f \cdot N^{b_f} + E \cdot \varepsilon'_f \cdot N^{c_f})} \quad (16)$$

d) Parameter P_{SWT} , for elastoplastic strains and mean stress, $\sigma_m \neq 0$:

$$P_{\text{SWT}} = \sqrt{\sigma_{\max} \cdot \left(\sigma'_f - \frac{1+R_\sigma}{2} \sigma_{\max} \right) \cdot N^{b_f} + E \cdot \varepsilon'_f \cdot N^{c_f}} \quad (17)$$

8. Modification of Smith-Watson-Topper's parameter

Smith-Watson-Topper's parameter, defined by the above stated equations, provides calculation results in a good correlation with the experimental results, in the case of the tensile mean stress. If the mean stress is compressive (negative), this correlation is not good enough and an additional correction is to be introduced. The modified Smith-Watson-Topper's parameter, which takes into account the sign of the mean stress, is defined by [7]:

$$P_{\text{SWT}} = \sqrt{\sigma_{\max} \cdot E \cdot \varepsilon_a} \quad \text{for } \sigma_m \geq 0 \quad (18)$$

and:

$$P_{\text{SWT}} = \sqrt{\sigma_a \cdot E \cdot \varepsilon_a} \text{ for } \sigma_m < 0 \quad (19)$$

Equations (18) and (19) may be transformed into non dimensional form:

$$\frac{P_{\text{SWT}}}{\sigma_{0.2}} = \sqrt{\frac{2}{1-R_\sigma}} \cdot \sqrt{\left(\frac{\sigma'_f}{\sigma_{0.2}}\right)^2 \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_{0.2}^2} \cdot N^{b_f+c_f}} ; \text{ for } \frac{1+R_\sigma}{2} \cdot \sigma_{\text{max}} > 0 \quad (20)$$

and

$$\frac{P_{\text{SWT}}}{\sigma_{0.2}} = \sqrt{\left(\frac{\sigma'_f}{\sigma_{0.2}}\right)^2 \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_{0.2}^2} \cdot N^{b_f+c_f}} ; \text{ for } \frac{1+R_\sigma}{2} \cdot \sigma_{\text{max}} \leq 0. \quad (21)$$

A special case appears if $\sigma_{\text{max}} = 0$, when the indeterminate form is to be avoided. In that case: $\sigma_m < 0$, which gives:

$$\frac{E \cdot \varepsilon_a}{\sigma_{0.2}} = \frac{(\sigma'_f)^2}{\sigma_a \cdot \sigma_{0.2}} \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_a \cdot \sigma_{0.2}} \cdot N^{b_f+c_f} ; \text{ for } \sigma_{\text{max}} = 0. \quad (22)$$

In the case of time independent loading, amplitude stress is: $\sigma_a = 0$. It can be supposed then that the fatigue life is not dependent on loading (where creep is neglected). For that loading case it can be supposed that the fatigue life is infinite.

9. Algorithm for fatigue life estimation

Fatigue life estimation to the initiation of a new crack, on the basis of previously calculated amplitude strain ε_a , amplitude stress σ_a and mean stress σ_m can be systematically shown in the form of algorithm shown in Table 1 [3]:

Ampl. stress σ_a	Maximal stress σ_{max}	Stress ratio R_σ	ESTIMATED FATIGUE LIFE N (OR EQUATION FOR N)	Remark
$\sigma_a = 0$	any	any	$N = +\infty$	static loading
$\sigma_a > 0$	$\sigma_{\text{max}} = 0$	any	$E \cdot \varepsilon_a = \frac{(\sigma'_f)^2}{\sigma_a} \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_a} \cdot N^{b_f+c_f}$	$\sigma_m < 0$
	$\sigma_{\text{max}} \neq 0$	$ R_\sigma \leq 1$	$E \cdot \varepsilon_a = \frac{(\sigma'_f)^2}{\sigma_{\text{max}}} \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_{\text{max}}} \cdot N^{b_f+c_f}$	$\sigma_m > 0,$ $\sigma_{\text{max}} > 0$
		$ R_\sigma > 1$	$E \cdot \varepsilon_a = \frac{(\sigma'_f)^2}{\sigma_a} \cdot N^{2b_f} + \frac{\sigma'_f \cdot E \cdot \varepsilon'_f}{\sigma_a} \cdot N^{b_f+c_f}$	$\sigma_{\text{min}} < 0,$ $\sigma_m < 0$

Table 1: Algorithm for fatigue life calculation, based on modified Smith-Watson-Topper's parameter

This algorithm is based upon the modification of Smith-Watson-Topper's parameter, which takes into account the sign of the mean stress. The algorithm also presumes that for the whole range of fatigue life cycles, regardless of that if it deals with the low- or high cycle regime (elastoplastic or purely elastic strains), expression for P_{SWT} is valid. A similar algorithm, but with somewhat different

coefficients in fatigue life equation, is valid if the approach based on Manson-Coffin equation is applied. Program LifeNISA provides possibilities to calculate fatigue life in accordance with this algorithm implementing both procedures.

However, it is very important to decide which component of strain or stress tensor, or equivalent strain or stress is to be used in fatigue life calculations, but this is beyond the scope of this paper (the reader may refer to [1] and [3]).

10. Numerical solution of fatigue life equation

Introducing the following designations:

$$x = \log(N) \quad (23)$$

$$y = \frac{E \cdot \varepsilon_a}{\sigma_{0.2}} \quad (24)$$

the fatigue life equation transforms to the following form, which is very convenient for numerical solution:

$$y = A \cdot e^{\alpha x} + B \cdot e^{\beta x} \quad (25)$$

The coefficients A , B , α and β are determined as follows:

A = coefficient of N^{2b_f} in Table 1;

B = coefficient of $N^{b_f+c_f}$ in Table 1;

$$\alpha = 2 \cdot b_f / \log(e) \quad (26)$$

$$\beta = (b_f + c_f) / \log(e) \quad (27)$$

Coefficients A and B are always non negative and exponents α and β are negative, so it can be shown that, for a positive ε_a , a solution of the equation always exists.

Equation (25) is solved numerically by means of Newton-Raphson's method. After that, fatigue life estimation is determined from (23):

Due to the fact that for these functions a small change of the argument, causes a significant change in the function value, it is convenient to compare $\log(N)$ values, instead of actual fatigue life values N .

Stresses and strains are functions of point and time, but their amplitudes and mean values are only functions of point (for a periodical loading). So, the fatigue life estimation is also a function of point, as well. This function may be presented in the form of level curves or diagrams in the selected section of the model.

11. Example

Specimen used in [1] and [3], for analysis of efficiency of crack arrest holes technique, was made of aluminium alloy AlCuMg2. These specimens were experimentally analysed in [8]. Mechanical properties of this material relevant for calculation of strains and stresses are: Young modulus of elasticity 73300 MPa, Poisson's coefficient 0.33, cyclic yield strength 440 MPa and exponent of cyclic strain 0.038. The specimen is a 10 mm thick rectangular plate, with scantlings 80x160 mm. One quarter of a specimen model is shown in Figure 2, with the layout of the crack and the crack arrest hole $\varnothing 8$ mm, bored in the tip of the crack. Boundary conditions arise from symmetry of the specimen. Eight-nodded plane stress and plane stress isoparametric finite elements were used for the specimen modelling by means of NISA II finite element analysis non-linear static module. External loading is a uniformly distributed cyclic sinusoidal loading, with zero mean load with the following amplitude values: $q=62.5$; 93.8 and 125 N/mm²

(which correspond to force amplitudes in the testing machine of 50, 75 and 100 kN). Loading is represented by arrows in Figure 2.

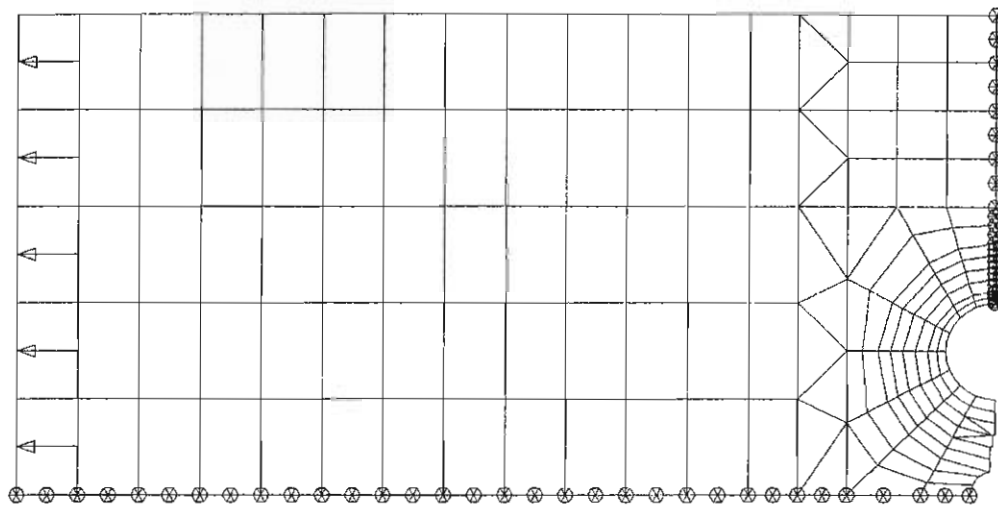


Figure 2: Specimen of AlCuMg2 used in fatigue life calculations

Results of fatigue life calculations, performed by LifeNISA, on the basis of stresses and strains previously calculated by NISA II are shown in Figure 3. Results are shown in forms of diagrams of $\log(N)$ valid for midsection of the specimen (rightmost border in Figure 2). This diagram shows curves obtained for plane stress and plane strain states separately, for the three above mentioned levels of loading amplitudes.

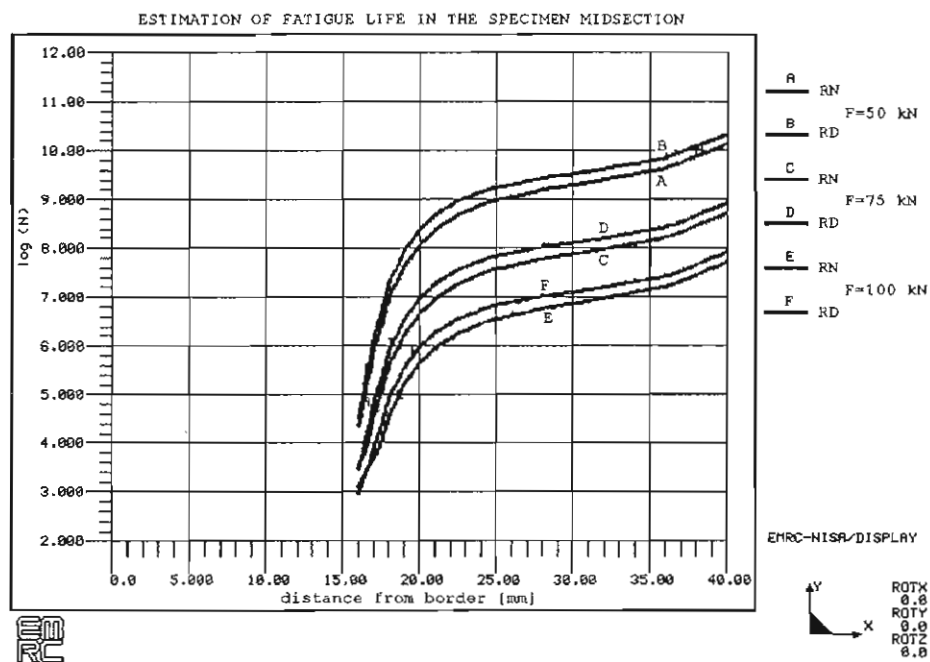


**Figure 3: Calculated fatigue life in the specimen's midsection for the three levels of loading amplitude:
RN -plane stress state; RD -plane strain state**

Conclusion

Fatigue life of a structure, or a structural part, plays a significant role in deciding about acceptance of a structural concept in design stage and even in later stages. It may be determined numerically in a calculation procedure based upon previously calculated stresses and strains. These calculations, originating from Manson-Coffin equation, or modified Smith-Watson-Topper parameter, take into account fatigue material properties. Calculation results comprise an estimation of fatigue life expressed in terms of number of loading cycles. Description of calculation algorithm, together with an illustrative example is also provided.

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THE NUMERICAL MODELING OF MIS-MATCH WELD JOINT ON FRACTURE BEHAVIOUR

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ABSTRACT

Safe use of welded structure is depending on fracture mechanics properties of weld joints. In this paper the fracture behaviour of macro different strength overmatch welded joints was analyzed. To this purpose, single edge notch bend (SENB) three point bend (3PB) specimens made with shallow crack were evaluated. The welded joints have been modeled for finite element - FE calculations, where the modify mesh elements of 2D modeling were applied. The verification of experimental and numerical results is done by using measured values of fracture behaviour as are crack tip opening displacement (CTOD), load line displacement (v_{LLD}) and crack mouth opening displacement (CMOD). The FE results shown that the cleavage fracture initiation commenced when the ability of further yield flow of weld metal has been limited by fusion line i.e. weld joint heat affected zone (HAZ). The analysis of experimentally achieved apparent fracture toughness values shown that fracture toughness values are most critical at the narrowest root region of X-grooved weld joint. Therefore, the fracture behaviour of weld joint is strongly effected by increasing of constraint in the cause of shape and width of weld joint.

1 Introduction

The recent investigations [1] of fracture behaviour of weld joint has been attempted to assessment of integrity of structure against collapse. The welded structures with strength differences between base material and weld joint metals have limited ability of deviation. In this case the constraint effect occurred. The constraint depends from the mode and velocity of loading, mechanical properties of the distinct weld joint region and about interaction of the local mis-matching weld metals. The gap of weld joint can also play significant effect on increasing of constraint. The prediction of cleavage fracture initiation is difficult in ductile to brittle transition region. Reason is that singularity of stress field at the crack tip during changing of dominance from small scale yielding (SSY) to large scale yielding (LSY) by unknown incremental crack extension. Considering that, the many researchers are using micromechanical failure models for accurate description of stress-strain field surrounding the crack tip [2,3].

The many of today used numerical conventional programs (e.i. BERSAFE, IDEAS, NISA, ANSYS) at two dimensional-2D finite element-FE calculations are assuming plane stress state or plane strain state for whole fracture mechanics specimens. The results of FE calculation at plane stress state are exhibiting greater displacements than at plane strain state. Experimental results are usually between this two extremes. This means that description of the stress/strain plane conditions at the crack tip play important role as mechanical properties of materials. The aim of this paper is present possibility to use of relative small number of elements in the 2D modeling of strength mis-match weldments for description stress-strain state at vicinity of crack tip. The verification are conformed by correlating numerical and experimental parameters of fracture mechanics.

2 Modified modeling in ductile-to-brittle transition region

Anderson [4] described varying of the crack tip stresses and strain conditions during monotonic loading of cracked specimen. In the case of small scale yielding (SSY), the fracture mechanics parameters K and J characterize crack tip condition. As the plastic zone increases in size - large scale yielding (LSY), the K dominated zone disappears, but J or $CTOD(\delta)$ are still approximately valid.

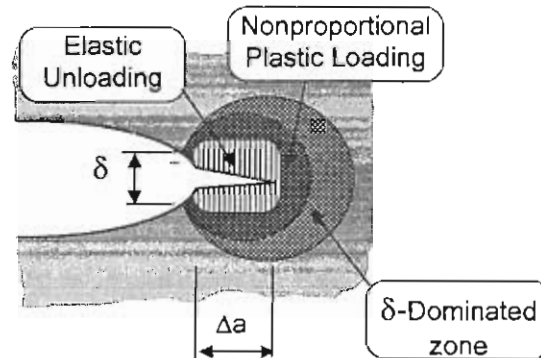


Fig. 1 Crack growth under δ - J controlled conditions

Figure 1 illustrates that material behind the growing crack tip has unloaded elasticity [5]. The material directly in front of the crack also violates the single-parameter assumption because the loading is highly nonproportional; i.e. the various stress components increase at different rates and some components actually decrease [4]. In order for the crack growth to be δ - J controlled, the elastic unloading and nonproportional plastic loading regions must be embedded within a zone of δ - J dominance. In small scale yielding, there is always a zone of δ - J dominance because the crack tip conditions are defined by the elastic stress intensity, which depends only on current values of load and crack size.

On the base of discussed effect of plasticity on the crack tip could be concluded follow:

- singularity of stress field at the crack tip during changing of dominance from SSY to LSY is permanent in δ - J dominance zone,
- stress field in vicinity of crack tip are changing from unloaded elastic to nonproportional plastic in the moment of fracture initiation process and during stable crack growth,
- during stable crack growth the achieved fracture toughness values becomes constant, and depends only on ligament size [6].

The moment of stable crack growth is difficult experimentally and also numerically determined, but the size of plasticity zone at the crack tip it is possible to calculate [7]. The analogy approach is enable to use in numerical modeling. Assumed is that strain hardening zone is large, relative to voids tearing process volume-zone and incremental crack extension (it is excluding data), size of strain hardening zone corresponding to size region where equivalent flow stress (i.e. σ_{eq} as Mises stress) is greater or equal to yield stress of material, in strain hardened material constraint increases as is according in plane strain state (i.e. $E^* = E/(1-\nu^2)$). If is this assumptions correct, than size of strain hardening zone has more significant role than incremental crack extension. This is important for description of fracture behaviour of material in ductile-to-brittle transition

region. Because the mechanical properties and size of stable crack extension vary with temperature and unstable fracture is initiated by trigger in LBZ, or transition between hard and soft weld metal or ferrite macrograin. Therefore, the size of ductile crack extension is difficult to determine. Hence, the aim of this work is determine the shape and the size of strain hardening zone and stress field at vicinity of the crack tip at the moment of unstable failure. One of possible modeling of fracture behaviour of material during FE calculation is reducing of ability to strain hardening of material at the vicinity of the crack tip, for elements where equivalent stress overcome yield stress of material [7]. The appropriation of the modified modeling will be confirmed if three characteristic values (δ_s , CMOD and v_{LLD}) of fracture behaviour exhibit at the same time good agreement between numerical and experimental results.

3 Numerical procedure

As it was mentioned, we wanted to apply modified modeling in the case of interaction different mechanical properties at same specimens. This case occurs at all welded structures. Hence, the three different types of materials in weld joints were considered (Table 1) for SENB BxB specimens, Fig. 2. As reference example was used specimens made from base metal (HSLA). Homogeneous weld joint was made by using consumable WM₁. Heterogeneous weld joints were made by using consumable WM₂ or WM₃ for two root passes and than gap was filled by consumable WM₁.

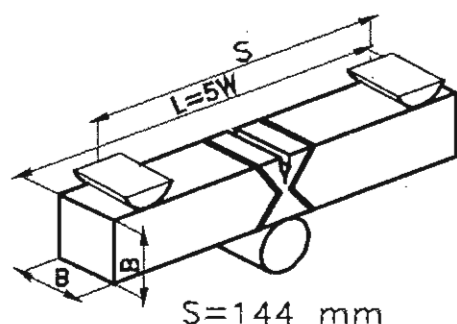


Fig.2 SENB BxB specimens made with shallow crack in the middle of mis-matching weld metal

Table 1 Mechanical properties of base metal-BM and all weld metals at the room temperature and testing temperature -10°C

material	Temp. °C	E MPa	R _{p0.2} MPa	R _m MPa	α -	σ_0 MPa	n -	M
Base metal	20°	201351	711	838	0.943	679	0.091	-
	-10°	208986	712	846	0.941	676	0.095	-
WM _{1-cap}	20°	205525	861	951	0.900	833	0.074	1.21
	-10°	211409	873	1041	0.897	833	0.107	1.22
WM _{2-root}	20°	205808	769	818	0.870	750	0.055	1.08
	-10°	205778	808	871	0.905	787	0.059	1.14
WM _{3-root}	20°	207485	623	677	0.926	604	0.058	0.88
	-10°	207594	649	706	0.935	629	0.058	0.91

M is mis-matching factor in Tab. 1, it is ratio between yield stress of weld metal and base metal.

Two-dimensional, plane-strain and modify plane-strain+stress finite element analyses are performed to obtain detailed fracture behaviour of specimens and equivalent stress in vicinity of crack tip with load increasing. Actually, we would simulated by FEM fracture behaviour during CTOD testing. Table 2. lists the planar geometry studied with different crack depth and different constituents of materials in weld joint.

An actual welded joint is rather complicated, both metallurgical and geometrically. For simplification in the following analysis the effect of heat affected zone, residual stresses of other kind of heterogeneity are omitted. e.i. fusion line is modeled as an interface in a bi or three-material system.

Table 2. Specimens BxB (B=36mm) with different crack lengths

Specimen	a (mm)	cap	root
Base metal	18.220	HSLA (HT80)	
Over-matching	9.912	WM ₁ M=1.22	
Mis-matching	14.386	WM ₁ M=1.22	WM ₂ M=1.14
Mis-matching	14.370	WM ₁ M=1.22	WM ₃ M=0.91

Due to symmetry only one-half (hatched part) of the specimen was modeled. At the surface of specimen are specified nodes-points, which correspond positions of measures points for CTOD(δ_s), CMOD and v_{LLD} , Fig. 4.

For constitutive model one axial stress-strain behaviour is described using the Ramberg-Osgood power law. The values of parameters in Ramberg-Osgood power law are approximated experimental measured data obtained by tensile testing specimens, Tab.1. The linear-elastic condition plane stress state was supposed as long as equivalent stress magnitude (von Misses) does not overcome the yield stress of material where is crack tip located. Thereafter, the elements where were magnitude of stress is greater than yield stress prevail to plain strain condition. This means that middle node is shifted to half side of edge of the element and model is prevail to $E^*=E/(1-\nu^2)$. The rest elements where magnitude of equivalent stress is lower or equal to yield stress a remain in plane stress condition. This modified modeling causes relaxation of equivalent stress at the crack tip and extension of strain hardening process zone, an example is shown in Fig. 3.



Fig. 3 Equivalent stress field at the crack tip for the uniform plane stress state and for the modified plane strain plus plane stress state at the same loading level

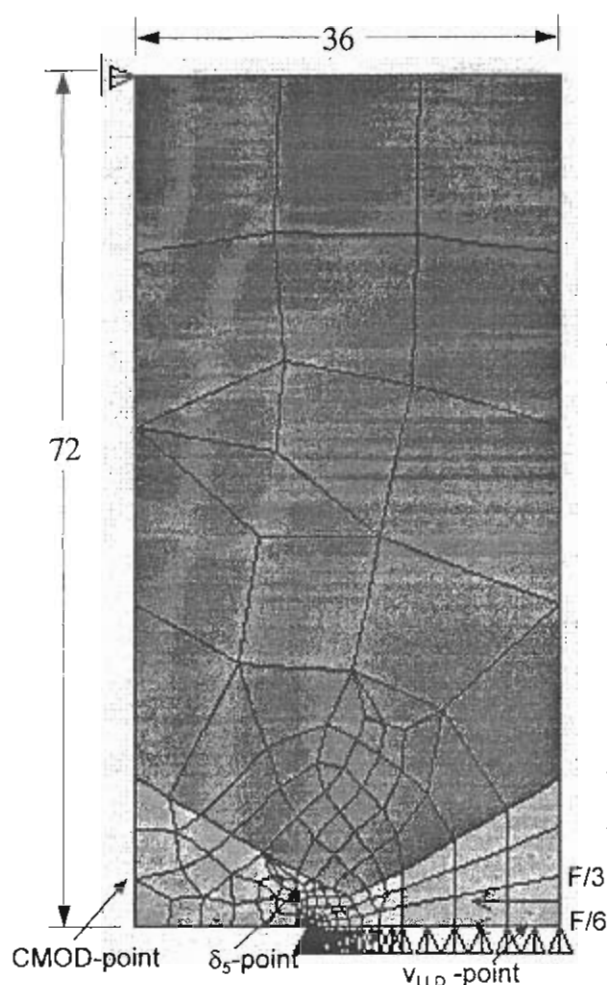


Fig.4 Mesh and boundary fixed supports on one half of BxB specimens

4 Results and discussion

The fracture behaviour of specimens have been calculated with quasistatic increasing of load up to the point of experimental instability. The stress-strain conditions at surface of specimens, displacements at specified points (δ_s , CMOD, v_{LLD}) and maximum equivalent stress σ_{eq} were calculated. In interest of comparison fracture behaviour between specimens made with different constituents materials and different crack depth a/W is convenient to interpret the results with load ratio F/F_y . The yield load F_y depends from geometry of specimens and kind of loading (e.i. bending, tensile), crack depth and yield stress of material [9]. Normalized load F/F_y is kind of utility load level, also named ratio of load carrying capacity.

The FE model has been tested through comparison between FE calculation results and experimental results on specimens made all from isotropic base metal. The comparisons between FE calculated displacements (CTOD, CMOD and v_{LLD}) and experimental measures values are showed very good agreement between those results. Actually, with FE results is possible to follow to curves obtained by experiments.

The comparisons between FE calculation and experimental results on BxB specimens made from weld joints were presented on Figs. 5 and 6.

In all specimens except BM after some amount of slow crack growth brittle fracture commenced. This behaviour characterize failure in ductile-to-brittle transition region.

Conventional small strain theory of plasticity is employed in the finite element analysis by using ANSYS 5.0 software [8]. The mesh was automatic generated with determined a radius $r=0.5$ mm of first row of crack-tip elements. Fig. 4. presents the mesh on one half of BxB specimens.

Analysis made on two-dimensional 8-nodes isoparametric crack-tip elements. This elements are essentially 4-sided elements which at the crack tip collapsed in triangle elements with three coincident nodes and for the crack tip stress concentration was assigned.

FEM calculation was performed by successive increase of force. The all nodes of elements in axis y (axis of specimen symmetry) and node in support position (x axis) were fixed. The loading is distributed in two nodes $F/2=F/6+F/3$ to avoid local stress and strain $F/3$ concentration, as is shown in Fig. 4.

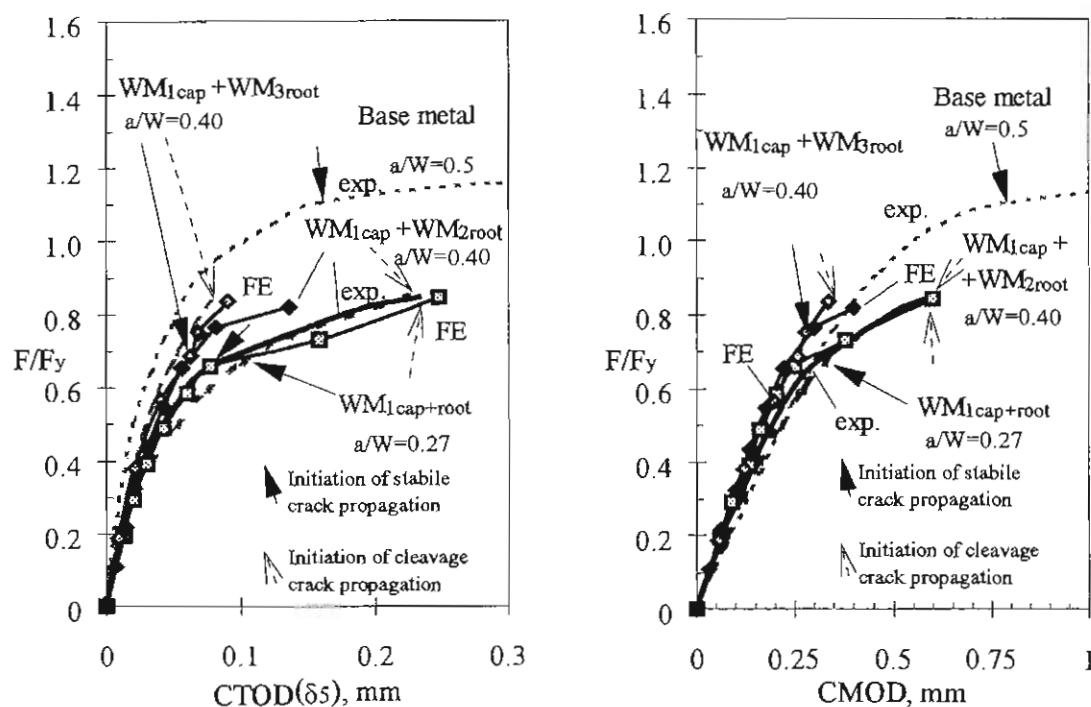


Fig. 5 F/F_y vs. $CTOD(\delta_s)$ and F/F_y vs. $CMOD$ in comparison between FE calculation results and experimental results on BxB specimens made from weld joints

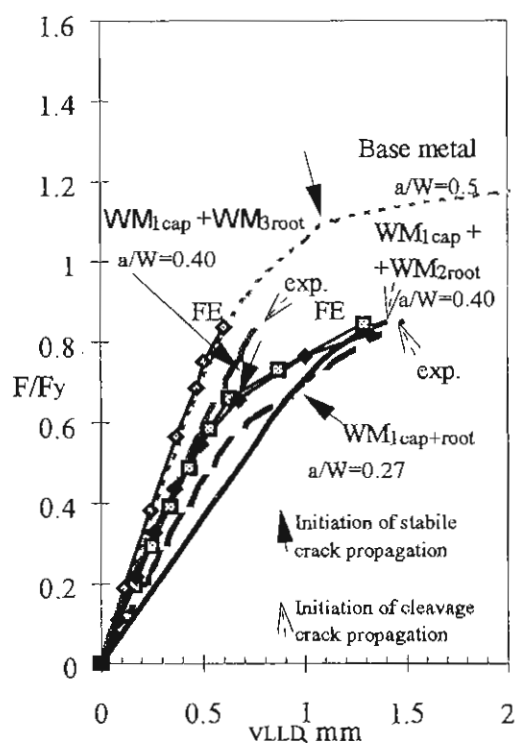


Fig. 6 F/F_y vs. v_{LLD} in comparison between FE calculation results and experimental results on BxB specimens made from weld joints

The agreement of FE calculation results and experimental results is significantly lower than as is in case of base metal. Important reasons are follow:

- 1) The effect of simplified of geometry in case of specimen made from heterogeneity weld on δ_s and v_{LLD} values. The δ_s value was measured at surface of specimen where is weld metal WM_{1cap} , while in FE calculation was performed in the middle section of specimen where is weld metal WM_{2root} or WM_{3root} . The reason for deviation of v_{LLD} values is that the main resistance to bending during experiment is caused by over-matched weld metal WM_{1cap} .
- 2) The size effect of fatigue plastic wake at crack tip on specimens has influence at value v_{LLD} displacement. Result is that fatigue crack length a_0 should replace with effective crack length a_{eff} relate to size of fatigue plastic wake.

As, was mentioned, the FE calculation with successive loading and changing plain stress to plain strain state of hardened mesh elements is completely independent from experimental results. This changing of state at crack tip relaxes stress and shifted the concentration point in direction of further plastic flow or later crack propagation. Actually is simulated same behaviour at crack tip as describe in introduction. Kirk and Dodds [10] are also obtained similar results by use the O'Dowd and Shih model. The good agreement between FE and experimental results for base metal (all values CMOD, v_{LLD} and CTOD(δ_s) are conformed this results.

Therefore, with changing state from plane stress to plane strain at crack tip elements the concentration of equivalent stress σ_{eq} decreases. Decreasing of equivalent stress σ_{eq} is possible to understand as finishing of hardening at vicinity of crack tip. But with further loading the σ_{eq} again start to increase, what caused (now away from crack tip) further plastic flow with hardening material and ductile or brittle fracture. Eventually, in ductile-to-brittle transition region the cleavage fracture interrupt the hardening and ductile tearing isolates region in volume of material. This is evident from fractographic observation of fracture surfaces [11].

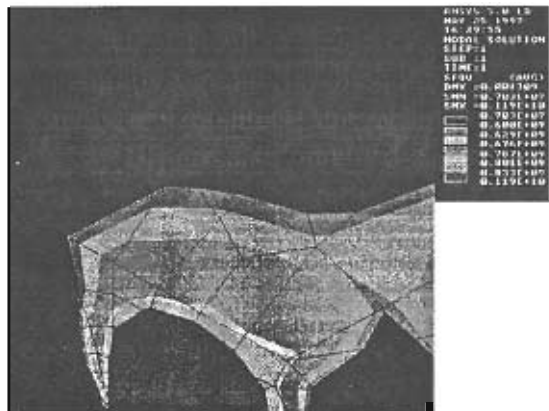


Fig.7 Stress field at the crack tip for homogenous weld joint $WM_{1cap+root}$

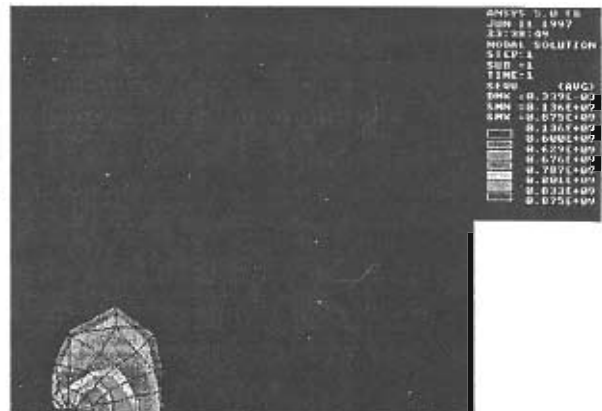


Fig.8 Stress field at the crack tip for heterogeneous weld joint $WM_{1cap}+WM_{2root}$



Fig.9 Stress field at the crack tip for heterogeneous weld joint $WM_{1cap}+WM_{3root}$

The observations of equivalent stress field at the crack tip confirmed that:

- homogeneous overmatch weld joint $WM_{1cap+root}$, Fig. 7 is contour for $\sigma_0=801$ MPa (curve between yellow and orange color for WM_{1root}),
- heterogeneous overmatch weld joint $WM_{1cap}+WM_{2root}$ Fig. 8 is contour for $\sigma_0=787$ MPa (curve between yellow and green color) and
- heterogeneous overmatch weld joint $WM_{1cap}+WM_{3root}$, Fig. 9 is contour for $\sigma_0=629$ MPa (curve between light blue and dark blue color).

Higher plastic flow in the WM_{3root} weld metal over narrowest root region has occurred. The contour of flow stress (von Mises, $\sigma_{eq}=629$ MPa) is penetrated to base metal, which did not start plastic flowing, because the yield stress of base metal is higher ($\sigma_{eq}=676$ MPa). This "irregularity" is consequence of higher yielding through narrowest width in the root region of weld joint.

5 Conclusions

Performed analyze on strength different weld joints are exhibiting understand good agreement between FE calculated displacements and experimental measured values. Through the analyze of results is evident that used FE calculation procedure is appropriate not just for isotropic base metal at ductile crack growth but also for mismatching weld joints in ductile-to-brittle transition region.

The conditions of stress fields at the crack tip have been determined in the moment of failure instability by using numerical finite element (FE) calculations. The comparison between numerical and experimental results shows that instability begins when ratio of load carry capacity (F/F_y) is lower than 1 at the all kind of weld joints, meanwhile, for base metal this ratio is greater than 1. This is consequence of increasing constraint effect of weld joints and interaction of weld metals regions during loading (LBZ, soft weld metal region, ferrite macrograin and different weld metals as well). As mentioned, the shape and width of weld joint are strongly effected by constraint effect in the weld joint. In the case of an locally overmatched HAZ with respect to the weld metal, the HAZ will act as an barrier against plastic deformation in the base metal. Because of HAZ, following the shape of weld gap, we can assume that highest constraint effect has to be near the narrowest width of weld joint of BxB specimen. Therefore, the most critical fracture behavior was found on series of BxB specimens with the pre-crack tip in the middle of X-groove multipasses weld joints. The initiation of instability began as the contour of plastic flow comes to near fusion line as is presented in Figs.7, 8 and 9. It seems that instability fracture began when the utility of plastic flow in weld gap is finished.

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FATIGUE LIFE PREDICTION OF WELDED JOINT BY SIMULATION OF SURFACE CRACK GROWTH

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ABSTRACT: The fatigue crack growth in butt welded joint subjected to amplitude loading is considered. Fracture mechanics K - concept applied to semi - elliptical surface crack and Paris law are used for fatigue life prediction. The influence of weld reinforcement and undercut are analysed. Finite Elements Method is used to determine local stress distribution in weld reinforcement area. Cycle-by-cycle simulation method is proposed for fatigue life prediction, if direct integration of Paris equation is not possible. Obtained results, comparison to literature and discussion are presented.

Keywords: Fatigue, welded joints, Paris law

1. INTRODUCTION:

Fatigue life prediction of welded joint by using the fracture mechanics concept is quite uncertain because of many reasons: complex geometry, stress concentration, heat affected zone, residual stresses, cracks, and of ... influences that cannot be successfully taken into account. Partially stochastic nature of crack initiation and crack growth disables, for now, the complete exact solution of this problem, but great number of investigations is devoted to particular solutions. Among all the parameters that can reduce the fatigue strength of welded joint, the weld toe geometry is probably the most important one. The aim of this paper is to analyse the difference in fatigue life due to weld toe geometry. Welded joints subjected to constant or variable amplitude loading usually fail due to initiation and growth of surface semi-elliptical crack in weld toe area. Therefore, the fracture mechanics model of surface crack and Paris law are used for fatigue life prediction.

2. THEORETICAL BACKGROUND

A great number of crack propagation laws can be found in literature, based on experimental values [1,2], crack tip plasticity [3,4], crack closure concept [5,6], etc. Among all these laws, the Paris law [1], defined by equation (1) is mostly used in practice, because of its simplicity and acceptable accuracy in practice:

$$\frac{da}{dN} = C(\Delta K)^m \quad (1)$$

where: da/dN - crack growth rate (mm/cycle),
 C, m - material constants,
 ΔK - stress intensity factors range,
 $\Delta K = K_{\max} - K_{\min}$. In the case of completely buried elliptical crack [7], K is defined by equation (2):

$$K = \frac{\sigma \sqrt{\pi a}}{\Phi} (\sin^2 \varphi + \cos^2 \varphi)^{1/4}, \quad (2)$$

where Φ is the second kind elliptic integral:

$$\Phi = \int_0^{\pi/2} \sqrt{1 - \sin^2 \alpha \sin^2 \varphi} d\varphi, \text{ with } \sin^2 \alpha = (c^2 - a^2)/c^2 \quad (3)$$

a, c - half crack width and depth, respectively.

Exact analytical solution of this integral is not found yet, but by using the series expansion [8], Φ can be approximated to an accuracy of about 5% by:

$$h = 1 - \left(\frac{a}{c}\right)^2, \quad n = (1 - \sqrt{1 - h^2})/h^2, \quad \Phi = \frac{\pi}{2(1 + n)} \left(1 + \frac{n^2}{4}\right) \quad (4)$$

3. PROCEDURE

Butt weld with reinforcement and various weld toe geometry, subjected to constant amplitude loading is considered. For semi-elliptical surface crack (and direct opening mode) stress intensity factor K has the form:

$$K_I = C_C \sigma \sqrt{\pi a} / \Phi, \quad (5)$$

where: σ - the nominal stress
 C_C - correction function that includes corrections for finite plate width, free front surface and stress concentration (6)

$$\begin{aligned} C_c &= [C_1 + C_2(a/B)^2 + C_3(a/B)^4] C_4 V_1 W_1 \\ C_1 &= 1.13 - 0.09(a/c) \\ C_2 &= -0.54 + 0.89 / (0.2 + (a/c)) \\ C_3 &= 0.5 - 1.0 / (0.65 + a/c) + 14(1.0 - a/c)^2 \\ C_4 &= 1 + [0.1 + 0.35(a/B)^2](1 - \sin \varphi)^2 \\ V_1 &= [\sin^2 \varphi + (a/c)^2 \cos^2 \varphi]^{1/4} \\ W_1 &= [\sec(\pi c \sqrt{a/B} / W)]^{1/2} \end{aligned} \quad (6)$$

This, quite complex function described in [7], depends of crack depth (a), crack width (c), plate width (W) and thickness (B). During the simulation, the value of this function is calculated for each cycle of loading. Stress concentration factors for different weld toe shapes are determined by FEM. Considering the high cycle fatigue, where great scale yielding is not expected, linear elastic model

with four - node plate elements is used. The mesh refinement in weld area is used in order to improve the accuracy of calculation. The typical meshes (because of symmetry, only a quarter of model) are shown on Fig. 1 - 2.

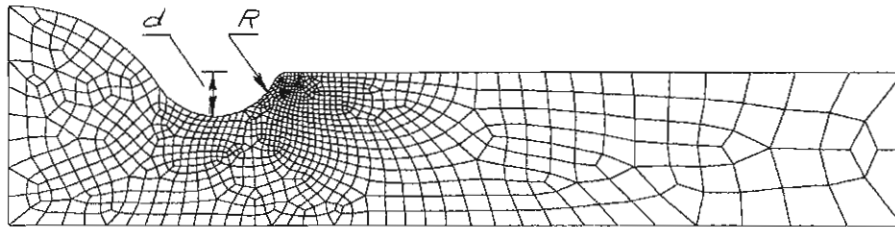


Fig. 1: Weld toe with undercut, a - type

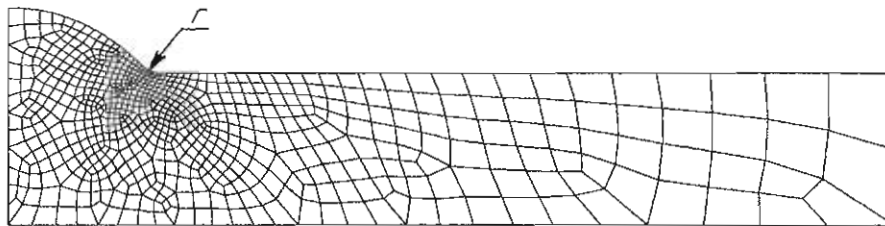


Fig. 2: Weld reinforcement with rounded toe, b - type

Typical stress distribution in reinforcement area is shown on Fig. 3 - 4. By using the FEM, stress concentration factor are determined for all shapes, and they vary from $K_t = 1.51$ to $K_t = 3.18$. The influence of stress concentration on fatigue life is showed on Fig. 5.

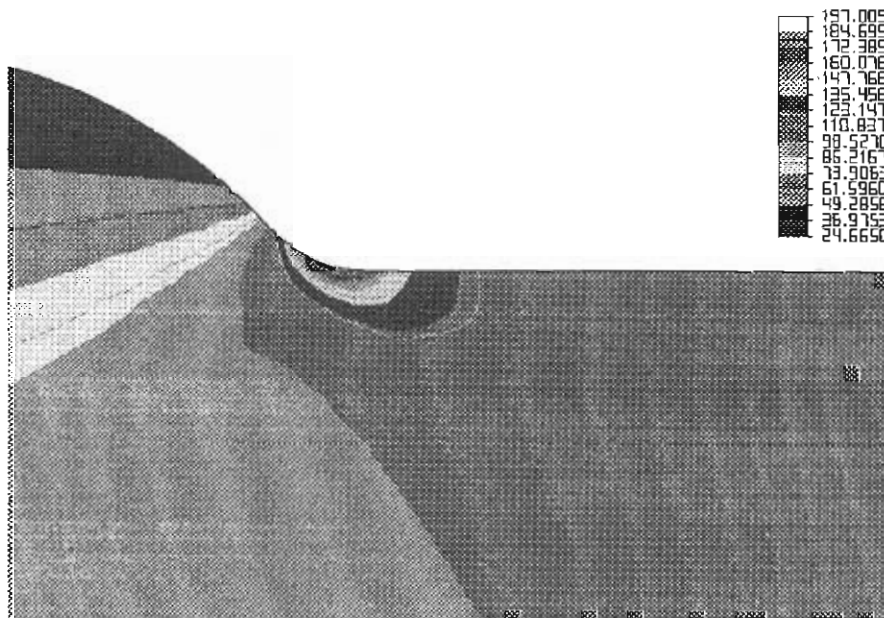


Fig. 3: Stress field in weld toe area - rounded toe

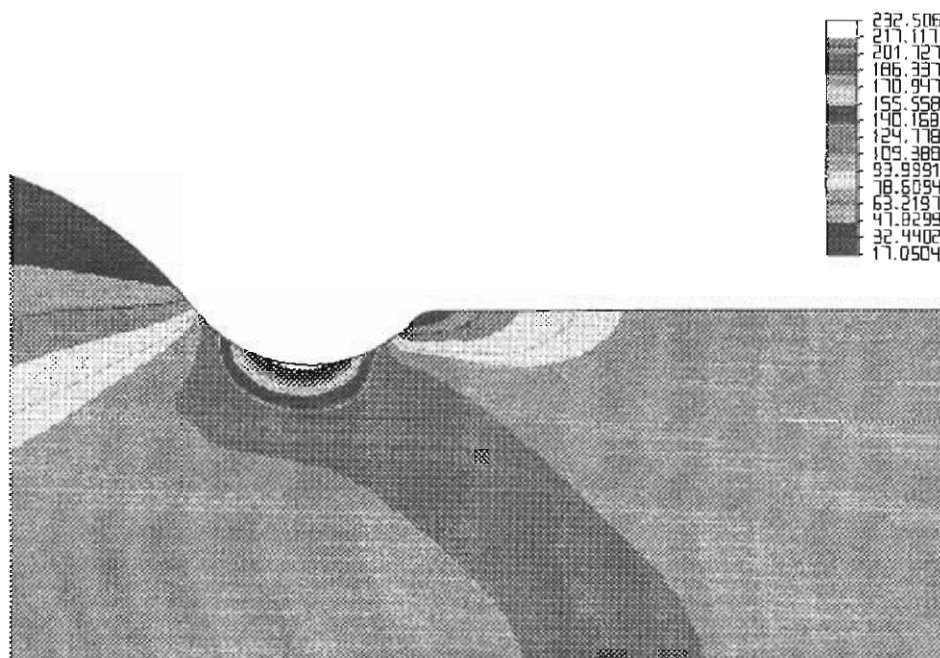


Fig. 5: Stress field in weld toe area - undercut

Undercut is considered as irregular groove situated along the weld toe (Fig.1), caused by careless electrode control or large heat input. This kind of defect reduces the fatigue strength of welded joint, so various shapes of undercut are analysed and compared to welds with regular rounded toe. The weld toe geometry is defined by: R - undercut radius, d - undercut depth (for a - type) and r - weld toe radius (for b - type). The reinforcement shape, aspect ratio a/c and smaller radius close to the end of undercut are kept constant. The model width is 20 mm and nominal stress is 105 MPa, with stress ratio $R = \sigma_{\min}/\sigma_{\max} = 0$. Fatigue thresholds, conditions of crack initiations and early crack growth are not analysed, for all the models initial crack depth is set to $a_0 = 0.1$ mm. Material constants were $C = 3 \times 10^{-13}$ and $m = 3$, as recommended for majority of welds in structural steel in [9]. The criteria of failure were: a) $K_I = K_{IC}$ (stress intensity factor equals the fracture toughness), b) crack depth equals half depth of the model (which always occurs earlier). Results of fatigue life prediction and dependence of weld toe shape are showed in Table 1.

Model type	R (mm)	d (mm)	r (mm)	Max. stress (MPa)	Cycles to failure
1.a	0.5	0.5	-	334	71.600
2.a	1	0.5	-	228	225.200
3.a	1	1	-	298	100.800
4.a	1.5	0.5	-	230	219.400
5.a	1.5	1	-	258	155.400
6.b	-	-	0.5	224	237.500
7.b	-	-	1	186	414.800
8.b	-	-	1.5	169	553.000
9.b	-	-	2	159	664.100

Table 1. The results of fatigue life prediction

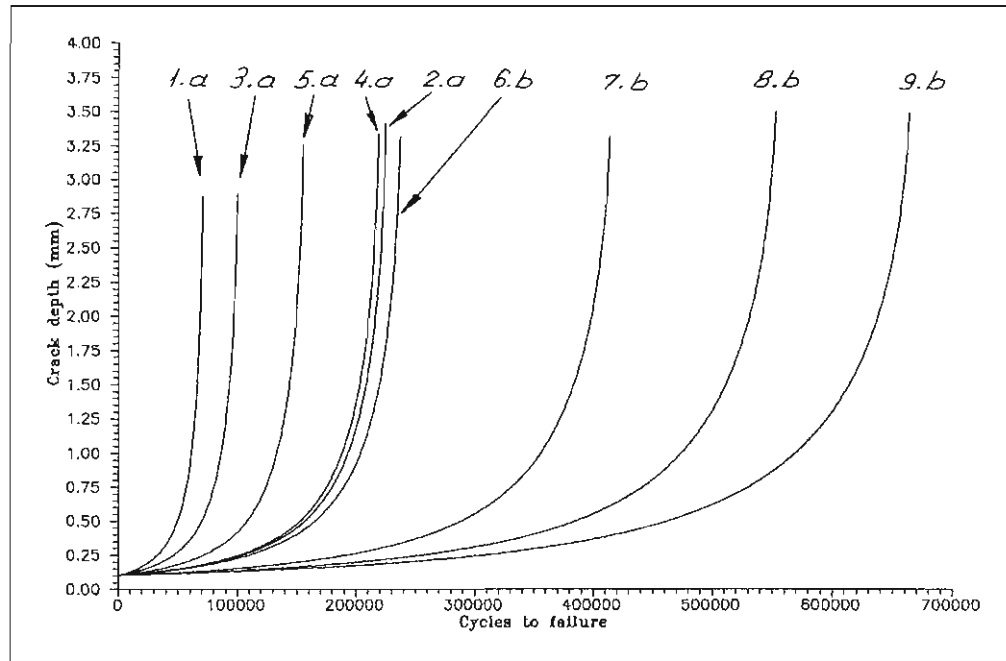


Figure 5. The results of crack growth simulation

The results of cycle - by - cycle crack growth simulation for all models is shown on Fig. 5., together with predicted fatigue life.

4. DISCUSSION

Generally, the fatigue life prediction can be done by integrating the Paris equation from initial size a_0 to critical size a_c . Separating the variables and integration of equation (1) leads to:

$$N = \frac{a_c^{(1-m/2)} - a_0^{(1-m/2)}}{\pi^{m/2} (1 - m/2) C_c^m C \Delta \sigma^m} \quad (6)$$

The equation (6) is valid for $m \neq 2$ and if correction function C_c is independent of crack length. In this case, correction function C_c depends of the crack length, so direct integration is not possible. Therefore, the cycle - by - cycle counting is used, and computer program is developed to facilitate all the necessary calculations. The simplified flow chart for calculation is shown on Fig. 6.

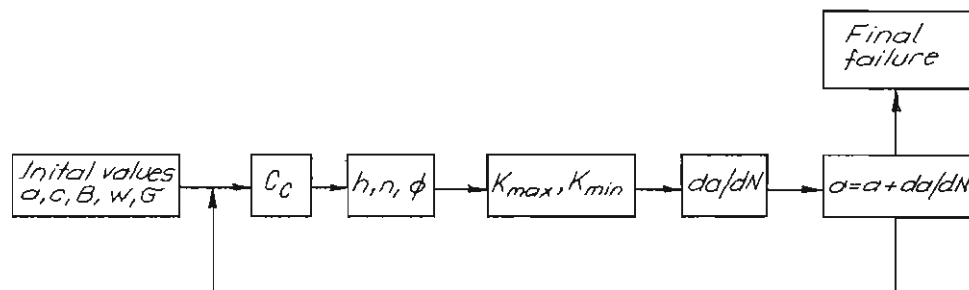


Fig. 6: Simplified flow chart of cycle by cycle counting

The aspect ratio a/c , according to references, during the crack growth changes from $a/c = 0.2$ (initial crack) to $a/c = 0.6$ (final failure) due to higher K_I value at the crack front. The constant aspect ratio $a/c=0.4$ is used as approximation of this process for fatigue life prediction.

The influence of cumulative errors due to small value of constant C and great number of calculations is analysed. The comparison of obtained results showed that these errors could be neglected, and the whole calculation could be shortened by keeping the crack growth rate constant during the period of for example 10 - 100 cycles, with cumulative error less than 1%.

5. CONCLUDING REMARKS

The obtained results show good agreement to the results found in literature and illustrate the importance of regularly rounded weld toe: extreme case, narrow crack - like undercut, reduces the fatigue life ten times. The best results showed b - type model (curved toe with radius $r = 2$ mm) what is in agreement to recommendations in literature. The whole method, by using the simplification showed in [8] and computer program for cycle - by cycle counting, it is easy to handle, and can be successfully used in practice. The accuracy depends of material constants C and m that are used, and the best way is to obtain these constants by experiment. The next step in this work will be attempt to include the effect of residual stress field in this model, and to generalise the model on different weld shapes.

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DETERMINATION OF FATIGUE STRENGTH PROPERTIES OF ALUMINIUM ALLOYS

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ABSTRACT. The paper presents the procedure and results of stress analysis and fatigue tests of aluminium alloy cast specimens (material AlSi7Mg0.6) with different surface treatments. The stress distributions on all types of specimens were estimated experimentally, by strain gauges, and numerically, by finite elements method (FEM). Six groups of cast specimens (different manufacturing procedures) were tested in mechanically driven bending test machine under constant amplitude and stress ratio $R = -1$. The Wöhler (S - N) curves were obtained after statistical analysis of results at all load levels.

This research is a part of COPERNICUS Project No. 1465, Contract No. CIPA - CT94 - 0219, European Commission in Brussels, coordinated by Prof. Dr.-Ing. Vatroslav Grubišić and Fraunhofer Institut für Betriebsfestigkeit.

Key words: fatigue strength, fatigue tests, aluminium alloys, surface treatment, strain gauges application, numerical stress analysis.

1. INTRODUCTION

A significant increase of aluminium alloy application is expected in the industrial countries due to the arising demands of light-weight design of cars, trains and ships. To support the usage of aluminium alloy structural components (good manufacturing properties, resistance to corrosion, strength, mass production ability and a high recycling potential) it is necessary to increase the knowledge of the possible improvement of strength properties by different manufacturing procedures. The investigation was carried out on the materials AlSi7Mg0.6 produced by permanent mold casting and pressure die casting. The specimens for the fatigue strength tests were prepared: as casted, as casted and shot peened, as casted and cold-formed, machined, machined and shot peened, machined and cold-formed.

Each partner institution in this Copernicus project (University of Ljubljana, Slovak Academy of Sciences, ŠKODA Plzen and Fraunhofer Institut Darmstadt) carried out its part of investigation, and the results of FESB - University of Split tests are presented here.

2. TEST SPECIMENS

Three groups of specimens were manufactured at FESB - University of Split, Laboratory for welding, and Škoda - Plzen, according to orders of Fraunhofer Institut für Betriebsfestigkeit, coordinator of this Project. The shapes and measurements of specimens are shown in Fig.1. Chemical compound and mechanical properties of the AlSi7Mg0.6 alloy are presented in Tab.1 and Tab.2.

The specimens from first group (group A1) were cut from different places of a wheel rim for commercial vehicles, and machined to required shape and dimensions. The second group specimens (group B1) were casted to final measures, and the specimens of the third group (group C1) were casted and fine ground to final shape and dimensions. The second half of specimens (groups A2, B2 and C2) are the same as specimens from first three groups, but cold formed by compression (degree of deformation approx. 8%).

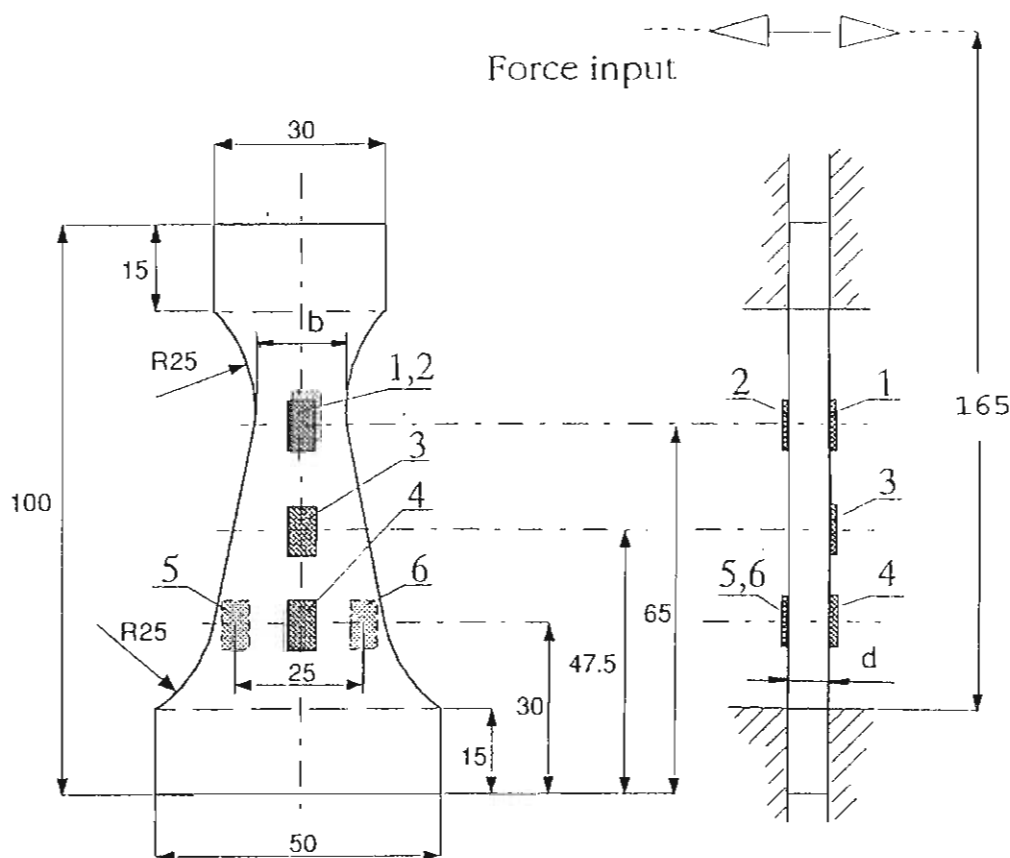


Fig.1 Test specimen and strain gauges application

Si	Mg	Mn	Ti	Fe	Cu	Reference
7,0	0,5-0,6	max. 0,05	max. 0,15	max. 0,15	max. 0,04	According Standards
7,2	0,58	0,00	0,1	0,06	0,002	Škoda Plzen
>6,5	0,58	0,01	0,09	0,09	<0,01	TLM Šibenik

Ultimate strength, R_m (MPa)	243
Yield point $R_{p0.2}$ (MPa)	223
Elongation A5 (%)	4
Contraction Z (%)	1,4
Module of elasticity E (MPa)	70000
Poisson ratio μ	0,3

Table 1 and Table 2 Chemical compound and mechanical properties of AlSi7Mg0.6 alloy

3. TEST PROGRAM

3.1. Stress analysis by strain gauges

The strains and stresses at specimens from all groups were analyzed experimentally by means of strain gauges. The 1.5/120LY13 (HBM) strain gauges were applied at six positions at one specimen from group C1, Fig.1, and two opposite positions on specimens from all other groups (strain gauges No. 1 and 2), because of different thickness and width of specimens and different positions in bending test machine during the fatigue tests.

For the calibration, the specimens with the strain gauges were loaded at four load levels in all possible sites in the bending test machine. Some results of calibration are shown in Fig. 2.

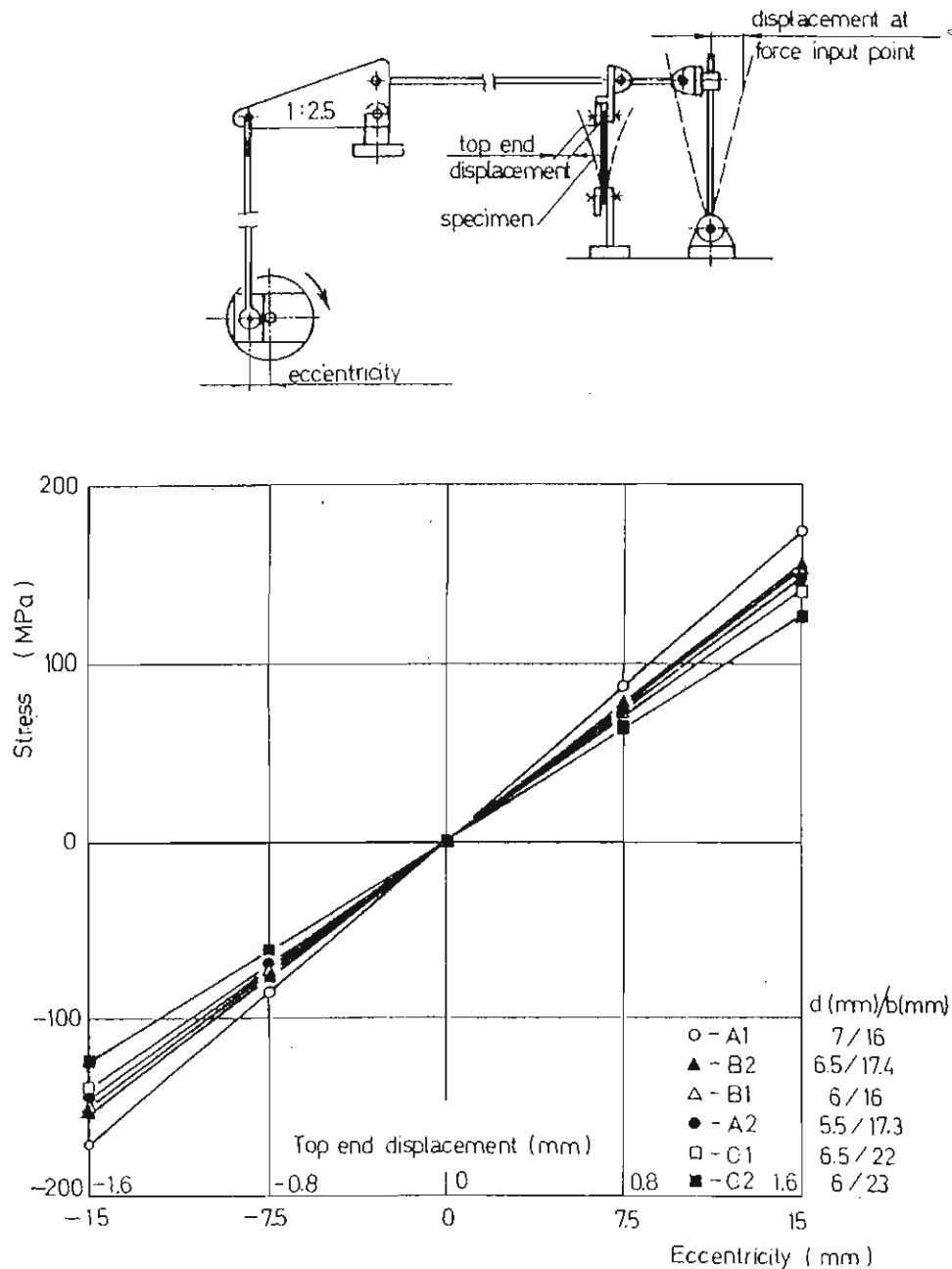


Fig.2 Calibration of bending specimens (strain gauge No.1, all types of specimens)

3.2. Numerical stress analysis

Numerical stress analysis was carried out by the finite elements method (FEM), applying commercial "Algor" program on 486 PC. Finite elements mesh, boundary and load conditions (top end displacements) are shown in Fig.3 and Tab.3..

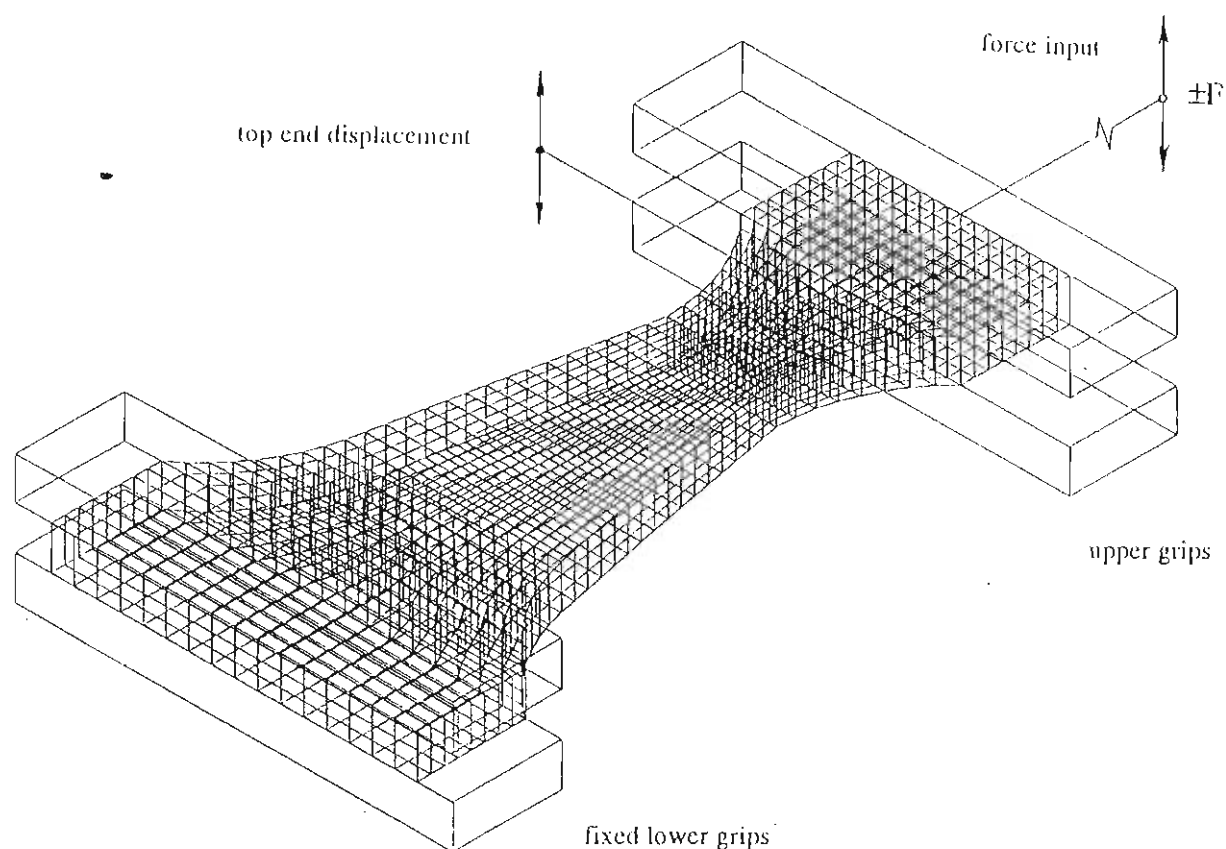


Fig.3 FEM mesh, boundary and load conditions

Thickness and type of specimen (mm)	Top end displacement (mm)	Displacement at force input line (mm)	Force (N)	Maximal stress (MPa)
5.5 (C2)	1.0	3.82	62.5	98.8
5.5 (C2)	2.0	7.64	125	197.7
6 (C1,B2)	1.0	4.00	81	107.7
6 (C1,B2)	2.0	8.00	162	215.5
6.5 (B1,A2)	1.0	3.86	103	116.6
6.5 (B1,A2)	2.0	7.72	206	233.3
7 (A1)	1.0	3.83	128	125.5
7 (A1)	2.0	7.66	256	251.0

Table 3 Results of FEM stress analysis for all types of specimens

3.3. Fatigue test machine

All fatigue tests were carried out at FESB-University of Split, Laboratory for fatigue strength. The plane bending cyclic loading was realized in BTM-1 machine, Fig.4, designed and manufactured at FESB-Split and Shipyard Split, Sector Testing. This machine is driven by electric AC motor and belt drive. The eccentric slider makes possible the cyclic displacement of four specimens simultaneously. Specimens are attached in the jaws by screws, loaded at the upper end, and fixed at the lower end. The cyclic displacements of upper end represent the fatigue loads within constant levels, $R = -1$. The optical-electronic devices for cycle counting and failure detection are mounted over each specimen.

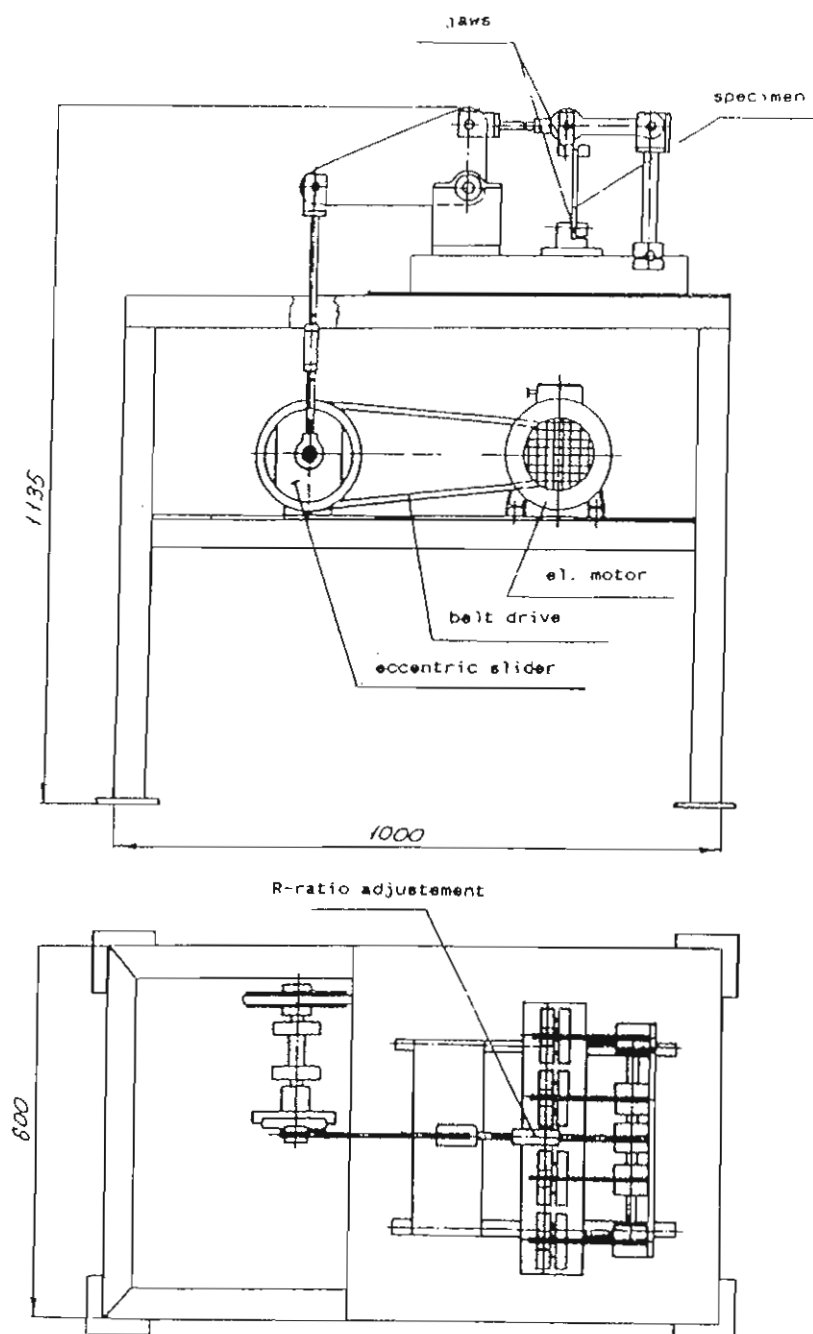


Fig.4 Scheme of the BTM - 1 fatigue test machine

4. TEST RESULTS

4.1. Results of stress analysis

The stress analysis showed different results for each group of specimens due to their different thickness. In both stress analysis, numerical and experimental, the maximum stresses were estimated in cross section of specimen with minimal width, what was caused by the position of load introduction, Fig.1 and 2. The strain gauges were applied in this, critical, cross section at both sides of tested specimens. The stresses in these positions were chosen as a nominal stress amplitudes in fatigue analysis, and all failures occurred there, Fig. 5.

All results of strain gauges application and all results of numerical stress analysis are presented in Copernicus Project report, and some of them are shown here in Tab. 3 and Fig. 6. Very closed stress results with both procedures direct to good analysis parameters and good designed specimens, without any sharp stress concentrations.

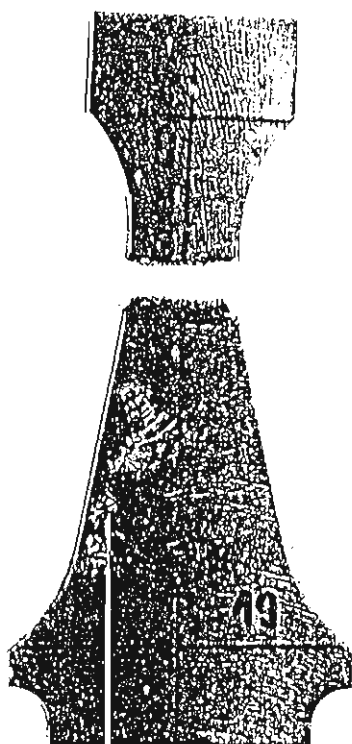


Fig.5 Broken specimen of group C

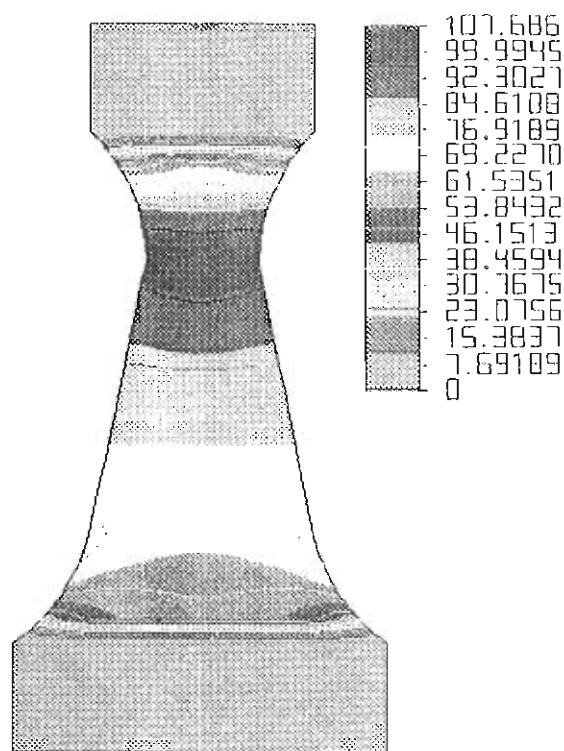


Fig.6 Stress analysis results for specimen C1 and top end displacement 1.0 mm

4.2. Fatigue test results

The fatigue test results of all groups of specimens were plotted in uniform double logarithmic S-N diagrams. The statistical analysis at every stress level represent Gaussian (log.) distribution with outlined probabilities of survival: $P=90\%$, 50% and 10% . These results were used for deriving the unified scatter band of S-N curves between $N=10^4$ and 2×10^6 cycles. The part of S-N curves above 2×10^6 cycles (endurance limit) was not determined, because only few test data were available for this area. Figure 7 and 8 show some results of fatigue tests. In particular the unified scatter bands in these figures are defined by the chosen stress amplitudes S_a , the slopes of the S-N curves k , and the calculated scatter ranges T_n at chosen stress levels.

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5. CONCLUSIONS

- ◆ The fatigue strength of specimens taken out from the wheel (A1) and individually casted (B1 and C1) show no significant differences.
- ◆ The cold deformation of specimens (by compression, A2, B2 and C2) resulted in decreasing fatigue strength at all three groups of specimens.
- ◆ Equal fatigue tests were performed at different laboratories to allow the comparison of the test results. Figure 9 shows the results of fatigue tests carried out at FESB University of Split and Slovak Academy of Sciences. Both test series show the same results.
- ◆ The results of fatigue tests with variable amplitudes, shot peened specimens and with fretting corrosion will be published by other Copernicus project partners.

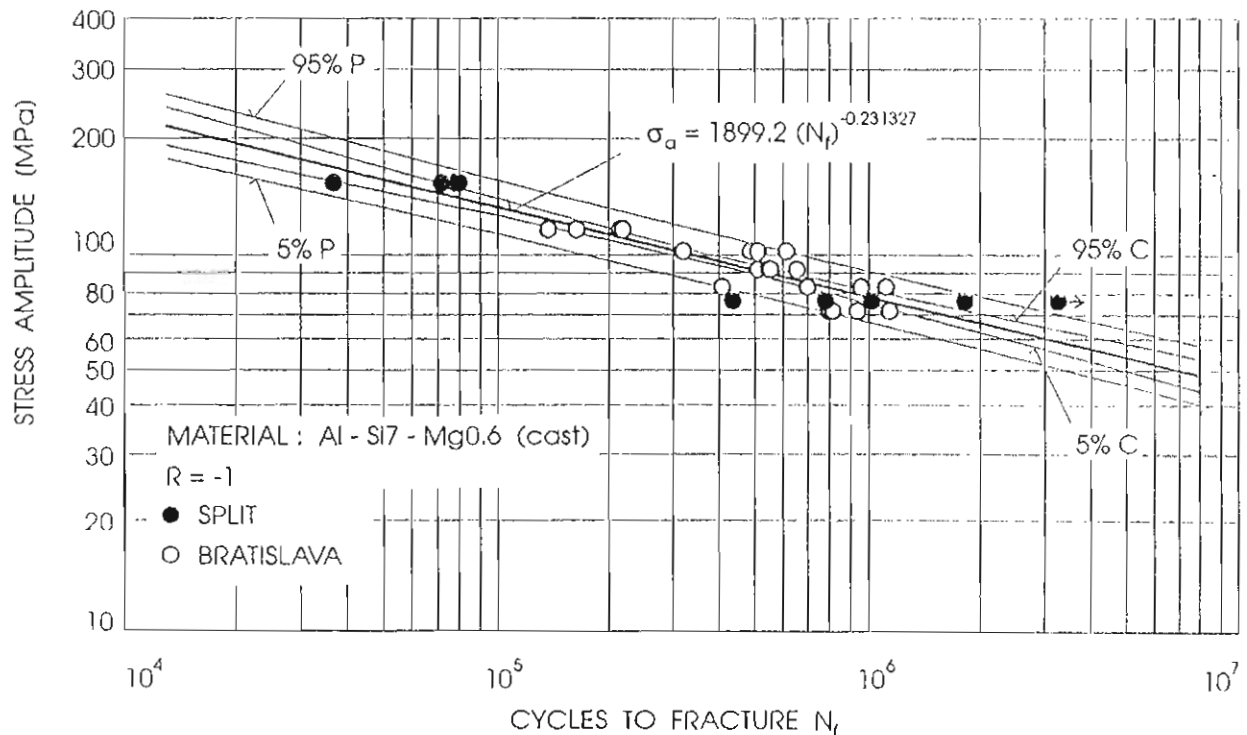


Fig.9 Comparison of fatigue test results from FESB-Split and SIAS-Bratislava

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Determination of Stress Intensities for Three-dimensional Cracks

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Summary

The increasing number of analytical and numerical methods, together with a development in a computer technology, is introduced in the past few years, but experimental verification of this results are often necessary.

A direct application of photoelasticity for SIF measurement is looking for a different approach then usual photoelastic measurements. The photoelastic effects in a vicinity of a crack tip is always shadowed by the influence of a plastic deformation, crack modeling errors, optical deformations caused by caustic effect, or the fact that this zone is too small for accurate measurements. This is especially true in three dimensional fracture models were stress freezing method is used.

The objective of this work is to develop a method for determining fracture parameters with full field photoelastic fringe pattern analysis in order to increase the accuracy of SIF measurements. Methods for determining SIF from full field fringe patterns which are based on the near-field equation for the stresses are not adequate.

The basis of the method is to use the fringe equation in a form of series with unknown coefficients (K_I , K_{II} , σ_{ox} , etc.). A statistical regression method (nonlinear data fit) is used to determine this coefficient from optical measurements.

A sample experimental result of a cracked shaft subjected to bending is presented on the end of this work. Methods for manufacturing models, loading, stress freezing, determining fringe values and data processing are described.

Introduction

Increasing the accuracy of stress intensity factor measurements is not only the question of measurement technique, but also depends upon data reduction methods. A number of methods were developed in the past, starting with the paper of Wells and Post and Irwin [1] and Irwin discussion of this paper [2]. This methods can be divided by the number and type of measurement data, and a difference between a number of data points and a number of unknown coefficients for stress distribution determination (a mathematical model of isochromatic fringe pattern) into:

1. single loop methods,
2. specific region data collection and
3. full field data collection.

Additional information which can be used to increase accuracy of the analysis is contained in the influence of the nonsingular terms in a mathematical model of a stress field. In order to use this terms the stress distribution used in this research is based on Alturi-Kobayashi [3] equations for stress components and nonsingular terms are used in modeling of the isochromatic fringe pattern. By using this model a data acquisition region is not restricted to the near crack tip region.

$$\begin{aligned}
\sigma_x &= \sum_{n=1}^{\infty} \frac{n}{2} A_{In} r^{\frac{n-2}{2}} \left[\left(2 + (-1)^n + \frac{n}{2} \right) \cos\left(\frac{n}{2}-1\right)\varphi - \left(\frac{n}{2}-1\right) \cos\left(\frac{n}{2}-3\right)\varphi \right] - \\
&\quad - \sum_{n=1}^{\infty} \frac{n}{2} A_{II n} r^{\frac{n-2}{2}} \left[\left(2 - (-1)^n + \frac{n}{2} \right) \sin\left(\frac{n}{2}-1\right)\varphi - \left(\frac{n}{2}-1\right) \sin\left(\frac{n}{2}-3\right)\varphi \right] \\
\sigma_y &= \sum_{n=1}^{\infty} \frac{n}{2} A_{In} r^{\frac{n-2}{2}} \left[\left(2 - (-1)^n + \frac{n}{2} \right) \cos\left(\frac{n}{2}-1\right)\varphi + \left(\frac{n}{2}-1\right) \cos\left(\frac{n}{2}-3\right)\varphi \right] - \\
&\quad - \sum_{n=1}^{\infty} \frac{n}{2} A_{II n} r^{\frac{n-2}{2}} \left[\left(2 + (-1)^n - \frac{n}{2} \right) \sin\left(\frac{n}{2}-1\right)\varphi + \left(\frac{n}{2}-1\right) \cos\left(\frac{n}{2}-3\right)\varphi \right] \\
\tau_{xy} &= \sum_{n=1}^{\infty} \frac{n}{2} A_{In} r^{\frac{n-2}{2}} \left[- \left((-1)^n + \frac{n}{2} \right) \sin\left(\frac{n}{2}-1\right)\varphi + \left(\frac{n}{2}-1\right) \sin\left(\frac{n}{2}-3\right)\varphi \right] + \\
&\quad + \sum_{n=1}^{\infty} \frac{n}{2} A_{II n} r^{\frac{n-2}{2}} \left[\left((-1)^n - \frac{n}{2} \right) \cos\left(\frac{n}{2}-1\right)\varphi + \left(\frac{n}{2}-1\right) \cos\left(\frac{n}{2}-3\right)\varphi \right]
\end{aligned} \tag{1}$$

In this equations stress intensity factors are introduced by the following relations: $A_{I1} = K_I / \sqrt{2\pi}$, $A_{II1} = -K_{II} / \sqrt{2\pi}$, and the σ_{0x} is related to the second term in mode I series as $A_{I2} = -\sigma_{0x} / 4$.

Evidently, in the vicinity of a crack tip a singularity $1/\sqrt{r}$ is predominant, and the influence of the other terms can be neglected.

For Mode-I this equations can be rewritten as:

$$\begin{aligned}
\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\varphi}{2} \left(1 - \sin \frac{\varphi}{2} \sin \frac{3\varphi}{2} \right) + \sigma_{0x} + \\
&\quad \sum_{n=3}^{\infty} \frac{n}{2} A_{In} r^{\frac{n-2}{2}} \left\{ \left[2 + (-1)^n + \frac{n}{2} \right] \cos\left(\frac{n}{2}-1\right)\varphi - \left(\frac{n}{2}-1\right) \cos\left(\frac{n}{2}-1\right)\varphi \right\} \\
\sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\varphi}{2} \left(1 + \sin \frac{\varphi}{2} \sin \frac{3\varphi}{2} \right) + \\
&\quad \sum_{n=3}^{\infty} \frac{n}{2} A_{In} r^{\frac{n-2}{2}} \left\{ \left[2 - (-1)^n - \frac{n}{2} \right] \cos\left(\frac{n}{2}-1\right)\varphi - \left(\frac{n}{2}-1\right) \cos\left(\frac{n}{2}-1\right)\varphi \right\} \\
\tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \cos \frac{3\varphi}{2} + \sigma_{0x} - \\
&\quad \sum_{n=3}^{\infty} \frac{n}{2} A_{In} r^{\frac{n-2}{2}} \left\{ \left[(-1)^n + \frac{n}{2} \right] \sin\left(\frac{n}{2}-1\right)\varphi - \left(\frac{n}{2}-1\right) \sin\left(\frac{n}{2}-1\right)\varphi \right\}
\end{aligned} \tag{2}$$

In three dimensional cracks the stress intensity factors varies along the crack border, but Kassir and Sih have shown that stress field around the crack border can be expressed in the same manner as in two dimensional cracks using a local Cartesian coordinate system (t, n, z) .

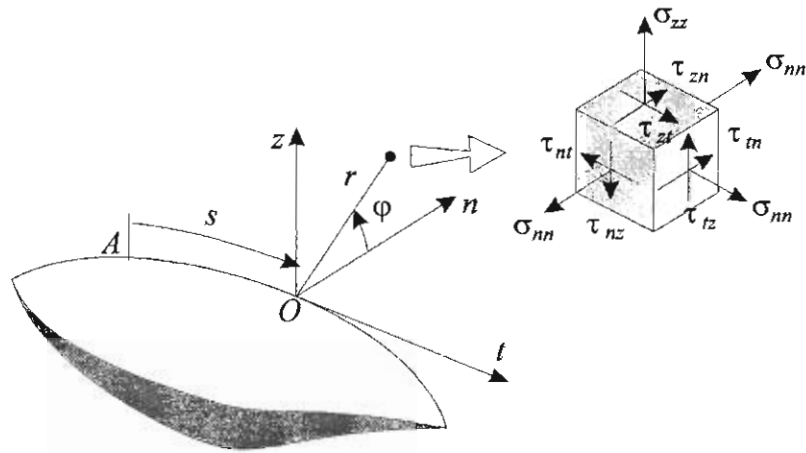


Figure 1. Local coordinate system in three dimensional cracks

All of the classical methods (Irwin, Bradley and Kobayashi, Smith, Sanford and Dally) for SIF determination using photoelastic data are based on the near crack tip equations for stress distribution. A disadvantage is that the accuracy of these methods increases as a region of data acquisition is smaller near crack tip. However, in this region the stress field is influenced by its three dimensionality (stress gradient through the thickness), crack tip blunting due to plastic deformations and modeling technique, etc. Further more, the data collection region is in reality very small (in practical cases the size of this region was between 0.2 to 4.0 mm)

To overcome this problems it is essential to use a higher order terms in the analytical equations of the stress field description as suggested by Sanford [4].

Data reduction method

The maximum in-plane shear stress can be expressed in terms of Cartesian stress components:

$$\tau_m^2 = \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2. \quad (3)$$

The stress optical law relates the fringe order N and the maximum shear stress as:

$$\tau_m = \frac{\sigma_1 - \sigma_2}{2} = \frac{N f_\sigma}{2t}, \quad (4)$$

which gives:

$$\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + (\tau_{xy})^2 = \left(\frac{N f_\sigma}{2t} \right)^2. \quad (5)$$

By denoting a left hand side of above equation by:

$$\tau_{mat}^2(A_{I1}, A_{I2}, \dots, A_{Ip}, A_{II1}, A_{II2}, \dots, A_{IIq}) = \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \quad (6)$$

where index *mat* stands for mathematical model of isochromatic fringes. The difference between mathematical model and measured data is:

$$\delta\tau^2 = \tau_{mat}^2 - \left(\frac{N_m f_\sigma}{2t} \right)^2 \quad (7)$$

This difference is obviously the function of unknown distribution coefficients A_{II} , $A_{I2}, \dots, A_{Ip}$, A_{II1} , $A_{II2}, \dots, A_{IIq}$ or shortly vector:

$$\mathbf{A} = [A_{I1} \quad A_{I2} \quad \dots \quad A_{Ip} \quad A_{II1} \quad A_{II2} \quad \dots \quad A_{IIq}]^T \quad (8)$$

For convenience of notation, we state the problem to be solved as follows: find vector $\mathbf{A}$ which minimize $\delta\tau^2$.

The solution procedure can be summarized through following steps:

1. Start with any initial vector $\mathbf{A}_1$ and an initial positive definite-symmetric matrix $[H_1]$. Usually $[H_1]$ is taken as the identity matrix.
2. Find the first search direction $\mathbf{S}_1$ as: $\mathbf{S}_1 = -[H_1] \nabla(\delta\tau^2)_1$ where $\nabla(\delta\tau^2)$ is the gradient vector evaluated at $\mathbf{A}_1$. Set the iteration number $i=1$.
3. Compute the new state vector: $\mathbf{A}_{i+1} = \mathbf{A}_i + \alpha_i^* \mathbf{S}_i$ where α_i^* is the minimizing step length along the search direction $\mathbf{S}_i$, that is α_i^* is chosen so that $f(\mathbf{A}_i + \alpha_i \mathbf{S}_i) = f(\alpha_i)$ attains at minimum at $\alpha_i = \alpha_i^*$,
4. Test for convergence of the process. If the magnitude of the gradient of the function at $\mathbf{A}_{i+1}$ is zero, that is, if: $|\nabla(\delta\tau^2(\mathbf{A}_{i+1}))| = 0$ stop the process. Otherwise, go to the next step.
5. Compute the new matrix $[H_{i+1}]$ as: $[H_{i+1}] = [H_i] + [M_i] + [N_i]$, where:
$$[M_i] = \alpha_i^* \frac{\mathbf{S}_i \mathbf{S}_i^T}{\mathbf{S}_i^T \mathbf{Y}_i}$$

$$[N_i] = -\alpha_i^* \frac{([H_i] \mathbf{Y}_i)([H_i] \mathbf{Y}_i)^T}{\mathbf{Y}_i^T [H_i] \mathbf{Y}_i}$$

$$\mathbf{Y}_i = \nabla(\delta\tau^2(\mathbf{A}_{i+1})) - \nabla(\delta\tau^2(\mathbf{A}_i))$$
6. Compute the new search direction $\mathbf{S}_{i+1} = -[H_{i+1}] \nabla(\delta\tau^2)_{i+1}$
7. Set the new iteration number as $i+1$, and repeat steps from 3. onward until the convergence in step 4. is satisfied.

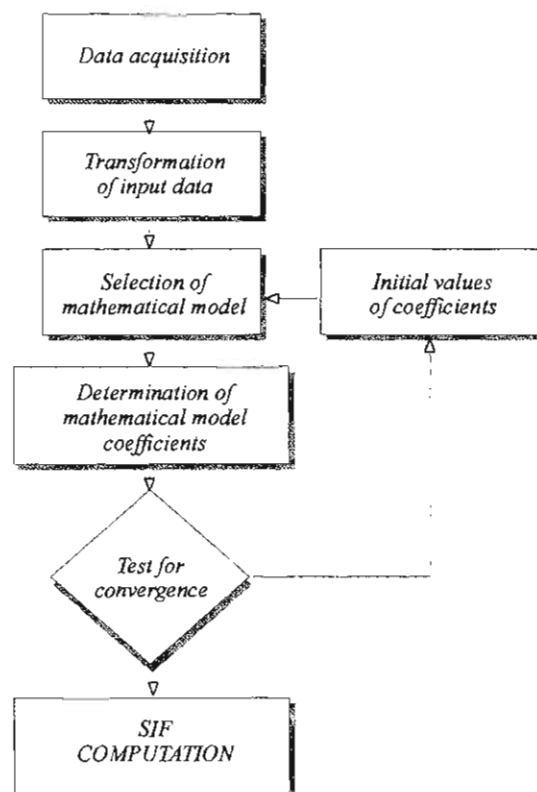


Figure 2. Iterative procedure for SIF determination by multiparametric minimization method

This procedure is implemented in the package of *Mathematica* 3.0 using NonlinearFit (Levenberg-Marquardt's method of maximal slope). Options in this package enables to change the initial value of unknown parameters, choose the interval for each parameter, change the accuracy of iteration, number of maximal iteration steps and so on.

Experiment

Model Manufacturing

Models were manufactured from epoxy resin (Araldit B with phthalic anhydride). In crack creation various methods were used:

1. Crack machining: cracks of a simple shape can be machined in a casted and machined model. This method can be used in a large scale model. Crack tip is not sharp enough to yield to $1/\sqrt{r}$ singularity. Accuracy of SIF determination can be improved using mathematical model with terms describing far field stress distribution.
2. Production of a natural crack starting from a sharp notch under controlled loading conditions. When producing a natural crack it is sometimes difficult to get a model with pre-defined crack shape due to prediction and control of crack propagation direction.
3. Production of crack during model casting by inserting thin metallic or non-metallic inserts. The shape of insert defines a crack shape and they are removed after polymerisation is completed. This method is very suitable when producing a 3D crack. Crack produced by this method is very accurate model of a real crack.

Stress freezing, loading, slicing

The frozen stress photoelastic technique is used to analyse cracks in three dimensions. Load levels are chosen to give sufficient isochromatic fringes in photoelastic slices. At stress freezing temperature ($>125^{\circ}\text{C}$) fracture toughness K_{Ic} for Araldite B with PA hardener is around $0.6 \text{ MPa}\sqrt{\text{mm}}$, but it varies with stress freezing temperature. Models were loaded with nominal $K_I=0.3\text{-}0.5 \text{ MPa}\sqrt{\text{mm}}$, which gives approximately two fringes per millimeter thickness. Slice thickness varied between 1.5 to 2.0 mm after cutting and polishing.

Fringe reading and data reduction

Digitized image obtained by scanning slides in a image scanner was used for data collection (this will be changed as soon as direct image digitalization in polariscope will be possible). Corel Draw software was used for image resizing and reading of isochromatic fringes order.

SIF computation was performed by multiparametric minimization method described. The number of parameters used in mathematical fringe model varied from 2 to 8.

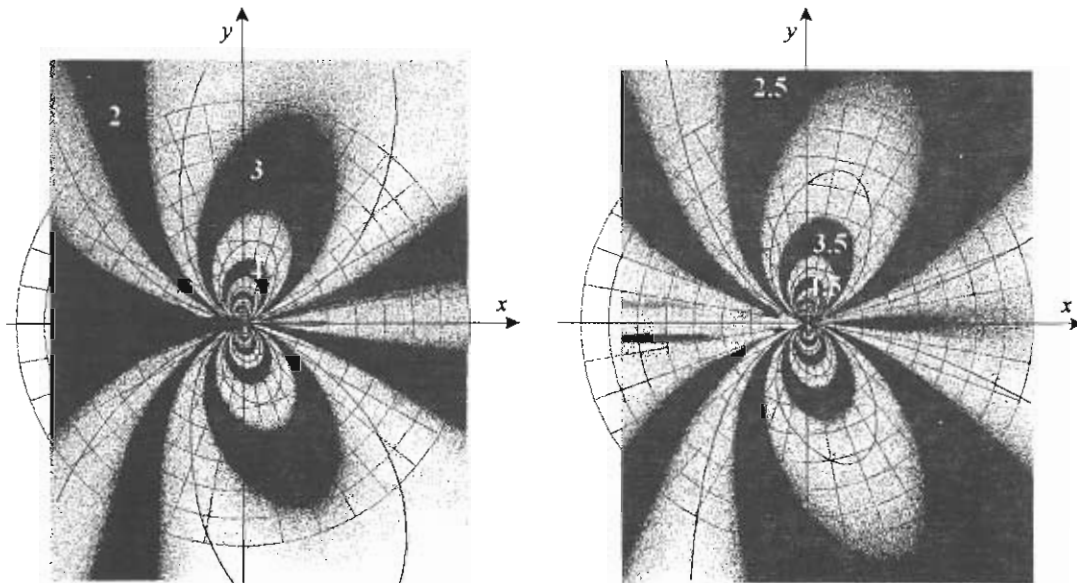


Figure 3. Isochromatic fringe pattern around crack tip and theoretically reconstructed fringes (6-parameters)

Sample model

A circular cracked shaft subjected to pure bending (four-point bending) was analyzed. The loading and geometry of a specimen is shown on Figure 4, and schematic view of slicing and isochromatic fringe pattern on Figure 5.

Experimental results for normalized stress intensity factor can be represented by polynomial equation and graphical representation of this results is shown on Figure 6.

$$Y\left(\frac{a}{d}\right) = 0.743 - 1.391\left(\frac{a}{d}\right) + 3.170\left(\frac{a}{d}\right)^2 + 4.985\left(\frac{a}{d}\right)^3 - 4.848\left(\frac{a}{d}\right)^4. \quad (9)$$

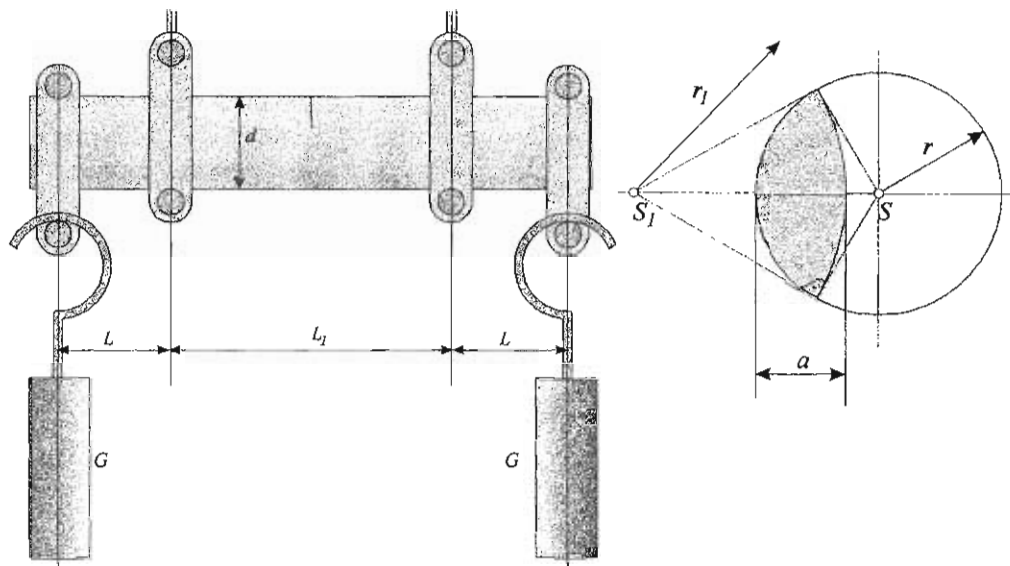


Figure 4. Model of a circular shaft under loading and crack geometry

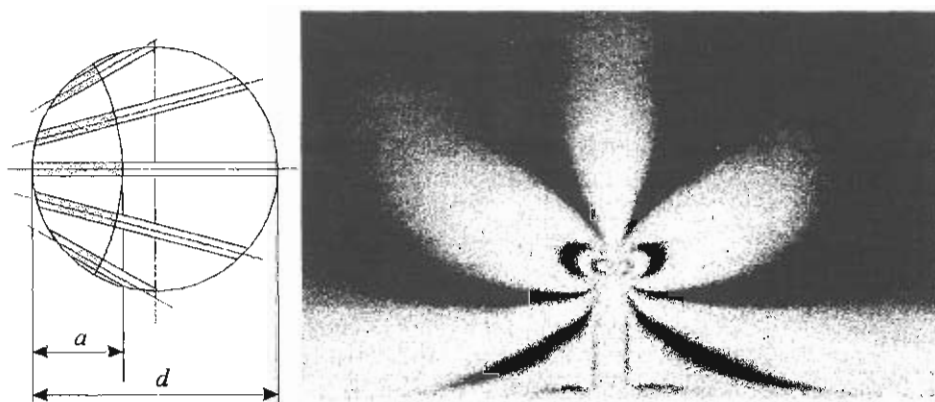


Figure 5. Schematic representation of model slicing and isochromatic fringe pattern in a model slice

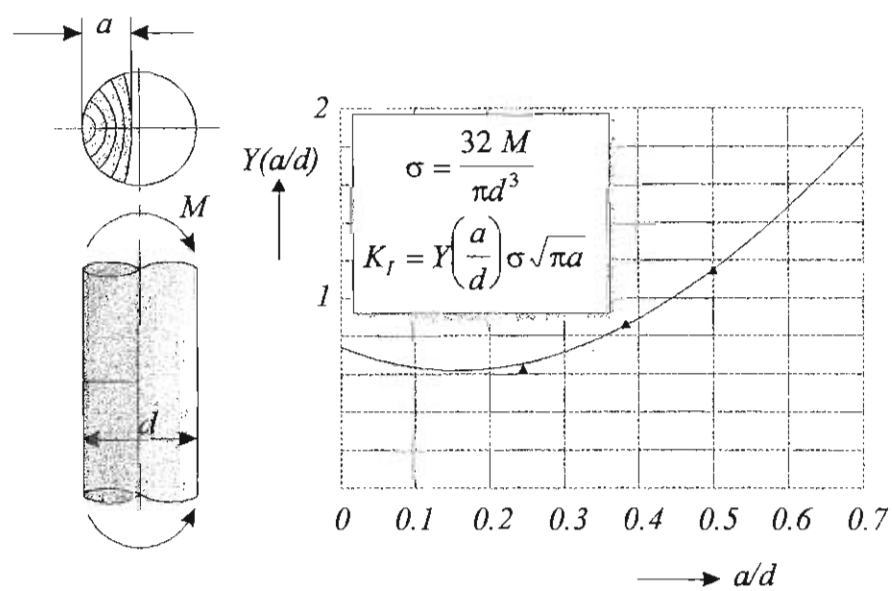


Figure 6. Experimental results for normalized stress intensity factor (circular shaft subjected to pure bending)

Conclusions

A full field photoelastic fringe pattern is used as a data input and mathematical multiparametric model describing a far field stress distribution is used for data reduction. Unknown parameters describing isochromatic fringe pattern are determined by minimization of difference between measured and predicted values of fringe orders. The accuracy of the method is improved by eliminating crack tip blunting influence due to machining and plastic deformations near crack tip.

With an increase in computer power (today's PC is faster than a main frame computer few years ago) and data acquisition systems (image digitalization and processing), a better mathematical model for stress distribution around crack tip and more sophisticated software can be used for data processing. But without a critical evaluation of the method this does not always mean the more accurate result.

Experimental results were compared with the results from literature, and with analytical results when possible. Through this work a *Mathematica* 3.0 by Wolfram Research was used to perform analytical as well as numerical computations.

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AN APPLICATION OF PLASTIC OVERPOTENTIAL FOR EVALUATION THE PLASTIC SHAKEDOWN BOUNDARY

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ABSTRACT:

A structure or structural components which operate (mainly within framework of nuclear power plant applications) in condition of plastic shakedown, have been the subject of extensive study. A suitable description of yield violation for the plastic shakedown behavior of structures is a plastic overpotential which has been introduced by Polizzotto, Ref. 6. The characteristics of plastic overpotential are very similar as those of the yield function. This paper obtains the nonratchetting condition for reversed plasticity using the concept of plastic overpotential. An example is presented as illustration of this approach.

1. INTRODUCTION

Today, the more recent investigations concerning with the behavior of elastic-plastic solid bodies under cyclic loading have understood the term "shakedown" or "plastic shakedown" as the steady state characterized by absence of plastic strain accumulation in a complete cycle. It has a more general meaning than the classical elastic shakedown which is defined as the long term structural behavior when plastic strain rate vanishes.

The phenomenon, frequent in practice, that structures can operate in condition of (plastic) shakedown has been found in many structural components of the Fast Nuclear Reactor, which are subjected to low mechanical loads and large and rapid variations in temperature. For certain configurations of mechanical loads and thermal gradients, the thermoelastic stress exceeds in magnitude twice yield stress.

In 1985, Ponter and Kardeniz, Ref. 8, developed the extended shakedown theory which are based, mainly, upon heuristic argument. The stationary plastic shakedown behavior of the structure is possible if an alternating plastic strain take place within one part of structure and no plastic strains occur within other part. More general theory, providing a necessary and sufficient condition for plastic shakedown of elastic-perfectly plastic solid body was established by Polizzotto, Refs. 5 and 6. Polizzotto has also introduced the concept of plastic overpotential or fictitious yield function as a suitable measure of the yield violations of elastic stress history. The fictitious yield surface has to satisfy the conditions which are very similar as those for real one.

In this paper, the nonratchetting condition for reversed plasticity is obtained from the yield equations for the plastic overpotential. This condition makes possible, the determination of plastic shakedown boundary without removing the alternating plasticity

regions of the body. Three bars structure is solved as an example to show the feasibility and efficiency of this approach.

2. CONSTITUTIVE EQUATIONS

Let us consider an elastic-perfectly plastic solid body of volume V referred to three orthogonal Cartesian axes $\mathbf{x} = (x_1, x_2, x_3)$, let it be bounded by the surface S which may be divided into a part S_u with constrained displacements and a part S_t on which surface tractions are prescribed ($S_u \cup S_t = S$). The body is loaded by body forces and imposed strains (e. g., thermal) in V , by tractions on S_t , all of which vary cyclically with time t and with the same time period Δt . All these actions can be collectively denoted as

$$\mathbf{F}(t) = \mathbf{F}^c(t) + \mathbf{F}^o \quad (1)$$

where $\mathbf{F}^c(t)$ denotes cyclically variable load (mechanical or kinematical) and $\mathbf{F}^o$ a constant load (mechanical).

According to the cyclic plasticity theory (see, e.g., Ref. 2) an elastic-plastic body subjected to cyclic loads responds in a non-periodic way during the transient period, but then, the stress response $\boldsymbol{\sigma}$ and the plastic strain rate response $\dot{\boldsymbol{\epsilon}}^p$ become cyclic, with the same period Δt of the loads, i.e.

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}(\mathbf{x}, t + \Delta t), \quad \dot{\boldsymbol{\epsilon}}^p(\mathbf{x}, t) = \dot{\boldsymbol{\epsilon}}^p(\mathbf{x}, t + \Delta t) \quad (2)$$

(steady-state response). The actual stress response $\boldsymbol{\sigma}(\mathbf{x}, t)$ is the sum of two parts, the one elastic stress response of body to $\mathbf{F}(t)$, $\boldsymbol{\sigma}^E(\mathbf{x}, t)$ (when the existence of a yield surface is ignored), the other one being the actual residual stress $\boldsymbol{\sigma}^R(\mathbf{x}, t)$. These stresses can be written as

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, t) + \boldsymbol{\sigma}^R(\mathbf{x}, t) \quad (3)$$

$$\boldsymbol{\sigma}^E(\mathbf{x}, t) = \boldsymbol{\sigma}^c(\mathbf{x}, t) + \boldsymbol{\sigma}^o(\mathbf{x}) \quad (4)$$

$$\boldsymbol{\sigma}^R(\mathbf{x}, t) = \boldsymbol{\rho}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}, t) \quad (5)$$

where $\boldsymbol{\sigma}^c(\mathbf{x}, t)$ and $\boldsymbol{\sigma}^o(\mathbf{x})$ are the elastic stress responses to $\mathbf{F}^c(t)$ and $\mathbf{F}^o$ respectively, $\boldsymbol{\rho}(\mathbf{x})$ denotes the residual stresses at the instant at which transient period terminates and the steady state begins, $\boldsymbol{\tau}(\mathbf{x}, t)$ denotes the residual stresses corresponding to the net plastic strains occurring in the steady cycle. The equations governing the steady-state response can be written in the form:

$$\phi(\boldsymbol{\sigma}) \leq 0, \quad \dot{\boldsymbol{\epsilon}}^p = \dot{\lambda} \frac{\partial \phi}{\partial \boldsymbol{\sigma}}, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} \phi(\boldsymbol{\sigma}) = 0 \quad \text{all in } V \times (0, \Delta t) \quad (6)$$

$$\mathbf{C}^T \boldsymbol{\sigma}^R = \mathbf{o}, \quad \dot{\boldsymbol{\epsilon}}^R = \mathbf{A} : \dot{\boldsymbol{\sigma}}^R + \dot{\boldsymbol{\epsilon}}^p \quad \text{all in } V \times (0, \Delta t) \quad (7)$$

$$\dot{\boldsymbol{\epsilon}}^R = \mathbf{C} \dot{\mathbf{u}}^R, \quad \Delta \boldsymbol{\epsilon}^p = \int_0^{\Delta t} \dot{\boldsymbol{\epsilon}}^p dt \quad \text{all in } V \times (0, \Delta t) \quad (8)$$

$$\Delta \mathbf{u}^R = \int_0^{\Delta t} \dot{\mathbf{u}}^R dt, \quad \Delta \boldsymbol{\epsilon}^p = \mathbf{C} \Delta \mathbf{u}^R \quad \text{all in } V \times (0, \Delta t) \quad (9)$$

$$\sigma^R n = 0 \quad \text{on } S_t \times (0, \Delta t) \quad (10)$$

$$\dot{u}^R = 0 \quad \text{on } S_u \times (0, \Delta t). \quad (11)$$

Here above $\Delta \epsilon^p$ denotes the increment of plastic strains compatible with the cycle net residual displacement Δu^R , λ is the plastic multiplier, C denotes the compatibility differential operator, A is the elastic compliance fourth-order tensor, n denotes the unit external normal to the boundary surface S and $\phi(\sigma)$ is the yield function by hypothesis convex.

The solution of Eqs. (6), (7), ... and (11) has the following properties:

- the plastic strain rates $\dot{\epsilon}^p$ are uniquely determined in whole V
- residual stresses σ^R are uniquely determined only within the region V_F , where $\dot{\epsilon}^p$ is not identically vanishing
- the steady-state solution is independent of any initial plastic strain as well as of the load cycle origin.

The uniquely determined plastic strain rates have three possible alternatives:

- elastic shakedown, when $\dot{\epsilon}^p = 0$
- alternating plasticity, when $\dot{\epsilon}^p \neq 0$ but $\Delta \epsilon^p = 0$
- incremental collapse (ratchetting), when $\Delta \epsilon^p \neq 0$.

3. PLASTIC OVERPOTENTIAL

Combining Eqs. (3), (4) and (5), the actual stress response can be written as

$$\sigma(x, t) = \hat{\sigma}(x, t) + \tau(x, t) \quad (12)$$

where $\hat{\sigma}$ denotes the post-transient stress which is defined as

$$\hat{\sigma}(x, t) = \sigma^e(x, t) + \sigma^o(x) + \rho(x). \quad (13)$$

The post-transient stress response depends on the value of load parameters. In case of elastic shakedown, $\hat{\sigma}$ lies inside the yield surface everywhere in V . Otherwise, $\hat{\sigma}$ exceeds the yield surface at least somewhere in V . A suitable description of the yield violation is the fictitious yield surface $\phi(\sigma) = y$ homothetic to yield surface $\phi(\sigma) = 0$ (similar as a isotropic hardening has transformed the actual yield surface into the expanded one), where y is defined as

$$y = \max_{t \in (0, \Delta t)} \phi[\hat{\sigma}_m(x, t)] \quad m \in (1, 2, \dots, n). \quad (14)$$

The scalar y is the plastic overpotential at x and the integral

$$Y = \int_V y dV \quad (15)$$

is the total plastic overpotential.

The total plastic overpotential has to be minimized with respect to post-transient self-stress field ρ

$$Y_{\min} = \min_{(y, \rho)} \int_V y dV$$

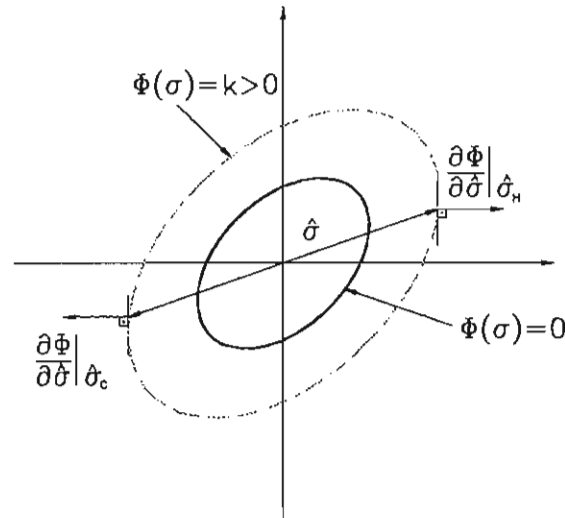


Figure 1 Yield surface and plastic overpotential

subjected to:

$$\begin{aligned} \phi(\hat{\sigma}_m) &\leq y & y \geq 0 & \quad \text{in} \quad V \\ C^T \rho &= 0 & & \quad \text{in} \quad V \\ \rho n &= 0 & & \quad \text{on} \quad S_t. \end{aligned} \quad (16)$$

The augmented functional for this problem can be written as

$$J = \int_V y \, dV + \sum_{m=1}^n \int_V l_m [\phi(\hat{\sigma}_m) - y] \, dV + \int_V \nu C^T \rho \, dV - \int_{S_t} \nu \rho n \, dS \quad (17)$$

where $l_m(x) \geq 0$ and $\nu(x)$ are scalar and vector Lagrangian multipliers respectively.

The first variation of functional, using Green-Gauss theorem

$$\begin{aligned} \delta J &= \int_V \delta y (1 - \Delta l) \, dV + \int_V \delta \rho : (\Delta \rho - C \nu) \, dV + \\ &\int_{S_u} n \delta \rho \nu \, dS + \sum_{m=1}^n \int_V \delta l_m [\phi(\hat{\sigma}_m) - y] \, dV + \int_V \delta \nu C^T \rho \, dV - \int_{S_t} \delta \nu \rho n \, dS \end{aligned} \quad (18)$$

gives desired Euler equations:

$$\phi(\hat{\sigma}_m) \leq y, \quad p_m = l_m \frac{\partial \phi}{\partial \hat{\sigma}} \Big|_{\hat{\sigma}_m}, \quad l_m \geq 0 \quad \text{in} \quad V \quad (19)$$

$$l_m [\phi(\hat{\sigma}_m) - y] = 0, \quad C^T \rho = 0 \quad \text{in} \quad V \quad (20)$$

$$\Delta l \equiv \sum_{m=1}^n l_m \leq 1, \quad \Delta \rho \equiv \sum_{m=1}^n p_m = C \nu \quad \text{in} \quad V \quad (21)$$

$$y \geq 0, \quad y(\Delta l - 1) = 0 \quad \text{in} \quad V \quad (22)$$

$$\rho n = 0 \quad \text{on } S_l \quad (23)$$

$$\nu = 0 \quad \text{on } S_u. \quad (24)$$

Hereabove, scalar Lagrangian multipliers ℓ_m take the meaning of plastic activation coefficients, Lagrangian multipliers ν of displacement field and p_m of fictitious plastic strain increments.

It is obviously, Eqs. (19) and (20), that the fictitious yield function $\phi(\sigma) = y$ satisfies equations similar to Eq. (6) for real yield function $\phi(\sigma) = 0$.

An alternating plasticity is the plastic deformation process such that, in one cycle, the plastic strain increment is equal zero

$$\dot{\epsilon}^p \neq 0, \quad \Delta \epsilon^p = \int_0^{\Delta t} \dot{\epsilon}^p dt = 0 \quad (25)$$

and its maximum amplitude remains below a certain limit.

In the framework of the plastic overpotential, it means, that the fictitious ratchet strain has to be also equal zero

$$\Delta p \equiv \sum_{m=1}^n p_m = 0 \quad n \geq 2. \quad (26)$$

Neutral configuration is called the position of the fictitious yield function characterized by Eq. (26) and can be denoted as

$$\phi(\sigma) = k > 0 \quad (27)$$

where k is the minimum value of y (see Figure 1).

When the shakedown is reached, Ref. 3, the structure can be composed of three kinds of regions: V_1 , V_2 and V_3 , where V_1 is an alternating plasticity zone, V_2 yields only in the heating half of cycles and V_3 yields only in the cooling half of cycles (Figure 2a).

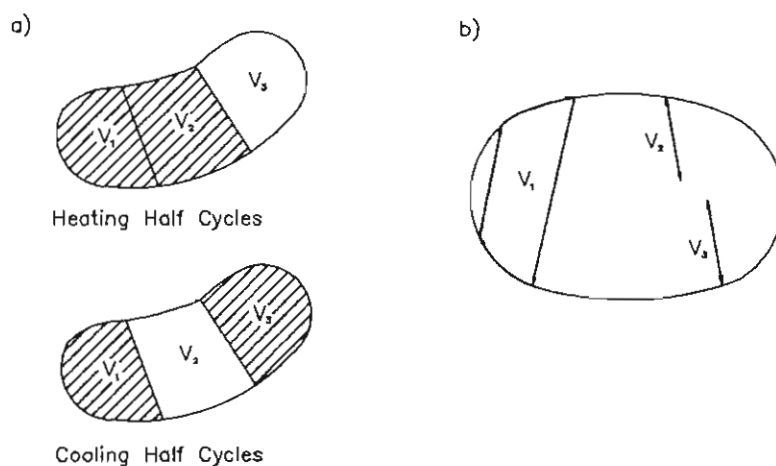


Figure 2 Possible behavior of structure a) division of the region b) yield surface

In the stress space (Figure 2b), it means that stresses in region V_1 hit the yield surface twice in complete cycle, whereas stresses in regions V_2 and V_3 hit the yield

surface only once and remain somewhere inside the yield surface during the other half of cycle:

$$\phi(\sigma_H^E + \sigma_H^R) = 0, \quad \phi(\sigma_C^E + \sigma_C^R) = 0 \quad \text{in } V_1 \quad (28)$$

$$\phi(\sigma_H^E + \sigma_H^R) = 0, \quad \phi(\sigma_C^E + \sigma_C^R) < 0 \quad \text{in } V_2 \quad (29)$$

$$\phi(\sigma_H^E + \sigma_H^R) < 0, \quad \phi(\sigma_C^E + \sigma_C^R) = 0 \quad \text{in } V_3. \quad (30)$$

The subscripts H and C are referred to the values for the heating and cooling half of cycles, respectively.

It can be proven, Ref. 6, that for loads above the elastic shakedown limit, regions V_2 and V_3 can exist only if alternating plasticity occurs in a region V_1 .

For an alternating plasticity zone V_1 , as a consequence of Eq. (22), a nonvanishing y implies that $\Delta \ell = 1$. Due to this statement and Eq. (26), as well as the convexity and symmetry with respect to zero of fictitious yield surface, are obtained

$$\frac{\partial \phi}{\partial \hat{\sigma}} \Big|_{\hat{\sigma}_H} = - \frac{\partial \phi}{\partial \hat{\sigma}} \Big|_{\hat{\sigma}_C} \quad \text{in } V_1 \quad (31)$$

and

$$\hat{\sigma}_H = -\hat{\sigma}_C \quad \text{in } V_1 \quad (32)$$

which means, that the nonratchetting condition for the post-transient stress has to be satisfied in every point of region V_1 .

4. NUMERICAL EXAMPLE

Three bars structure has been studied to illustrate the application of plastic overpotential for determination the plastic shakedown limit load.

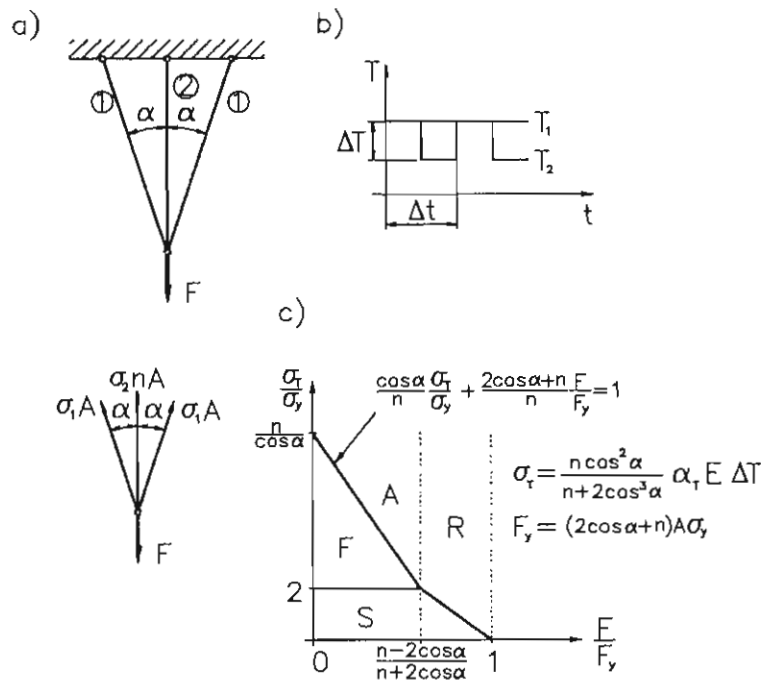


Figure 3 Three bars structure a) geometry and loading b) thermal loading history c) limit load domains

The geometry and loading are shown in Figures 3a and 3b; the bars 1 of cross section A have a constant uniform temperature T_1 , the bar 2 of cross section nA has an uniform temperature $T_2 = T_2(t)$, cyclically variable with period Δt and temperature drop ΔT . A constant mechanical load F is applied to the structure.

The equilibrium equation, compatibility condition and stress-strain relationship are respectively:

$$2A\sigma_1 \cos\alpha + nA\sigma_2 = F \quad (33)$$

$$\varepsilon_1 = \varepsilon_2 \cos^2\alpha \quad (34)$$

$$\varepsilon_1 = \frac{\sigma_1}{E} + \varepsilon_1^p, \quad \varepsilon_2 = \frac{\sigma_2}{E} - \alpha_T \Delta T + \varepsilon_2^p. \quad (35)$$

Elastic responses to constant load F and to cyclically variable temperature for heating and cooling half of cycle are obtained from Eqs. (33), (34) and (35),

$$\sigma_1^0 = \frac{\cos^2\alpha}{n+2\cos^3\alpha} \frac{F}{A}, \quad \sigma_2^0 = \frac{1}{n+2\cos^3\alpha} \frac{F}{A},$$

$$\sigma_{1H}^C = -\sigma_T, \quad \sigma_{2H}^C = \frac{2\cos\alpha}{n} \sigma_T, \quad \sigma_{1C}^C = \sigma_{2C}^C = 0,$$

where E is Young's modulus, α_T is thermal expansion coefficient and $\sigma_T = \frac{n\cos^2\alpha}{n+2\cos^3\alpha} E \alpha_T \Delta T$ is a max thermoelastic stress. In the case of plastic shakedown, bars 1 incurs alternating plasticity during the cycles, while bar 2 yields in tension during the heating half and remains elastic during the cooling half of cycles. Referring Figure 2 we now have two regions V_1 (bars 1) and V_2 (bar 2). Applying Eqs. (32) and (29) for two bar problem

$$\sigma_1^0 + \sigma_{1H}^C + \rho_1 = -(\sigma_1^0 + \rho_1) \quad \text{in bars 1,} \quad (36)$$

$$\sigma_2^0 + \sigma_{2H}^C + \rho_2 = \sigma_y \quad \text{in bar 2}$$

and Eq. (33) for constant residual stresses

$$2\rho_1 \cos\alpha + n\rho_2 = 0, \quad (37)$$

the following expression for the plastic shakedown boundary is obtained, Figure 3c,

$$\frac{2\cos\alpha + n}{n} \frac{F}{F_y} + \frac{\cos\alpha}{n} \frac{\sigma_T}{\sigma_y} = 1, \quad (38)$$

where σ_y is yield stress and $F_y = (2\cos\alpha + n)A\sigma_y$ is a plastic collapse limit load. Eq. (38) is in agreement with the previous results.

5. CONCLUSION

In an elastic-perfectly plastic solid body subjected to cyclic/steady loads above the elastic shakedown limit may exist the plastic shakedown. In this case the body is divided in two regions; the region where it responds elastically and the region where an alternating plasticity takes place. For the assessment of the plastic shakedown limit load were used methods in which the alternating plasticity zone had been removed and an elastic shakedown limit load was solved only for the elastic zone, Refs. 6 and 8. Another approach to the same problem is by implementing some properties of the alternating plasticity process. Accordingly, this paper obtains the plastic shakedown boundary by using nonratchetting condition for the reversed plasticity region.

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Simulation of actual fatigue at aluminium components under conditions of fretting and plane-reverse-bending

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Abstract

Due to the arising demands by the manufacturing of vehicles a significant increase of request for technically applied light metallic materials is expected. To support the utilisation of aluminium an international project Copernicus has been established. Within this project the influence of manufacturing and operational parameters on fatigue strength of some aluminium alloys will be investigated. A short preview of results obtained by FEM analyses of contact problem and operational strength tests of aluminium components under conditions of fretting and plane-reverse-bending is presented in this paper. The simulation presents an approach to the real operating conditions on large (consisting of several parts) wheel in the region of fixing. An aim of our numerical analyses is to inspect the expected stress-strain state in the contact, to obtain the basis for various most influencing experimental parameters adjustment and to obtain the influence level of some observed technological and specific operational parameters on endurance limit using operational strength tests.

1. Introduction

Aluminium and aluminium alloys have good manufacturing properties, corrosion resistance, fatigue strength, mass production ability and a high level of recycling potential so they are one of the most applied metal materials. The objective of research within Copernicus project 'Improvement of Strength Properties at Aluminium Components by Manufacturing Procedures' (CIPA-CT94-0219), established by European Commission, is to increase the knowledge about influence of different manufacturing procedures on improvement of the strength properties of aluminium components and accordingly to diminish the frequently applied expensive processes. These basic investigations are made by some independent research groups and laboratories. Partners in this project are Fraunhofer-Institut für Betriebsfestigkeit (LBF) from Darmstadt (D) as a co-ordinator, LAVEK laboratory from the Faculty of Mechanical Engineering from Ljubljana (SLO), Slovak Academy of Sciences (SK) and Skoda Vyzkum Ltd. from Plzen (CZ).

Operational strength of aluminium components (under special conditions of fretting and plane-reverse-bending) is a function of the material strength, surface roughness and the residual stress state so that often expensive processes are applied to improve the strength properties. Knowing the possible improvement of the strength properties using more cost-effective and reliable procedures and by the implementation of concurrent design more economical manufacturing of products might be ensured. Because of weight reduction and loading capacity enlargement the consumption of material and energy would be also more cost-effective.

2. Materials and manufacturing of specimens

The investigated materials were permanent mould casted and pressure dye casted aluminium alloy $AlSi7Mg0,6wa$ and permanent mould casted $AlSi11MgSr$. Comparing the achieved results of strength properties the influence of two different casting technologies would be possible. Investigated aluminium alloys are suitable for manufacturing of wheels of different size.

For the determination of the influence of different manufacturing procedures on the strength properties specimens were manufactured getting the next properties: as casted, as casted and shot-peened, finished, finished and shot-peened. Metallurgical investigations have indicated that the chemical composition and the grain-structure of the aluminium alloys correspond to the according standards.

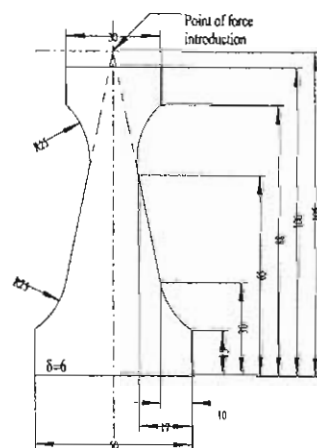


Fig. 1: Test specimen

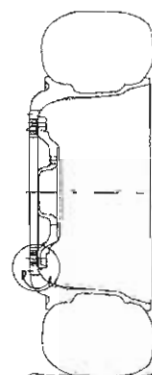


Fig. 2: Wheel

3. Modelling of an actual situation

In automotive industry the utilisation of aluminium wheels is steel increasing. Larger wheels are usually combined from wheel rim and disc, connected and fastened by screws (Fig.2). During exploitation variable stress-strain state occur in the material, locally in the place of screw fixing also additional pressure component, perpendicular to the contact area. It was found out that these special operating conditions generate failure (crack) in this place earlier as we expected on the basis of strength

properties test results under plane bending. We also want to simulate the effect of dynamic strength reduction due to special loading state experimentally since to approach actual operating conditions. For this purpose the special testing facility and the test specimen (Fig.1) were designed. By the introduction of bending force in proposed point we get uniform bending (tensile-compressive) stress field in the observed central part of the specimen.

4. FEM analysis of the stress-strain state in the contact between clamps and specimen

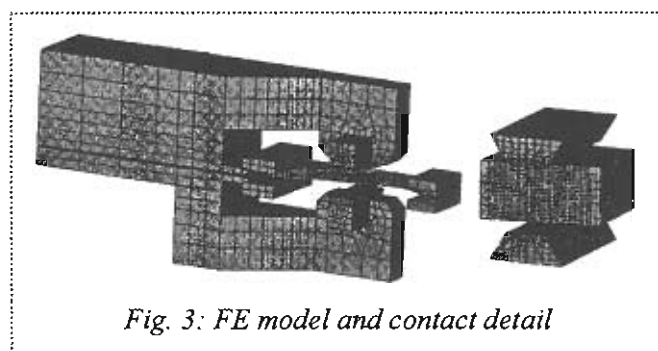


Fig. 3: FE model and contact detail

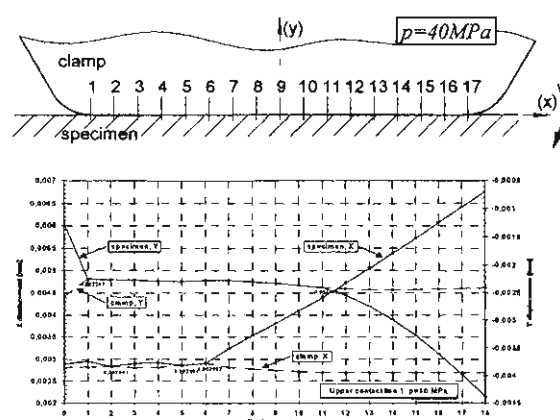


Fig. 4: Relative displacements in the contact

and specimen and also the distribution of the stress fields in the contact area. All this variables are important for crack initiation and propagation of fretting fatigue cracks. Analyses also supported more appropriate selection of parameters for measurement of surface roughness and coefficient of friction and most important the primary settings of experimental parameters on the test-rig.

The stress-strain state in the contact is due to the special operating conditions under bending only partly predictable so it was decided to build up the FE model (Fig.3). To compute stresses and strains in the contact actual geometry and assumed coefficient of friction in the contact are used and loads of adequate sizes are imposed at

various mesh locations. The adjustable parameters in the FE model are clamping (contact) force, bending force and coefficient of friction. FEM stress and deformation analysis for example of clamping pressure 40MPa, bending force 240N and coefficient of friction 0.14 are presented in Fig.4 and Fig.5.

On the basis of numerical analysis results for our model it is possible to conclude that the magnitude of contact pressure greatly influences the relative displacement magnitude between clamps

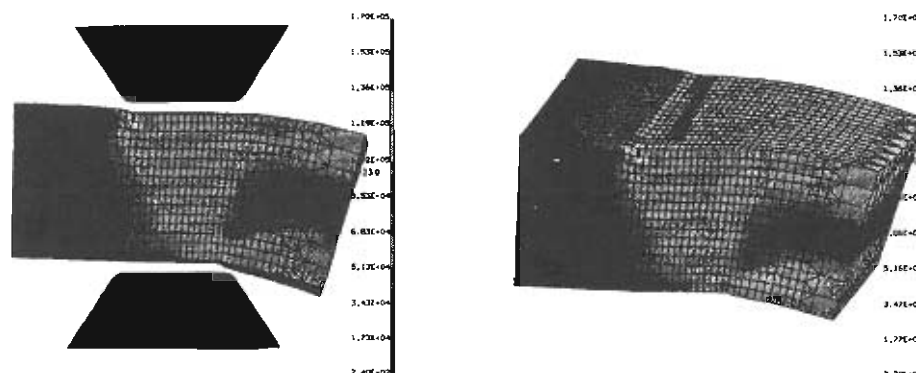


Fig. 5: Results of stress (Mises) and strain analysis in the contact ($p=40\text{MPa}$)

5. Starting points for designing the test facility

For the fatigue prediction based on the local strain approach the strain controlled Wöhler (S-N) curves are to be determined. To achieve the influence of the casting and manufacturing procedures fatigue tests on the aluminium specimens made by different

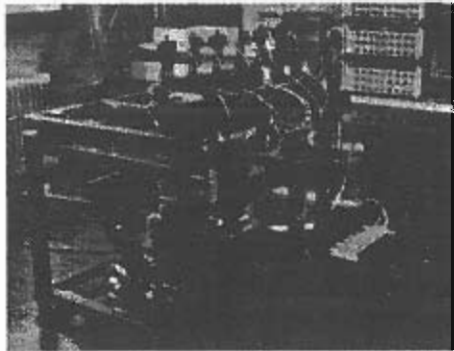


Fig. 6: Test facility

technologies have to be carried out under constant and variable amplitude loading and under different fretting conditions. To accomplish the planned test program in much shorter time testing up to six specimens at the same time should be possible. Stress-strain level in the specimen is defined by the control of the specimen free end displacement. The load levels are selected concerning the expected test fatigue life at the constant amplitude tests of 10^5 to 2×10^6 cycles is attained. The tests under variable amplitude ranges are to be carried out in the form of eight-step block loading. Fretting is to

be produced by clamping the footprints of flat contact geometry to the specimen using special frame that enables clamping pressure control during the test. For this purpose the special test facility for strength properties tests under conditions of fretting and plane-reverse-bending was built (Fig.6).

6. Basic features of test facility and measuring chain

Test facility makes feasible to test up to six specimens at the same time (Fig.7), having the same conditions on each specimen. The greatest transverse displacement in the point of bending load introduction is equal to the maximal bolt eccentricity on the flywheel and equals to 9mm. The tests are carried out under a displacement-controlled condition of $R=-1$ at a frequency range from 0 to 15Hz.

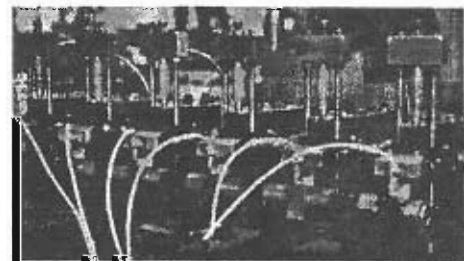


Fig. 7: Setting of specimens

Amplitude adjustment during test process is achieved using special mechanism for eccentricity control, built-in to the flywheel body. Distinctive features of the mechanism are great amplitude adjustment ability at very short response time and also lower cost in comparison with the servo-hydraulic testing machine. The purpose of the sturdy frame design is to prevent the influence of possible vibrations on probes. There is also possible to adjust the mean stress ratio by shifting the fixed end of the specimen. An asynchronous AC motor ($P=3\text{kW}$) is used for driving the flywheel while the mechanism for load amplitude adjustment is driven with a computer controlled DC motor. The design of the test bench enables nearly perfect sinusoidal loading even after failure of some specimens during the test. To verify the dynamic behaviour of the testing machine for expected operational conditions (loading frequency, variable amplitudes and stiffness of specimens) modal analyses have been additionally carried out.

For test and measurement process control and for test conditions adjustment specially designed transducers have been added to the machine on every single testing place. The loading range (amplitude, mean) in a specimen and the state of plane pressure between specimen and clamp are measured indirectly by means of specially designed transducers for estimation of a bending and clamping force. Furthermore it is possible to examine

actual life of each specimen (load cycle counting) and fatigue crack initialisation and growth. All transducers have been calibrated over the whole range of their implementation. Transducers are via units for signal conditioning (amplifiers and filters), multiplexer and A/D card connected with the processing unit (PC) (Fig.8). The software for computer control of the testing machine is developed in LabVIEW using graphical programming language G (Fig.9).

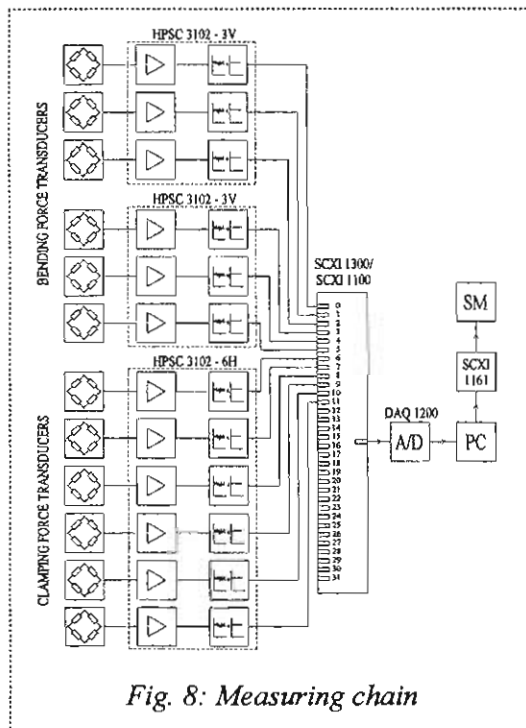


Fig. 8: Measuring chain

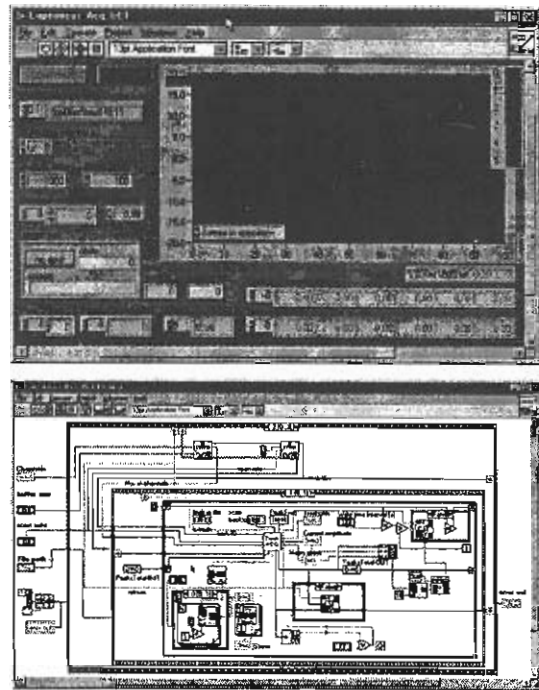


Fig. 9: Application for test-facility control

7. Determination of fatigue life

Considering the complexity of the problem and test results analyses only a part of

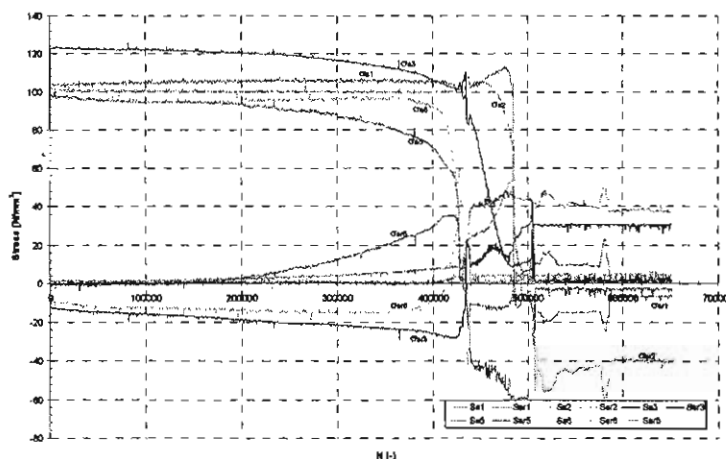


Fig. 10: Amplitude and mean stress

obtained results will be presented in this paper, whereas all other results would be a topic of a special paper. Examples in the diagram on Fig.10 indicate the amplitude and mean stress values during a whole test (till the last specimen failed). It is possible to observe the process of crack initiation and propagation and thus to determine the fatigue life of each specimen.

8. Conclusion

This article in short presents the simulation process of actual fatigue at aluminium components under conditions of fretting and plane-reverse-bending using FEA of the contact problem and experimental design. The design of the test facility has proven

successfully for it enables stable test conditions and high feasibility of operational parameters regulating during the test. It is possible to determine the initialisation and growth of the crack using diagrams of amplitude and mean stress course during the test. S-N curves for investigated materials and the influence of fretting conditions will be discussed separately in another article.

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STRUCTURAL ANALYSIS OF A R. C. CHIMNEY UNDER BOTH STATIC AND TRANSIENT THERMO-MECHANICAL LOADING

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SUMMARY

The practical use of the specialised computer package BETHEV44 is illustrated with the analysis of a 150 m tall chimney of a thermo-electric plant. The package is based on finite element non-linear modelling for the analysis of reinforced concrete structures under both static and transient (thermal & mechanical) loading conditions. The material model is prepared to simulate the dominant non-linear behaviour of concrete (e.g. tensile cracking, non-linear multiaxial compressive behaviour, etc.) and reinforcement. Steel reinforcing bars or meshes are modelled by special bar or membrane elements embedded in concrete elements. Using the software described, the R.C. chimney, built over 25 years ago, is analysed simulating the various working and ambient conditions in order to determine the cause of the structural cracking as well as to provide a reliable base for a structural repair. Selected results of the analysis are presented and briefly described showing the merit of applying BETHEV44.

1. INTRODUCTION

The specialised computer package BETHEV44 (Damjanić 1994), based on a finite element non-linear modelling, is developed for the thermo-mechanical analysis of reinforced concrete structures under both time-dependent static and transient loading conditions. The solution technique involves concurrently solving an uncoupled set of equations (the transient heat transfer equations and the incremental mechanical equilibrium equations) within a time interval ($\Delta t_n = t_{n+1} - t_n$) to provide the temperature, displacement, strain and stress increments.

$$K(u_n) u_n + C(u_n) \tilde{u}_n + f(u_n, t_n) = 0 \quad (1)$$

$$\int_{\Omega} [{}_n B^T \Delta \sigma_n(u_n, t_n)] d\Omega + \Delta f_n(t_n) = 0 \quad (2)$$

In Eq. (1) u_n and $\tilde{u}_n$ are, respectively, the temperature field and temperature gradient vector at the time t_n , f represents the thermal loading which may vary with respect to both temperature and time, and $K(u_n)$ in $C(u_n)$ are the thermal and heat capacity matrices. The first part of Eq. (2) represents the incremental vector of internal forces and the second one is the external mechanical load which may be time dependent.

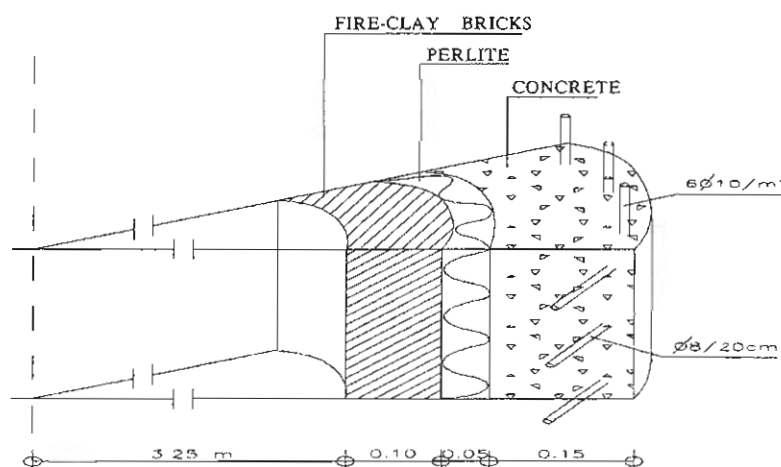


Figure 1. Typical cross-section of the chimney (Section 140-150m)

Since all the material properties may be temperature, time and strain dependent, the above equations are highly non-linear, and an appropriate incremental-iterative solution procedure is applied.

Apart to temperature dependent material properties, the mechanical material model is prepared to simulate the dominant non-linear behaviour of concrete and reinforcement. The model can simulate tensile cracking, crack spreading and eventual closing, as well as non-linear multiaxial compressive behaviour of concrete. Steel reinforcing bars or meshes are modelled by special bar or membrane elements embedded in concrete elements. The steel behaviour is idealised using a uniaxial elasto-plastic model resisting only the axial force in the reinforcement bar or plane direction.

The practical engineering use of the briefly described software is illustrated with the analysis of a 150 m tall chimney of a thermo-electrical plant in Slovenia (Damjanič 1996). The chimney was built over 25 years ago. Its structure is made as a vertical reinforced concrete tube with the thickness varying from 40.0 cm at the chimney bottom to 15.0 cm at the top. The structural thickness is reduced every 10.0 m in the vertical direction. It is reinforced with vertical and circumferential reinforcement which is only located close to the outer surface of the chimney. Internal surface of the structure should be insulated (according to the original design) with a granulated perlite protected by the fire-clay brick wall. Typical cross section of the chimney is shown in Fig. 1. During the last 25 years a largely spreaded system of vertical cracks were observed in the reinforced concrete structure. It was apparent that the cracks had occurred due to severe thermal conditions. Gas temperature was sometimes up to 160 °C (although the new ecological rules prescribes that the maximum gas temperature should not exceed 100 °C). In addition to that the chimney's internal insulation material has been found to be worn, i.e. it disappeared in cca 1/3 of every vertical section.

Using the software briefly described above, the chimney is analysed simulating the various working and ambiental conditions. The analysis has been performed in order to determine the cause of the structural cracking as well as to provide a reliable base for an efficient structural (including insulation) repair of the chimney. Some interesting results of the analysis are presented and briefly described showing the merit of applying BETHEV44.

2. THE CONCEPT OF THE NUMERICAL ANALYSIS

Since early 80-ies it was proven that the transient thermal loading can be a more critical one than steady state conditions (Owen et al. 1983, Damjanić 1983). Also, the use of a non-linear material modelling is necessary if a realistic response of reinforced concrete structures is needed (Damjanić 1984, 1986). BETHEV44 satisfies all the mentioned requirements.

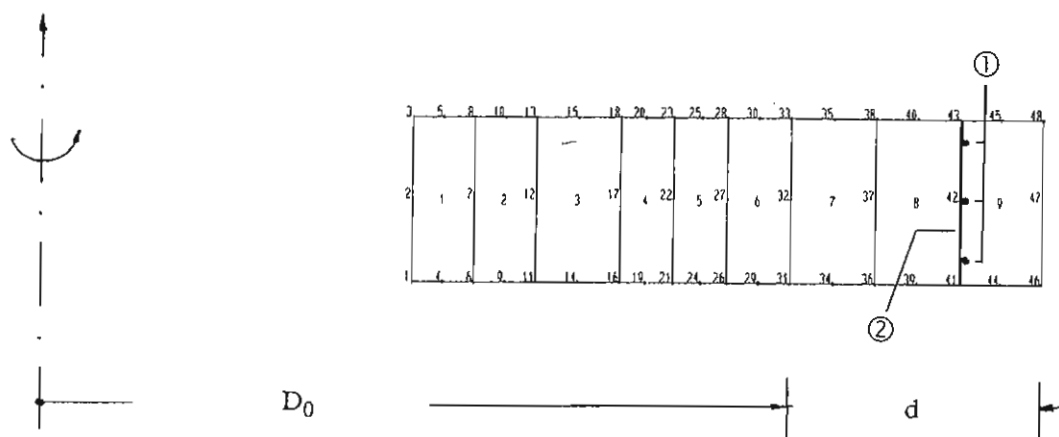
Several numerical examples were performed in order to obtain the most critical strain/stress states of the structure. Various combinations of mechanical (self weight, dead load, wind) and temperature loading conditions for several chimney sections were applied. The influence of variations of internal (gas) temperature simulating the working conditions in combination with external ambient temperature (simulating different weather conditions) were tested. The analysis were performed taking into account the following assumptions: (i) existing reinforced concrete is repaired (concrete is compacted with epoxy injection materials), and (ii) the insulation is correctly repaired according to the original design. Typical material properties used in the analysis (for the vertical section at the height 140-150 m) are given in the tables 1 and 2. In order to simplify the problem solution, the numerical analysis was carried out separately for typical cross sections in axisymmetric or 2D finite element representation whenever possible.

Table 1: Mechanical properties

Concrete MB25		
Young's modulus	$E = 2.000.00$	kN/cm ²
Poisson's ratio	$\nu = 0.15$	
Compressive strength	$F_c = 2.50$	kN/cm ²
Tensile strength	$F_t = 0.21$	kN/cm ²
Mass density	$g = 00025$	kg/cm ³
Thermal expansion parameter	$\alpha_t = 0.00001$	1/K
Max. Tension stiffening strain	$_{max}\epsilon_t = 0.0023$	
Shear retention equivalent strain	$_{max}\epsilon_{sh} = 0.0030$	
Crushing strain	$\epsilon_{cu} = 0.004$	
Reinforcement		
Young's modulus	$E = 21.000.00$	kN/cm ²
Yield stress	$\sigma_y = 0.15$	kN/cm ²
Hardening parameter	$H' = 2.50$	kN/cm ²
Thermal expansion parameter	$\alpha_t = 0.00001$	
Eqv. thickness of vertical bars (6Ø10/m')	$t_v = 0.050$	cm/m'
Eqv. thickness of circum. bars. (Ø8/20cm)	$t_h = 0.025$	cm/m'

Table 2: Thermal properties

Concrete		
Thermal conductivity	$k = 2.33$	W/m-K
Specific heat capacity	$c = 0.96$	kJ/kg-K
Convection coef.: concrete/air	$\alpha_c = 21.00$	W/m ² -K
Perlite		
Thermal conductivity	$k = 0.05$	W/m-K
Specific heat capacity	$c = 0.84$	kJ/kg-K
Fire-clay bricks		
Thermal conductivity	$k = 1.56 (x1.85 \times 10^{-4}/K)$	W/m-K
Specific heat capacity	$c = 0.950$ at 20°C	kJ/kg-K
	$c = 1.050$ at 700°C	kJ/kg-K



Elements 1-3: Fire-clay bricks
Elements 6-9: Reinforced concrete

Elements 4-5: Insulation (granulated perlite)
Reinforcement elements: ① circumferential bars
② vertical bars

Figure 2. Finite element mesh for the axisymmetric analysis

3. RESULTS OF THE NUMERICAL ANALYSIS

The results of three typical numerical examples, as parts of the chimney analysis are presented. The first one represents the the top section under transient thermal load in the F.E. axisymmetric idealisation and the second one represents the same section in 2D F.E. idealisation under asymmetric steady state thermal and mechanical loads. The third example represent the axisymmetric analysis of the chimney crown.

3.1 EXAMPLE I: TRANSIENT ANALYSIS OF THE SECTION (140-150 M) IN THE AXISYMMETRIC REPRESENTATION

In Fig. 2 is shown the finite element idealisation which is used for the axisymmetric analysis of all chimney sections. Only reinforcement and the sizes D_0 and d_0 are accordingly adjusted. The same mesh is used for both the thermal and analysis. However, a structural influence of the elements 1-3 (representing fire-clay bricks) and the elements 5-6 (representing insulation) are neglected in the mechanical part of the analysis. In the analysis, reinforcement bars are idealised with a special 1D finite elements.

Mechanical load represents selfweight and dead load above the analysed section. Thermal time dependent load is given as a variation of the internal (gas) temperature shown in Fig. 3. External thermal boundary condition is represented by the ambiental air temperature. Precisely, for the summer case I(a): air temperature is $T_a = +40^\circ\text{C}$, and for the winter case I(b): air temperature is $T_a = -10^\circ\text{C}$. The numerical analysis, in case without insulation layer, predicts cracks throughout the reinforced concrete structure. However, supposing that the concrete is correctly compacted and repaired together with the repair of the insulation layer, the numerical results are acceptable and new cracking is not predicted. These partial results are shown in the figures 4 and 5, and the tables 3 and 4.

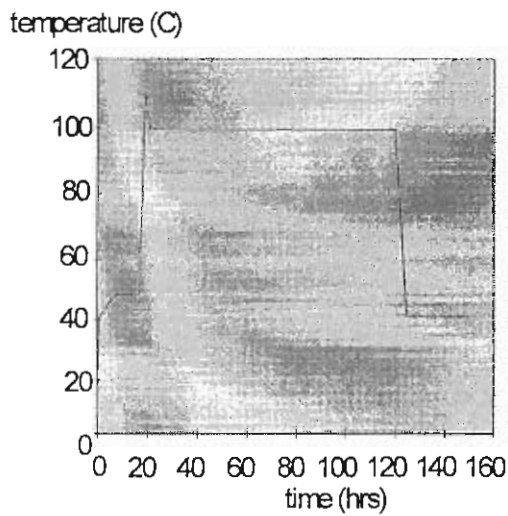


Fig. 3 Internal (gas) temperature variation with time for Example I

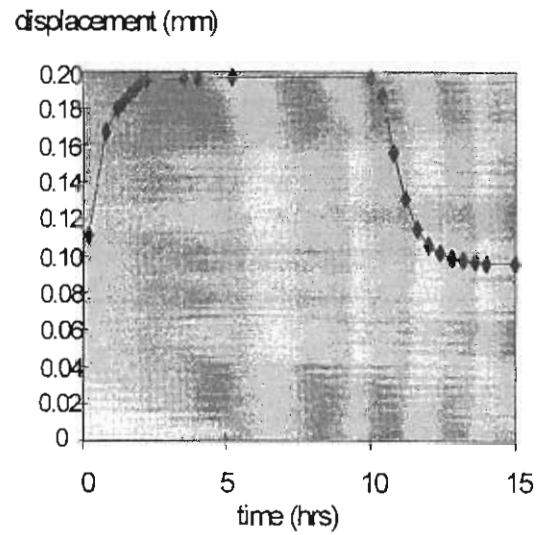


Fig. 4 Radial horizontal displacement response of the external surface of Example I(a)

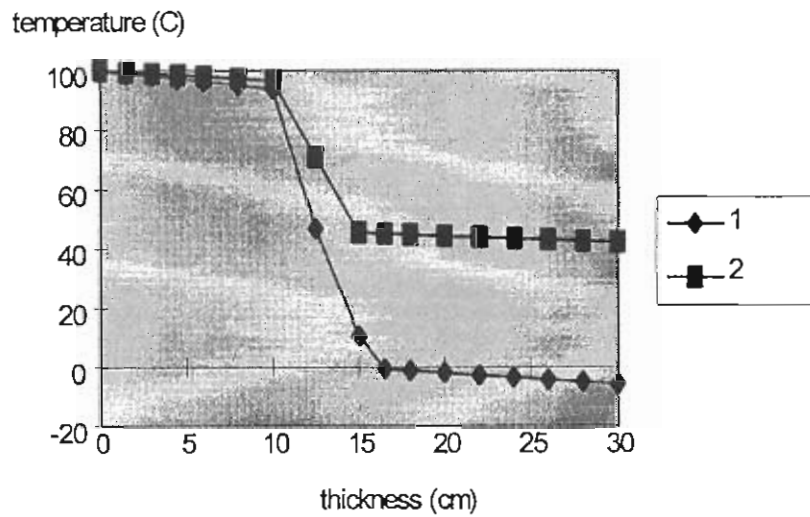


Figure 5 Temperature distribution through the wall layers for Examples I(a) and I(b)

Table 3: Extreme stresses and displacements of Example I(a)

Stress in concrete	max σ_B /tension/	+1.98 MN/m ²	min σ_B /compress./	-4.50 MN/mm ²
Stress in circumferential reinforcement	max σ_A /tension/	+1.05 MN/m ²	min σ_A /tension/	+12.36 MN/m ²
Radial displacements /mm/	internal surface, d_{B26}	+0.192 mm	external surface, d_{B46}	+1.980 mm

Table 4: Extreme stresses and displacements of Example I(b)

Stress in concrete	max σ_B /tension/	+2.01 MN/m ²	min σ_B /compress./	-0.43 MN/m ²
Stress in circumferential reinforcement	max σ_A /compress./	-0.85 MN/m ²	min σ_A /compress./	-11.81 MN/m ²
Radial displacements /mm/	internal surface, d_{B26}	-0.179 mm	external surface, d_{B46}	-0.180 mm

3.2 EXAMPLE II: STEADY STATE ANALYSIS OF THE SECTION (140-150 M) IN THE 2-DIMENSIONAL REPRESENTATION

Due to some asymmetric loading conditions like wind and ambient temperature at sunshine and shaded chimney surface, one half of the chimney horizontal cross-section is analysed in the 2D plane-strain formulation. In Fig. 6 is shown the deformed and undeformed half cross section for the case II(a). In this case, the internal thermal load is the maximum gas temperature, $T_g = 100^\circ\text{C}$, and the external is the ambient temperature which varies from $T_a = +40^\circ\text{C}$ (for $\varphi = 0^\circ$) to $T_a = 0^\circ\text{C}$ (for $\varphi = 180^\circ$). Fig. 7 presents the deformed and undeformed half cross section for the case II(b).

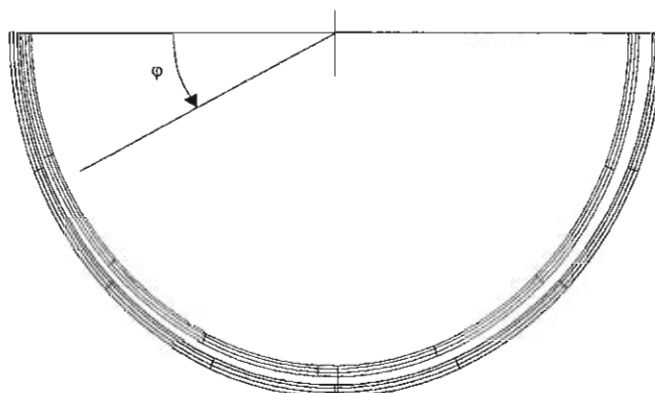


Figure 6 Undeformed and deformed half cross section for the case II(a).

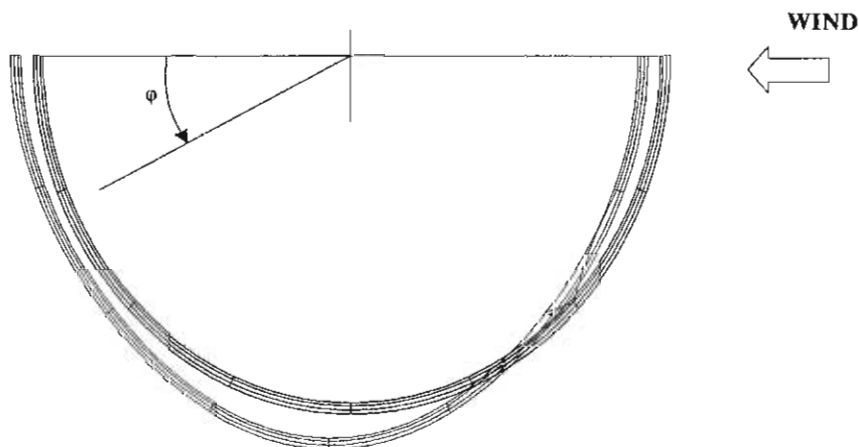


Figure 7 Undeformed and deformed half cross section for the case II(b).

The loading conditions in this case is the same as in the case II(a), however, an appropriate horizontal wind pressure is here added (wind pressure is given according to EUROCODE). The numerical results for repaired chimney evidently show that the displacements, and concrete and reinforcement stress are acceptable. New cracks have no to be expected

3.3 EXAMPLE III: ANALYSIS OF THE CHIMNEY'S CROWN

The crown at the top of the chimney is analysed under the identical loading conditions as applied in Example I: with the case (a) for summer ambiental temperature and (b) for winter temperature. In this case the existing reinforcement could not prevent radial (vertical) crack openings, therefore, an additional reinforcement is designed. The four carbon tracks (SiKa-CFK type 50) at the external crown surface are adopted as additional. The isotherms and deformed radial cross-section of the chimney's crown for the cases (a) and (b) are shown in Fig. 8.

REMARK: The results here presented are taken from the practical consultancy project (Damjanić 1996).

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NUMERICAL FIRE RESISTANCE TESTING OF LIGHTWEIGHT CONCRETE AND NORMAL WEIGHT CONCRETE ELEMENTS

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ABSTRACT

This paper presents a numerical fire resistance testing of completely reinforced lightweight concrete and reinforced normal weight concrete structural members. The fire action is applied in accordance to normalized ISO-834 diagram. Thermal and mechanical properties of steel, lightweight concrete and normal weight concrete are linear for each level of temperature loading. The simple parallel fire test was done for the lightweight concrete material. The results show fire resistance of a few common cast structural elements like TT-plate, T and I beams, simple plate and wall panel made from lightweight concrete or normal weight concrete. These results can be taken as relevant engineer's approval of fire resistance of tested member.

SAŽETAK

U ovome je radu izložena primjena numeričkog ispitivanja na analizu požarne otpornosti potpuno armiranih lakobetonskih i armirano betonskih konstruktivnih elemenata. Kao požarna akcija usvojeno je tipsko opterećenje ISO - 834. Za laki beton, normalni beton i čelik pretpostavljena su linearna svojstva materijala pri svim temperaturnim stanjima. Za lake stiro betone numerički model je baždaren na jednostavnim fizikalnim pokusima. Rezultati analiza su pokazani za konstruktivne elemente koji se učestalo proizvode te se mogu smatrati vrlo pouzdanim inženjerskim dokazom stupnja požarne otpornosti tih konstrukcija.

1. INTRODUCTION

More frequent everyday demands for the analysis of the fire resistance are caused by the more clearer investors' wishes as well as the use of more precise regulations such as EUROCODE, BS and DIN. The development of new engineering structures and materials has the special influence on the analysis of the fire resistance. The new type of structures called *completely reinforced lightweight concrete structures* which appeared a few years ago in Croatia represent completely new field in the analysis and testing of the fire resistance [1, 2]. Although, the field test on fire action is the most reliable proof of the structural resistance due to its high price, hardly repeatable results in the same conditions and the high prices of the equipment, the numerical tests including calibration tests on small specimens represent very good and valuable engineering answer to the complex question of the fire resistance of structures.

The three criteria were tested in this paper: (1) mechanical resistance under fire conditions, criterion R; (2) resistance on fire penetration by hot gases or flame, criterion E; and (3) resistance on fire transfer through the body represented by our structure, criterion I. Criterion R considers fire loading expressed in fire time duration under which the failure of the mechanical resistance and stability of the structure occur. Criterion E represents penetration of hot gases or flame through the holes, cracks or other openings. Criterion I is achieved when the average temperature on the non-exposed surface does not exceed 140°C and the maximum rise does not exceed 180°C.

2. NUMERICAL MODEL OF THE HEAT TRANSFER

The governing heat conduction equation which is solved here is:

$$\rho c \frac{\partial T}{\partial t} = \nabla(k \nabla T) + Q$$

where the principal variable is the temperature T , with the heat capacity (ρc), conductivity (k) and the heat source term (Q). During the fire action heat capacity and conductivity of both concrete and steel are temperature independent, so the problem is materially linear.

A two-dimensional mesh of the 8-noded parabolic finite elements is used to define the domain of interest. The boundary constraints consist of specified nodal temperatures, specified boundary heat flux or convective edge flux, where the flux is proportional to the difference between boundary and ambient temperature. These data are taken later in accordance with ISO-834 standard. As the heat conduction equation is transient, a time integration must be performed to obtain the solution at given time steps.

Numerical tests in these analyses were performed with the software packages TEMP [3] and μ Field [4].

The lightweight concrete specimen, 7 × 12 × 25 cm, was tested under the temperature of up to 400°C. The obtained test results were compared with the numerical ones and were found very similar. In this way both the material properties and the numerical model were tested.

3. FIRE TESTING OF THE STRUCTURAL MEMBER

3.1 Global model of the mechanical resistance and stability of structure

The mechanical resistance of the structure in the fire conditions is evaluated supposing constant fire load during the fire and the decrement of the bearing capacity and stiffness of the material and cross-section. The failure occurs in the time point when the cross-sectional resistance forces fall at the level of acting forces.

The model of the structural resistance is created combining the method of characteristic cross-sections and substructures. The structure is divided into substructures, i.e. members, under the rule which state that the failure of the member causes failure of the whole structure. In this model, by means of iterating, the redistribution of the inside forces on the level of substructures is evaluated due to the change of the cross-section stiffness during the fire action.

The failure of the member is evaluated according to the bearing capacity of the characteristic cross-sections. The characteristic cross-sections are those most loaded in which the bearing capacity decrement is the quickest.

The coefficient of safety of the cross-section in the fire loading changes in accordance with the initial one, decrement of the bearing capacity due to the rise of the temperature and due to the redistribution of the inside forces. If the starting coefficient of safety of some cross-section is given as SF_i^0 , if the decrement of the bearing capacity is represented with the function $r_i(t)$ and if the function of the redistribution of inside stresses is described with $p(t)$, then the coefficient of safety in fire conditions can be obtained as:

$$SF_i(t) = SF_i^0 r_i(t) p(t)$$

what is shown in Figure 1.

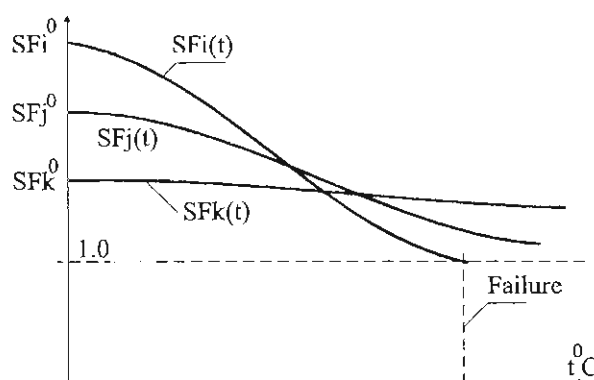


Fig. 1 SF due to fire loading

3.2 Model of the fire transfer through the structure

According to the facts described in Section 1, special attention is paid to those cross-sections where the largest increase of the temperature at the outer side is expected.

3.3 Model of the fire penetration by hot gases or flame

In general, in concrete structures the penetration appears in the case of large cracks. In lightweight concrete structures with plastic properties as EPS concrete with mass lower than 900 kg/m^3 the penetration occurs practically at the moment of the failure of the structure. In other types of concrete with ductile properties in the tensile region large cracks appear at the loading coefficient level of 1.3; in the fire conditions, i.e. for the fire action level when the cross-section resistance decreases, cracks appear at the level of $1.3/1.7 \approx 0.75$.

4. EXAMPLES

The numerical fire resistance tests of the common cast structural elements like TT-plate, I-beam, T-beam and wall panel were performed for both lightweight concrete and normal weight concrete. In all four examples, the used lightweight concrete has the following properties: mass is 1000 kg/m³, heat capacity is 960000 (kg/m³ × J/kgK) and conductivity is 0.25 W/mK.

Example 1: Reinforced lightweight and normal weight concrete TT-plates

All three criteria, R, E and I, are examined for normal weight concrete and lightweight concrete roof TT-plates. The fire load acts on the lower side.

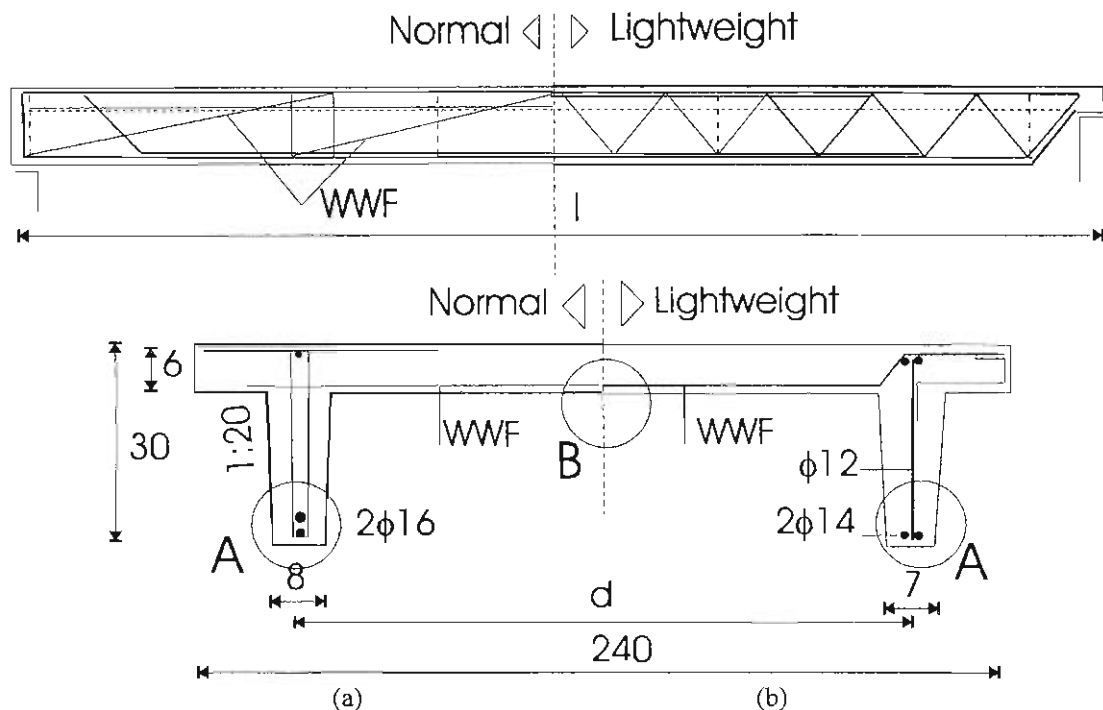


Fig. 2 Sections of the TT-beam

The resistance R is evaluated for the cross-section in the middle, as the appropriate one, and the local point in the cross-section is the lower part of the rib, detail A in Figure 2. Dimensions and properties of the cross-section are shown in Figure 2 and the temperature fields for characteristic time periods for the rib of the TT-concrete plate are shown in Figure 3a. The cross-section resistance is defined with the failure of the tensile reinforcement. For the case of full loading under fire conditions the ratio of working and failure loading is $1/1.7=0.59$, so, the cross-section bearing capacity exists until the reduction of bearing capacity under fire conditions reaches the level of 59%. In accordance with the temperature profile shown in Figure 3a, and taking into the account precedent criterion, the resistance of the cross-section exists approximately up to the time $t=30$ minutes. In the case of the lightweight concrete TT-plate the temperature profile is shown in Figure 3b. According to the precedent criterion the bearing capacity exists approximately up to the time $t=60$ minutes.

Fire transfer and fire penetration for the concrete plate are evaluated in the middle of the connecting plate and in the middle cross-section in the local point B. The characteristic temperature profile for the plate of the concrete TT-plate is shown in Figure 4a. Resistance R exists for approximately $t=30$ minutes, while resistance E exists for approximately $t=30$ minutes and resistance I for approximately $t=15$ minutes.

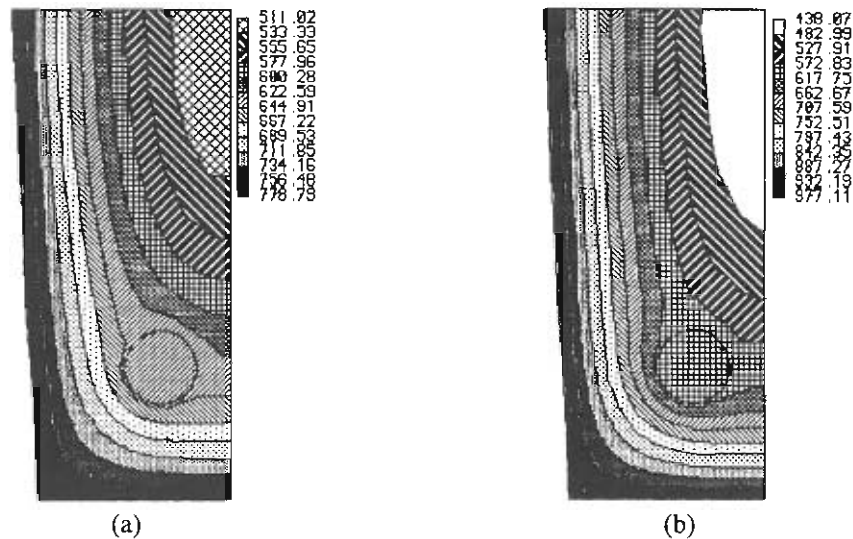


Fig. 3 Temperature profile for TT-plate rib

In case of lightweight concrete plate the temperature profile is shown in Figure 4b. Resistance R exists for approximately $t=90$ minutes, resistance E for approximately $t=90$ minutes, and resistance I for approximately $t=45$ minutes.

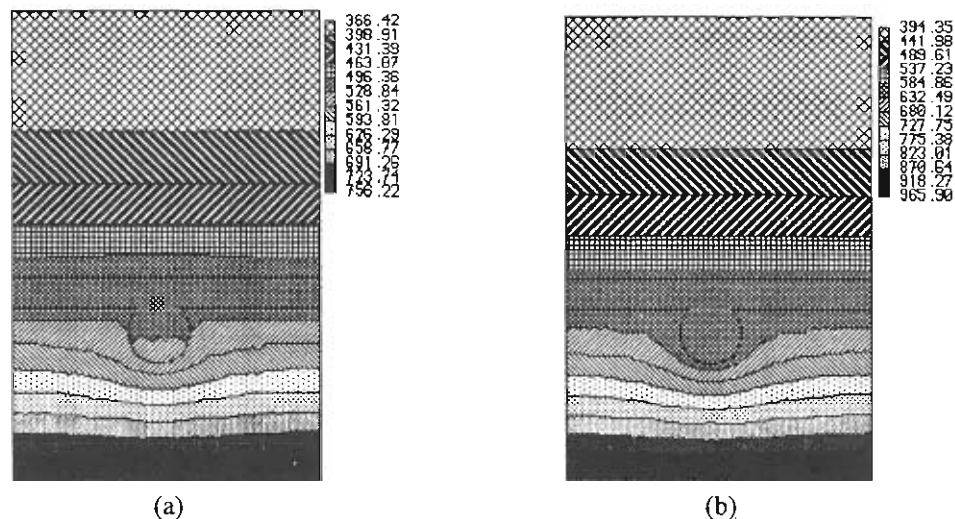


Fig. 4 Temperature profile for TT-plate slab

In case of increasing the weight of the lightweight body the time of each resistance decreases.

Example 2: Reinforced lightweight and normal weight concrete I-beam

The criterion R is evaluated for normal weight concrete and lightweight concrete I-beam, as shown in Figure 5. The fire loading acts on the lower and vertical sides. The resistance R is evaluated for the cross-section in the middle, as the appropriate one, and the local point in the cross-section is in the lower part of the rib, detail A in Figure 5. The temperature fields for characteristic time periods in the case of concrete beam are shown in Figure 6a.

The resistance R of the cross-section exists approximately up to the time $t=25$ minutes. In the case of the lightweight concrete beam the temperature profile is shown in Figure 6b. The resistance R of the cross-section exists approximately up to the time $t=120$ minutes.

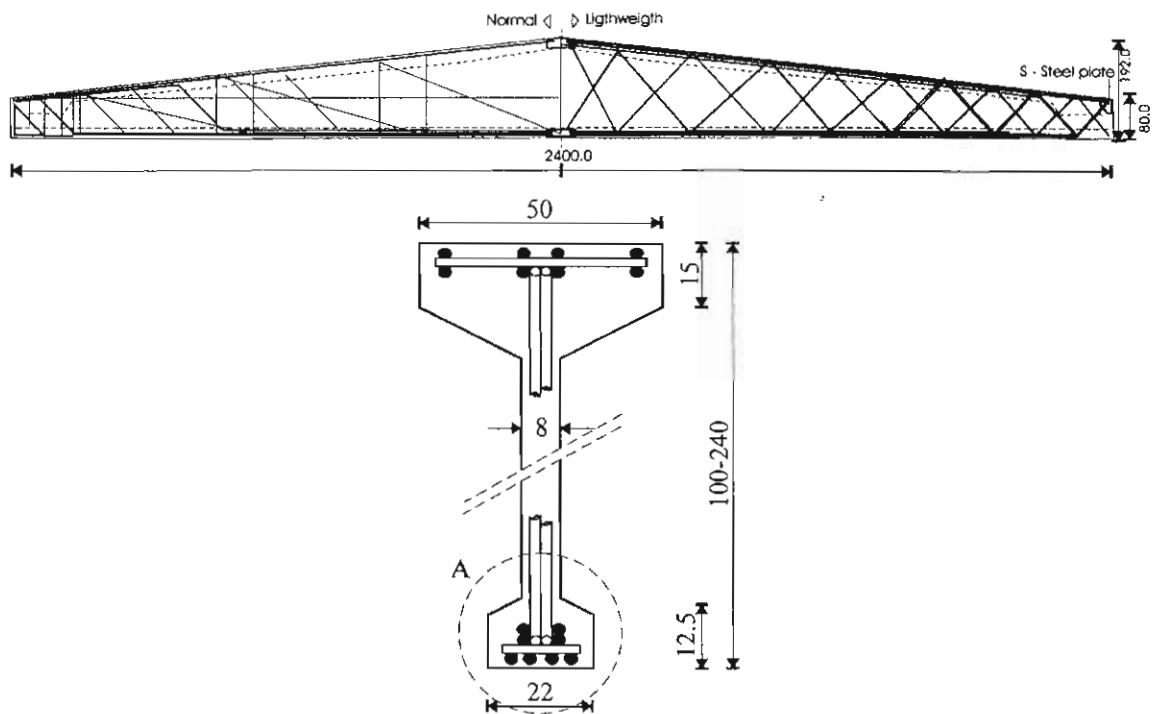


Fig. 5 View of the I-beam

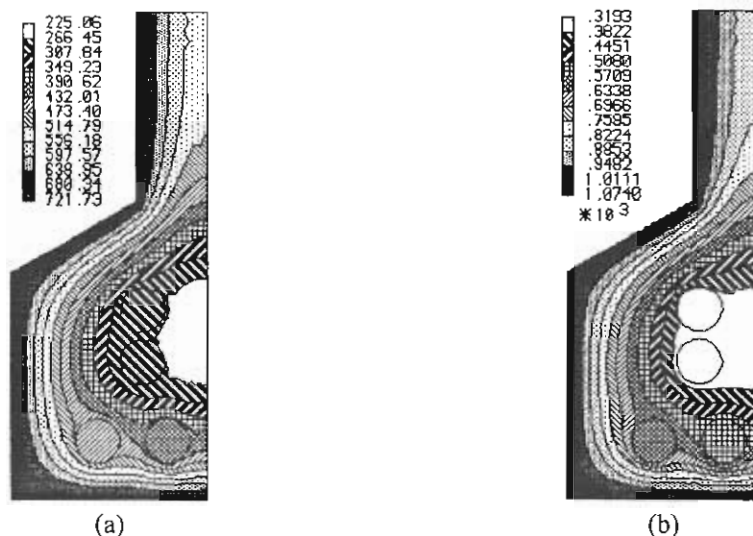


Fig. 6 Temperature profile for I-beam

Example 3: Reinforced lightweight and normal weight concrete T-beam

The criterion R is evaluated for normal weight concrete and lightweight concrete T-beam, as shown in Figure 7. The fire loading acts on the lower and vertical sides. The resistance R is evaluated for the cross-section in the middle, as the appropriate one, and the local point in the cross-section is in the lower part of the rib, detail A in Figure 7. The temperature fields for characteristic time periods in the case of concrete beam are shown in Figure 8a.

The resistance R of the cross-section exists approximately up to the time $t=45$ minutes. In the case of the lightweight concrete beam the temperature profile is shown in Figure 8b. The resistance R of the cross-section exists approximately up to the time $t=90$ minutes.

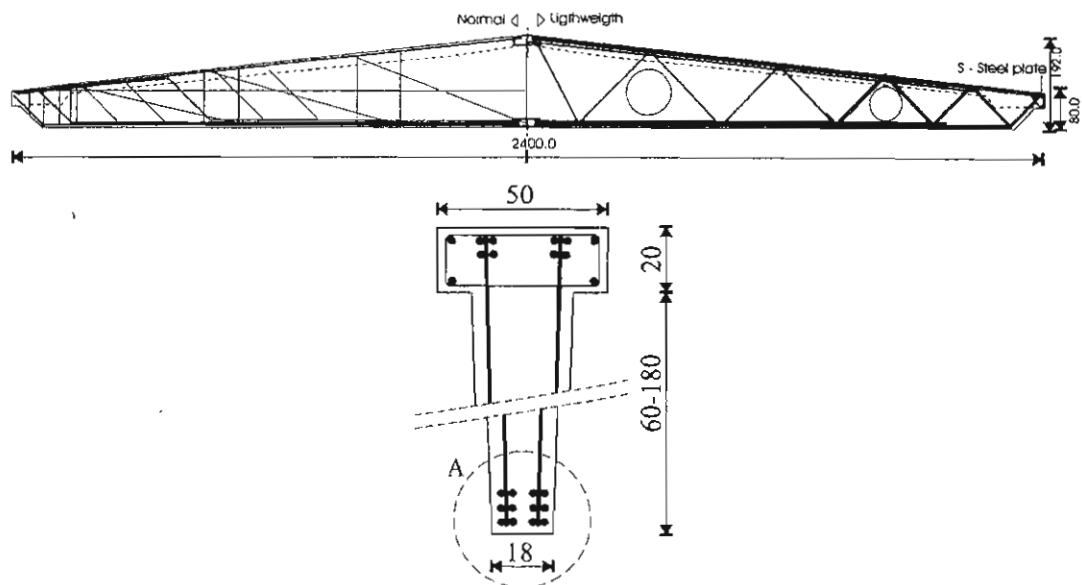


Fig. 7 View of the T-beam

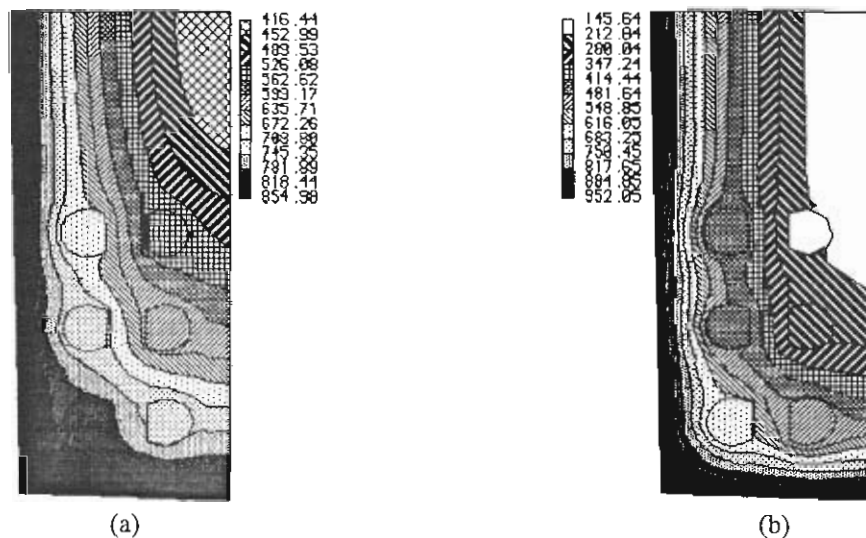


Fig. 8 Temperature profile for T-beam

Example 4: Reinforced lightweight and normal weight concrete wall panel

All three criteria, R, E and I, are evaluated for normal weight concrete and lightweight concrete wall panel shown in Figure 9. The fire load acts on the lower side.

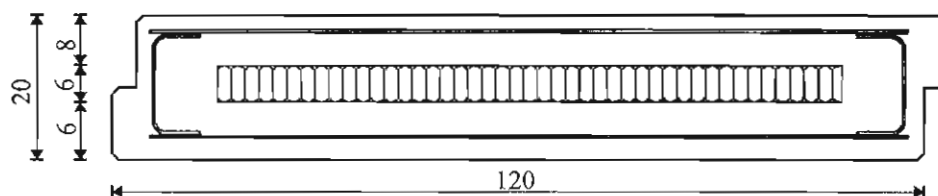


Fig. 9 Section of the wall panel

The resistance R is evaluated for any horizontal cross-section. Dimensions and properties of the cross-section are shown in Figure 9. The temperature fields for characteristic time periods in the case of the concrete wall panel are shown in Figure 10a. The resistance R of the cross-section exists approximately up to the time $t=60$

minutes. The resistance E exists approximately up to the time $t=60$ minutes and the resistance I approximately also up to the time $t=60$ minutes.

In the case of lightweight wall panel the temperature profile is shown in Figure 10b. The resistance R of the cross-section exists approximately up to the time $t=90$ minutes. The resistance E exists approximately up to the time $t=90$ minutes and the resistance I approximately also up to the time $t=90$ minutes.

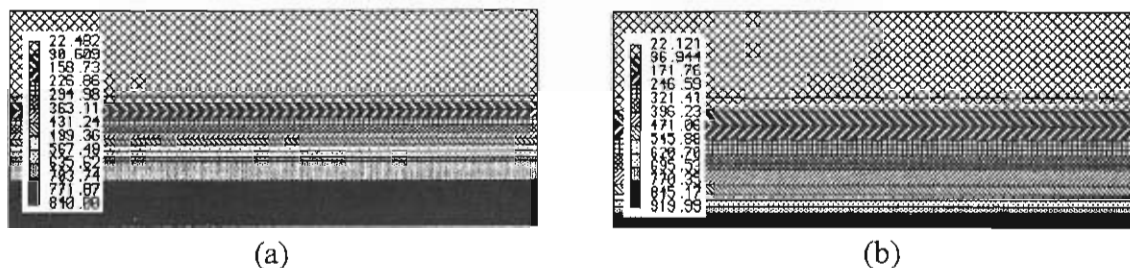


Fig. 10 Temperature profile in the wall panel

5. CONCLUSION

The numerical fire resistance testing of completely reinforced lightweight concrete and reinforced concrete structural members is shown in this paper. According to the obtained test results on the real elements as well as the appropriate numerical results and their comparison, the latter can be accepted with full confidence.

The fire resistances of the lightweight concrete and normal weight concrete, according to the R, E and I criteria, show great differences; the lightweight concrete structures are in a much better position (approximately two times).

The special case arises with the lightweight concrete made of expanded polystyrene aggregate, with a total mass less than 800 kg/m^3 . In the early phase, at the temperature of approximately 90°C , polystyrene starts to expand and destroys the lightweight concrete body. Thus, this kind of lightweight concrete must not be used as a structural material.

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ACKNOWLEDGMENTS

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Volumetric Constraint in Microplane Model for Concrete Materials

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Abstract

Arc-length method is a well-known tool for post-peak analysis of structures with a lot of implementation examples available in literature. However, most of the examples deal with either geometric or hardening-material nonlinearities. Authors have tried to implement the method in microplane model (for concrete) which is a softening material model. Originally microplane model is based on initial stiffness procedure and displacement control to perform post-peak analysis. Introduction of the usual arc-length procedure has resulted in serious instabilities in the post-peak regime. In order to stabilize the procedure a variable had to be chosen to serve as a *guidance variable* in determination of arc-length parameters. This has significantly improved the performance of computer code and allowed some simplifications, too. Consistent linearization of constraint equation further stabilized the procedure. Some numerical examples illustrate the behavior of the model.

1 Introduction

As it is well known concrete belongs to the so called "softening materials", i.e. materials that exhibit reduction of bearing capacity once that certain "peak level" of loading has been exceeded. The main reason for that kind of behaviour is creation of microcracks and cracks after the certain level of loading. In this paper we are interested in post-peak analysis of a structure, i.e. in a phase when structure has deformed considerably but still has some bearing capacity. This type of analysis is important in collapse investigation and safety factor determination.

There are several procedures that make possible determination of load-displacement curves after the maximum load has been reached. One of the most powerfull is the "arc length" method. The basic idea of the arc length method has been presented by Ramm [1980] and Crisfield [1981] followed by many other authors who had pointed out that choice of the constraint equation brings a great difference in the convergence of the solving procedure. The existing microplane computer code is based on the initial stiffness and modified Newton methods and the implementation of an arc length procedure has to take that into consideration.

Several possible implementations of the constraint equation have been tried out, like constant sphere arc, normal plane iterations and orthogonality constraint. After some refinement of the procedure sphere arc managed to pass beyond the peak load but remained unstable and unreliable. At this stage of development it showed necessary to switch to the displacement control method as described by Ramm [1980]. This approach has brought us to the stable and reliable procedure. The main characteristic of the method can be observed on our numerical examples.

2 Arc Length Method

Microplane model under consideration is a materially nonlinear model with special capabilities for analysis of concrete. The classical procedure for solving the emerging nonlinear equilibrium equation is Newton's method. In this version of the microplane computer code initial stiffness method has been adopted, i.e. stiffness matrix $\mathbf{K}$ is formed only at the beginning of the calculation and is left unchanged during the incremental iteration procedure. The tedious task of developing the consistent microplane formulation that would give us the tangent stiffness matrix is left for some time in the future.

For the sake of arc length method the equilibrium equation can be written

as an incremental iteration scheme

$$\mathbf{K}^{(i)} \Delta \mathbf{u}^{(i)} - \mathbf{P} \Delta \lambda = (\lambda^m + \lambda^{(i)}) \mathbf{P} - \mathbf{R}(\mathbf{x}^{(i)}), \quad (1.a)$$

$$\lambda^{(i+1)} = \lambda^{(i)} + \Delta \lambda^{(i)}, \quad \mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} + \Delta \mathbf{u}^{(i)} \quad (1.b)$$

where λ is an additional unknown that represents the load scaling factor. In order to balance the number of unknowns and equations we are introducing an additional constraint equation which can be written in its general form as

$$f(\mathbf{x}, \lambda) = 0 \quad (2)$$

This constraint equation can take various forms and precisely that makes the difference between convergence and stability properties of the arc length method when solving various problems.

In order to improve the stability of the procedure consistent linearization of the resulting system of equations can be performed. The resulting system of equations is nonsymmetric but it can be reduced to the solution of the original problem together with the solution of an additional constraint equation.

We have followed Schweizerhof and Wriggers [1986] and have consistently linearized the system of equations so that the constraint equation becomes

$$\Delta \lambda^{(i)} = -\frac{1}{\alpha^{(i)}} (f^{(i)} - \mathbf{v}^{(i)} \Delta \mathbf{u}^{(i)}) \quad (3)$$

which is separated from the (modified) equilibrium equation

$$\left[\mathbf{K}^{(i)} + \frac{1}{\alpha^{(i)}} \mathbf{P} \mathbf{v}^{(i)} \right] \Delta \mathbf{u}^{(i)} = -\mathbf{G}^{(i)} - \frac{f^{(i)}}{\alpha^{(i)}} \mathbf{P} \quad (4)$$

Update procedures are developed which permit us to solve one system of equations at a time.

2.1 Constraint equation

In our work we have chosen the constraint equation that corresponds to Crisfield's method with iterations performed along the elliptic arc. This is obtained using the constraint equation described in Kožar [1991]

$$\mathbf{x}^T \cdot \mathbf{x} + \beta \lambda^2 \mathbf{P}^T \cdot \mathbf{P} - s^2 = \alpha \quad (5)$$

where β is a control parameter between 0 and 1. As it can be seen α can also take various values from 0 to 1 thus changing the shape of iteration. We have chosen $\alpha = 1$ and $\beta = 1$.

2.2 Update procedure

If we introduce our specific constraint equation in the general expressions given above, then we have to solve the following equations

$$\mathbf{K}^{(i)} \Delta \mathbf{u}_I^{(i)} = \mathbf{P}, \quad \mathbf{K}^{(i)} \Delta \mathbf{u}_{II}^{(i)} = -\mathbf{G}^{(i)}(\mathbf{x}^{(i)}) \quad (6.a)$$

For load increment parameter λ we obtain the quadratic equation

$$\Delta \lambda_{1,2}^{(i)} = \frac{-B \pm \sqrt{B^2 - AC}}{A} \quad (6.b)$$

where

$$\begin{aligned} A &= \sqrt{\alpha + \Delta \mathbf{u}_I^{(i)} \cdot \Delta \mathbf{u}_I^{(i)}}, & B &= \mathbf{x}^{(i)} \cdot \Delta \mathbf{u}_I^{(i)} + \alpha \Delta \lambda^{(i)}, \\ C &= \Delta \mathbf{u}_{II}^{(i)} \cdot \Delta \mathbf{u}_{II}^{(i)} + \alpha \Delta \lambda^{(i)^2} - s^2 \end{aligned} \quad (6.c)$$

Since one gets two solutions out of a quadratic equation a procedure for selection of the right one has to be established. Scalar product between previous and current displacement increment is actually a measure of the angle between the current and the next increment step

$$\delta = \Delta \mathbf{u}^{(i-1)} \cdot \Delta \mathbf{u}^{(i)} \quad (7)$$

According to Bellini and Chulya [1987] one should allays take $\Delta \mathbf{u}^{(i)}$ which gives larger δ since that ensures smooth advancement along the load displacement curve. A slightly modified update procedure has been used in our computer code

$$\begin{aligned} \Delta \mathbf{u}^{(i)} &= \Delta \lambda^{(i)} \Delta \mathbf{u}_I^{(i)} + \Delta \mathbf{u}_{II}^{(i)} \\ \mathbf{x}^{(i+1)} &= \mathbf{x}^{(i)} + \Delta \mathbf{u}^{(i)}; & \lambda^{(i+1)} &= \beta \lambda^{(i)} + \Delta \lambda^{(i)} \end{aligned} \quad (8)$$

The above procedure has been programed and numerical examples have been solved. As it could be seen in section 4 one-dimensional example performs very well but the tree-dimensional microplane code exhibits significant instabilities after the peak loading has been reached.

3 Displacement Control in Microplane Model

Difficulties in the pure arc length procedure have motivated us to introduce a "guidance variable" in a way similar to the one described by Ramm [1981]. One displacement in one node (say Δu_k) is selected and while its value is kept constant (as prescribed) all the other variables are adjusted until the equilibrium is reached. The Δu_k is not prescribed in advance but its value from the first iteration in an increment is kept constant throughout the increment.

In order to improve performance of the method consistent linearization of the constraint equation was performed. Following Schweizerhof and Wriggers [1986] we are left with

$$\mathbf{v}^{(i)} \Delta \mathbf{u}^{(i)} + \alpha^{(i)} \Delta \lambda^{(i)} = -f(\mathbf{x}^{(i)}, \lambda^{(i)}) = -f^{(i)} \quad (9)$$

where the following abbreviations are taken for directional derivatives

$$\mathbf{v}^{(i)} = \frac{d}{d\epsilon} f(\mathbf{x}^{(i)} + \epsilon \mathbf{u}^{(i)}, \lambda^{(i)})|_{\epsilon=0}$$

$$\alpha^{(i)} = \frac{d}{d\epsilon} f(\mathbf{x}^{(i)}, \lambda^{(i)} + \epsilon \Delta \lambda^{(i)})|_{\epsilon=0}$$

For our constraint equation

$$f(\mathbf{x}^{(i)}, \lambda^{(i)}) = \mathbf{x}^{(i)T} \cdot \mathbf{x}^{(i)} + \beta^2 \lambda^{(i)2} - s^2 = 0 \quad (10.a)$$

we have

$$\mathbf{v} = 2\mathbf{x}^{(i)}; \quad \alpha = 2\lambda^{(i)} \quad (10.b)$$

3.1 Update procedure

We have performed consistent linearization in this procedure and our additional equation has changed. Now, there is no quadratic equation any more and we calculate the load increment parameter according to

$$\Delta \lambda^{(i)} = - \frac{f(\mathbf{x}^{(i)}, \lambda^{(i)}) + 2\mathbf{x}^{(i)} \Delta \mathbf{u}_{II}^{(i)}}{2(\beta \lambda^{(i)} + \mathbf{x}^{(i)} \Delta \mathbf{u}_I^{(i)})} \quad (11)$$

Update procedure remains the same as described for the arc length method. The above procedure has been programed and numerical examples have been solved.

4 Numerical examples

4.1 One-dimensional example

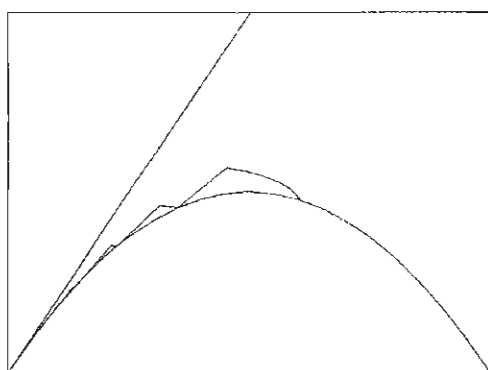


Fig.1a. Arc length method.

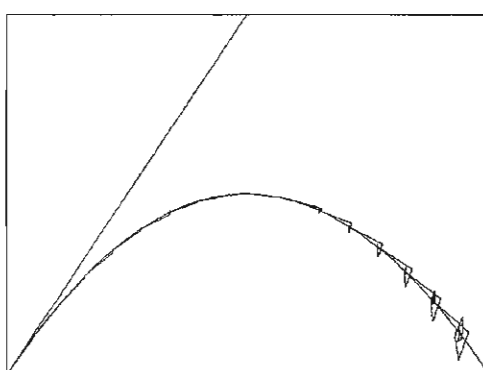


Fig.1b. Consistent linearisation.

One-dimensional example serves only as an illustration of the solution method. It is not a microplane code but solution of a rather simple parabolic equation $y(x) = b - (x - a)^2$. First we have tried to solve the above equation using arc length method and iteration along an elliptic arc. As it could be seen from the Fig.(1a) it was difficult to get far beyond the peak point and trace the curve along the full path. The second example employs consistent linearization procedure. Fig.(1b) shows that this time procedure converged along the whole path of the curve.

The same examples have been repeated using the full Newton procedure (with exact tangent equation) without success. As it can be seen consistent linearization gives superior results, but both procedures fail under the full Newton scheme.

4.2 Three-dimensional example

The three-dimensional example is a concrete brick under uniform tension.

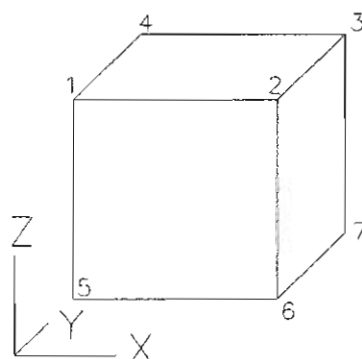


Fig.2. Concrete brick example.

Compared are load-displacement curves for three different cases

- standard microplane code with prescribed values of displacements for all the nodal variables,
- arc length procedure with iteration along circular or elliptical arc,
- displacement control procedure with one of the unknown variables selected as a guidance variable.

All results are compared on the following graph. Unmodified arc length method oscillated around the peak load and showed instability for the initial stiffness method used in the analysis. Consistently linearized constraint

equation with one unknown taken as a guidance variable produced satisfying results. The same drawback of the method as mentioned by Ramm [1981] is clearly pronounced, i.e. snap back could not be traced (observe the straight part of the load displacement curve on the Fig.(3)). However, it should not represent a serious obstacle in analysis of post-peak behaviour of concrete structures since post peak behaviour is determined along its full path.

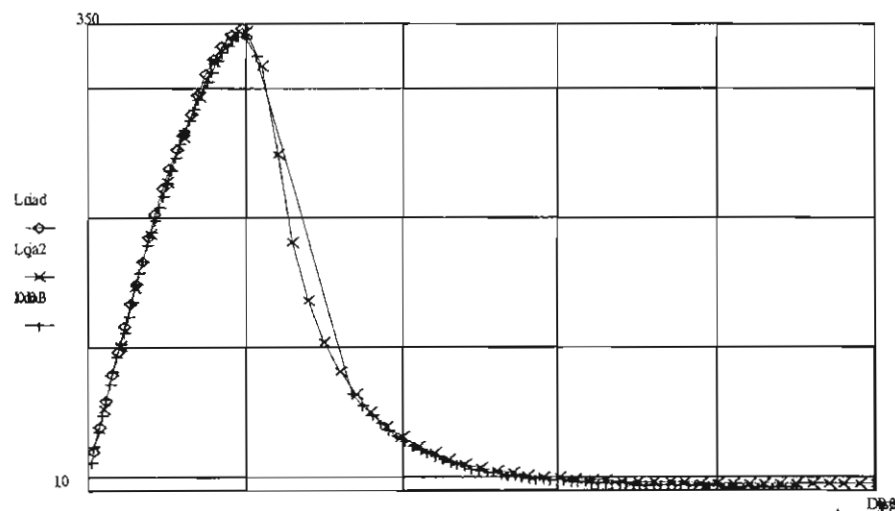


Fig.3. Load-displacement diagrams.

5 Conclusion

There is a pronounced interest in post-peak analysis of "softening" structures that retain load bearing capacity after the peak load has been reached. In our example that analysis is performed through use of the existing microplane computer code by prescribing nodal displacements in order to obtain load displacement curve for the region of interest (which regularly includes post peak area). However, in some examples nodal displacements could not be prescribed in advance and arc length method has been investigated as a logical solution to the problem. Unfortunately, even after consistent linearization arc length method that gives good results in problems of geometrical instability could not provide us with stable and satisfying results.

Introduction of a single "guidance" variable stabilized the arc length method and the way of implementation removed the need for a prescribed value of that or any other unknown variable (nodal displacement). Introduction of the consistently linearized constraint equation provided the method with an additional stability and accuracy.

The comparison of the mentioned numerical procedures is given on the example of a concrete brick in tension.

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MODELS FOR NUMERICAL ANALYSIS OF STONE STRUCTURES

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ABSTRACT

We have recently witnessed the destruction of many existing and valuable stone structures in ancient urban cores of European cities due to warlike situations or earthquakes. Assuming that we are interested in safety aspect of existing structures under actual and increasing loading conditions, we have become aware of the tasks imposed by present situation, i.e., to preserve such structures which usually have some historical importance. While some small structures will be reconstructed by simple civil engineering and stone-cutting works, other more significant structures or their details will require complex analyses. This paper presents some aspects of numerical modelling of stone structures, namely three numerical models based upon finite element techniques which can be used to perform numerical analyses of stone structures. These three models are: (1) standard linear model; (2) linear model with contact (interface) elements; and (3) non-linear elasto-plastic (orthotropic) model. The application of these models will be presented on some characteristic details of stone structures.

1. INTRODUCTION

We have recently witnessed the destruction of many existing and valuable stone structures in ancient urban cores of European cities, especially in Croatia, due to warlike situations and earthquakes. Assuming that we are interested in safety aspects of existing structures under actual or new (usually increased) loading conditions and in preserving architectural and cultural heritage, we have become aware of the tasks imposed by present situation, i.e., to preserve such structures. While some small structures will be reconstructed by simple civil engineering and stone-cutting works, other more significant structures or their details will require complex analyses. Reliable numerical models are necessary to assess and strengthen existing stone structures. This paper presents three numerical models based upon finite element techniques which can be used to perform numerical analyses of stone structures.

The presented models include the following: (1) standard linear model; (2) linear model with specific contact (interface) elements for the joints; and (3) non-linear elasto-plastic (orthotropic) model. Each of the mentioned models can be applied to a certain extent considering the accuracy of the approach and type of the problem. Consequently, stone structures can be generally analyzed according to the *macro-modeling* or *micro-modeling* principles.

In *macro-modeling* the stone structures are treated as a homogenous anisotropic continuum with average properties of stone units (blocks) and joints, both for in each element and for the entire finite element mesh which is used to discretize the structure. This approach, with the model of smeared cracks, makes it possible to reproduce successfully the effects of material non-linearity and progressive local failure, so that the effects of local failure on the blocks and/or joints are "smeared" over the finite element either fully or over its respective part. This technique is sufficient for the analysis of stone structures as large panels, globally, and it is applicable up to that level of accuracy when local effects become significant - e.g. in the region adjacent to a concentrated load where stress gradients become high or when details of the stress field and/or displacement should be solved. Summarily, macro-models are applicable when the structure is composed of solid walls with sufficiently large dimensions so that the stresses across or along a macro-length will be essentially uniform.

Higher level of accuracy of the approach to the analysis of stone structures can be obtained by *micro-modeling* where the stone blocks are modelled by the continuum elements, whereas the joints, as a source of weakening and non-linearity, are separately modelled by contact (interface) elements. Hence, contact (interface) elements are used as materially and geometrically non-linear element at the place of geometrical discontinuities or joints, while the remaining part is treated on the linear or non-linear analysis.

Clearly, macro-modeling is more practice oriented due to the reduced time and memory requirements as well as user-friendly mesh generation. This type of modeling is most valuable when a compromise between accuracy and efficiency is needed.

The application of the presented models will be illustrated by the analysis of several characteristic details of stone structures, namely, stone cantilever (structural part of the Mediterranean balconies) and semi-circular opening in the stone wall. All the examples were analyzed for the case of plane stress state. More detailed results of these analyses as well as the analyses of some other characteristic structural details in the stone structures can be found in the Ref. [1].

2. THREE MODELS FOR THE STONE STRUCTURES ANALYSIS

In this paper three numerical models based upon finite element techniques which can be used to perform numerical analyses of stone structures are going to be presented. These three models are: (1) standard linear model; (2) linear model with contact (interface) elements; and (3) non-linear elasto-plastic (orthotropic) model.

2.1 Linear model

The application of the general or standard linear model with finite element method based on the displacement on stone structures is limited, i.e., it can be applied only within the limit of elastic behaviour, until the appearance of the first cracks. Nevertheless it can be applied in predicting the behaviour for the low levels of loading, which is a frequent case for the existing stone structures. It also represents the first step in the non-linear analysis which enables a quick insight into the problem and represents the "mental control" of the solution. Consequently, it is recommended that a linear analysis should be performed first.

Subsequently, only the basic relations are presented without considering the details of the linear model, which represent the standard of the numerical analysis and has been described in details in literature [2, 3]. Considering the limits of the model applicability, it should be noted that elastic linear behaviour (in 2-D models) for stone is valid only within the von Mises initial yield surface in biaxial compression-compression stress state, i.e., within the tension cut-off failure surface in biaxial tension-tension and tension-compression. Beyond these limits the elastic linear model cannot be applied and elasto-plastic model becomes relevant (in biaxial compression-compression up to the failure surface), as well as orthotropic non-linear model for cracked stone (in all stress state beyond the failure surface defined by von Mises failure envelope and tensile cut-off).

Elastic behaviour (for the plane stress state) is described by the incremental stress-strain relationship:

$$\begin{Bmatrix} d\sigma_x \\ d\sigma_y \\ d\tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} d\epsilon_x \\ d\epsilon_y \\ d\gamma_{xy} \end{Bmatrix} \quad (1)$$

or:

$$d\sigma = \mathbf{D} d\epsilon \quad (1')$$

where σ_x and σ_y are normal stresses in the global x and y directions; τ_{xy} shear stress; E and ν elastic constants, namely modulus of elasticity and Poisson's ratio; ϵ_x and ϵ_y strains in x and y directions; γ_{xy} shear strain; $\mathbf{D}$ elastic material matrix; $d\sigma$ and $d\epsilon$ incremental stress and strain vectors.

Thus, it is possible to define the material model, which together with the standard finite element procedure based on the displacement and frontal algorithm for solving the system of equations, after respective post-processing, form an entire model.

The application of the linear model is illustrated by the analyses of a stone cantilever, Figures 1(a), 1(b) and 1(c), and a semi-circular opening in the wall, Figure 2, where displacements and stress fields are shown.

2.2 Linear model with interface elements

This model deals with the stone blocks and their joints separately. The previously mentioned linear model can be applied to the continuum elements which form the blocks. The 1-D non-linear interface element was described in details in Refs. [4, 5].

Figure 2(a) presents the principle for the model development. Each stone block consists of several continuum elements, while interface elements which model the joints as geometrical discontinuities, are placed along the block boundaries (determined by bold lines), so that relative displacements between the blocks are made possible. This can be noted from the presentation of displacements, Figure 3. In the stone cantilever the contact zone is at the boundary between the first and the second row, and between the second and the third row of finite elements from Figure 1(a). In both examples the characteristic vertical displacement (boundary point of the cantilever, i.e., the top of the opening) is greater than in macro-modeling.

2.3 Non-linear elasto-plastic (orthotropic) model

The appearance of cracks due to tensile stresses in the system is one of the main causes of material non-linearity in stone structures. Consequently it is necessary to introduce into the model, in addition to the average non-linear behaviour (of the bond) unit-joints, also the appropriate treatment of the cracks behaviour, which enable the analysis of the local failure with the propagation of cracks under incremental loading. Thus, it is possible to respect the principle that accurate mathematical description of the mechanical behaviour of the material is a necessary condition for any calculations for the structures, and, at the same time, it is possible to satisfy the requirement that stone structures are studied at a level higher than the one possible with the linear model. In addition to the expansion of the cracking zones it is possible to solve the problems of the critical and yield loads, actual safety factor, the weakest points and yield zones, the failure mechanism - for specific levels of loads in incremental approach.

With this objective in mind it was possible to develop the model for the non-linear analysis of stone structures according to the principle of macro-modeling and smeared cracks, in accordance with similar models in the world [6, 7], according to the basic elasto-plastic model [8]. The model was presented with more details in Ref. [1].

Until the failure criterion (defined by von Mises parabola through the point of uniaxial compressive strength f_c' and tensile cut-off through uniaxial tensile strength f_t' in the system of principal stresses) standard elasto-plastic J_2 model is valid. The material matrix $\mathbf{D}$ from equation (1) is substituted by elasto-plastic $\mathbf{D}_{ep}$, which is of practical importance only in biaxial compression-compression. After reaching the failure criterion it is considered that cracks are formed both in tension-tension and tension-compression, and crushing occurs in compression-compression. The material becomes orthotropic, with orthotropy axes n - t normal and tangential with regard to the direction of the crack.

For a single crack, the incremental stress-strain relationship in the local n - t coordinate system can be expressed as:

$$d\hat{\sigma} = \hat{\mathbf{D}}_c d\hat{\varepsilon} \quad (2)$$

in which $d\hat{\sigma} = \{d\sigma_n, d\sigma_t, d\tau_{nt}\}^T$ and $d\hat{\varepsilon} = \{d\varepsilon_n, d\varepsilon_t, d\gamma_{nt}\}^T$. The local tangent stiffness matrix $\hat{\mathbf{D}}_c$ of the cracked material is generally of the following form:

$$\hat{\mathbf{D}}_c = \begin{bmatrix} E_{nn} & 0 & 0 \\ 0 & E_{tt} & 0 \\ 0 & 0 & G_{nt} \end{bmatrix} \quad (3)$$

where E_{nn} , E_{tt} and G_{nt} are the tangent moduli according to the correspondent 1-D diagrams, either tensile or compressive, and the respective principle strains, in the n and t directions. Poisson's effects in the cracking zone are neglected. Actual values depend upon the "zone" of principle stresses, i.e., the type of cracking and local failure (tension-tension, tension-compression, compression-compression).

The transformation into the global system follows after the determination of the local matrix:

$$\mathbf{D}_c = \mathbf{T}^T \hat{\mathbf{D}}_c \mathbf{T} \quad (4)$$

where $\mathbf{T}$ is the transformation matrix (Ref. [9]) dependent upon the cracks direction with regard to the global coordinate directions. Finally, the relationship stress-strain in the global system is:

$$d\sigma = \mathbf{D}_c d\epsilon \quad (5)$$

If the laws of softening are added, so that the cracked zones extend due to the redistribution of stresses, it is possible to develop a complete material model which should be implemented in the standard displacement-based incremental-iterative finite element formulation. Because of the softening behaviour, the conventional Newton-Raphson and modified Newton-Raphson iteration schemes are not feasible and, consequently, the displacements are evaluated with the initial stiffness method.

Some results, namely displacements and stresses with crack positions, for different load factors F , obtained by a specially adapted post-processor, are presented in Figures 1(d), 1(e) and 4.

3. CONCLUSION

The application of the previously presented models depends upon the required level of accuracy of the approach and actual structure. There is no general solution which could be applied to all stone structures.

The linear model is sufficient only for a quick insight into the structure state for low levels of loading or as an approximate calculation and a first phase of non-linear analysis. However, more detailed approaches require the use of micro-modeling. General non-linear models will yield more data than linear models at the macro-modeling level. Macro-models will be satisfactory when normal stresses are dominant, but they will not be sufficient if failure occurs due to shear and in seismic analyses.

Some more informative and/or detailed information on the different numerical as well as on the experimental techniques which can be employed in the numerical analyses and in the field investigation procedures of the stone or generally masonry structures can be found in the Refs. [10-16].

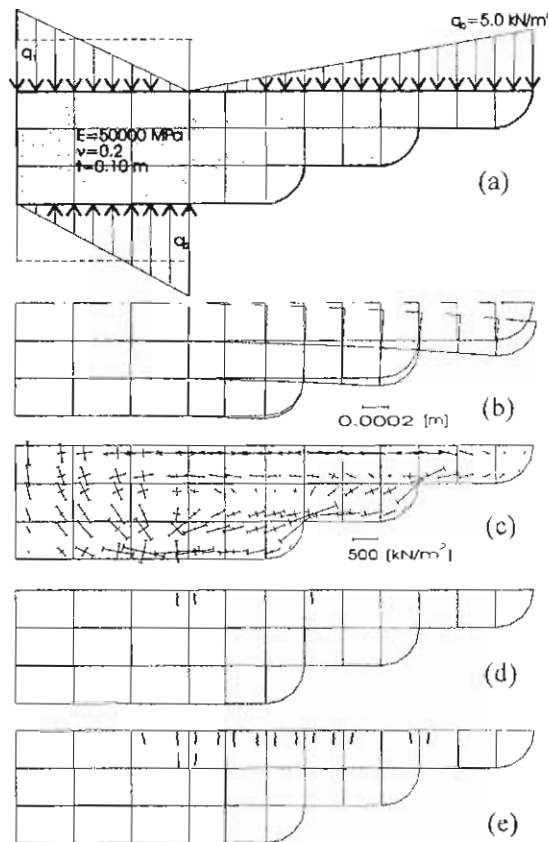


Figure 1. Stone cantilever:

- (a) discretization;
- (b) displacements for elastic level ($F = 1.0$)
- (c) stresses for elastic level ($F = 1.0$)
- (d) cracks for $F = 25.0$
- (e) cracks for $F = 27.0$

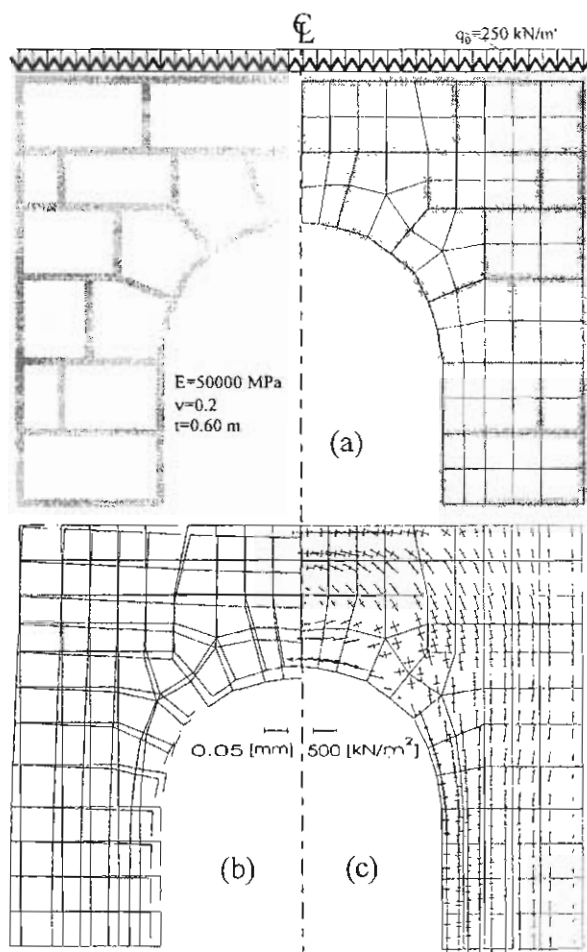


Figure 2. Semi-circular opening:

- (a) discretization and distribution of stone blocks;
- (b) displacements for elastic level ($F = 0.4$)
- (c) stresses for elastic level ($F = 0.4$)

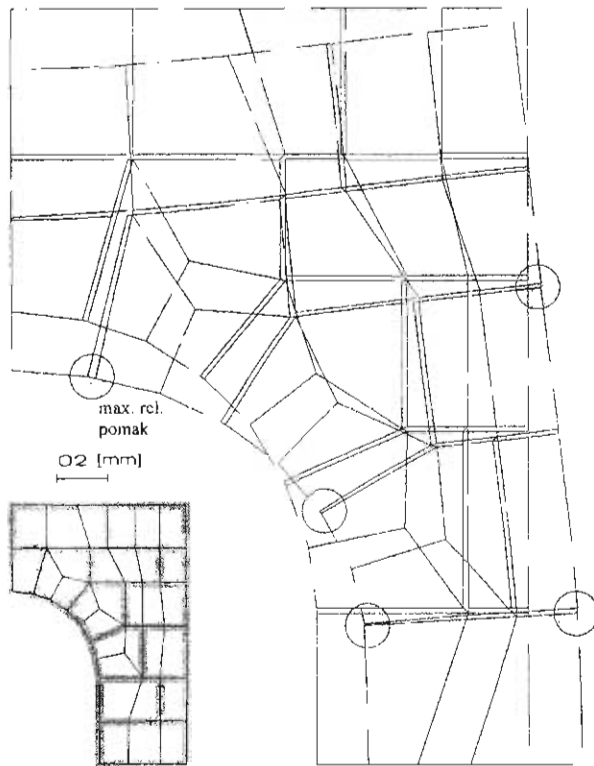


Figure 3. Semi-circular opening:
linear model with interface
elements
displacements

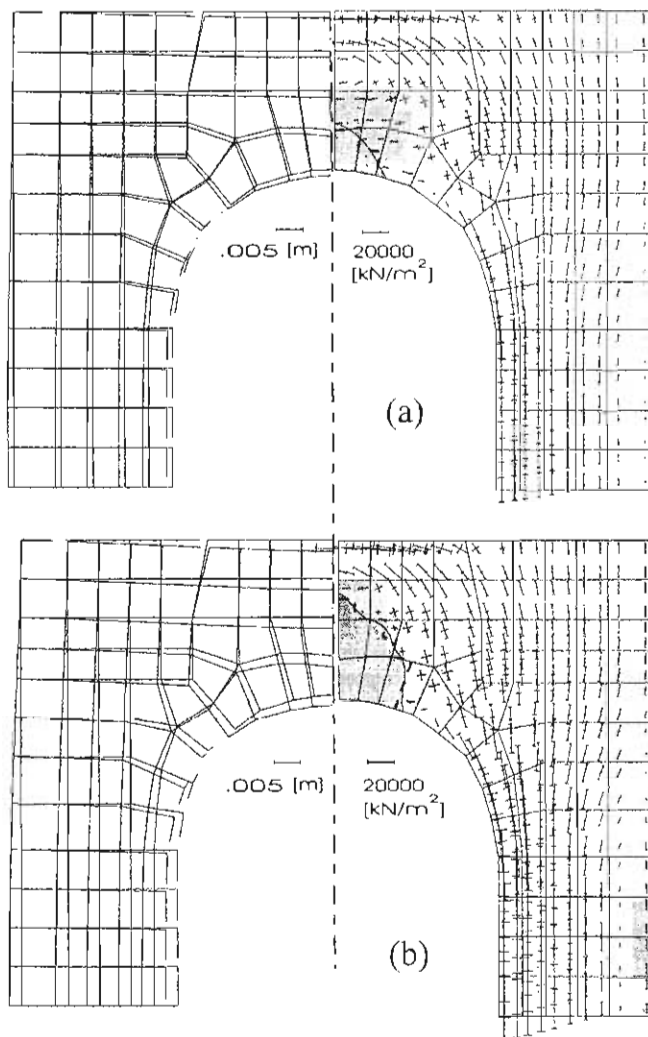


Figure 4. Semi-circular opening:
non-linear model
(a) displacements and
stresses with cracks for
 $F = 14.0$
(b) displacements and
stresses with cracks for
 $F = 20.0$

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FINITE ELEMENT FORMULATIONS FOR COLD ROLLING PROCESS

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Abstract: Numerical analysis of cold rolling process of thin plates by finite element method numerical procedure as a plane strain problem is solved. Large strain elastoplastic updated Lagrangian Hencky formulation was used, and in that way material and geometrical non-linearities are included in analysis. Mixed formulation of finite element method based of "displacement-pressure" variables for the description of the material incompressibility is used. Friction on the contact surface between rolls and plate is simulated with friction layer technique. Above mentioned formulation and technique were incorporated into the development of computer program named ROLLING. Efficiency of used formulation is validated by analysis carried out with mentioned software and by comparison with known experimental investigation. Additional verification is provided by comparison with other results available in the literature.

1. INTRODUCTION

Cold rolling process is one of the most frequent metal forming processes. It is characterized by high productivity rates, so a small improvement in process modelling can have a significant impact on both production expenses and product quality. Exact prediction of mechanical data like stresses, forces, velocities, final shape, etc. is prime goal of such modelling. Not so long ago only analytical modelling was used. Analytical models can give limited results, but since modern industry requires details what these models fail to give, analysis must be carried out by adequate numerical method, preferably finite element method.

There are several problems which using efficient model based on finite element method have to be solved:

- Process includes large elastoplastic strains, so an adequate finite strain algorithm must be used. The earliest large strain models were hypoelastic models, and they have been based on rate-type formulations and additive decomposition of strain in elastic and plastic parts. Hyperelastic models consider total strain formulation and multiplicative decomposition of deformation gradient. Since hyperelastic formulation has certain advantages [1, 2, 3], it has to be implemented into finite element model instead of hypoelastic formulation.
- Material subjected to large strains exhibits incompressible behaviour. Pure displacement-based finite element method can not be used since it fails to predict pressure part of stress tensor. So, incompressible behaviour must be simulated in some way. For instance, Lagrange multipliers, penalty method, etc., can be used.
- In contact region between roll and workpiece exists so-called neutral point. From the entrance to the neutral point the workpiece velocity is smaller than the roll velocity, but from this point to the exit it is greater. In neutral point workpiece velocity is

equal to roll velocity. That is, friction forces change their direction in neutral point. Since neutral point position is generally not known in advance, then friction forces are also not known. Furthermore, classical expression $T = \mu N$ is not accurate enough, so other friction laws must be used.

2. FINITE ELEMENT DISPLACEMENT / PRESSURE FORMULATION FOR LARGE STRAINS

Material that undergoes process involving large strains exhibits incompressible behaviour. The displacement-based finite elements can provide accurate description of such behaviour. However, the required number of elements to obtain requested accuracy is far greater than in the similar problem that involves compressible behaviour.

Basic problem that displacement-based finite element method encounters is that hydrostatic pressure in stress tensor is calculated from the volumetric strain. Since volumetric strain is equal to zero in total incompressibility, and bulk modulus tends to infinity, then it is obvious that displacement-based finite elements fail to predict pressure part of stress tensor within a totally (or near) incompressible material.

The key step in solving aforementioned problem is a proper modification of incremental potential to include pressure as independent variable beside displacement variable [4]. Such modified potential must fulfill three basic requirements:

1. Pressure calculated from the modified potential must be equal to the separately interpolated pressure since pressure calculated from displacement must be eliminated.
2. Variation of modified potential with respect to pressure must give constraint equation relating separately interpolated pressure to pressure calculated from displacement.
3. If the pressure calculated from displacement is equal to separately interpolated pressure then modified potential must be equal to the potential in the case of displacement-based finite elements. This ensures that displacement-based finite element solution is included in formulation as a special case.

Mentioned modification of potential in the case of updated Lagrangian formulation can be stated as follows:

$$\delta \left(\int_{\tau_V} ({}^t\bar{W} + {}^tQ) d^tV \right) = {}^t\mathfrak{R} \quad (1)$$

where ${}^t\bar{W}$ represents potential corresponding to displacement-based case, ${}^tQ$ is modification which is a function of displacement and separately interpolated pressure ${}^t\bar{p}$. This modification happens to be:

$${}^tQ = -\frac{1}{2\kappa^*} ({}^t\bar{p} - {}^t\bar{p})^2 \quad (2)$$

where are:

${}^t\bar{p}$ - pressure calculated from displacement field

$\kappa^* = \kappa \det {}^t_0\mathbf{X}$

κ - bulk modulus

${}^t_0\mathbf{X}$ - deformation gradient from initial moment till time t .

Variation of proposed modified potential gives governing finite element equations:

$$\begin{pmatrix} {}^t\mathbf{KUU} & {}^t\mathbf{KUP} \\ {}^t\mathbf{KPU} & {}^t\mathbf{KPP} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} {}^{t+\Delta t}\mathbf{R} \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} {}^t\mathbf{FU} \\ {}^t\mathbf{FP} \end{pmatrix} \quad (3)$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{p}}$ are vectors of the increment in the nodal point displacements and element internal pressure variables. Other matrices can be calculated as:

$$\left. \begin{aligned} {}^tFU_i &= \int_V {}^t\tau_{kl} \frac{\partial {}^te_{kl}}{\partial \hat{u}_i} dV \\ {}^tFP_i &= \int_V \frac{1}{\kappa^*} ({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_i} dV \\ {}^tKUU_{ij} &= \int_V {}^tCUU_{klrs} \frac{\partial {}^te_{kl}}{\partial \hat{u}_i} \frac{\partial {}^te_{rs}}{\partial \hat{u}_j} dV + \int_V {}^t\tau_{kl} \frac{\partial^2 {}^te_{kl}}{\partial \hat{u}_i \partial \hat{u}_j} dV \\ {}^tKUP_{ij} &= \int_V {}^tCUP_{kl} \frac{\partial {}^te_{kl}}{\partial \hat{u}_i} \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_j} dV \\ {}^tKPP_{ij} &= \int_V -\frac{1}{\kappa^*} \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_i} \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_j} dV \end{aligned} \right\}, \quad (4)$$

where are:

$$\left. \begin{aligned} {}^t\tau_{kl} &= {}^t\bar{\tau}_{kl} - \frac{1}{\kappa^*} ({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial {}^t\tilde{p}}{\partial {}^te_{kl}} \\ {}^tCUU_{klrs} &= {}^t\bar{C}_{klrs} - \frac{1}{\kappa^*} \frac{\partial {}^t\tilde{p}}{\partial {}^te_{kl}} \frac{\partial {}^t\tilde{p}}{\partial {}^te_{rs}} - \frac{1}{\kappa^*} ({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial^2 {}^t\tilde{p}}{\partial {}^te_{kl} \partial {}^te_{rs}} \\ {}^tCUP_{kl} &= \frac{1}{\kappa^*} \frac{\partial {}^t\tilde{p}}{\partial {}^te_{kl}} \end{aligned} \right\} \quad (5)$$

Pressure is generally interpolated:

$${}^t\tilde{p} = \sum_{i=1}^q g_i {}^t\hat{p}_i. \quad (6)$$

Investigations carried out by Sussman and Bathe [4] show that the best behaviour exhibits 9/3 finite element (notation implies to 9 nodes for displacements and 3 for pressure). This also means linear pressure interpolation is used. Consequently, this finite elements were used in analysis.

3. LARGE STRAIN ELASTOPLASTIC UPDATED LAGRANGIAN HENCKY FORMULATION

An effective large strain formulation have to properly include material nonlinearity, geometric nonlinearity and use of adequate stress and strain measures. It is also crucial to use hyperelastic formulation since such formulation do not include integration of rate-type quantities. So, in contrast to hypoelastic formulation these numerical errors are avoided. Besides integration errors, hypoelastic stress-strain description leads to nonphysical response in purely elastic cycle motions. This effect can be negligible when both the elastic stretches and stretch rates are small, but is naturally an undesirable phenomenon. Furthermore, micromechanics observations have confirmed that natural approach is based on hyperelastic material description with multiplicative decomposition of deformation gradient into elastic and plastic parts.

This paper is not concerned with details of development of material model and time integration scheme of the resulting constitutive equations. Only computational procedure

that summarizes basic steps of such hyperelastic formulation will be given briefly in the following text [2, 3, 5]:

1. The trial elastic state

- Evaluate the trial elastic deformation gradient:

$$\mathbf{X}_*^E = {}^{t+\Delta t} \mathbf{X} ({}_0^t \mathbf{X}^P)^{-1} \quad (7)$$

- Perform the polar decomposition:

$$\mathbf{X}_*^E = \mathbf{R}_*^E \mathbf{U}_*^E \quad (8)$$

Polar decomposition is carried out in the following way:

Calculate the right Cauchy-Green strain tensor:

$$\mathbf{C} = (\mathbf{X}_*^E)^T \mathbf{X}_*^E \quad (9)$$

Determine eigenvalues and eigenvectors of the right Cauchy-Green strain tensor:

$$\mathbf{C}\mathbf{P} = \mathbf{P}\mathbf{C}'$$

Eigenvectors are placed in columns of $\mathbf{P}$ matrix, while $\mathbf{C}'$ is diagonal matrix with corresponding eigenvalues.

Calculate:

$$\mathbf{R}' = \mathbf{X}'(\mathbf{C})^{-1/2} \quad ; \quad \mathbf{U}' = (\mathbf{C})^{1/2} \quad (10)$$

Calculate rotation and stretch matrix:

$$\mathbf{R}_*^E = \mathbf{P}\mathbf{R}'\mathbf{P}^T \quad ; \quad \mathbf{U}_*^E = \mathbf{P}\mathbf{U}'\mathbf{P}^T \quad (11)$$

- Calculate the trial elastic Hencky strain tensor:

$$\mathbf{E}_*^E = \ln \mathbf{U}_*^E \quad (12)$$

Here is necessary to perform a spectral decomposition of matrix $\mathbf{U}_*^E$.

Eigenvalues of $\mathbf{U}_*^E$ matrix are placed into diagonal matrix $\mathbf{\Lambda}$, while eigenvectors are in matrix $\mathbf{R}_L$. Hencky strain tensor is then calculated as follows:

$$\mathbf{E}_*^E = \mathbf{R}_L (\ln \mathbf{\Lambda}) \mathbf{R}_L^T \quad (13)$$

- Determine the trace of stress tensor:

$$\text{tr } \bar{\boldsymbol{\tau}}_* = 3\kappa \text{tr } \mathbf{E}_*^E \quad (14)$$

- Calculate the trial stress deviator tensor:

$$\bar{\mathbf{S}}_* = 2\mu \mathbf{E}_*^E \quad (15)$$

- Calculate the trial equivalent stress

$$\bar{\sigma}_* = J^{-1} \sqrt{\frac{3}{2} \bar{\mathbf{S}}_* \cdot \bar{\mathbf{S}}_*} \quad \text{where is: } \bar{\mathbf{S}}_* \cdot \bar{\mathbf{S}}_* = (\bar{S}_*)_{ij} \cdot (\bar{S}_*)_{ij} \quad i, j = 1..3 \quad (16)$$

2. Check whether the solution corresponds to elastic condition:

- If $\bar{\sigma}_* < {}^t \bar{\sigma}_y$ then the solution step was elastic so there are:

$$\left. \begin{aligned} {}^{t+\Delta t} \bar{\sigma} &= \bar{\sigma}_* \quad ; \quad {}^{t+\Delta t} \bar{\sigma}_y = \bar{\sigma}_y \quad ; \quad {}^{t+\Delta t} \mathbf{E}^E = {}^{t+\Delta t} \mathbf{E} \\ \bar{\boldsymbol{\tau}}_* &= {}^{t+\Delta t} \bar{\mathbf{S}} + \frac{1}{3} \text{tr}(\bar{\boldsymbol{\tau}}_*) \mathbf{I} - {}^t \bar{p} \mathbf{I} \quad ; \quad {}^{t+\Delta t} \bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_* \\ {}^{t+\Delta t} \boldsymbol{\tau} &= ({}^{t+\Delta t} J)^{-1} \mathbf{R}_*^E \bar{\boldsymbol{\tau}}_* (\mathbf{R}_*^E)^T \end{aligned} \right\} \quad (17)$$

Continue with step 4.

- Otherwise (plastic solution) continue with step 3.

3. Plastic solution

- Use effective stress function to calculate ${}^{t+\Delta t}\bar{\sigma}$ i ${}^{t+\Delta t}\bar{\epsilon}^P$.

This means solving for root of following function:

$$\Phi({}^{t+\Delta t}\bar{\epsilon}^P) = \frac{1}{{}^{t+\Delta t}J} \sqrt{\frac{3}{2}} {}^{t+\Delta t}\bar{S}({}^{t+\Delta t}\bar{\epsilon}^P) \cdot {}^{t+\Delta t}\bar{S}({}^{t+\Delta t}\bar{\epsilon}^P) - {}^{t+\Delta t}\bar{\sigma}({}^{t+\Delta t}\bar{\epsilon}^P) = 0 \quad (18)$$

- Calculate the deviator of Cauchy stress matrix ${}^{t+\Delta t}\bar{\tau}$.

$$\lambda = \frac{3}{2} \frac{{}^{t+\Delta t}\bar{\epsilon}^P - {}^t\bar{\epsilon}^P}{{}^{t+\Delta t}J {}^{t+\Delta t}\bar{\sigma}}; \quad {}^{t+\Delta t}\bar{S} = \frac{\bar{S}_*}{1 + 2\mu\lambda} \quad (19)$$

- Calculate Cauchy stress matrix ${}^{t+\Delta t}\bar{\tau}$:

$${}^{t+\Delta t}\bar{\tau} = {}^{t+\Delta t}\bar{S} + \frac{1}{3} \text{tr}(\bar{\tau}_*) \mathbf{I} - {}^t\bar{p} \mathbf{I} \quad (20)$$

- Calculate Cauchy stress tensor

$${}^{t+\Delta t}\tau = ({}^{t+\Delta t}J)^{-1} \mathbf{R}_*^E {}^{t+\Delta t}\bar{\tau} (\mathbf{R}_*^E)^T \quad (21)$$

- Update plastic deformation gradient:

$${}^{t+\Delta t}_0 \mathbf{X}^P = \exp(\lambda {}^{t+\Delta t}\bar{S}) {}^t \mathbf{X}^P \quad (22)$$

- Continue with step 4.

4. Evaluate residual vectors FU and FP:

$${}^tFU_i = \int_V {}^t\tau_{kl} \frac{\partial {}^te_{kl}}{\partial \hat{u}_i} dV; \quad {}^tFP_i = \int_V \frac{1}{\kappa} ({}^t\bar{p} - {}^t\tilde{p}) \frac{\partial {}^t\tilde{p}}{\partial \hat{p}_i} dV \quad (23)$$

4. FRICTION MODEL

Experimental observations have shown that for a given roll-workpiece interface and lubricating condition over most of the region of contact, the shear stress at the interface is an essentially constant fraction m of the shear yield stress of the metal undergoing deformation.

Since position of neutral point, i.e., point where frictional forces change their direction, is not known in advance, imposition of friction forces can not be done in a simple way. To override this problem, Hartley *et al.* have developed so-called friction layer technique [6, 7, 8]. This technique requires forming an extra layer of fictitious elements at the interface between rolls and workpiece. The upper nodes of these finite elements are fixed to the roll while lower nodes are fixed to the workpiece (Fig. 1.). Since friction layer nodes fixed to the roll can not move parallel to the workpiece surface, the effect is to apply a shear force to the interface nodes acting in the opposite direction of their movement. Material properties of the friction layer elements are multiplied by stiffness matrix multiplier β that is function of m :

$$\beta = \frac{m}{1-m}. \quad (24)$$

It can be shown that this modification of material properties means multiplying stiffness matrix of each friction layer finite element by β . There can exist two extreme cases: if $m=0$ ($\beta=0$) then sliding between rolls and workpiece is unavoidable and workpiece can not be pulled into the gap; if $m=1$ ($\beta \rightarrow \infty$) there can not be relative movement between rolls and workpiece - this is the case of sticking friction. In cold rolling sticking does not occur, while during the hot rolling such effects are often. This ensures that applied frictional forces correspond to experimental observations.

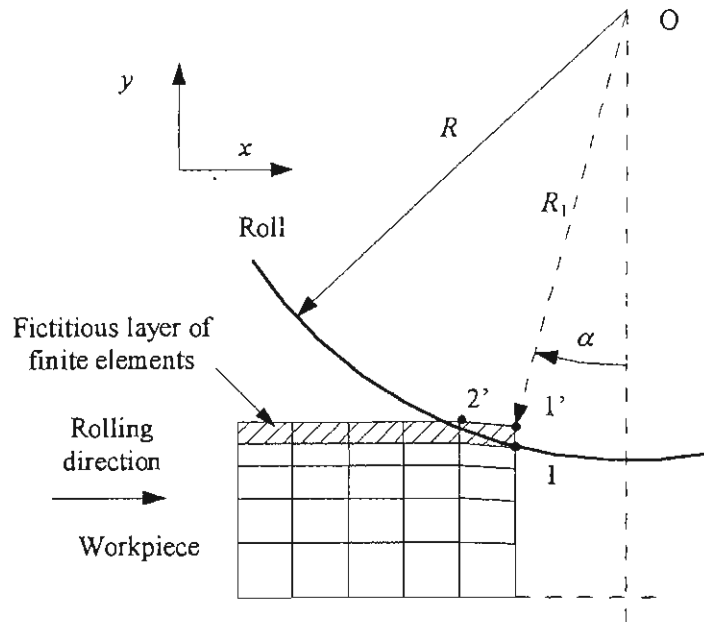


Fig..1. Friction layer technique

5. APPLICATION- COLD ROLLING OF ALUMINIUM

Computer software ROLLING based on previous formulation was developed and used for analysis of cold rolling process of aluminium workpiece. Accuracy of proposed model is verified by the comparison with experimental results of Al-Salehi *et. al.* [9]. Geometrical and material properties are selected according to experiment (Tab. 1).

Tab. 1. Geometrical and material properties

Property	Symbol	Value
Initial height	h_i	5.385 mm
Output height	h_o	6.274 mm
Roll radius	R	79.375 mm
Initial workpiece length	ℓ_0	16 mm
Elasticity modulus	E	70 GPa
Friction factor	m	0.25
Yield stress	σ_{y0}	50.3 MPa
Poisson's ratio	ν	0.30

On the basis of above data reduction is 14.17%. Plastic behaviour of aluminium is described by the following constitutive equation [10]:

$$\sigma = 50.3 \left(1 + \frac{1}{0.05} \varepsilon_p \right)^{0.26} \quad (25)$$

Problem was analyzed as a plane strain problem. Rolls were assumed to be rigid. Initial finite element mesh is presented on Fig. 2.

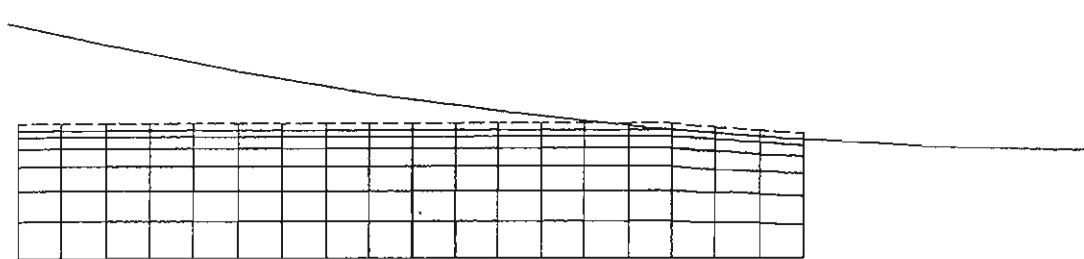


Fig. 2 Initial finite element mesh

Obtained field of relative velocities is presented on Fig. 3. Velocities are calculated with respect to the neutral point. It is clearly seen that material is moving slower than roll before neutral point, while after neutral point its velocity is greater than roll velocity.



Fig. 3. Relative velocity field

Obtained pressure distribution is shown in Fig. 4. On the basis of comparison with experimental observations and finite element model of other authors, it is seen that proposed model with sufficient accuracy can describe pressure at the interface between roll and workpiece.

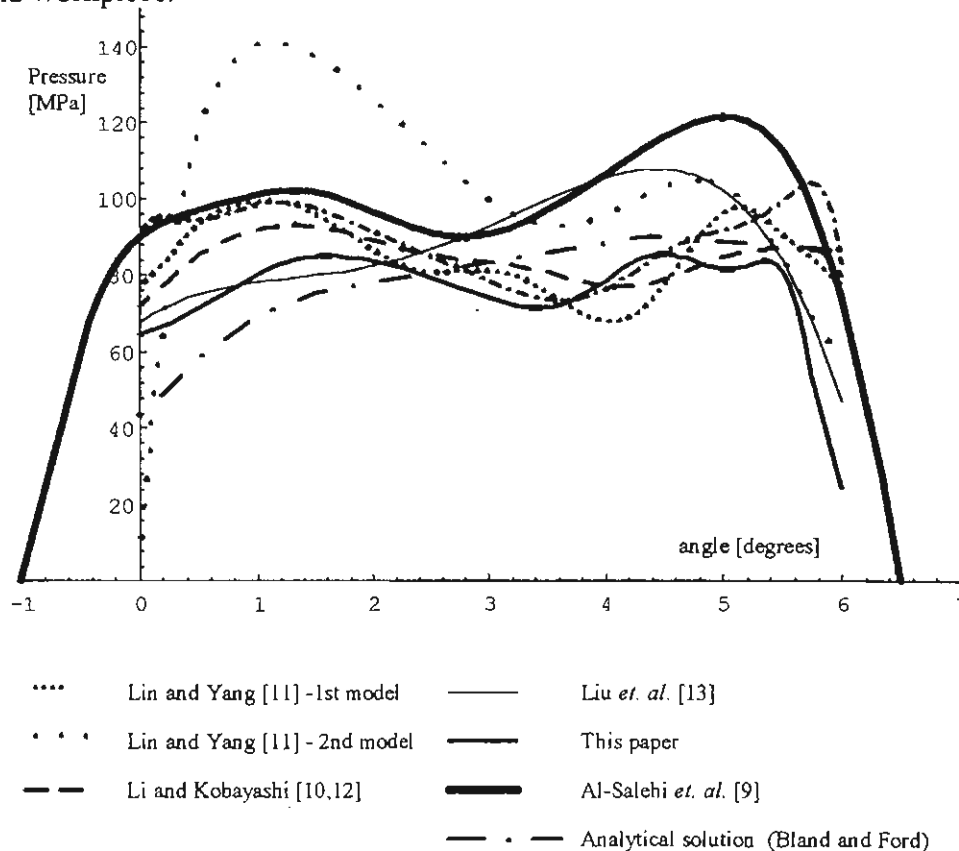


Fig. 4. Pressure distribution

6. CONCLUDING REMARKS

The most important aspects of the finite element formulations significant for description of cold rolling process is presented. Comparison with experimental investigations and with other numerical investigations confirms accuracy of proposed model. General character of these formulations enables future improvement in cold rolling process as well as application to other metal forming operations.

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The calculus of displacements and stresses

for the 1870 kW suburban train

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Abstract

This paper presents the results - displacements and stresses - made by a statical analysis with finite elements to the 1870 kW suburban train. The structure was divided in 19650 nodes and 19113 shell elements. The problem has 117900 equations and was performed on a PC computer with Cosmos/M software.

1. Introduction

The suburban train is a train with electric traction and it is use for people transport. The strength structure of this train are composed by some components:

- the chassis;
- the lateral wall;
- the roof.

Regarding the strength calculus with finite elements, they are many posibilites to define the structure:

- with beam elements;
- with beam and shell elements;
- with shell elements.

In this paper it is presented a model with shell elements.

2. The model

The structure was divided in 19650 nodes and 19113 shell elements and the problem has 117900 equations. It must be noted that the model is only for a quarter of this structure, because it is symetric regarding geometry, loading and boundary conditions.

The model was made only with shell elements as shown in fig. 1.

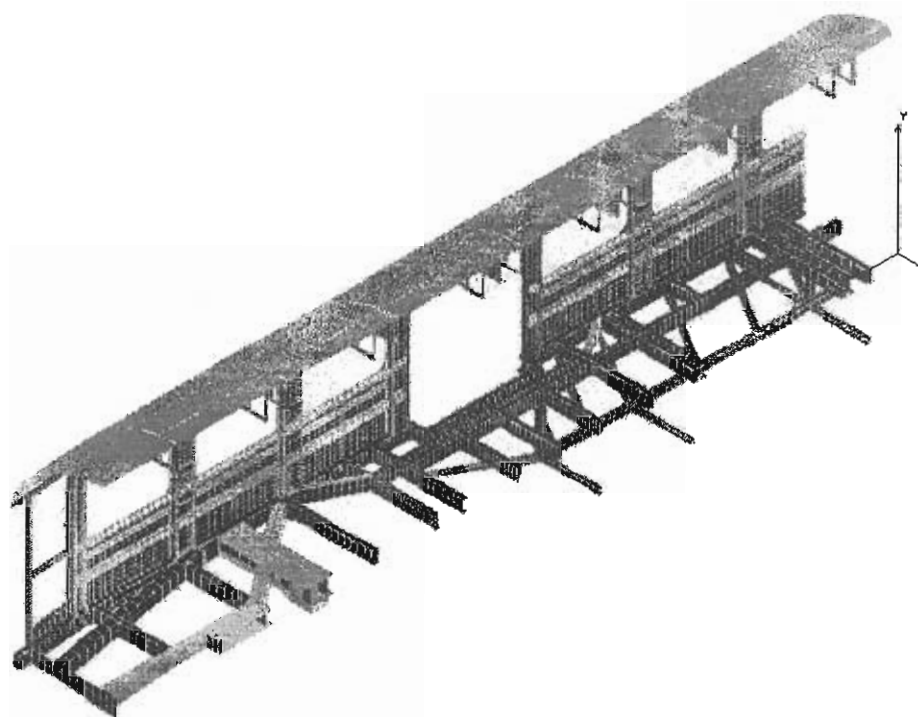


Fig. 1. The finite element model with shell elements for 1870 kW suburban train

The material for this structure has the proprieties show in the table nr. 1

Table nr. 1

Material contant	Ex [Pa]	NUxy	Dens [kg/m ³]
Value	2.1E+11	0.28	7.7E+03

- Ex the elasticity modul;
- NUxy Poisson ration;
- Dens the density of the material;

In the table nr. 2 can be show the geometric proprieties of this structure: the thickness of the shell elements; each number of the real constant corespond with the number show in figure nr. 1, where it is associated with a color.

Table nr. 2

Real constant	Element type	Thickness [m]
1	SHELL4	0.006
2	SHELL4	0.008
3	SHELL4	0.010
4	SHELL4	0.005
5	SHELL3	0.010
6	SHELL4	0.025
7	SHELL4	0.012
8	SHELL4	0.0025
9	SHELL4	0.004
10	SHELL3	0.006
11	SHELL3	0.0025
12	SHELL4	0.005

13	SHELL4	0.002
14	SHELL4	0.003
15	SHELL4	0.0055
16	SHELL3	0.003
17	SHELL4	0.0015
18	SHELL3	0.0015
19	SHELL3	0.002
20	SHELL3	0.004

It must be know that there two kinds of boundary conditions:

- for the link with the bogie;
- to simulate only a quarter of this very complex structure; the nodes on the symetric planes must be fixed with the usual conditions.

Int the table nr. 3 can be see this conditions.

Tabelul 2.4

	U_x	U_y	U_z	R_x	R_y	R_z
<i>Nodes in plane XOY</i>	0	0	1	1	1	0
<i>Nodes in plane YOZ</i>	1	0	0	0	1	1
<i>Nodes in link region</i>	1	1	1	1	1	1

U_x - OX translation

R_x - OX rotation

1- degree of freedom fixed;

U_y - OY translation

R_y - OY rotation

0- degree of freedom free;;

U_z - OZ translation

R_z - OZ rotation.

The link of the body with the bogie can be see in the figure nr. 2; in this region all the degree of freedom are fixed.

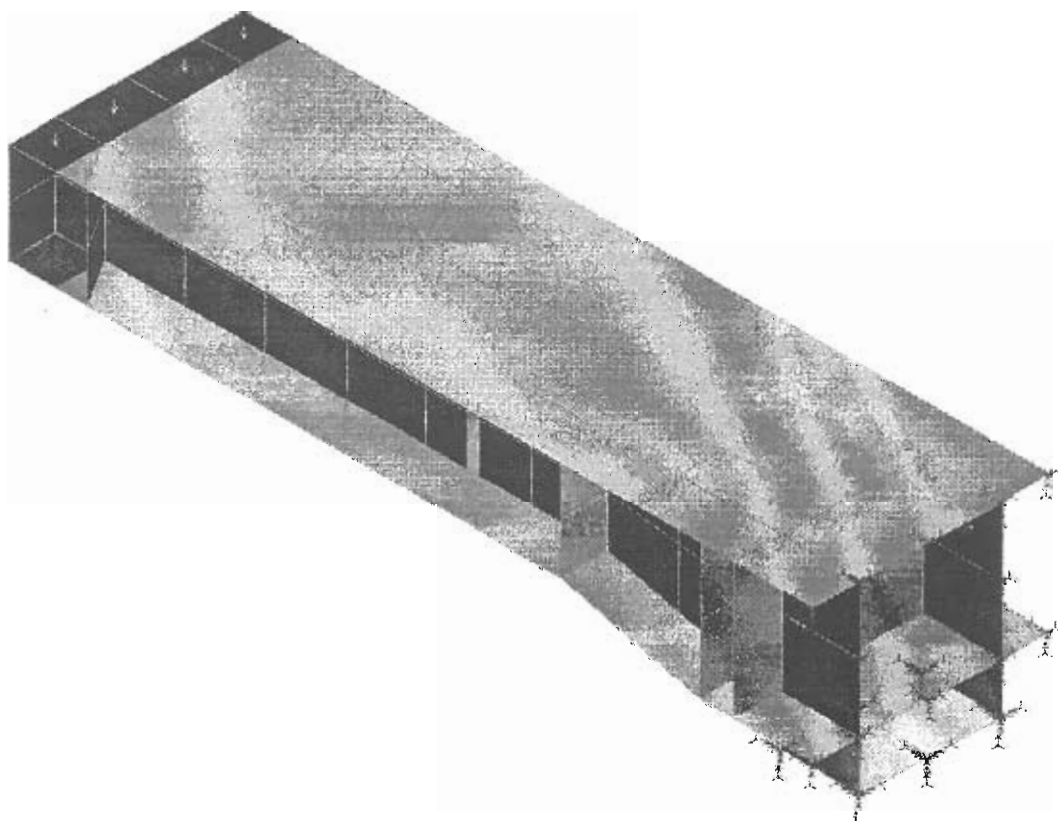
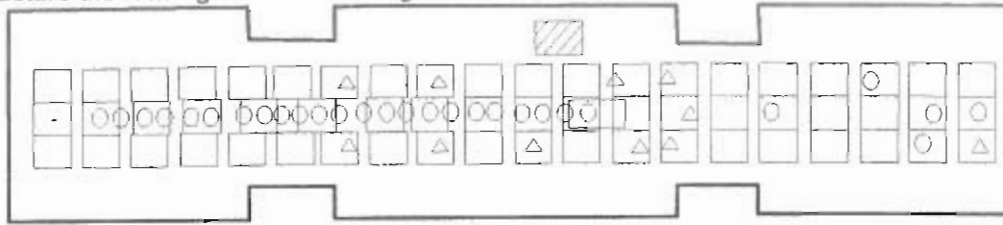


Fig. 2 Boundary conditions and loading for 1870 kW suburban train

The loading was made in concordance with International Reports UIC 625-7; for this kind of structure the loading can be see in figure 3.



LEGEND

- weight 780 Kg on chassis
- weight 50 Kg on chassis
- weight 780 Kg chassis
- weight 780 Kg on roof

Fig. 3 Loading on the 1870 kW suburban train

After a very simple calculus, it was performed the next pressure:

- on the chassis: $p_c = 48961 \text{ MPa}$;
- on the roof $p_r = 1904 \text{ MPa}$.

The model was analysed on a Pentium PC computer with 64 Mb RAM with Cosmos/M software and Fast Finite Element option. This probleme needed 35 Mb swap space on hard-disk and was performed in 90 minutes.

3. Results

At this time, two kinds of results was performed and will be show:

- displacements;
- stresses.

All this results will be compared with maximum values given by the international standards
Displacements

The international paper ORE - UIC Raport B 7/RP 7 define the maximum displacement for this kind of structure to 18.92 mm. In the figure nr. 4 can be see a general picture with displacements

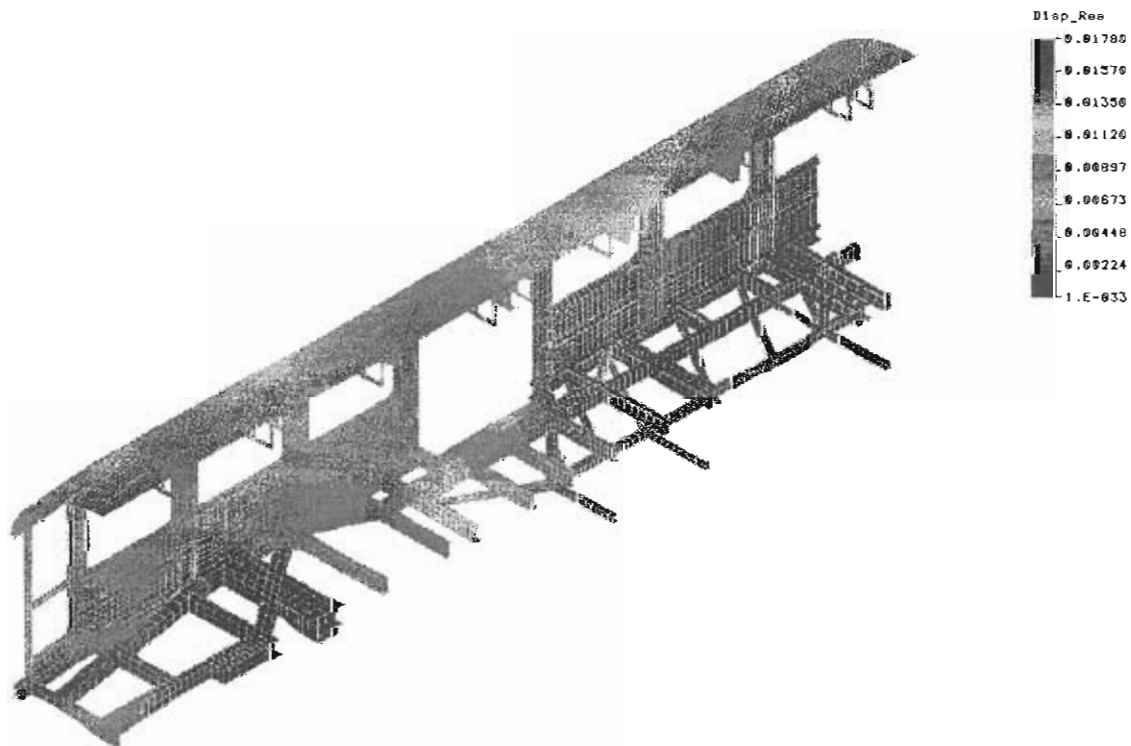


Fig. 4 Displacements on the 1870 kW suburban train

It can be observed that the maximum value for this structure calculated with finite element method is 17.90 mm, value that is less than the maximum. So, the structure does not violate the displacement criteria.

Stresses

The international paper ORE - UIC Report B 12/RP 17 defines the maximum Von Mises stresses for this kind of structure to 200 MPa. In the figure nr. 5 can be seen a general picture with Von Mises stresses.

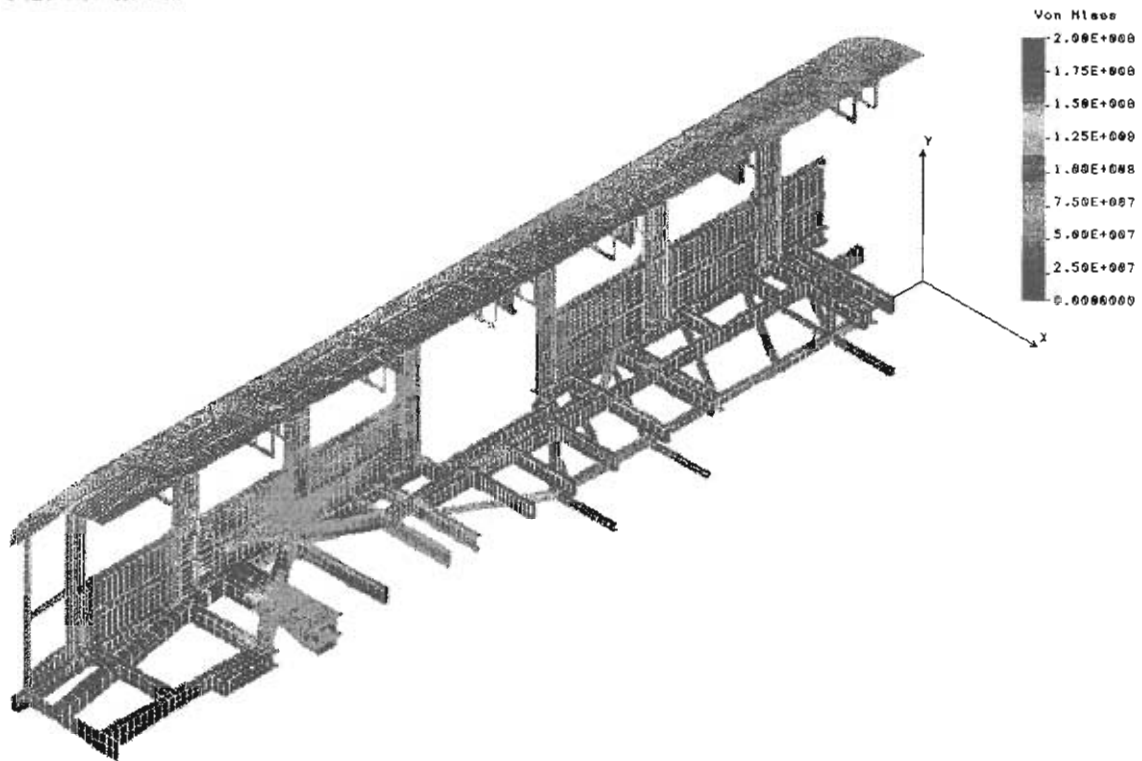


Fig. 5 Von Mises stresses on the 1870 kW suburban train

It can be seen that, generally, the values are smaller than the maximum value. They are also two regions where the value must be studied.

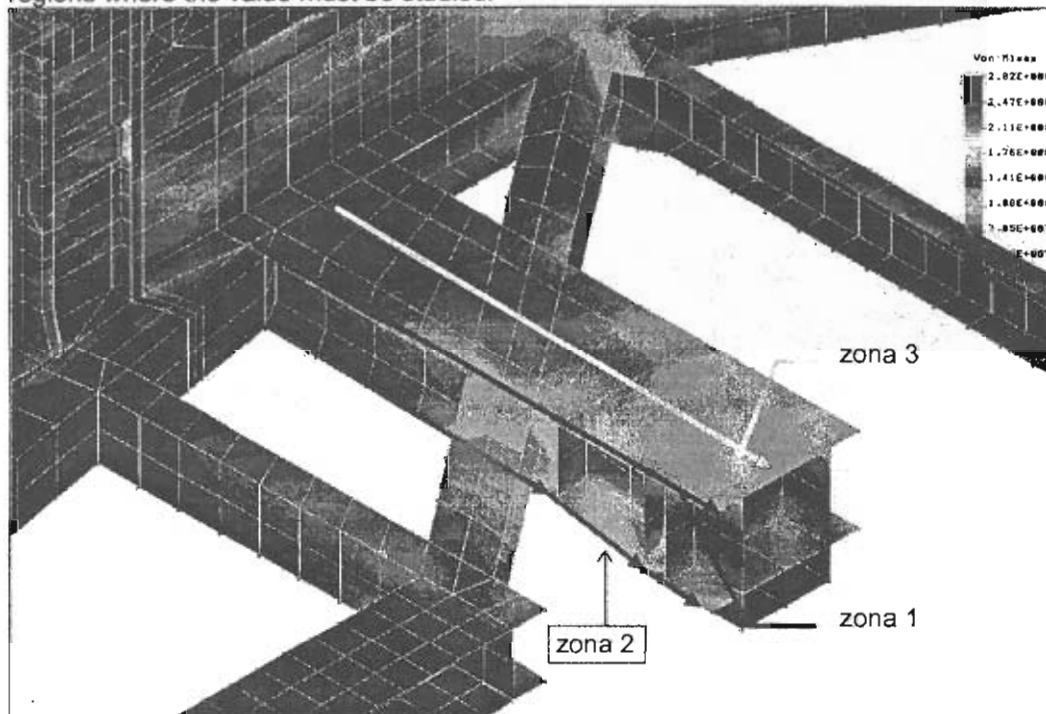


Fig. 6 Von Mises stresses on the region with the link with the bogie

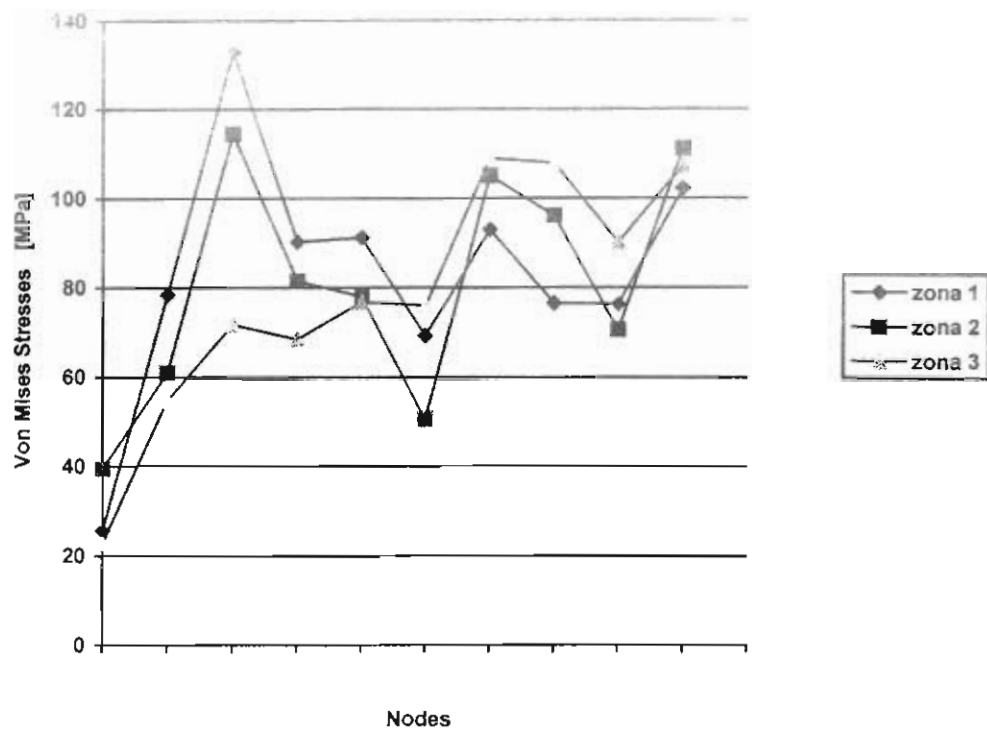


Fig. 7 The variation of Von Mises stresses on the region with the link with the boguie

It can be see in figures nr. 6 and 7 the value of Von Mises stresses on the region with the link with the body. The value of the stress are near the maximum value, but not greater.

In the figures 8 and 9 can be see the picture with Von Mises stresses in the door region where the value are greater then the maximum.

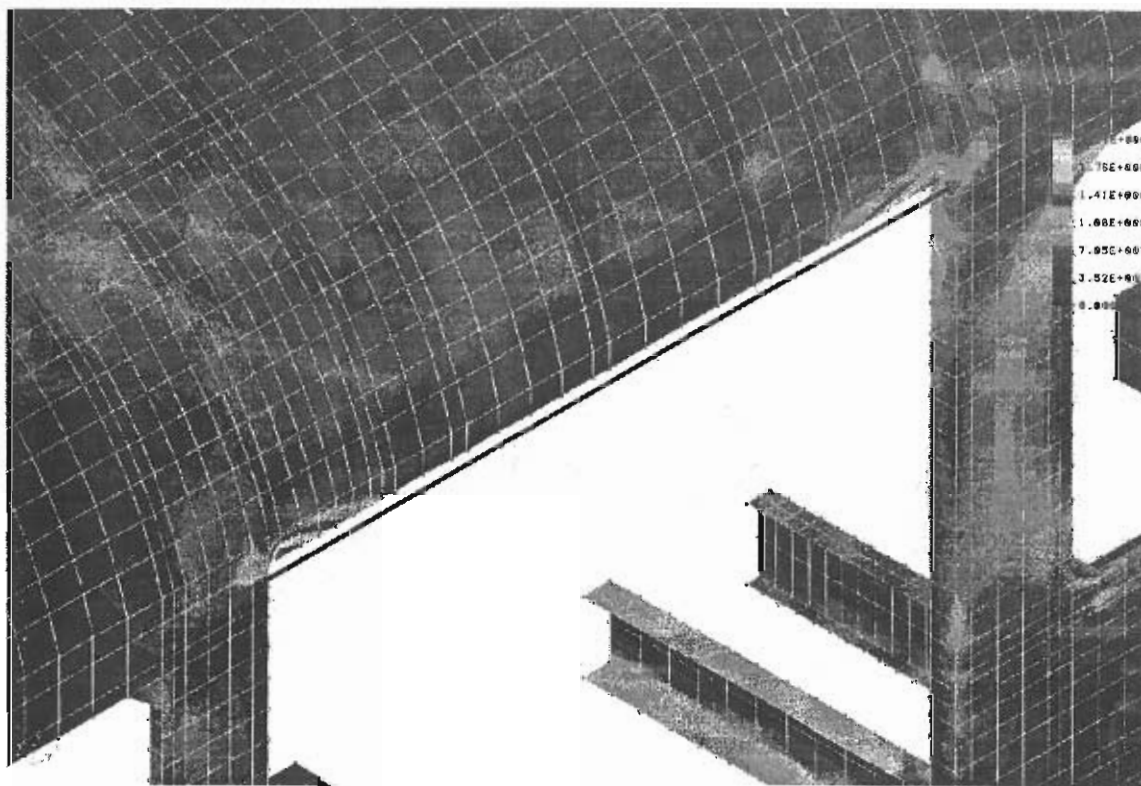


Fig. 8 Von Mises stresses on the door region

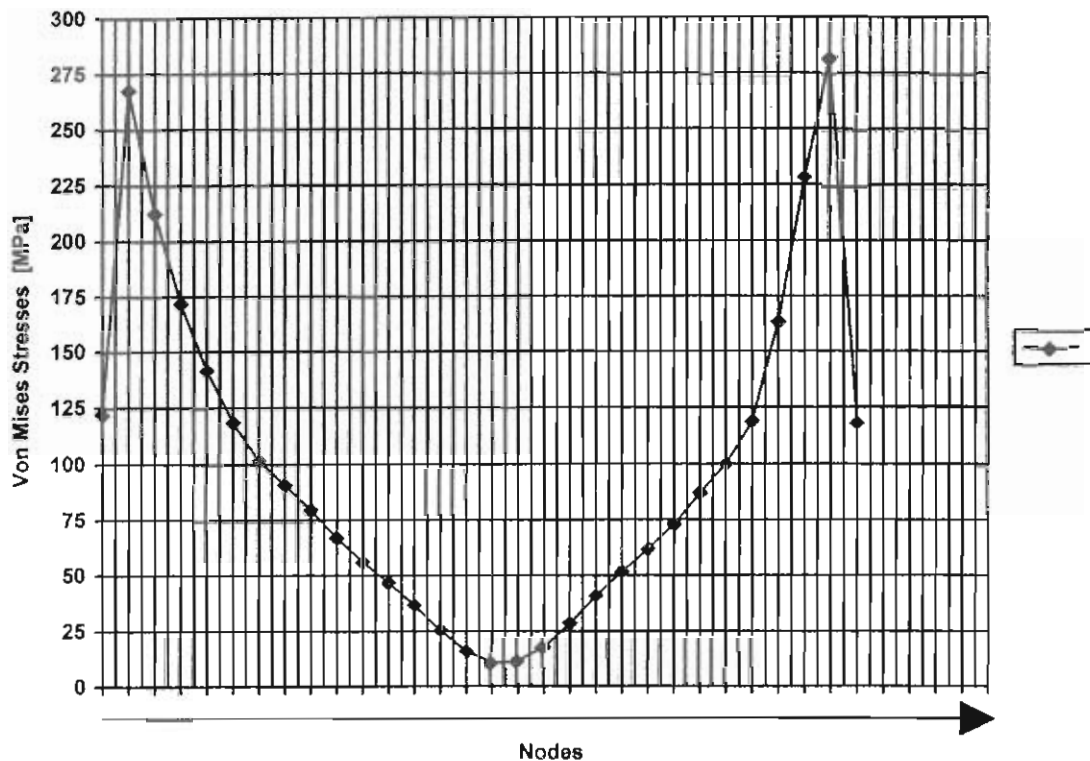


Fig. 9 The variation of Von Mises stresses on the door region

It can be seen that the maximum value in this region is greater with 40 % than the maximum value. This value in this position is normal, because the global section here is less than in the rest.

4. Conclusions

The body of the suburban train must be calculated to find the displacements and stresses in the condition defined by the next international papers:

- UIC 625-7 for loading;
- ORE - UIC Raport B 7/RP 7 for maximum displacement;
- ORE - UIC Raport B 12/RP 17 for maximum Von Mises stresses.

In this condition and with a geometry defined as in figure 1, was performed a linear static analysis with the next results:

- maximum displacement: 17.90 mm less than the maximum admissible 18.92 mm;
- maximum Von Mises stresses: 280 MPa greater than the maximum admissible 200 MPa.

In this condition, for the future, it is necessary to do:

- a comparison between experimental and numerical results;
- some changes in the structure to decrease the Von Mises stresses less than maximum admissible;
- an optimisation of this structure to make a smaller weight of the structure.

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THE STRESSES AND STRAINS ANALYSIS IN THE BASIC DESIGN OF THE PROPELLER BLADE

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Abstract

In the paper, some problems referring to automatically done of the mesh, in order to determine the stress and strain values in the propeller blade are presented. Also hydrodynamic characteristics determination of the propeller is presented.

An aided computer propeller design methodology used in the preliminary stage of ship design is presented. But the main decisions are taken by the designer, who may to estimate very quickly the ship propulsion.

Static and dynamic analysis of the stress state in the blade is made with a specialized computer program, based on FEM, adapted for this kind of structure.

1. Introduction

The ship propulsion is made with one or more propeller rotated by one or two engines.

In the preliminary ship design, the estimation of the ship propulsion characteristics is very important because this implies the determination of the engine power and the total ship weight and finally the ship efficiency.

In the propeller design a compromise between the distribution of the blade thickness and the blade cavitation and the hydrodynamic efficiency must be made but in the same time the fulfillment of the blade strength does not to avoid.

2. The preliminary propeller design

In this paper a methodology for preliminary propeller design is presented. Based on this methodology a computer soft for quickly propeller characteristics estimation was made.

For the preliminary propeller design a minimum number of information about the ship it is necessary. The ship's data help to estimate the coefficient of influence of the interaction ship hull - propeller, the wake (w) and the suction (t), as well as to estimate the ship's frictional resistance (if it is necessary). The precision of the preliminary estimation increase if much more information about the ship there are. For the information in absence, the classical formulas to determinate these characteristics are used.

This methodology is an aided computer propeller design used in the preliminary stage of ship design. But the main decisions are taken by the designer, who may to estimate very quickly the ship propulsion.

The main propeller characteristics which are determinate are in the following:

- the main propeller geometrical characteristics: diameter, rotation, pitch, disk ratio, propeller efficiency in still water;
- the blade geometry, that is: expanded outline, the outline of the blade cross sections on the main relative radiuses, the projected outline in the both views, the intersection between the blade and the propeller hub. Finally a mesh which describes the blade space surface is obtained;
- the inertial characteristics of the blade and propeller.

The preliminary propeller design consists of estimation of the thrust coefficient K_T and torsion moment coefficient, K_Q , based on polynomial representation for the propeller series B Wageningen, which are very used for the sea-going ships propulsion.

If the main data referring to the propeller are taken, the exact design is made.

3. FEM analysis

3.1. Blade mesh

The 3D blade mesh, automatically generated, is presented in fig.1.

The mesh is concerned of "parallel" and "meridian" lines. "Parallel" lines are placed at relative radios. Meridian lines are placed at the same relative abscise from maximum thickness line. These contours depend of the propeller series. In this methodology any series can be used.

For a reasonably mesh it is suggesting to use a 8 nodes element. Also, with this element the inertial characteristics of blade (and propeller to) are computed.

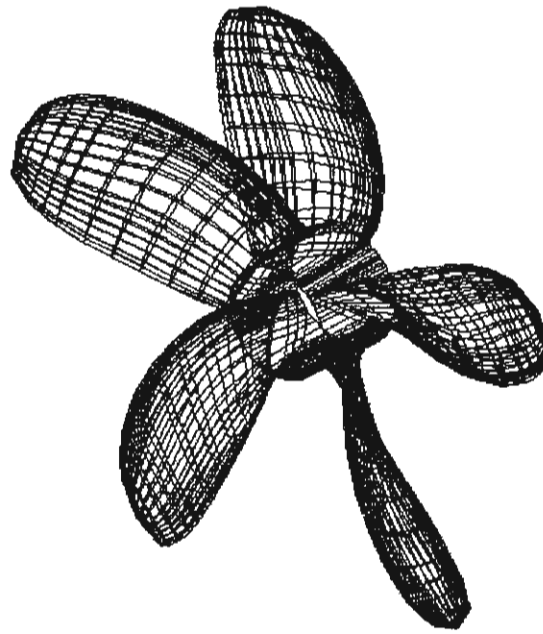


Fig.1

3.2. Element type

It is used a 8 nodes FE, named, in [1], Zienkiewicz - Irons - Brick element.

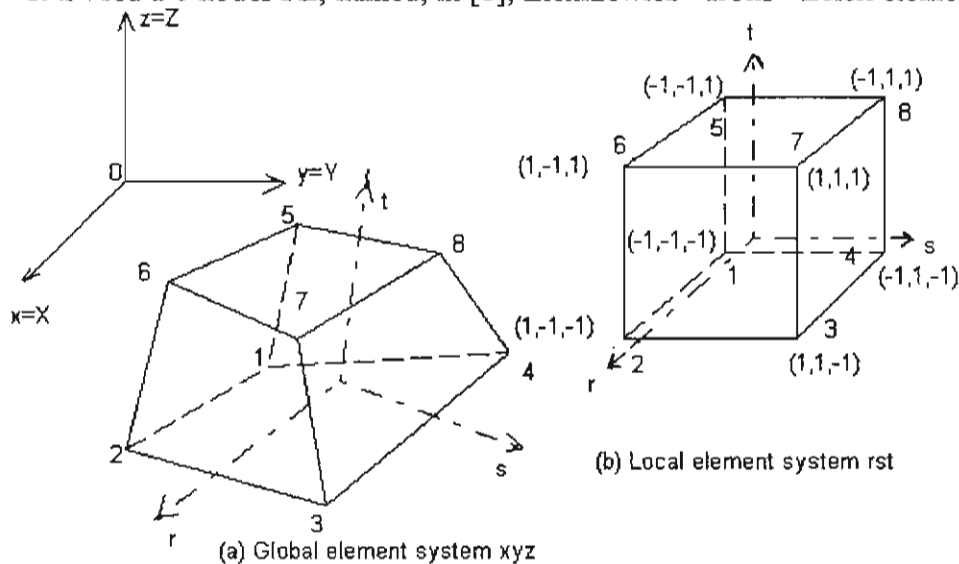


Fig.2

This element has 3 degrees of freedom per node (the displacements u , v and w in Cartesian system of axis x , y and z). This is an isoparametric element, so the trial function for his geometry and his displacement fields are of the same order.

The hexahedron element can be transformed from global system xyz into local system rst at one unit length side cube.

In fig. 3 it is illustrated the elements numbering mode (example for the first element: numbering is 2, 4, 3, 1, / 42, 44, 43, 41) and the local system attaching manner (r axis is generated by 2 - 4 direction, s axis by, 4 - 3 direction, t axis by 2 - 42 direction).

$$\bar{\epsilon} = \underset{6 \times 24}{B} \underset{24 \times 1}{\bar{Q}}^{(e)} \quad (6)$$

Element strain matrix B is:

$$\underset{6 \times 24}{B} = [B_1 \quad B_2 \quad \dots \quad B_8] \quad (7)$$

where B_i is for each node.

For integrate computing, the natural system is used. Relation between elementary volumes is:

$dV = dx \, dy \, dz = \det [J] \, dr \, ds \, dt$ where $[J]$ is Jacobian matrix so that:

$$[k^{(e)}] = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 B^T C B \det[J] \, dr \, ds \, dt \quad (8)$$

Since the matrix $[B]$ is an implicit function of rst , a numerical method has to be used to evaluate the multiple integral. The two-point Gaussian quadrature is the most efficient method.

$$[k^{(e)}] = \sum_{r=R_1}^{R_2} \sum_{s=S_1}^{S_2} \sum_{t=T_1}^{T_2} \left[(B^T C B \det[J]) (R_i, S_j, T_k) \right] \quad (9)$$

where $\left[(B^T C B \det[J]) (R_i, S_j, T_k) \right]$

means $\left[(B^T C B \det[J]) \right]$ computed in the point $r=R_i, s=S_j, t=T_k$ ($i, j, k = 1, 2$)

and $R_1 = S_1 = T_1 = -1/\sqrt{3}$, $R_2 = S_2 = T_2 = 1/\sqrt{3}$ Gauss integrating station.

4. The stresses and strains analysis of the propeller blade

The stress and strain analysis of the propeller is made using the substructure method.

Because a propeller has a number of Nb (from 4 to 8 or more) identical blades, this is the reason for which the substructure method is the best used method.

So, the propeller is divided into a number of substructures equal to $Nb+1$, representing the all blades and the propeller hub.

The blade and the propeller hub are divided into the solid elements using the mesh made in the stage of propeller design.

The characteristics matrices of the all blades are made once. The complete set of equilibrium equations for the substructure-blade regarded as a free body may be written in matrix form as

$$K\Delta = F \quad (10)$$

where K is the stiffness matrix, Δ represents a column matrix of displacements corresponding to external forces F .

The column matrix of boundary displacements common to the substructure-blade and the substructure-propeller hub is denoted by Δ_b , and the matrix of interior displacements (each of which occurs at interior point of only one substructure) is Δ_i . If the corresponding effective external forces are denoted by matrices F_b and F_i , can be written in partitioned form as

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{Bmatrix} \Delta_b \\ \Delta_i \end{Bmatrix} = \begin{Bmatrix} F_b \\ F_i \end{Bmatrix} \quad (11)$$

Applying the techniques used in the substructure method, finally it is obtained the forces - displacements equation related to the boundary nodes

$$\mathbf{K}_{bb} \Delta_b = \mathbf{F}_b \quad (12)$$

For every blade, the own characteristics matrices are made by transforming of these with an rotational matrix and then coupling of them with the propeller hub matrices, so forming the matrices for the whole propeller.

The transformation matrix is done using the number of blades, Nb .

Noting α the position angle of the blade, the transformation matrix for an node coordinate is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \quad (13)$$

For the Nb boundary nodes, the blade transformation matrix is obtained as follows

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & 0 & 0 & 0 \\ 0 & \mathbf{A} & 0 & 0 \\ & & \ddots & \\ 0 & 0 & & \mathbf{A} \end{bmatrix} \quad (14)$$

The displacements of the blade for its position are

$$\Delta = \mathbf{T} \Delta' \quad (15)$$

By substituting the global displacements in the equation (12) the substructure-blade equation in global system is obtained

$$\mathbf{K}'_{bb} \Delta'_b = \mathbf{F}'_b \quad (16)$$

where

$$\mathbf{K}'_{bb} = \mathbf{T}^T \mathbf{K}_{bb} \mathbf{T} \quad (17)$$

$$\mathbf{F}'_b = \mathbf{T}^T \mathbf{F}_b \mathbf{T} \quad (18)$$

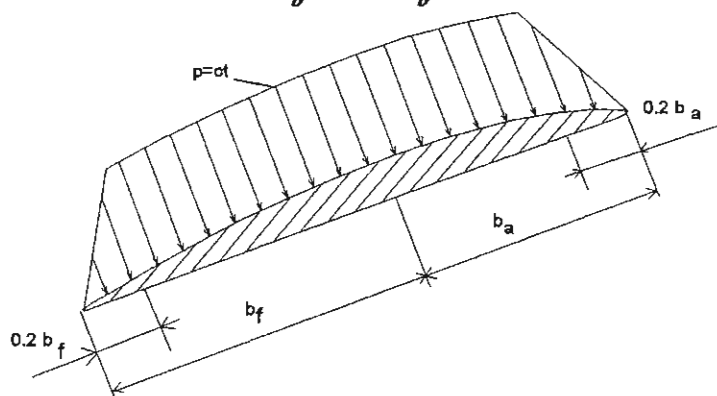


Fig.4

The nodal forces are determined by considering that the result of the thrusting forces, T , and the results of the torque forces, F , acting on the elementary blade surface, are distributed as it is presented in Fig.4.

In Fig. 4 the distribution of these forces is made in accordance with [2]. The thrusting forces are directed to the X axis (propeller hub axis) and the torque forces are directed to the Y axis.

The characteristics matrices of the substructure-blades are obtained in the same time with the assembling operation.

Finally, the propeller equation is obtained, and the nodal boundary displacements of the substructure-blade are determined.

5. Results

5.1. Nodal displacements

From FEM computing the nodes displacements are obtained.

In fig. 4 these displacements are present by 200 times amplified.

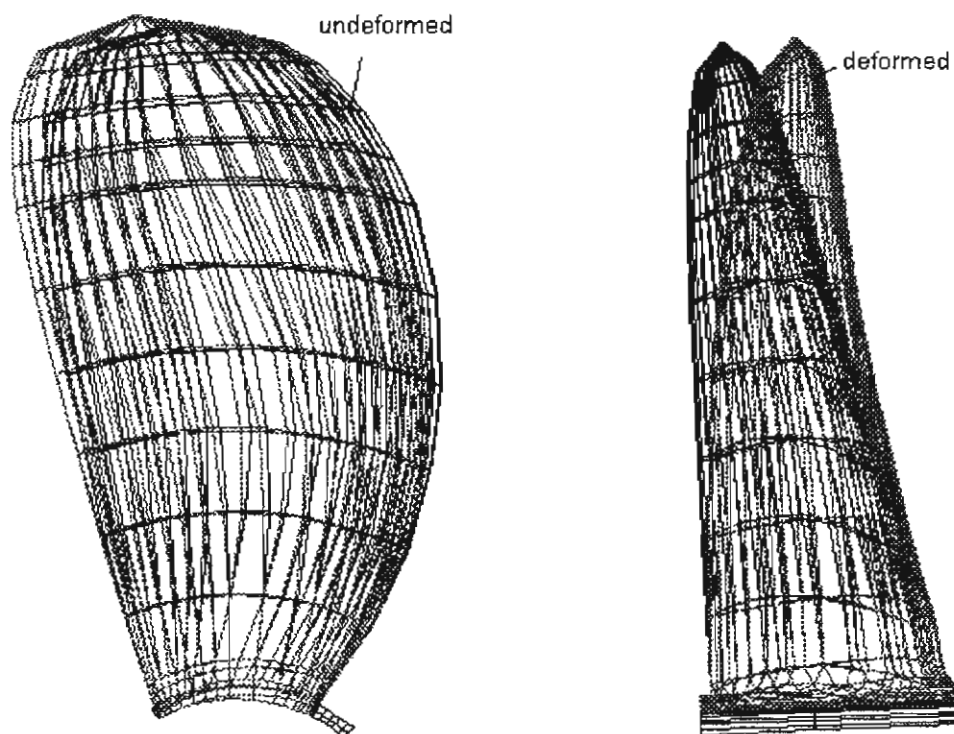


Fig. 5

5.2. Stresses

The stresses are computed in the center of the element faces. Maximum stress is on global Z axis (it is expected, because the blade is bending in YOZ plane).

Maximum stress values are in the areas of the elements 26 and 27, on suction face, in hub-blade link zone. There are compression stress (computing is for $r=0$, $s=-1$, $t=0$ mean center of element suction face).

For the most loading elements the following values are resulting :

	σ_{xx}	σ_{yy}	σ_{zz}	σ_{xy}	σ_{yz}	σ_{zx}	[kPa]
El. 26	-16.1	-17.5	-38.9	-0.7	0.1	-0.4	
El. 27	-17.5	-18.7	-42.0	-0.7	-0.1	-0.6	
El. 28	-17.7	-17.9	-42.0	-1.0	0.7	-0.1	

Computing of the equivalent stress is made with the von Mises resistance hypothesis.

The maximum stress, 36.205 kPa, is obtained for the element 27.

It is obtaining a safety coefficient of about 5.6 for the bending compression stress and 5.8 for the equivalent stress (for admissible stress = 90% from flow stress).

6. Conclusions

This study has some limitations:

- in safety coefficient consideration must be taken into consideration the dynamics stress and strain witch are not included in this work;
- blade geometry is considering from B-Wageningen series but can be used other geometry described on coefficients;
- boundary condition can be improved by including the hub in study.

Computing using FEM confirms the value of safety coefficient in rules' limits for propeller design and, in the same time, opens the way for other studies like as:

- optimization of the ratio between effective thrust and weight;
- the combined effect of hub stress (from technology) and blade stress (from operation).

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STRESS ANALYSIS OF A COMPLEX SHELL STRUCTURE

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ABSTRACT

The stress analysis of a complex shell which is a part of a butterfly shutter that is widely used in hydrotechnics is presented in this paper. The analysis includes the working load conditions corresponding to a pressure of 135 (m) of a water-vapor. The loads are obtained by measuring.

The theoretical analysis is realized by using the finite element method and the ALGOR SUPERSAP computer program.

The shape of the shell structure that belongs to the butterfly shutter is very complex. In this paper it is accepted as a combination of spherical and cylindrical shells strengthened by ribs.

The analysis carried out according to:

- the hypothesis of the maximum shear stress (the Coulomb's hypothesis),
- the Huber-Von Mises-Hencky hypothesis

The results of the analysis are presented in graphical representation using isostress lines.

1. INTRODUCTION

The shutters are elements that compose the base group of equipment in hydrotechnics. They are widely used in assembly operations and in overhamling. Their role is to shut, partially or completely, the apertures in water deliveries.

The butterfly shutters are used as overhamling shutters which shut the supply pipes with diameters in range from 0,5 to 8,5 (m), and pressure from 600 to 170 (m) of a water-vapor.

The shutter analyzed in this paper is in fact a complex shell structure very close in its form to the form of lens.

In general, the structure is composed by:

- cylindrical part, where the axle is settled down,
- spherical part which takes over the loads mostly,
- ribs, which make the structure more rigid.

The real working conditions, where the high pressure, beat of pressure and cavitations are presented, make the structure to be loaded very complex.

2. MODELLING OF THE BUTTERFLY SHUTTER STRUCTURE

The real shutter, that is built in installations of the hydroenergetic plant Globocica in Republic of Macedonia, has these dimensions: diameter $D=2200$ (mm), shell thickness $\delta=40$ (mm), altitude $H=445$ (mm), shaft diameter $d=340$ (mm), Fig. 2.1.

The purpose of this work was to design the model of the shutter close to the real one. The modeling of the geometry of the shutter carried out in these steps:

- a) determination of mathematical expressions for the sphere, cylinder and their reciprocal breach,
- b) partial modeling of cylindrical part of the shutter and connection of segments in a complete,

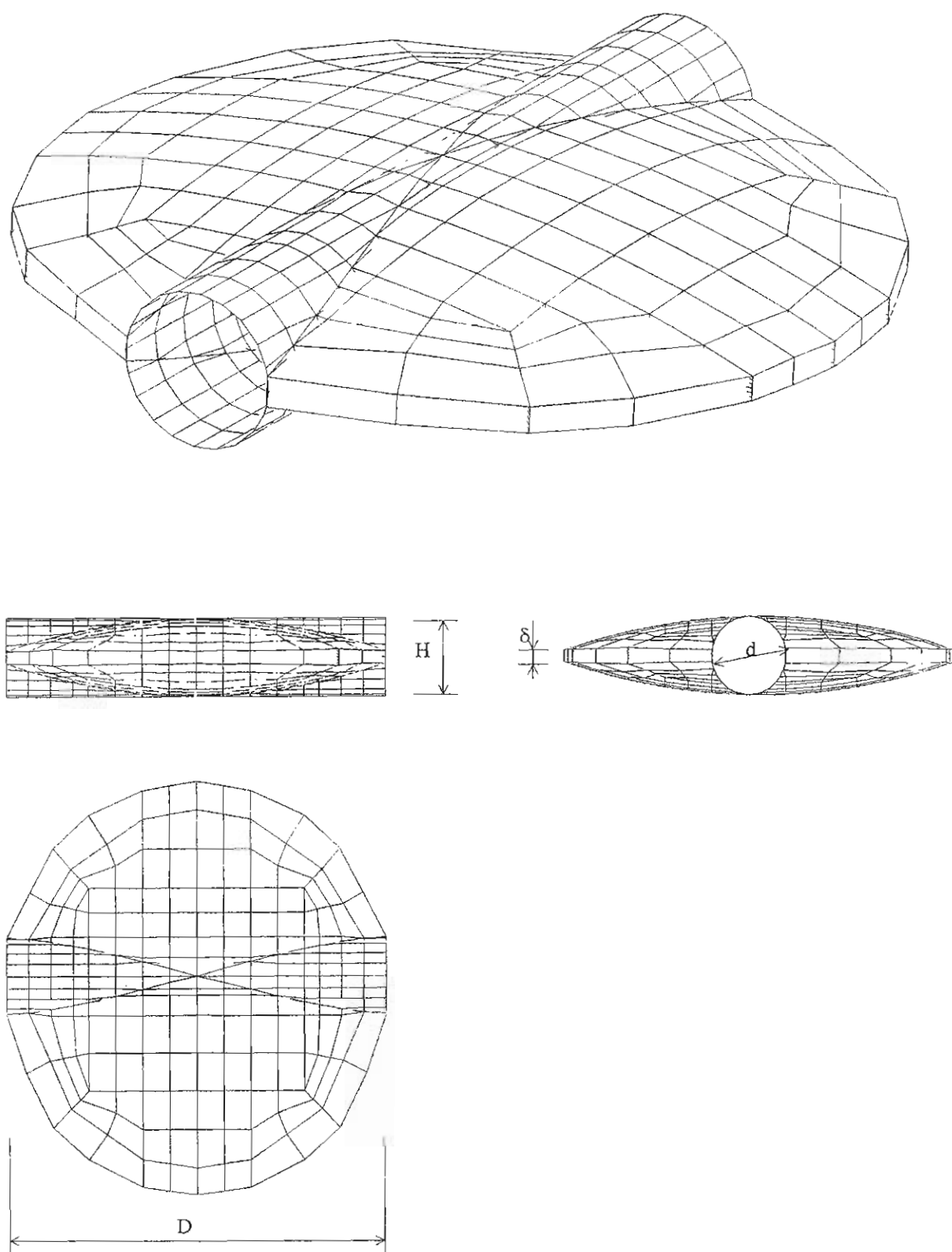


Fig. 2.1. The shutter structure and its elements

- c) partial modeling of the upper and lower sphere of the shutter and connection of segments in a complete,
- d) partial modeling of ribs and connection of segments in a complete,
- e) connection of the sphere, cylinder and ribs to each other in a complete structure,
- f) modeling of shaft and connection with the rest part of the shutter model

3. MODELLING IN FINITE ELEMENT METHOD

The shell quadrilateral finite element for the purposes of this work was chosen. Its displacements in each node are presented in Fig. 3.1

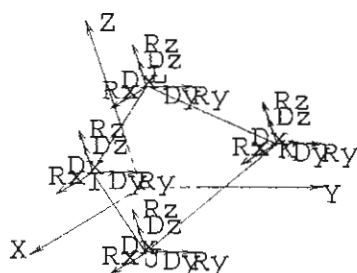


Fig. 3.1. The displacements of each node of the finite element

where is:

R_x, R_y, R_z - rotations about x, y, z axes respectively,

D_x, D_y, D_z - translations along x, y, z axes respectively.

The shell element stress components as a final result of the analysis are presented in Fig. 3.2.

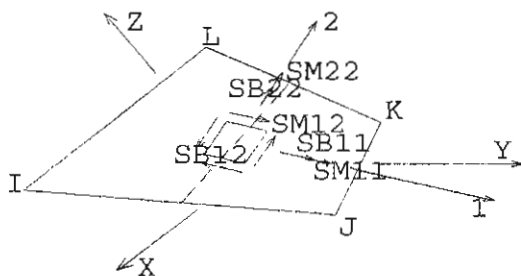


Fig. 3.2. The stress components

where is:

$SM11, SM22$ - membrane stress components in directions 1 and 2,

$SB11, SB22$ - bending stress components in directions 1 and 2,

$SM12, SB12$ -shear stress components.

4. ANALYSIS BY THE ALGOR SUPERSAP PROGRAM

The application of the ALGOR SUPERSAP program was in direct relation with the modeling of the shutter presented in Part 2 of this paper.

The first step of the analysis consists of a preparation of accessory programs for computing the sphere and cylinder nodal points, and points of their breach. It carried out in a format demanded by ALGOR SUPERSAP program.

The second and fourth phases of Part 2 of this paper were realized by the MSHGEN program.

The third phase of Part 2 of this paper was realized by the RADGEN program.

The connection of segments in phases b, c, d, e, f of the Part 2 of this paper was realized by the PUSH and GLUE programs.

The complete input data were realized by the AEDIT program.

The upper and the lower plane of the shutter are of interest in this work. Their network of finite elements is presented in Fig.4.1.

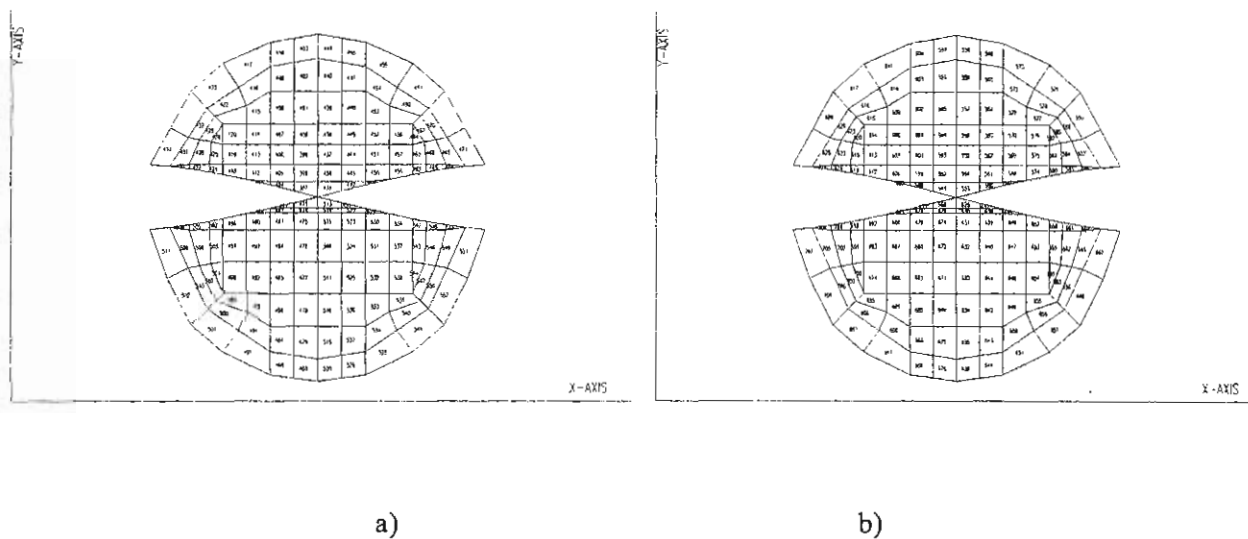


Fig.4.1. Network of finite elements of : a)The upper plane, b) The lower plane

5. RESULTS OF THE ANALYSIS

The results of the analysis are presented by using the graphic representation through isostress lines.

The stress state of the upper plane of the shutter, according to the hypothesis of maximum shear stress is presented in Fig. 5.1a, and according to the Huber-Von Mises-Hencky hypothesis in Fig. 5.1b.

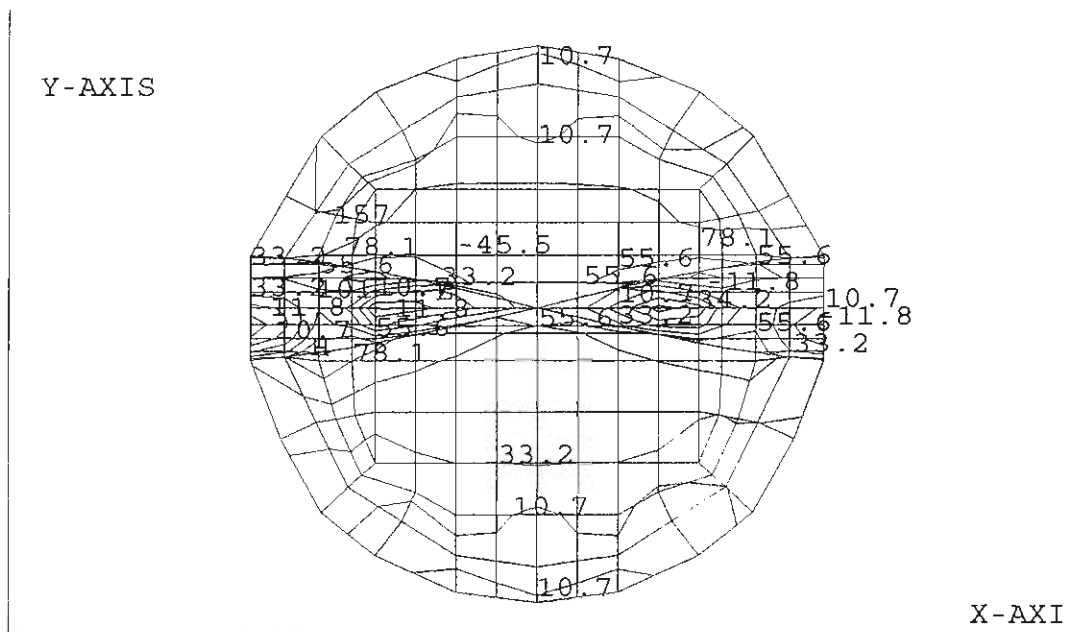


Fig. 5.1a. Isostress lines for the upper plane by the hypothesis of maximum shear stress

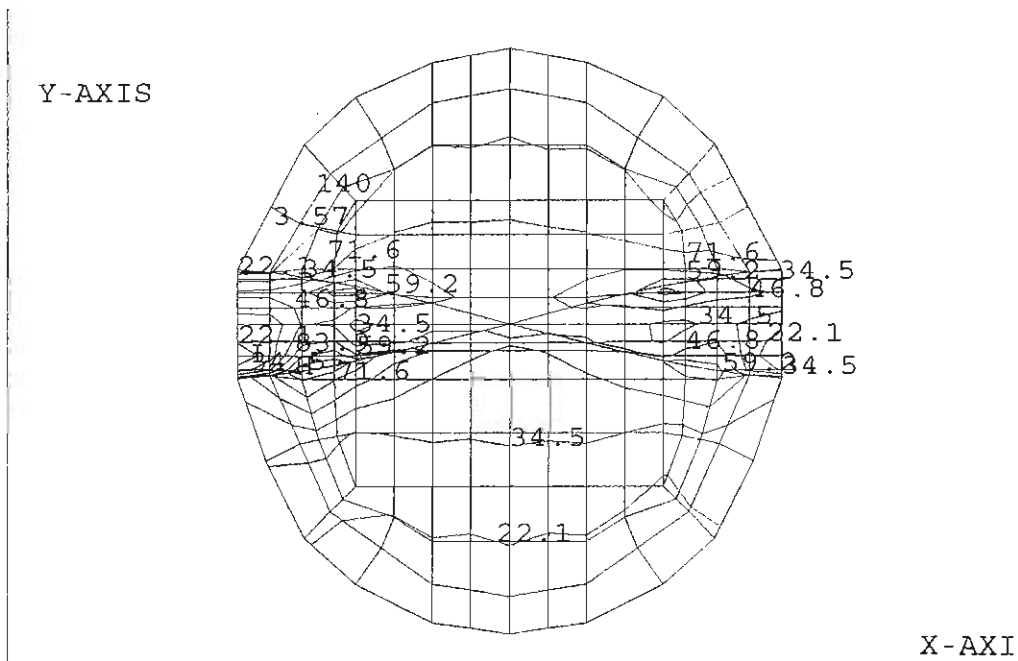


Fig. 5.1b. Isostress lines for the upper plane by Huber-Von Mises-Hencky hypothesis

[illegible]

Y-AXIS

8.39

19.6

64.5

42.1

30.8

19.6

8.39

30.8

64.5

42.1

30.8

19.6

X-AXIS

334

6. ANALYSIS OF RESULTS AND CONCLUSIONS

The results obtained by the analysis with the ALGOR SUPERSAP program show that:

- the maximum values of the stresses, according to both, the hypothesis of maximum shear stress and the Huber-Von Mises-Hencky hypothesis are located at the same location which coincides with the breach round about the wedge,
- the maximum stress value of the upper plane of the shutter according to the hypothesis of maximum shear stress is $\sigma_{\max} = 140(N/mm^2)$ and according to the Huber-Von Mises-Hencky hypothesis is $\sigma_{\max} = 157(N/mm^2)$. The difference between the stress values according to the two mentioned hypothesis is 10,8%,
- the maximum stress value of the lower plane of the shutter according to the hypothesis of maximum shear stress is $\sigma_{\max} = 126(N/mm^2)$ and according to the Huber-Von Mises-Hencky hypothesis is $\sigma_{\max} = 138(N/mm^2)$. The difference between the stress values according to the two mentioned hypothesis is 8,7%.

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A new approach on stress analysis of thin-walled beams

Marjan Stanek and Goran Turk †

Abstract

A straight thin-walled beam of constant cross-section has been studied considering small strains and isotropic homogeneous elastic material. A set of governing differential equations of equilibrium for a straight thin-walled beam have been derived. Unlike governing equations known from the literature, a sixth order differential equation for torsional rotations has been obtained. The latter differential equation has been solved for a cantilever beam and the solution has been compared to the solutions of the fourth order differential equation.

1. Introduction

We consider a straight beam built of elastic, homogeneous and isotropic material with elastic modulus E and shear modulus G . Cross-section $\mathcal{A}_x$ may be open or closed with one or more cells and is constant along the longitudinal axis x . The other two axes y and z lay in the cross-section plane. Furthermore, a coordinate system describing a thin wall of the cross-section is defined: ζ is the coordinate along centerline of the wall, while η is the coordinate perpendicular to the wall. Only small strains are taken into account. It is customary that normal strains ε_{yy} and ε_{zz} and shear strain ε_{yz} are neglected ($\varepsilon_{xx} \neq 0$, $\varepsilon_{xy} \neq 0$, $\varepsilon_{xz} \neq 0$, $\varepsilon_{yy} \approx 0$, $\varepsilon_{zz} \approx 0$, and $\varepsilon_{yz} \approx 0$).

2. Displacements

Based on assumption stated in previous section, displacements u_y and u_z of an arbitrary point $T(x, y, z)$ of the beam may be expressed in terms of displacements v_0 and w_0 of a chosen point $T_0(x, y_0, z_0)$ and rotation ω_x

$$\begin{aligned}u_y(x, z) &= v_0(x) - \omega_x(x) (z - z_0), \\u_z(x, y) &= w_0(x) + \omega_x(x) (y - y_0).\end{aligned}\quad (1)$$

On the other hand the differential of the displacement $u_x(x, y, z)$ is

$$du_x = \varepsilon_{xx} dx + \left[2\varepsilon_{xy} - \frac{dv_0}{dx} + \frac{d\omega_x}{dx} (z - z_0) \right] dy + \left[2\varepsilon_{xz} - \frac{dw_0}{dx} - \frac{d\omega_x}{dx} (y - y_0) \right] dz. \quad (2)$$

From the theorem of the total differential the following compatibility conditions are derived

$$\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} - \frac{d\omega_x}{dx} = 0, \quad (3)$$

in arbitrary point of $\mathcal{A}_x$ and

$$\oint_{\mathcal{C}_i} (\varepsilon_{xy} dy + \varepsilon_{xz} dz) = A_{ni} \frac{d\omega_x}{dx} + \frac{1}{2} \oint_{\mathcal{C}_i} \left(\frac{dv_0}{dx} dy + \frac{dw_0}{dx} dz \right), \quad i = 1, \dots, N. \quad (4)$$

for all cells of closed cross-section where $\mathcal{C}_i$ represent the inner line of the i_{th} cell. Shear strains are further decomposed into the part corresponding to the uniform twisting ε_{xy}^u (or ε_{xz}^u) and to the part of non-uniform twisting and shear forces ε_{xy}^{ns} (or ε_{xz}^{ns}) ($\varepsilon_{xy} = \varepsilon_{xy}^u + \varepsilon_{xy}^{ns}$

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or $\varepsilon_{xz} = \varepsilon_{xz}^u + \varepsilon_{xz}^{ns}$). It is customary that ε_{xz}^{ns} and the second term of the right hand side of Eqn. (4) are neglected for the determination of stress function φ . Therefore, the determination of stress function φ is done by a standard procedure from the well known second order partial differential equation $\partial^2 \varphi / \partial y^2 + \partial^2 \varphi / \partial z^2 + 2 = 0$. Eqn. (2) may now be expressed in terms of stress function $\varphi(y, z)$ and warping function $\phi_0(y, z)$

$$du_x = \varepsilon_{xx} dx - \left(\frac{dv_0}{dx} dy + \frac{dw_0}{dx} dz \right) + \frac{d\omega_x}{dx} d\phi_0 + 2(\varepsilon_{xy}^n dy + \varepsilon_{xz}^n dz), \quad (5)$$

where warping function is defined by the following equations

$$\frac{\partial \phi_0}{\partial y} = \frac{\partial \varphi}{\partial z} + (z - z_0), \quad \frac{\partial \phi_0}{\partial z} = -\frac{\partial \varphi}{\partial y} - (y - y_0) \rightarrow \phi_0 = \phi_0(y, z). \quad (6)$$

Eqn. (6) is now integrated from the point T_R where the displacements and rotations are known to the arbitrary point T . Again, the influence of non-uniform twisting and shear force to displacement u_x have been neglected.

$$u_x = \bar{u}_x(x) - \frac{dv_0}{dx}(y - y_R) + \frac{dw_0}{dx}(z - z_R) + \frac{d\omega_x}{dx}(\phi_0 - \phi_{0R}),$$

$$\bar{u}_x(x) = u_x(T_R) + \int_{T_R}^T \varepsilon_{xx}|_{y=y_R, z=z_R} dx. \quad (7)$$

Warping function ϕ_0 may be expressed by stress function φ

$$\phi_0 = \phi_{0R} + \int_{T_R}^T \left(\frac{\partial \varphi}{\partial z} dy - \frac{\partial \varphi}{\partial y} dz \right) + \int_{T_R}^T [(z - z_0) dy - (y - y_0) dz]. \quad (8)$$

If the coordinates y_0 and z_0 correspond to conditions

$$\int_{A_x} \phi_0 dA_x = 0, \quad \int_{A_x} y \phi_0 dA_x = 0, \quad \int_{A_x} z \phi_0 dA_x = 0, \quad (9)$$

they define a shear center and are denoted by y_s in z_s . The corresponding warping function is ϕ_s . Finally, we express the displacements in Eqns. (7) and (1) by y_s , z_s , v_s , w_s and ϕ_s to obtain

$$u_x = \bar{u}_x - \frac{dv_s}{dx}(y - y_R) - \frac{dw_s}{dx}(z - z_R) + \frac{d\omega_x}{dx}(\phi_s - \phi_{sR}),$$

$$u_y = v_s - \omega_x(z - z_s),$$

$$u_z = w_s + \omega_x(y - y_s). \quad (10)$$

The warping function ϕ_s and coordinates y_s and z_s may be written as

$$\phi_s = \phi_P - \frac{I_{\phi pz}}{I_z} y - \frac{I_{\phi py}}{I_y} z - \frac{S_{\phi p}}{A_x}, \quad y_s = y_P - \frac{I_{\phi py}}{I_y}, \quad z_s = z_P + \frac{I_{\phi pz}}{I_z}, \quad (11)$$

where the auxiliary warping function ϕ_P is

$$\phi_P = \int_{T_R}^T \left(\frac{\partial \varphi}{\partial z} dy - \frac{\partial \varphi}{\partial y} dz \right) + \int_{T_R}^T [(z - z_P) dy - (y - y_P) dz] \quad (12)$$

and

$$\begin{aligned} I_{\phi p y} &= \int_{\mathcal{A}_x} \phi_P z dA_x, \quad I_{\phi p z} = \int_{\mathcal{A}_x} \phi_P y dA_x, \\ S_{\phi p} &= \int_{\mathcal{A}_x} \phi_P dA_x, \quad \phi_{sR} = -\frac{S_{\phi p}}{A_x} - (z_P - z_s) y_R - (y_P - y_s) z_R. \end{aligned} \quad (13)$$

Furthermore it is assumed that the axes y and z are principle axes of the cross-section.

3. Stresses

3.1 Normal stresses

From the Hooke's law for elastic material and Eqn. (7) the normal stress may be written as

$$\sigma_{xx} = E \frac{\partial u_x}{\partial x} = E \left[\frac{d\bar{u}_x}{dx} - \frac{d^2 v_s}{dx^2} (y - y_R) - \frac{d^2 w_s}{dx^2} (z - z_R) + \frac{d^2 \omega_x}{dx^2} (\phi_s - \phi_{sR}) \right] \quad (14)$$

where N_x , M_y and M_z are internal axial force and bending moments, whereas bi-moment is

$$M_\phi = \int_{\mathcal{A}_x} \sigma_{xx} \phi_s dA_x = E I_{\phi\phi} \frac{d^2 \omega_x}{dx^2}, \quad I_{\phi\phi} = \int_{\mathcal{A}_x} \phi_s^2 dA_x. \quad (15)$$

If the expressions for internal forces are included into Eqn. (14) the normal stress reads

$$\sigma_{xx} = \frac{N_x}{A_x} - \frac{M_z}{I_z} y + \frac{M_y}{I_y} z + \frac{M_\phi}{I_{\phi\phi}} \phi_s. \quad (16)$$

3.2 Shear stresses

In a thin-walled cross-section there is linear and constant part of shear stress. The linear shear stress occurs due to uniform twisting in open section

$$\sigma_{x\zeta}^{l,os} = G \frac{d\omega_x}{dx} 2\eta = \frac{M_x}{I_x^{os}} 2\eta, \quad (17)$$

where torsional moment M_x and torsional moment of inertia I_x^{os} are

$$M_x = \int_{\mathcal{A}_x} (\sigma_{xz} y - \sigma_{xy} z) dA_x = \frac{d\omega_x}{dx} G I_x^{os}, \quad I_x^{os} = \frac{1}{3} \int_{\mathcal{S}} t^3(\zeta) d\zeta. \quad (18)$$

$\mathcal{S}$ represents the centerline of the wall, t is the thickness of the wall.

The constant shear stress in an open thin-walled cross-section $\sigma_{x\zeta}^{c,os}$ occurs due to internal shear forces N_y and N_z , axial loading $\mathcal{P}_x$, distributed moment loading $\mathcal{M}_y$ and $\mathcal{M}_z$, and non-uniform twisting M_ϕ

$$\begin{aligned} \sigma_{x\zeta}^{c,os} &= \frac{1}{t} \left[\mathcal{P}_x \frac{A_x^*}{A_x} - \mathcal{P}^* - \frac{(N_y + \mathcal{M}_z)}{I_z} S_z^*(\zeta) - \frac{(N_z + \mathcal{M}_y)}{I_y} S_y^*(\zeta) - \frac{dM_\phi}{dx} \frac{S_\phi^*}{I_{\phi\phi}} \right] = \\ &= \frac{1}{t} (q_b^{os} + q_n^{os}). \end{aligned} \quad (19)$$

Asterisk (*) corresponds to the part of cross-section $\mathcal{A}_x^*$. In the last equation the shear flux q is introduced as the force acting along the wall of thin-walled cross-section. For a constant shear stress the shear flux is simply $q = \sigma_{x\zeta}^c t$. The term S_ϕ^* is

$$S_\phi^* = \int_{\mathcal{A}_x^*} \phi_s dA_x. \quad (20)$$

The constant shear stresses in a closed thin-walled cross-section is due to bending, uniform and non-uniform twisting. However, the determination of the shear fluxes q_i in a closed cross-section is statically indeterminate problem. The method of forces is used for this purpose. The beam with N cells is virtually dissected by N virtual cuts so that an open cross-section is created (see Fig. 1). The influence of the cuts is replaced by shear fluxes

$$q_i = X_i \bar{q}_i, \quad i = 1, 2, \dots, N \quad (21)$$

where $\bar{q}_i$ ($|\bar{q}_i| = 1.0$) represents the direction and location of shear flux, whereas X_i represents its value. In Fig. 1 shear fluxes corresponding to three cuts of the three cells cross-section are shown. The location of the shear flux q_i is determined by the centerline $\mathcal{S}_i$ of the wall.

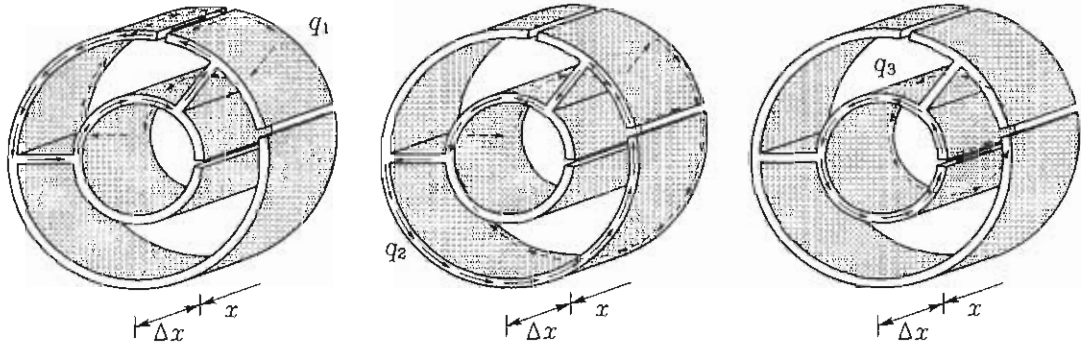


FIG. 1. Three cells cross-section and corresponding shear fluxes q_1 , q_2 in q_3 .

Therefore the shear flux in the closed cross-section is composed of two parts (open section part and closed section part):

$$q = q^{os} + q^{cs} = q^{os} + \sum_{i=1}^N q_i = q^{os} + \sum_{i=1}^N X_i \bar{q}_i. \quad (22)$$

The values of shear fluxes X_i ($i = 1, 2, \dots, N$) are determined from the kinematic condition requiring the relative longitudinal displacement of the cut to be zero. For the segment of the beam of length Δx the principle of complementary virtual work is $\Delta(\delta W_e^*) = \Delta(\delta W_i^*)$. The segment is loaded by virtual shear fluxes δq_i with the same direction and location as shear fluxes q_i : $\delta q_i = \delta X_i \bar{q}_i$ ($i = 1, 2, \dots, N$). The complementary virtual work of external forces $\Delta(\delta W_e^*)$ for the beam segment is

$$\Delta(\delta W_e^*) = - \sum_{i=1}^N \int_{\Delta x} \delta q_i u_{xi}^l dx + \sum_{i=1}^N \int_{\Delta x} \delta q_i u_{xi}^r dx - \omega_x^- \sum_{i=1}^N \oint_{\mathcal{S}_i} \delta q_i r d\zeta + \omega_x^+ \sum_{i=1}^N \oint_{\mathcal{S}_i} \delta q_i r d\zeta, \quad (23)$$

where r is the distance from the chosen point 0 and the tangent to the centerline of the wall. The displacements u_{xi}^l and u_{xi}^r correspond to the both sides of the cut, whereas the rotations $\omega_x^-(x)$ and $\omega_x^+(x + \Delta x)$ correspond to both sides of the beam segment. If the length of the segment Δx goes to zero the specific complementary virtual work $d(\delta W_e^*)/dx$ can be derived

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta(\delta W_e^*)}{\Delta x} = \frac{d(\delta W_e^*)}{dx} = \sum_{i=1}^N \delta X_i \bar{q}_i (u_{xi}^r - u_{xi}^l) + \frac{d\omega_x}{dx} \sum_{i=1}^N \delta X_i \oint_{\mathcal{S}_i} \bar{q}_i r d\zeta. \quad (24)$$

Considering that $\sigma_{x\eta} = 0$, the complementary virtual work of internal forces $\Delta(\delta W_i^*)$ for the same beam segment is

$$\Delta(\delta W_i^*) = \Delta x \int_{\mathcal{A}_x} 2\varepsilon_{x\zeta} \delta\sigma_{x\zeta} dA_x. \quad (25)$$

If the length of the segment goes to zero, then the specific complementary virtual work $d(\delta W_i^*)/dx$ is

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta(\delta W_i^*)}{\Delta x} = \frac{d(\delta W_i^*)}{dx} = \int_{\mathcal{A}_e} 2 \varepsilon_{x\zeta} \delta \sigma_{x\zeta} dA_x = \int_{\mathcal{S}} \frac{\sigma_{x\zeta}}{G} \delta \sigma_{x\zeta} t d\zeta = \sum_{i=1}^N \delta X_i \oint_{\mathcal{S}_i} \frac{q \bar{q}_i}{G t} d\zeta. \quad (26)$$

From the principle of complementary virtual work and taking into account that the relative displacement along the longitudinal cut is zero ($u_{xi}^r - u_{xi}^l = 0$) follows

$$\sum_{j=1}^N X_j \oint_{\mathcal{S}_j} \frac{\bar{q}_j \dot{q}_j}{G t} d\zeta = - \oint_{\mathcal{S}_i} \frac{\bar{q}_i q_b^{os}}{G t} d\zeta + \frac{d\omega_x}{dx} \oint_{\mathcal{S}_i} \bar{q}_i r d\zeta + \frac{d^3\omega_x}{dx^3} \oint_{\mathcal{S}_i} \frac{\bar{q}_i E S_\phi^*}{G t} d\zeta, \quad i = 1, 2, \dots, N. \quad (27)$$

The solution of the system of linear equations (27) is decomposed into three parts (bending, uniform twisting and non-uniform twisting)

$$X_i = X_{i,b} + X_{i,u} \frac{d\omega_x}{dx} + X_{i,n} \frac{d^3\omega_x}{dx^3}. \quad (28)$$

Constant shear stresses in the closed thin-walled cross-section is therefore equal to

$$\sigma_{x\zeta}^{c,cs} = \frac{1}{t} \sum_{i=1}^N \left(X_{i,b} + X_{i,u} \frac{d\omega_x}{dx} + X_{i,n} \frac{d^3\omega_x}{dx^3} \right) \bar{q}_i = \frac{1}{t} \sum_{i=1}^N X_i \bar{q}_i = \frac{q^{cs}}{t}. \quad (29)$$

The total shear stress may now be written as a sum (see eqns. (17), (19) and (29))

$$\sigma_{x\zeta} = \sigma_{x\zeta}^{l,os} + \sigma_{x\zeta}^{c,os} + \sigma_{x\zeta}^{c,cs}. \quad (30)$$

4. Strains

Normal strain ε_{xx} is derived from Eqn. (10)

$$\varepsilon_{xx} = \frac{d\bar{u}_x}{dx} - \frac{d^2 v_s}{dx^2} (y - y_R) - \frac{d^2 w_s}{dx^2} (z - z_R) + \frac{d^2 \omega_x}{dx^2} (\phi_s - \phi_{sR}). \quad (31)$$

The shear strain is (see eqns. (17), (19) and (29))

$$\varepsilon_{x\zeta} = \varepsilon_{x\zeta}^l + \varepsilon_{x\zeta}^c = \frac{d\omega_x}{dx} \eta + \frac{1}{2Gt} \left[q_b^{os} - E S_\phi^* \frac{d^3\omega_x}{dx^3} + \sum_{i=1}^N \left(X_{i,b} + X_{i,u} \frac{d\omega_x}{dx} + X_{i,n} \frac{d^3\omega_x}{dx^3} \right) \bar{q}_i \right]. \quad (32)$$

5. Equilibrium equations

Equilibrium equations are derived by the principle of virtual work $\delta W_e = \delta W_i$, where the virtual work of external forces δW_e is

$$\delta W_e = \int_{\mathcal{A}_{x0}} \vec{p} \cdot \delta \vec{u} dA_{x0} + \int_{\mathcal{A}_{xL}} \vec{p} \cdot \delta \vec{u} dA_{xL} + \int_{\mathcal{O}} \vec{p} \cdot \delta \vec{u} d\mathcal{O}, \quad (33)$$

whereas the virtual work of internal forces δW_i is

$$\delta W_i = \int_V [\sigma_{xx} \delta \varepsilon_{xx} + 2 (\sigma_{x\zeta}^l \delta \varepsilon_{x\zeta}^l + \sigma_{x\zeta}^c \delta \varepsilon_{x\zeta}^c)] dV. \quad (34)$$

The influence of shear forces is neglected in the derivation of equilibrium equations, therefore the term $\bar{\sigma}_{x\zeta}^c \delta \bar{\varepsilon}_{x\zeta}^c$ is used in δW_i instead of $\sigma_{x\zeta}^c \delta \varepsilon_{x\zeta}^c$

$$\begin{aligned}\bar{\sigma}_{x\zeta}^c &= \frac{1}{t} \left[q_n^{os} + \sum_{i=1}^N \left(X_{i,u} \frac{d\omega_x}{dx} + X_{i,n} \frac{d^3\omega_x}{dx^3} \right) \bar{q}_i \right], \\ \delta \bar{\varepsilon}_{x\zeta}^c &= \frac{1}{2Gt} \left[-E S_\phi^* \frac{d^3(\delta\omega_x)}{dx^3} + \sum_{i=1}^N \left(X_{i,u} \frac{d(\delta\omega_x)}{dx} + X_{i,n} \frac{d^3(\delta\omega_x)}{dx^3} \right) \bar{q}_i \right].\end{aligned}\quad (35)$$

After some derivation and several integrations by parts the following governing differential equations for thin-walled beam are derived

$$\begin{aligned}\frac{dN_x}{dx} + \mathcal{P}_x &= 0, \\ \frac{d^2 M_y}{dx^2} + \mathcal{P}_z + \frac{d\mathcal{M}_y}{dx} &= 0, \\ \frac{d^2 M_z}{dx^2} - \mathcal{P}_y + \frac{d\mathcal{M}_z}{dx} &= 0, \\ \frac{d^2 M_\phi}{dx^2} - \frac{dM_{sv}}{dx} - \frac{d^3 M_{pr}}{dx^3} - \mathcal{M}_x - \mathcal{P}_y z_s + \mathcal{P}_z y_s + \frac{d\mathcal{M}_\phi}{dx} &= 0.\end{aligned}\quad (36)$$

The corresponding natural boundary conditions at both ends are also derived from the principle of virtual work. At $x = 0$ the natural boundary conditions are:

$$\begin{aligned}N_{x0} + R_{x0} &= 0, \quad \frac{dM_{z0}}{dx} - R_{y0} + \mathcal{M}_{z0} = 0, \quad \frac{dM_{y0}}{dx} + R_{z0} + \mathcal{M}_{y0} = 0, \\ M_{z0} + M_{z0}^R &= 0, \quad M_{y0} + M_{y0}^R = 0, \quad \frac{dM_{pr0}}{dx} - M_{\phi0} + M_{\phi0}^R = 0, \\ \frac{dM_{\phi0}}{dx} - \frac{d^2 M_{pr0}}{dx^2} - M_{sv0} - M_{x0}^R - R_{y0} z_s + R_{z0} y_s + \mathcal{M}_{\phi0} &= 0, \quad M_{pr0} = 0,\end{aligned}\quad (37)$$

whereas at $x = L$ the natural boundary conditions are:

$$\begin{aligned}N_{xL} - R_{xL} &= 0, \quad \frac{dM_{zL}}{dx} + R_{yL} + \mathcal{M}_{zL} = 0, \quad \frac{dM_{yL}}{dx} - R_{zL} + \mathcal{M}_{yL} = 0, \\ M_{zL} - M_{zL}^R &= 0, \quad M_{yL} - M_{yL}^R = 0, \quad M_{\phi L} - \frac{dM_{prL}}{dx} - M_{\phi L}^R = 0, \\ \frac{d^2 M_{prL}}{dx^2} - \frac{dM_{\phi L}}{dx} + M_{svL} - M_{xL}^R - R_{yL} z_s + R_{zL} y_s - \mathcal{M}_{\phi L} &= 0, \quad M_{prL} = 0,\end{aligned}\quad (38)$$

where $\mathcal{P}_y$, $\mathcal{P}_z$, and $\mathcal{M}_x$, are distributed load in direction y , direction z , and distributed moment load about axis x , respectively. $\mathcal{M}_\phi$ is moment due to warping of the cross-section. The reactions at $x = 0$ are R_{x0} , R_{y0} , R_{z0} , $M_{\phi0}^R$, M_{x0}^R , M_{y0}^R , and M_{z0}^R , whereas the reactions at $x = L$ are R_{xL} , R_{yL} , R_{zL} , $M_{\phi L}^R$, M_{xL}^R , M_{yL}^R , and M_{zL}^R . Furthermore, the following notations have been used in differential equations (36) and boundary conditions (37) and (38)

$$\begin{aligned}M_{sv} &= \frac{d\omega_x}{dx} \left(G I_x^{os} + \sum_{i=1}^N X_{i,e} \oint_{\mathcal{S}_i} \bar{q}_i r d\zeta \right), \\ M_{pr} &= \frac{d^3\omega_x}{dx^3} \left(\int_{\mathcal{S}} \frac{E^2 S_\phi^{*2}}{Gt} d\zeta - \sum_{i=1}^N X_{i,n} \oint_{\mathcal{S}_i} \frac{E S_\phi^* \bar{q}_i}{Gt} d\zeta \right) + \\ &\quad + \frac{d\omega_x}{dx} \left(\sum_{i=1}^N X_{i,n} \oint_{\mathcal{S}_i} \bar{q}_i r d\zeta - \sum_{i=1}^N X_{i,e} \oint_{\mathcal{S}_i} \frac{E S_\phi^* \bar{q}_i}{Gt} d\zeta \right).\end{aligned}\quad (39)$$

The differential equations governing axial deformation and bending in two directions are equal to those stated in the literature. On the other hand, the differential equation governing twisting differs from those cited in the literature. The last differential equations (36) expressed in terms of rotations ω_x is a sixth order differential equation, unlike the traditional fourth order differential equation

$$\begin{aligned} & \frac{d^6 \omega_x}{dx^6} \left(\sum_{i=1}^N X_{i,n} \oint_{\mathcal{S}_i} \frac{E S_\phi^* \bar{q}_i}{G t} d\zeta - \int_{\mathcal{S}} \frac{E^2 S_\phi^{*2}}{G t} d\zeta \right) + \\ & + \frac{d^4 \omega_x}{dx^4} \left(E I_{\phi\phi} + \sum_{i=1}^N X_{i,e} \oint_{\mathcal{S}_i} \frac{E S_\phi^* \bar{q}_i}{G t} d\zeta - \sum_{i=1}^N X_{i,n} \oint_{\mathcal{S}_i} \bar{q}_i r d\zeta \right) - \\ & - \frac{d^2 \omega_x}{dx^2} \left(G I_x^{os} + \sum_{i=1}^N X_{i,e} \oint_{\mathcal{S}_i} \bar{q}_i r d\zeta \right) = \mathcal{M}_x + \mathcal{P}_y z_s - \mathcal{P}_z y_s - \frac{d\mathcal{M}_\phi}{dx}. \end{aligned} \quad (40)$$

While natural boundary conditions are derived from the principle of virtual work, the kinematic (essential) boundary conditions have to be defined otherwise. The choice of kinematic boundary conditions governing axial deformation and bending is standard and won't be discussed here. Since the differential equation (40) is of sixth order, three boundary conditions at each end of the beam are needed. The kinematic boundary conditions for clamped end are derived from the condition that the rotation ω_x (twisting) and displacements u_x (warping) are restrained. The first kinematic boundary condition is obvious: $\omega_x = 0$. The second kinematic boundary condition follows from Eqn. (10): $d\omega_x/dx = 0$. The second derivative $d^2\omega_x/dx^2$ can not be zero since this would imply that σ_{xx} is zero (see Eqn. (14)) which is obviously not true. Similarly the third derivative $d^3\omega_x/dx^3$ can not be zero since this implies that $\sigma_{x\zeta}$ is zero (see Eqn. (30)) which is also not true. Therefore, the condition $d^4\omega_x/dx^4 = 0$ has been chosen as the third kinematic boundary condition.

6. Numerical example

A cantilever thin-walled beam of length $L = 500$ m has been solved by three different differential equations: our sixth order equation, fourth order equation given by Oden [2], and fourth order equation given by Kollbrunner [1]. The free end of the beam is loaded by a concentrated moment $M_{xL}^R = 500$ kNm. The cross-section with one cell is constant along the beam axis and is shown in Fig. 2.

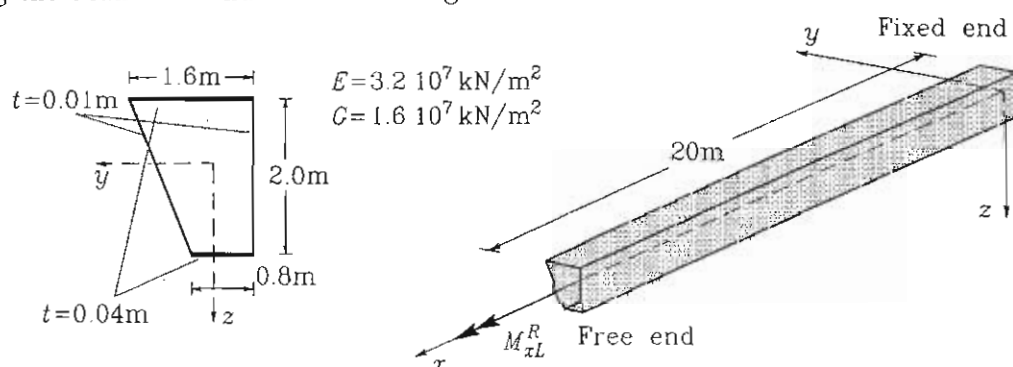


FIG. 2. Cantilever beam loaded by a concentrated moment.

Rotation at free end $\omega_x(L)$, and its second and third derivatives at clamped end $\omega_x''(0)$ and $\omega_x'''(0)$ evaluated from three different equations are compared in Tab. 1.

Tab. 1: Comparison between the three methods

Method	$\omega_x(L)$	$\omega_x''(0)$	$\omega_x'''(0)$
6 th order	0.012627	0.0010262	-0.001018
4 th order [2]	0.012494	0.0010333	-0.001656
4 th order [1]	0.012660	0.0005101	-0.000750

The differences between the rotation at the free end are very small, while the differences of the second and third derivative of rotation are more notable. Fig. 3 shows the differences in normal and shear stresses given in kN/m² at the clamped end. Normal stresses evaluated by Oden's equations and the 6th order equations virtually coincide. Normal stresses evaluated by Kollbrunner's equations are slightly different (by 6%). The differences in results of shear stresses are much more evident. The results obtained by Oden's equations are 62% larger than present results, while the difference of Kollbrunner's results is as high as 25% compared to the results obtained by 6th order equation.

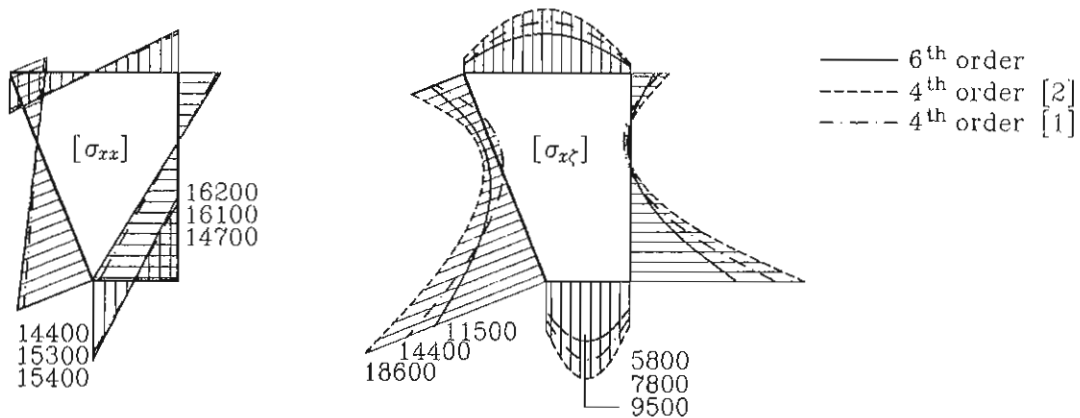


FIG. 3. Normal and shear stresses at $x = 0$.

7. Conclusions

From the principle of virtual work the sixth order differential equation governing twisting has been found. While the difference in rotations ω_x is not very large, the difference in shear stresses are much more pronounced. In order to determine which of the three methods gives the best approximation of the actual behavior of the structure, a better shell mathematical model should be employed.

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An approximate solution for stress distributions in girders with multi-cell cross-sections

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SUMMARY

Prismatic thin-walled girders with multi-cell cross-section can be considered as an assembly of membrane strips. In this paper an analytical solution for stress distributions in the girder flanges is given. The flanges are considered as wide membrane strips, loaded by line loads in the longitudinal direction (shear forces). The plane stress problem is solved approximately, assuming that changes of the cross-section contour dimension and the normal stress in the cross-section contour direction can be ignored in the shear strain-displacements relation and the strain in the longitudinal direction-stresses relation, respectively. A comparison with the finite element calculation is given.

1. INTRODUCTION

Prismatic thin-walled girders with multiple cell cross-sections, subjected to transverse bending, can be found in many structures (in civil engineering, shipbuilding, aeronautical industry, etc.). The ratio of the girder length, l , to the cross-section contour dimension, b , can be some times very small: < 5 (Fig. 1a,b).

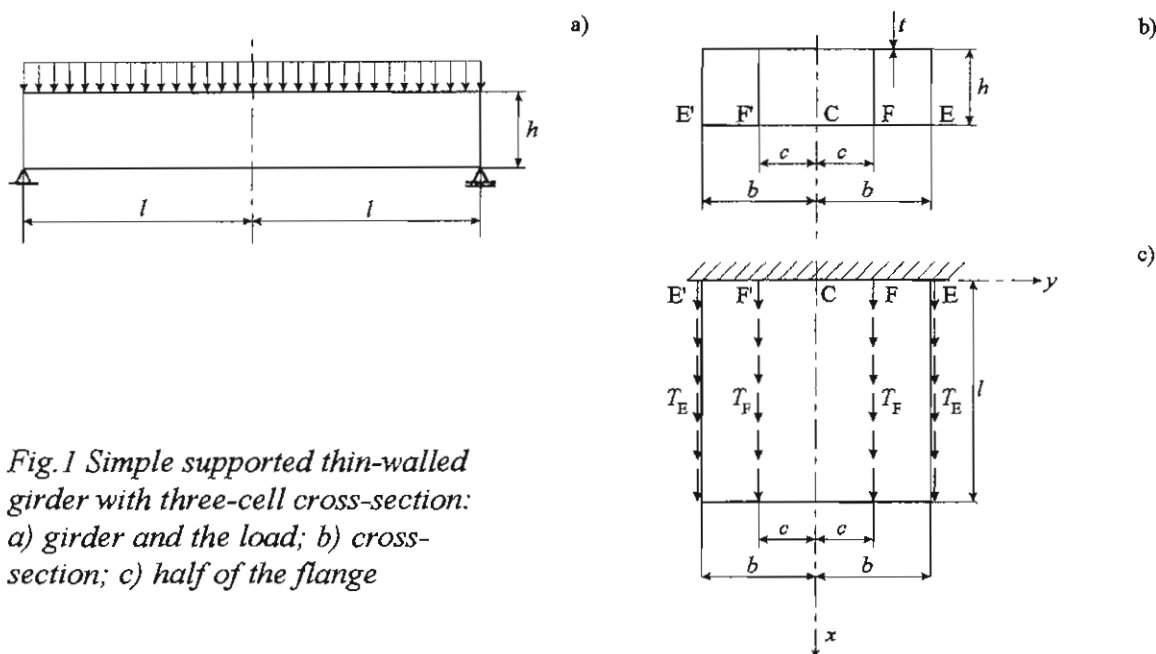


Fig.1 Simple supported thin-walled girder with three-cell cross-section:
a) girder and the load; b) cross-section; c) half of the flange

The girders can be considered as an assembly of membrane strips, loaded along the strip longitudinal edges [1]. The problem leads to a numerical procedure to determine membrane forces and displacements in the strips, by satisfying boundary conditions along

the strips longitudinal edges [1,2]. The standard finite element formulation leads to a model of the whole girder, using membrane elements [3,4].

In this work the distributions of the stresses and displacements in the girder flanges (Fig. 1c) will be considered. A simple analytical approach will be provided; by introducing assumptions about the stresses and the strains.

Some analytical solutions can be found for simple girders with one cell cross-sections [3-6], with the comparisons with the finite element method [3] and the method of plane theory of elasticity [6].

2. ASSUMPTIONS

Flanges of prismatic thin-walled girders with multi-cell cross-section are considered. The cross-sections of flanges have double axis of symmetry. The load is given by line loads in the longitudinal direction (shear forces) $T = T(x)$, symmetrically distributed in the y-direction (Fig. 1c). The flange thickness may be variable in the y-direction, $t = t(y)$.

A state of plane stress in the flanges will be simplified, by following assumptions:

1. Changes of dimensions in the y-direction are small compared to displacements in the x-direction, that can be ignored in the shear strain-displacements relation.
2. The normal stresses in the y-direction are small compared to the normal stresses in the x-direction, that can be ignored in the strain in the x-direction-stresses relation.

3. STRAIN-DISPLACEMENT AND STRESS-STRAIN RELATIONS

According to Assumption 1, the shear strain-displacements relation becomes

$$\gamma_{xy} = \frac{\partial u}{\partial y}, \quad (1)$$

where $\gamma_{xy} = \gamma_{xy}(x, y)$ is the shear strain and $u = u(x, y)$ is the displacement in the x-direction. According to Assumption 2, the stress-strain relations (Hooke's law) can be expressed as:

$$\sigma_x = E\varepsilon_x, \quad \tau_{xy} = G\gamma_{xy}, \quad (2)$$

where $\sigma_x = \sigma_x(x, y)$ is the normal stress in the x-direction, $\tau_{xy} = \tau_{xy}(x, y)$ is the shear stress and $\varepsilon_x = \varepsilon_x(x, y)$ is the strain in the x-direction:

$$\varepsilon_x = \frac{\partial u}{\partial x}; \quad (3)$$

E and G are modulus of elasticity and shear modulus, respectively. The stresses σ_x and τ_{xy} are assumed to be active -- associated with the strains ε_x and γ_{xy} , respectively. The normal stress in the y-direction, $\sigma_y = \sigma_y(x, y)$, is assumed to be reactive -- can be obtained only from the equilibrium condition.

The equilibrium conditions can be written as follow (Fig. 2):

$$\int_{y^*} \frac{\partial \sigma_x}{\partial x} t dy + q_x^* - \tau_{yx} t = 0, \quad \int_{y^*} \frac{\partial \tau_{xy}}{\partial x} t dy - \sigma_y t = 0, \quad (4)$$

where

$$q_x^* = q_x^*(x, y) = T_E + T_F; \quad (5)$$

y^* is the coordinate from the longitudinal edge, where $\sigma_y = 0$.

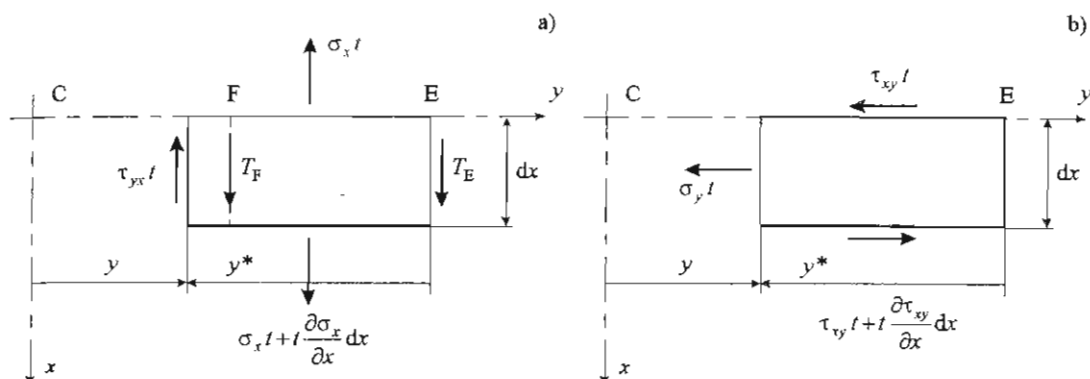


Fig.2 Equilibrium of a portion of the flange: a) in the x -direction; b) in the y -direction

The total line load is

$$q_x = 2q_x^*(x, b) = 2(T_E + \sum T_F), \quad (6)$$

where b is the half of the flange contour length. Equations (4) may then be rewritten as

$$\tau_{yx} = \tau_{xy} = \frac{1}{t} \left(\int_{A^*} \frac{\partial \sigma_x}{\partial x} dA + J^* q_x \right), \quad \sigma_y = \frac{1}{t} \int_{A^*} \frac{\partial \tau_{xy}}{\partial x} dA, \quad (7)$$

where A^* is the area of the portion of the cross-section, defined by y^* ;

$$J^* = J^*(y) = \frac{q_x^*}{q_x}. \quad (8)$$

The compatibility condition, according to (1) and (3), and (2), may be expressed as

$$\frac{\partial \varepsilon_x}{\partial y} = \frac{\partial \gamma_{xy}}{\partial x}, \quad \frac{\partial \sigma_x}{\partial y} = \frac{E}{G} \frac{\partial \tau_{xy}}{\partial x} \quad (9)$$

Taking into account (1), (2) and (3), the following expressions can be written:

$$u = u_C + \frac{1}{G} \int_0^y \tau_{xy} dy, \quad \sigma_x = E \frac{du_C}{dx} + \frac{E}{G} \int_0^y \frac{\partial \tau_{xy}}{\partial x} dy, \quad (10)$$

where $u_C = u_C(x)$ is the displacement of the cross-section centroid (the origin of the y co-ordinate).

The equilibrium conditions yields:

$$\int_A \sigma_x dA = N, \quad \frac{dN}{dx} = -q_x. \quad (11)$$

where $N = N(x)$ is the normal force (internal force); A is the cross-section area.

The strains in the y-direction, $\varepsilon_y = \varepsilon_y(x, y)$, can be obtained from the stress-strain relation (Hooke's law)

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \quad (12)$$

where ν is Poisson's ratio.

4. STRESSES AND DISPLACEMENTS

The following expression for shear stress can then be obtained:

$$\tau_{xy} = \left(J^* - \frac{A^*}{A} \right) \frac{q_x}{t}, \quad A^* = A^*(y) = \int_{A^*} dA. \quad (13)$$

The normal stress in the x-direction can be obtained as

$$\sigma_x = \frac{N}{A} + \frac{E}{G} \kappa \frac{dq_x}{dx} + \frac{E}{G} \left(\int_0^y \frac{J^*}{t} dy - \frac{1}{A} \int_0^y \frac{A^*}{t} dy \right) \frac{dq_x}{dx}, \quad (14)$$

where

$$\kappa = \frac{1}{A^2} \int_A \left(\frac{A^*}{t} \right)^2 dA - \frac{1}{A} \int_A \left(\int_0^y \frac{J^*}{t} dy \right) dA; \quad \int_A \left(\frac{A^*}{t} \right)^2 dA = \int_A \left(\int_0^y \frac{A^*}{t} dy \right) dA. \quad (15)$$

The normal stress in the y-direction is obtained as

$$\sigma_y = \frac{1}{t} \left(\int_{A^*} \frac{J^*}{t} dA - \frac{1}{A} \int_{A^*} \frac{A^*}{t} dA \right) \frac{dq_x}{dx}. \quad (16)$$

The obtained stresses satisfy the equilibrium conditions (4) and the compatibility condition (9)₂ if

$$\frac{dq_x}{dx} = \text{const}. \quad (17)$$

Then:

$$EA \frac{du_C}{dx} - \frac{EA}{G} \kappa \frac{dq_x}{dx} = N, \quad EA \frac{d^2 u_C}{dx^2} = -q_x \quad (18)$$

The displacement u_C can be expressed as

$$u_C = u_p + u_a, \quad \frac{du_C}{dx} = \frac{du_p}{dx} + \frac{du_a}{dx}. \quad (19)$$

where

$$\frac{du_p}{dx} = \frac{N}{EA}, \quad \frac{d^2 u_p}{dx^2} = -\frac{q_x}{EA}, \quad (20)$$

and

$$\frac{du_a}{dx} = \frac{\kappa}{G} \frac{dq_x}{dx}. \quad (21)$$

The displacement u_p corresponds formally to the displacement u_C when $q_x = \text{const.}$ The displacement u_a is an additional displacement due to shear:

$$u_a = \frac{\kappa}{G} q_x, \quad (22)$$

The displacement u then is given by

$$u = u_p + \frac{\kappa}{G} q_x + \frac{1}{G} \left(\int_0^y \frac{J^*}{t} dy - \frac{1}{A} \int_0^y \frac{A^*}{t} dy \right) q_x. \quad (23)$$

The boundary conditions (at cross-sections) can be defined as follows:

-- built-in end ($x = 0$)

$$u_p(0) = 0; \quad (24)$$

-- simple supported end ($x = l$)

$$\frac{du_p(l)}{dx} = 0 \quad (N(l) = 0), \quad (25)$$

where l is the flange length.

At the built-in end the displacement u is equal to zero if

$$q_x(0) = 0. \quad (26)$$

Then $\tau_{xy} = 0$, $\varepsilon_y \neq 0$.

At the simple supported end σ_x is equal to zero if

$$\frac{dq_x(l)}{dx} = 0. \quad (27)$$

Then $\varepsilon_y = 0$, $\tau_{xy} \neq 0$.

If $q_x = \text{const.}$, at the built-in end will be $u \neq 0$ (only $u_p = 0$), $\tau_{xy} \neq 0$, $\varepsilon_y \neq 0$. At the simple supported end will be $\sigma_x = 0$, $\varepsilon_y = 0$, $\tau_{xy} \neq 0$. The conditions for built-in end are not fully satisfied.

If $\frac{dq_x}{dx} = \text{const.}$, at built-in end will be $u = 0$, $\tau_{xy} = 0$, $\varepsilon_y \neq 0$. At the simple supported end is $\sigma_x \neq 0$ (only σ_x of N is equal to zero), $\varepsilon_y \neq 0$, $\tau_{xy} \neq 0$. The conditions for the simple supported end are not fully satisfied.

5. EXAMPLES

A flange built-in at one end and simple supported at the other is loaded along the longitudinal sections by line loads $T = T(l) \frac{x}{l}$.

The load is $q_x = 2T$. The cross-section functions are shown in Fig.3.

The cross-section properties are

$$\int_A \left(\int_0^y \frac{A^*}{t} dy \right) dA = \frac{2}{3} b^3 t, \quad \int_A \left(\int_0^y \frac{J^*}{t} dy \right) dA = \frac{bc}{3} \left(2 - \frac{c}{b} \right), \quad \kappa = -\frac{b}{12t} \left[1 - 3 \left(\frac{c}{b} \right)^2 \right].$$

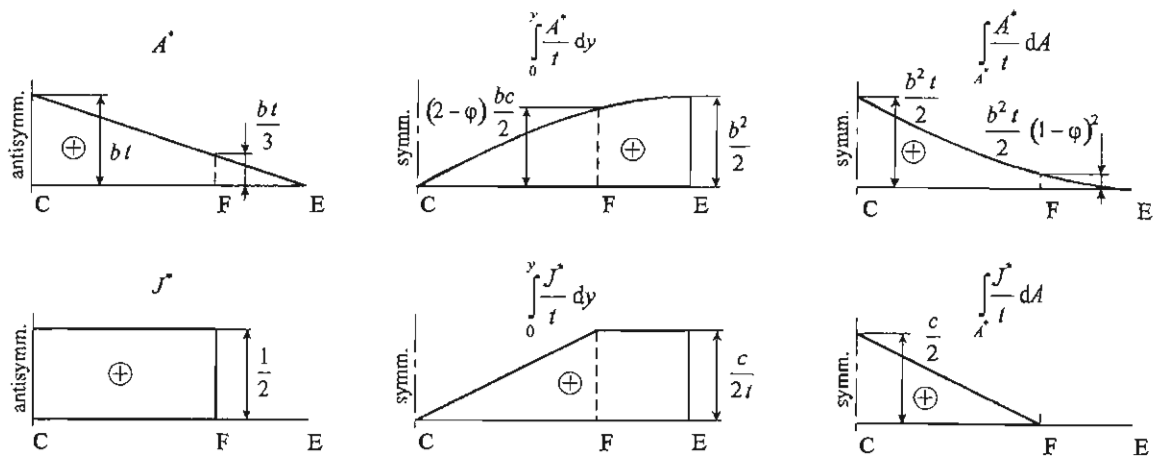


Fig.3 Flange loaded along longitudinal sections: distribution of the cross-section functions.

The stresses τ_{xy} , σ_x and σ_y are shown in Fig.4, where

$$N = \frac{q_x(l)}{2} \left[1 - \left(\frac{x}{l} \right)^2 \right].$$

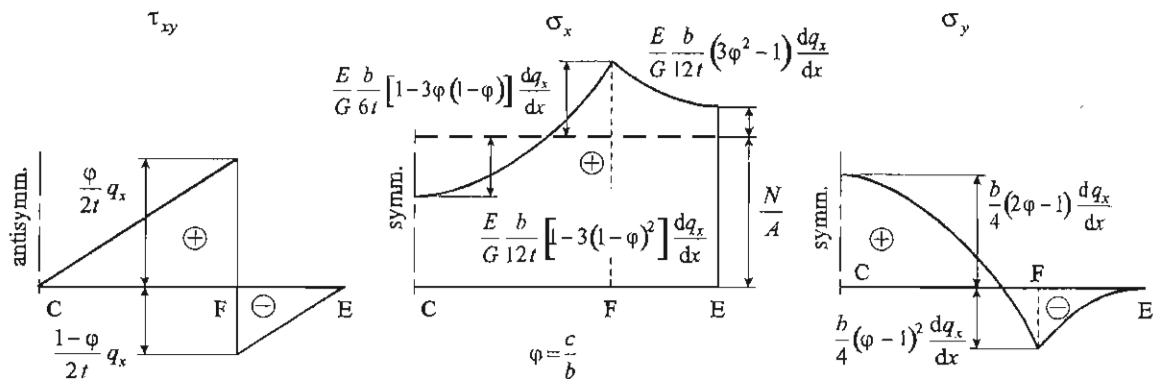


Fig.4 Flange loaded along longitudinal sections (F): distributions of the stresses along the cross-section contour

If $c = b$, the flange loaded along the longitudinal edges (E) may be obtained. Then $\kappa = -\frac{b}{12t}$. If $c = 0$, the flange loaded along the central longitudinal section (C) may be obtained: $\kappa = \frac{b}{6t}$, $q_x = T$.

By superposing the distributions for the load cases "F", "E" and "C" the stress distributions for any multiple cell cross-section can be obtained. "E" and "C" give the distributions for girders with two-cell cross-sections; "F" and "E" give the distributions for girders with three-cell cross-sections, etc.

6. NUMERICAL EXAMPLES

A flange of three-cell cross-section is considered (Fig. 1b), where $c/b = 2/3$. The flange load is

$$T_E = T_E(l) \frac{x}{l}, \quad T_F = T_F(l) \frac{x}{l}, \quad T_F(l) = T_E(l).$$

The results of calculation are given for points C, F and E: $l/b = 2$ and $l/b = 3$; $E/G = 2,6$; $\nu = 0,3$; Fig. 5, 6 and 7.

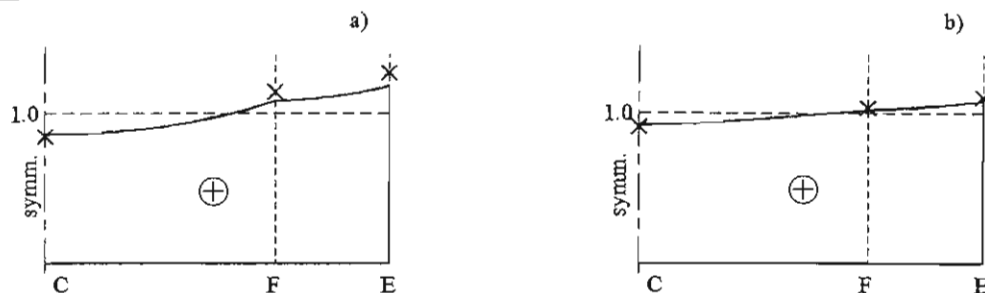


Fig. 5 Distribution of $\frac{\sigma_x A}{N}$ at built-in end for the girder with three-cell cross-section

(— finite element method, $\times$ analytical): a) $\frac{l}{b} = 2$, b) $\frac{l}{b} = 3$.



Fig. 6 Distribution of $\frac{\sigma_y A}{N}$ at built-in end for the girder with three-cell cross-section

(— finite element method, $\times$ analytical): a) $\frac{l}{b} = 2$, b) $\frac{l}{b} = 3$.

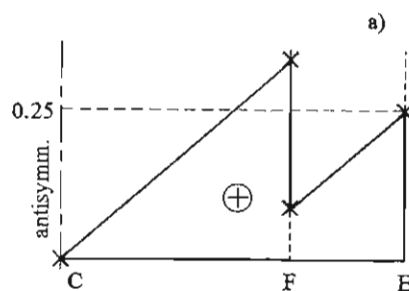


Fig. 7 Distribution of $\frac{\tau_{xy} t}{q_x}$ at $x = l/2$ for the girder with three-cell cross-section

(— finite element method, $\times$ analytical)

The results of calculation by finite element method are shown by full line. One quarter of the flange is modelled ($b \times l$), by membrane elements, as shown in Fig. 8.

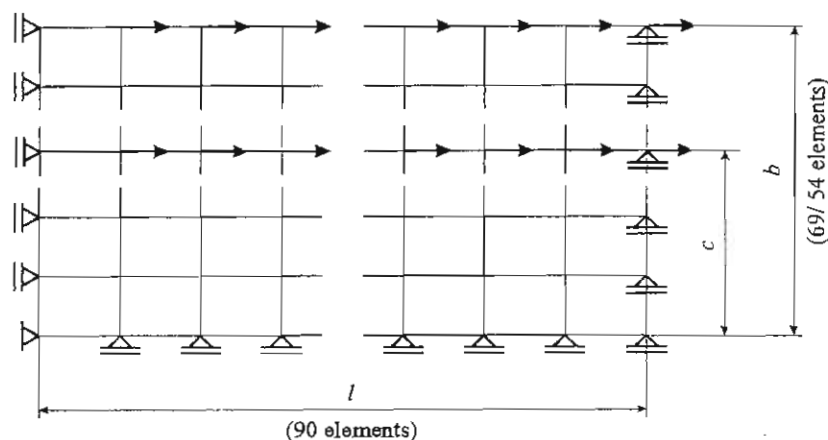


Fig.8 Finite element model

6. CONCLUSION

The analytical solution for stresses in the flanges of prismatic thin-walled girders with multiple cell cross-sections is given. Linear distribution of the line loads along the flange length (the uniform distribution as a special case) is assumed. The expressions for stresses are given in a simple form -- acceptable for engineering purposes (preliminary calculations in civil engineering, shipbuilding, etc.).

The comparison with the finite element analysis (using membrane elements) shows a good agreement of the obtained results -- even for extremely short girders. The maximum stress σ_x is 6.1% is higher than the maximum stress obtained by finite element modelling, for the flange length to width ratio of 2, and 3.2%, for the flange length to width ratio of 3. The maximum stress σ_y is 0.6% lower, for the ratio of 2, and 0.4%, for the ratio of 3.

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THEORY OF THE TORSION TEST EVALUATION

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ABSTRACT. This original method solves the state of strain and state of stress in some general point of the torsion-straining cylindrical specimen. It makes possible to express both small and large (plastic) deformations in the region where Cauchy's equations have no validity. This method reflects rotation of the main strain ϵ_i and main stress σ_i direction in some angle α_i . This theory enables to compute the components of the main strains $\epsilon_1, \epsilon_2, \epsilon_3$, strain rate, main stresses $\sigma_1, \sigma_2, \sigma_3$ and also to calculate the effective strain S_e , effective strain rate $S_{\dot{\epsilon}}$ and effective stress S_{σ} . Using this method a constitutive equation $S_{\sigma} - S_{\dot{\epsilon}}$ can be determined.

Keywords: Constitutive equation, effective strain, effective strain rate, effective stress, torsion test

INTRODUCTION. Theory of plasticity supposes, that the dependence: effective stress S_σ versus effective strain S_ϵ (called also constitutive equation) is identical - despite it is calculated from a tensile or compressive test. Consequently S_σ - S_ϵ dependence must be the same also if it is computed from a torsion test.

But this statement is invalid for „classical“ method of the torsion test evaluation. S_σ is less, S_ϵ is greater than S_σ and S_ϵ calculated from the tensile / compressive test. I am sure that this disproportion can be eliminated using new, more accurate method.

It was shown [1, 2], that the main strain components cannot be calculated from the Cauchy's equations in the zone where large plastic strain is present. But this main strains can be determined by some means of the analytical geometry.

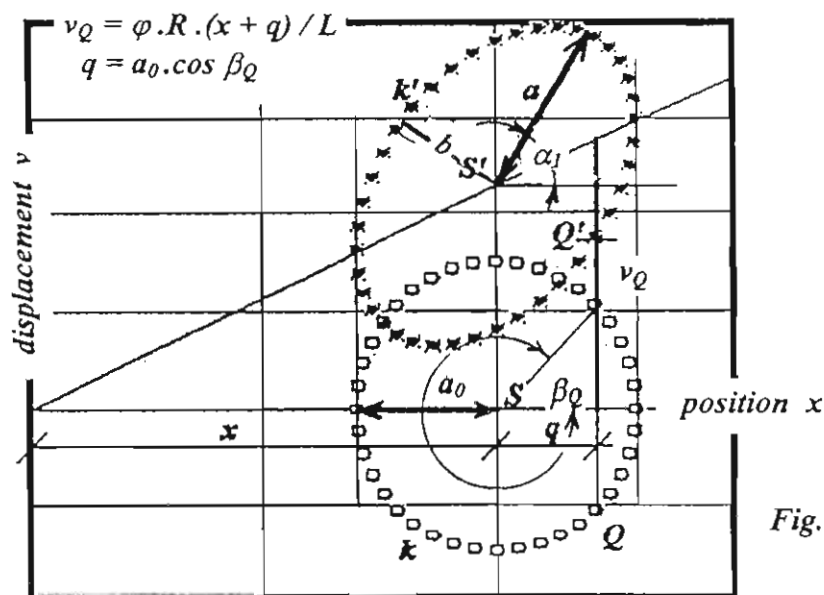


Fig. 1 - Displacement v versus position x

DISPLACEMENT. The cylindrical specimen of the length L , radius R_l is twisted in angle φ , thus some general point S is displaced in the displacement $\mathbf{v}$ to the new position S' :

$$(1) \quad v = \frac{R \cdot \varphi}{L} \cdot x = Z \cdot x \quad \dots \quad R \in \langle 0; R_l \rangle.$$

Let us consider a fictive circle k - with its centre S , radius a_0 - on the surface of the cylinder incorporated into the specimen. Consequently torque, this circle changes into ellipse k' . Axes of these ellipse a, b enable to determine main strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and also to compute the directions of these main strain axes - angles $\alpha_1, \alpha_2, \alpha_3$.

MAIN STRAIN ε_1 . It can be deduced a dependence of the angle α_1 as a function of Z parameter:

$$(2) \quad \alpha_1 = \arctg \left(\frac{Z}{2} + \sqrt{\frac{Z^2}{4} + 1} \right)$$

and also another dependence of the ε_1 as a function of Z :

$$(3) \quad \varepsilon_1 = \lim_{a_0 \rightarrow 0} \left(\ln \frac{a}{a_0} \right) = \frac{1}{2} \cdot \ln \frac{2 + \frac{Z^2}{2} + Z \cdot \sqrt{\frac{Z^2}{4} + 1}}{2 + \frac{Z^2}{2} - Z \cdot \sqrt{\frac{Z^2}{4} + 1}}$$

where parameter Z depends on L, φ and generally on the distance R of the S point to the specimen axis:

$$(4) \quad Z = \frac{R \cdot \varphi}{L}.$$

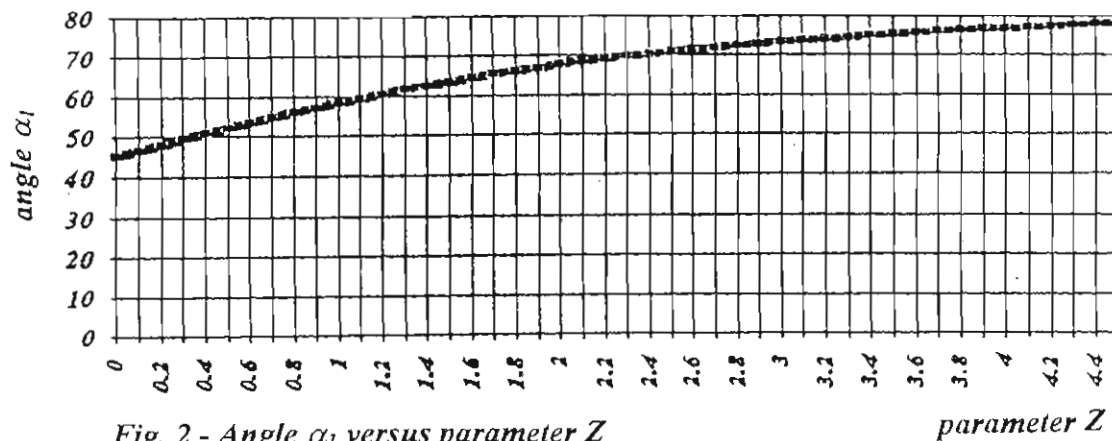


Fig. 2 - Angle α_1 versus parameter Z

From (2),(3) it is clear, that α_1 and ε_1 are non-linear functions of the parameter Z .

Equation (3) informs us that ε_1 changes its direction during plastic strain development. Initially the direction of the ε_1 lies in angle $\alpha_1 = 45$ degrees to the specimen axis - as it is known also from the basic course of the strength of material. This angle increases during plastic strain evolution, the limiting value achieves 90 degrees for the infinitely great twist angle φ . Equations (2) and (3) substitute Cauchy's (geometrical) equations and they are satisfied both for small and large strain.

The specimen volume is constant during plastic straining, thus $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ and because of condition $\varepsilon_3 = 0$ it is possible to write formula for effective strain.

EFFECTIVE STRAIN (STRAIN INTENSITY) - S_ϵ . The strain intensity S_ϵ can be calculated from this term:

$$(5) \quad S_\epsilon = \frac{\sqrt{2}}{3} \cdot \sqrt{(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2} =$$

$$= \frac{1}{\sqrt{3}} \cdot \ln \frac{2 + \frac{Z^2}{2} + Z \cdot \sqrt{\frac{Z^2}{4} + 1}}{2 + \frac{Z^2}{2} - Z \cdot \sqrt{\frac{Z^2}{4} + 1}} \quad \dots \text{because of } \epsilon_2 = -\epsilon_1; \quad \epsilon_3 = 0$$

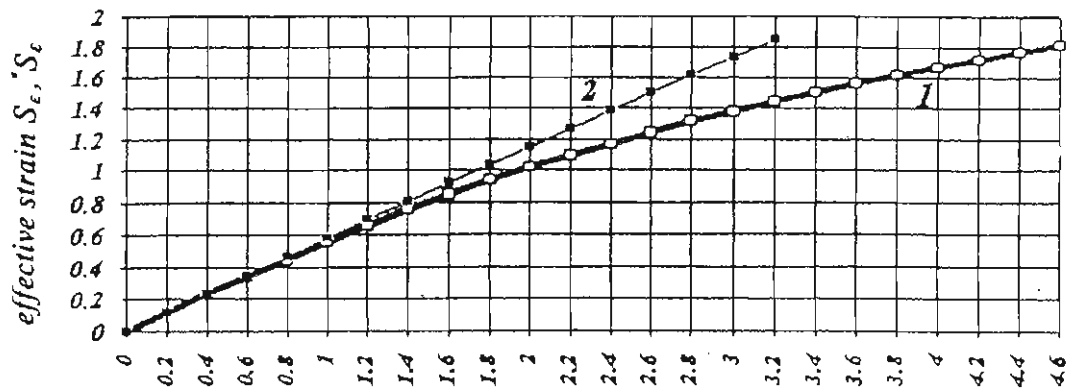


Fig. 3 - Effective strain S_ϵ versus parameter Z parameter $Z = R \cdot \phi / L$
1 - new method; 2 - classical method

S_ϵ is also non-linear function of the twist angle ϕ (or Z). Also the classical method of $*S_\epsilon$ calculation has been depicted in fig. 3:

$$(6) \quad *S_\epsilon = \frac{1}{\sqrt{3}} \cdot \frac{R \cdot \phi}{L}.$$

The curve 1 reflects the new method, the line 2 is drawn for the classical one. Comparing equations (5) and (6) it can be said that the classical $*S_\epsilon$ gives very close value to the S_ϵ but only for little (elastic / plastic) strain; however the error of the $*S_\epsilon$ value in the large strain region has been too high.

EFFECTIVE STRAIN RATE (STRAIN RATE INTENSITY) - S_u . Strain rate intensity is defined by the formula: $S_u = dS_\epsilon/dt$, where t represents time. Thus:

$$(7) \quad S_u = \frac{\partial S_\epsilon}{\partial Z} \cdot \frac{\partial Z}{\partial \phi} \cdot \frac{\partial \phi}{\partial t} = \frac{R \cdot \omega}{\sqrt{3} \cdot L \cdot \sqrt{\frac{Z^2}{4} + 1}}.$$

It is clear that S_u varies in dependence on the strain growth. Fig. 4 compares the ratio $S_u / *S_u$ where $*S_u$ has been calculated using the "classical" method formula:

$$(8) \quad *S_u = \frac{R \cdot \omega}{\sqrt{3} \cdot L}.$$

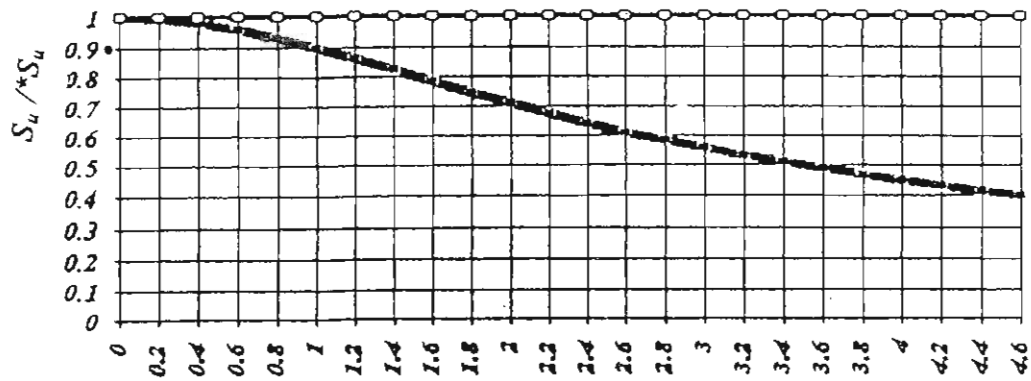


Fig. 4 - S_u / S_u^* versus parameter Z parameter $Z = R \cdot \varphi / L$

Comparing (8) and (7) it can be said that the S_u value is not constant nor yet for a constant value of the angular velocity ω .

STATE OF STRESS is bounded with the state of strain by the constitutive equation $S_\sigma = f(S_\varepsilon)$. The directions of the main stresses $\sigma_1, \sigma_2, (\sigma_3=0)$ are the same as the main strains ones $\varepsilon_1, \varepsilon_2, (\varepsilon_3=0)$. Because ε_1 changes its direction according to α_1 angle - see equation (2) - also σ_1 changes its direction in the same style. This situation is depicted in fig.5.

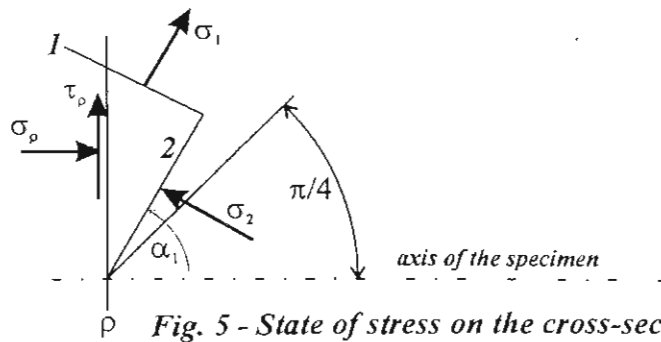


Fig. 5 - State of stress on the cross-section plane ρ

Components of the stress σ_ρ and τ_ρ are present on the cross-section plain ρ :

$$(9) \quad \sigma_\rho = \sigma_1 \cdot \cos^2(\alpha_1) + \sigma_2 \cdot \cos^2(\alpha_2) + \sigma_3 \cdot \cos^2(\alpha_3)$$

$$(10) \quad \tau_\rho = \{ \sigma_1^2 \cdot \cos^2(\alpha_1) + \sigma_2^2 \cdot \cos^2(\alpha_2) + \sigma_3^2 \cdot \cos^2(\alpha_3) - \left[\sigma_1 \cdot \cos^2(\alpha_1) + \sigma_2 \cdot \cos^2(\alpha_2) + \sigma_3 \cdot \cos^2(\alpha_3) \right]^2 \}^{1/2}$$

where α_1 value is computed from (2), together with $\alpha_2 = \alpha_1 + \pi/2$, $\alpha_3 = 0$. Using equations (2, 3, 9, 10, 13) it is possible to deduce:

$$(11) \quad \sigma_\rho = \frac{S_\sigma}{\sqrt{3}} \cdot \left(1 - 2 \cdot \frac{1 + \frac{Z^2}{2} + Z \cdot \sqrt{\frac{Z^2}{4} + 1}}{2 + \frac{Z^2}{2} + Z \cdot \sqrt{\frac{Z^2}{4} + 1}} \right)$$

$$(12) \quad \tau_{\rho} = \frac{S_{\sigma}}{\sqrt{3}} \cdot \left(2 \cdot \frac{\frac{Z}{2} + \sqrt{\frac{Z^2}{4} + 1}}{2 + \frac{Z^2}{2} + Z \cdot \sqrt{\frac{Z^2}{4} + 1}} \right)$$

where the effective stress S_{σ} is defined by the formula:

$$(13) \quad S_{\sigma} = \frac{1}{\sqrt{2}} \cdot \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{3} \cdot \sigma_1$$

... because of $\sigma_2 = -\sigma_1; \quad \sigma_3 = 0$

It is clear that σ_{ρ} and τ_{ρ} varies in dependence upon the strain development. First - S_{σ} depends on the S_{ϵ} (and S_{ϵ} depends on the Z parameter). Second - despite of the S_{σ} - S_{ϵ} dependence - the share stress τ_{ρ} (on the cross-section plain ρ) varies due to rotation of the main stress σ_1 (in consequence of the angle α_1 change) thus τ_{ρ} varies also in dependence upon the Z parameter. This dependence is depicted in fig.6, where values $\sqrt{3} \cdot \sigma_{\rho} / S_{\sigma}$ and $\sqrt{3} \cdot \tau_{\rho} / S_{\sigma}$ versus Z parameter have been displayed. Both $\sqrt{3} \cdot \sigma_{\rho} / S_{\sigma}$ and $\sqrt{3} \cdot \tau_{\rho} / S_{\sigma}$ are non linear functions of Z parameter.

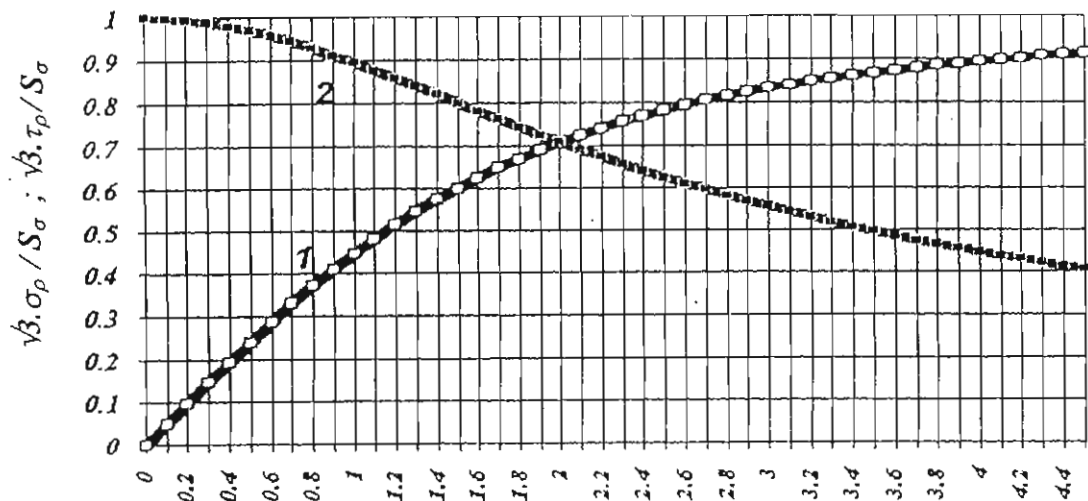


Fig. 6 - $\sqrt{3} \cdot \sigma_{\rho} / S_{\sigma}$ (curve 1) and $\sqrt{3} \cdot \tau_{\rho} / S_{\sigma}$ (curve 2) versus parameter Z

Measuring torque M_k then value of the $\tau_{\rho}(S_{\epsilon}, Z)$ must in every moment fit equation of equilibrium:

$$(14) \quad M_k = 2 \cdot \pi \cdot \int_{R_0}^{R_1} R^2 \cdot \tau_{\rho} \cdot dR$$

where R_0 is the inside radius of the specimen.

APPLICATION. A cylindrical specimen with hollow - external diameter $2R_1=8.8$ mm, inside diameter $2R_0=6$ mm, length $L=3.2$ mm - is twisted by 500 revolutions per minute. The specimen (made from the steel grade 14109 - by the ČSN standard) was tested under temperature $T=20^{\circ}\text{C}$. Measured torque M_k versus time t has been depicted in fig.7. Measured twist angle φ versus time t has been displayed in fig. 8.

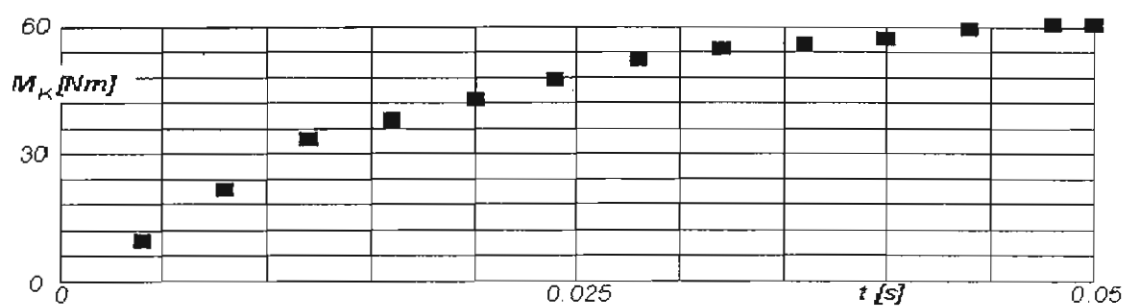


Fig. 7 - Torque M_k versus time t

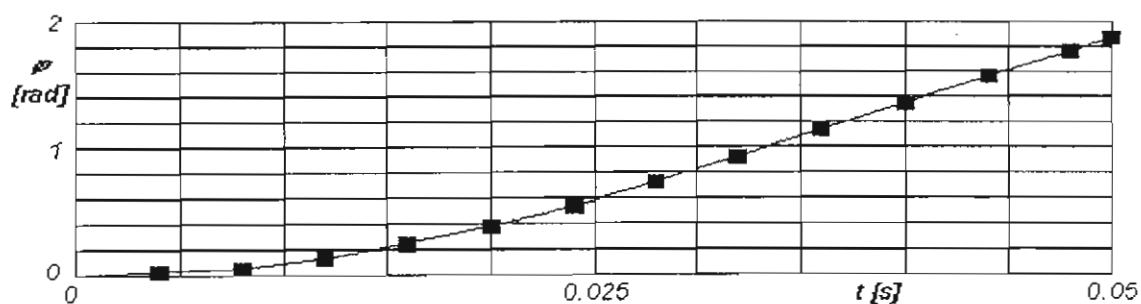


Fig. 8 - Twist angle α_l versus time t

Computed $S_\sigma - S_\epsilon$ dependence has been drawn in fig.9. Differences between computed M_{kvyt} and measured M_k torque are clear from fig.10.

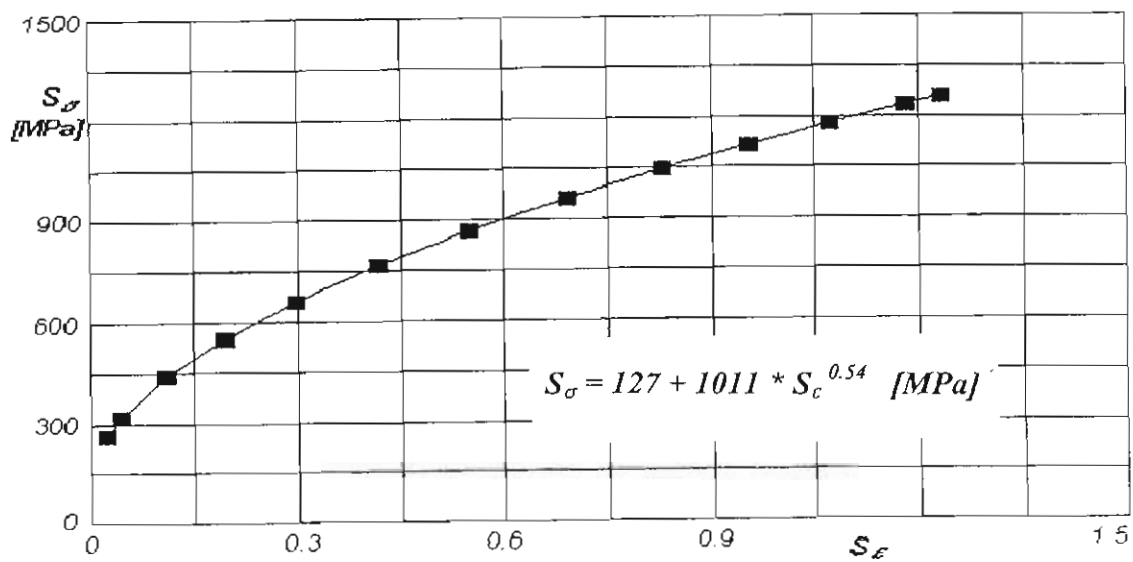


Fig. 9 - S_σ versus S_ϵ (for 14 109 steel grade by ČSN)

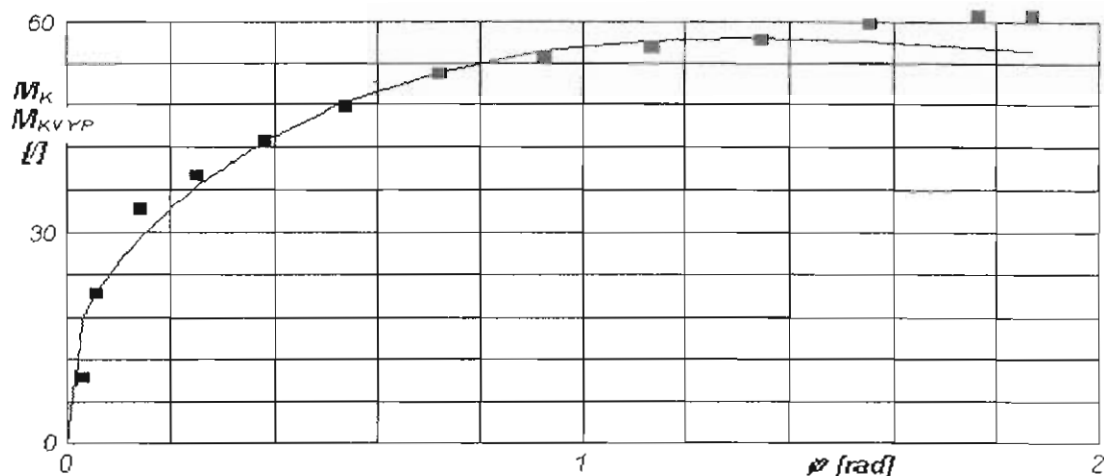


Fig. 10 - Torque M_k , M_{kvyp} versus twist angle φ

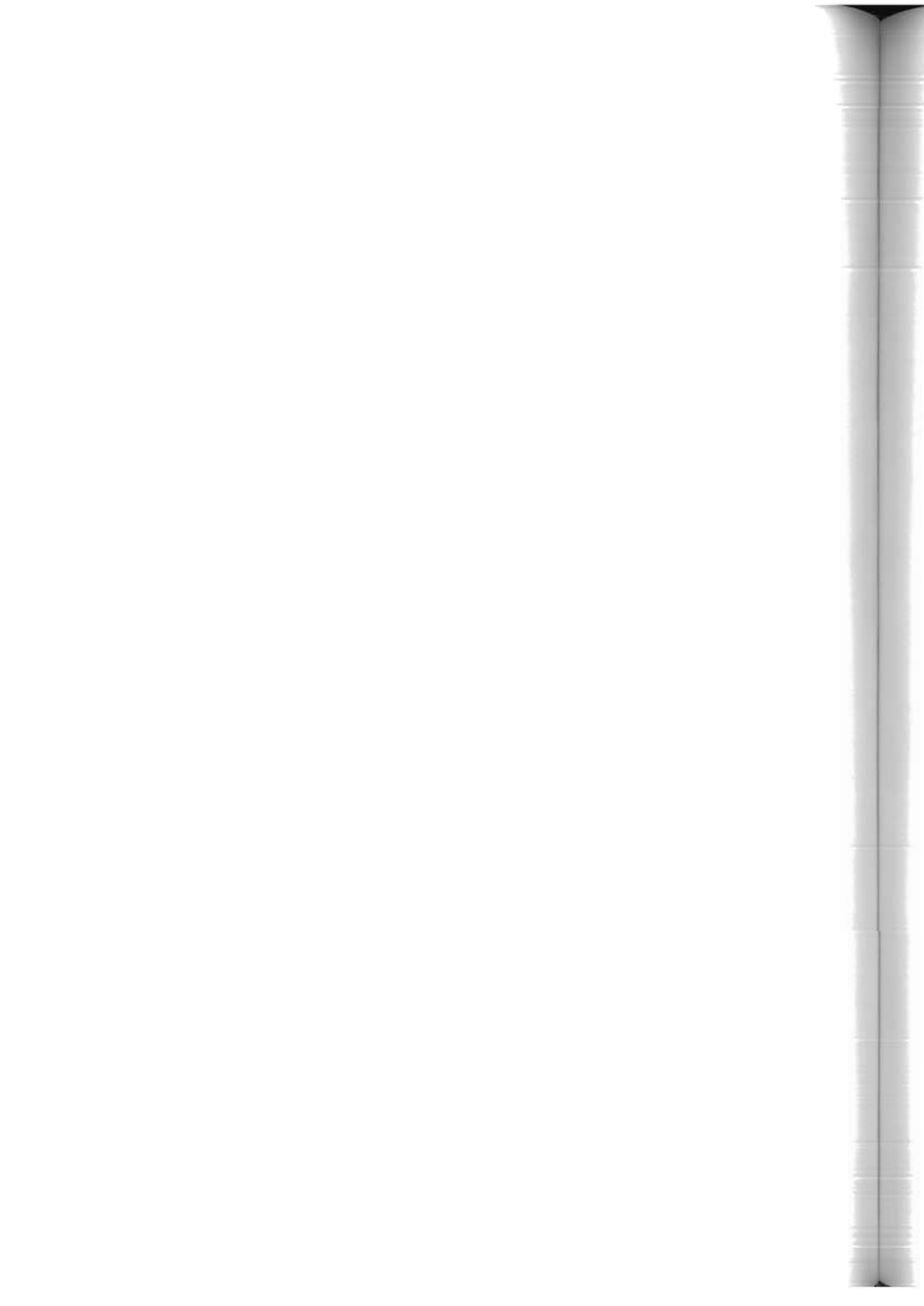
CONCLUSIONS

- 1) New method has been derived for the strain and strain rate parameter calculation. This method replaces „classical“ one (see equations 6, 8), based on the Cauchy's geometrical equations.
- 2) The direction of the main strain ε_I varies in angle α_I - see equation (3), fig.2.
- 3) This main strain ε_I rises in accordance with equation (3). The effective strain S_ε can be calculated from equation (5) - see also fig. 3.
- 4) The effective strain rate S_u is non-linear function of the parameter Z - see equation (7). S_u varies despite the angular velocity ω can be constant - see equation (7) and fig. 4.
- 5) Shear stress τ_ρ in the cross-section plain ρ (perpendicular to the specimen axis) varies with the development of the Z parameter - see (12), fig. 6.
- 6) Using this new method S_σ - S_ε dependence has been computed - see fig.9 [4].
- 7) Differences between computed M_{kvyp} and measured M_k torque are clear from fig.10, where computed curve $M_{kvyp}=f(\varphi)$ has been drawn together with measured points of the torque M_k .

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EXPERIMENTAL DETERMINATION OF CREEP PARAMETERS IN NUMERICAL CALCULATIONS

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Key words: plastic deformation, finite element method, creep, relaxation

ABSTRACT

Tensile test experiments under force control and under displacement control have been carried out with the force changes and time shift recorded. Both with loading under force control and with loading under displacement control, short-term stops (of 5 minutes each) at a specified load level have been realised. At the same time relaxation (creep) of the stressed beam has been observed.

In a specific way these data have been applied to the analysis of creep when simulating this experiment by the finite element method.

The analysis showed good accordance between calculated and measured values.

The finite element method analysis takes particularly into consideration the influence of the true stress-strain curve and deformation-time curve with constant force and stress-time curve with constant elongation of specimen. Comparison with measured values makes it possible to determine parameters which define creep by numerical calculation.

1 Introduction

The fact is that when compressive action of the force is relieved elastic relaxation of the plastically deformed workpiece takes place.

Determination of: mechanical properties of material, deformation rate and duration, magnitude of the stamp compressive force, depth of plastic deformation and tool geometry, all play an important part in the final form of the workpiece.

This paper analyses several of the factors that may have effect on the above mentioned factors. It is the aim of the paper to offer, first of all, a preliminary calculation by finite element method. Such a calculation can be used to define, with acceptable reliability, within specified limits, several variables important for plastic deformation.

2 Mechanical properties of material

2.1 True stress-strain curve

Several specimens have been tested for this purpose. Force and elongation have been measured in a standard way while a special gauge has been used to measure the round specimen decrease of diameter during the test. Transversely, these measurements were performed, first of all, to determine the Poisson's factor and define the current diameter of the loaded specimen so that the true stress could be calculated later.

From hot rolled steel plate TStE 500, round specimens were made 6 mm in diameter and 30 mm in length. Mean values are shown in the diagram in **figure 2.1**. Mean value of the Poisson's factor $\mu = 0,32$ and of the module of elasticity $E = 206855 \text{ N/mm}^2$.

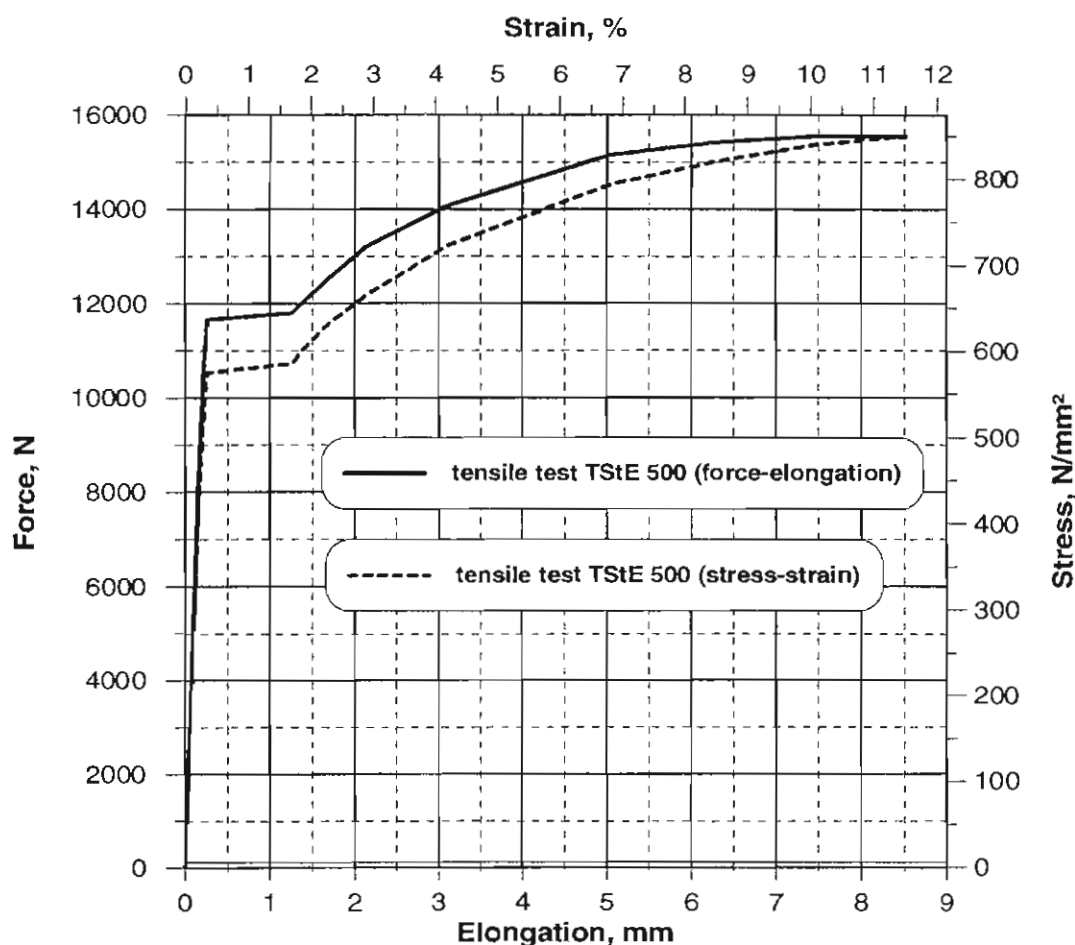


Figure 2.2 Mean value of the force-elongation and stress-strain curves

According to the experiments and based on the measuring of the specimen's transverse deformations, the true stress can be calculated as follows [4,5]:

$$\sigma_w = \frac{F}{A_0} \left(1 + \frac{\Delta l}{l_0} \right) = \sigma (1 + \epsilon_x) \quad (2.1)$$

This expression was later used whenever it was necessary to convert conventional to true stress. For technical reasons it was not possible to measure transverse deformation when determining relaxation parameters so that the true stresses in the direction of the specimen's longitudinal axis were calculated by means of the above given expression.

2.2 Parameters of relaxation (creep)

As is known from experience, time period i.e. rate of plastic deformation affects the extent of permanent deformation [3]. To determine these parameters is a complex task of elasto-plasticity. Due to a number of influential factors, it was necessary to limit the investigations to those parameters with the strongest effect on the calculations. In the stressed area above the yield point and below tensile strength two states were investigated.

In order to determine the material relaxation parameters 8 rectangular specimens were made (10x2 mm) from the direction of plate rolling and the direction transverse to the direction of rolling. As the plate was in normalized state, it was found that the differences

in the values of mechanical properties could be nearly neglected and were in fact indistinguishable due to the limited accuracy of the measuring instruments (1% acc.).

2.2.1 Relaxation (creep) with constant force

These experiments were performed in such a way that during the tensile test the rise of force was stopped and kept so for 5 minutes in the area above the yield point.. The tensile test was carried out "under the force control". Force, elongation and time were measured. By the equation (2.1) conventional stress was converted into true stress. A part of the force - elongation curve is shown in **figure 2.2**, while the detail "A" is shown enlarged in **figure 2.3** and in the form of the elongation - time curve in **figure 2.4**.

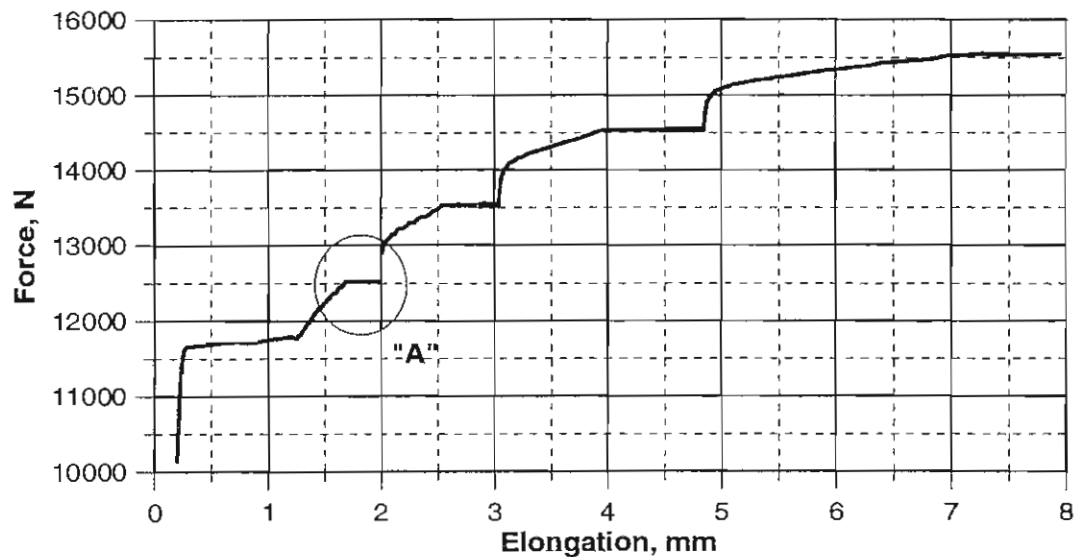


Figure 2.2 - Part of the force - elongation curve (test under force control)

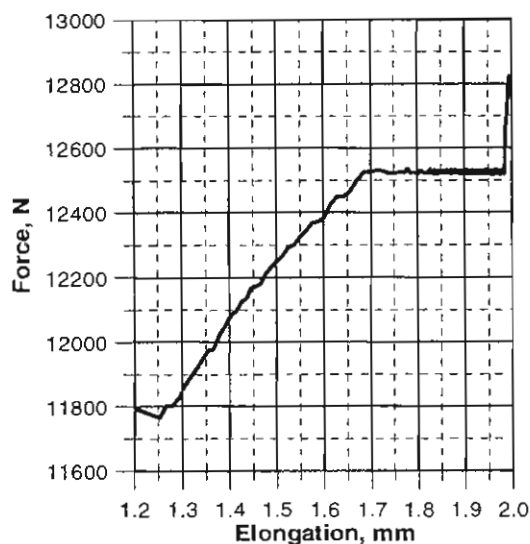


Figure 2.3 - Force - elongation curve (Detail "A" in figure 2.2)

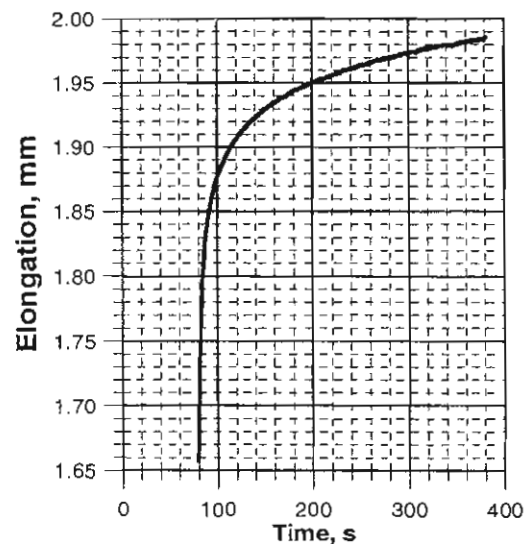


Figure 2.4 - Elongation - time curve (Detail "A" in figure 2.2)

2.2.2 Relaxation (creep) with constant elongation

These experiments were performed in such a way that during the tensile test the increase of elongation was stopped and kept so for 5 minutes in the area above the yield point. The tensile test was carried out "under the displacement control". Force, elongation and time were also measured.

A part of the force - elongation curve is shown in **figure 2.5** while the detail "A", greatly enlarged, shown in **figure 2.6**, is shown in the form of the elongation - time curve in **figure 2.7**, and in the form of the force - time curve in **figure 2.8**.

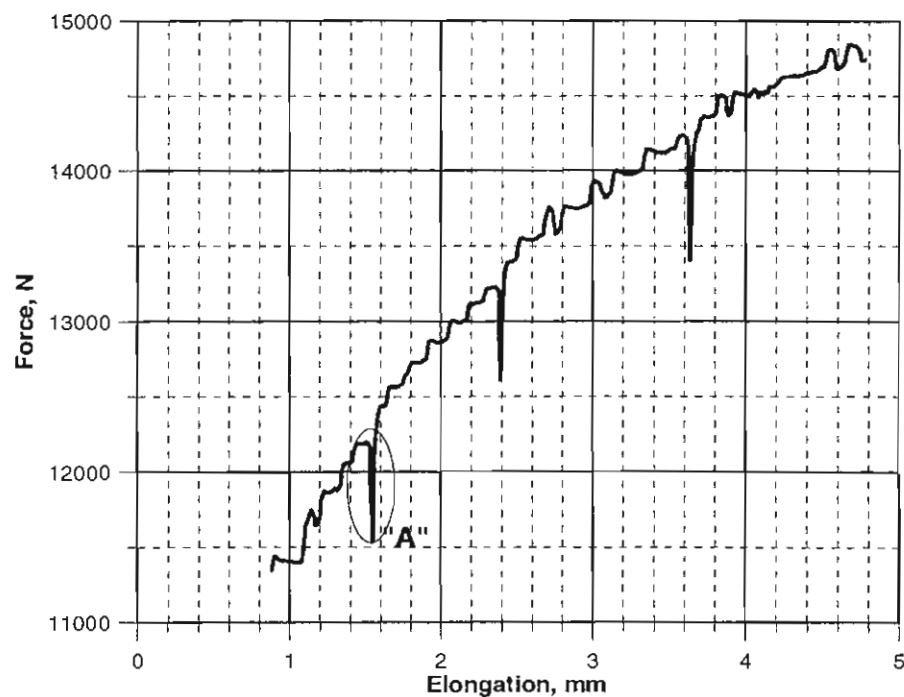


Figure 2.5 - Part of the Force - elongation curve (Test under displacement control)

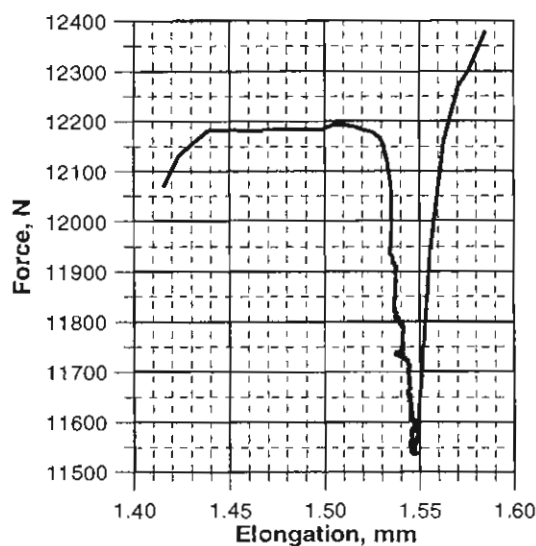


Figure 2.6 - Force - elongation curve (Detail "A" in figure 2.5)

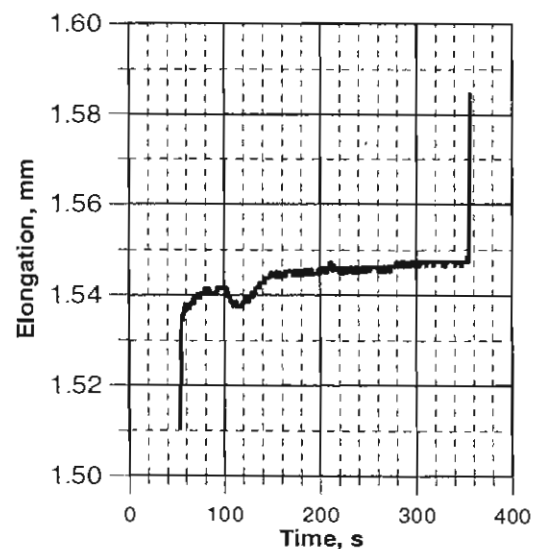


Figure 2.7 - Elongation - time curve (Detail "A" in figure 2.5)

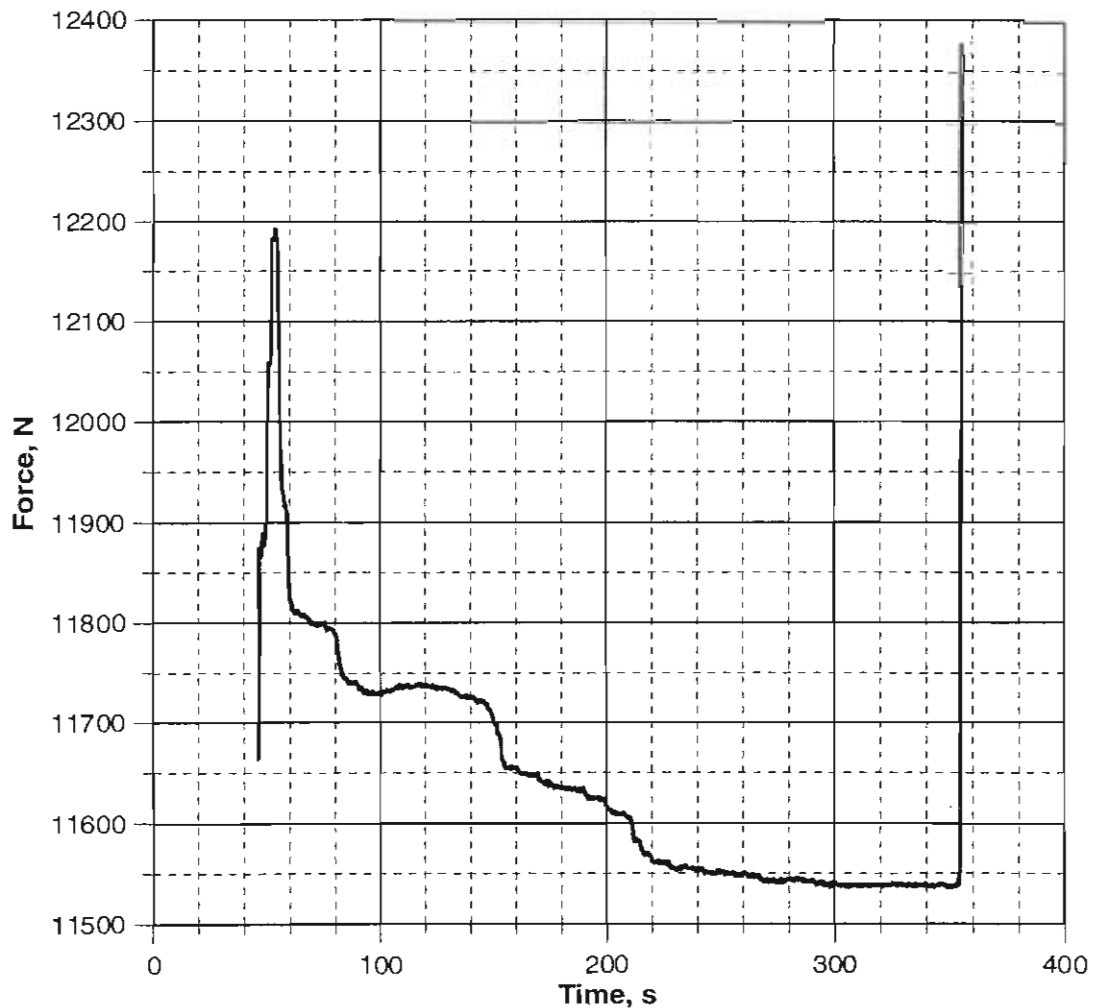


Figure 2.8 - Force - time curve(Detail "A" in figure 2.5)

3 Numerical calculation of creep by finite element method

FEM numerical calculation considering:

- large plastic deformations,
- effect of hardening,
- true stress - strain curve,
- isotropic properties of materials and
- creep (relaxation during slow deformation processes)

is a complex task. Exceptionally wide experience in the use of program packages on the one hand and a more powerful computer on the other, are necessary for the calculations of this kind.

The present calculations were carried out on PC Pentium P133 40 MB RAM and by ANSYS 5.3 for WIN95. Some simplifications had to be made.

1. The orthotropic features of materials were considered in view of the negligible differences in mechanical properties in the direction of and transversally to the direction of rolling. Thus many other difficulties connected with the limitations of the program were avoided.

2. Inclusion of the effect of creep (relaxation) due to successive and slow loading (as in practice) was possible in ANSYS program in several ways [1,2]. Only some of 15 different approaches to calculate creep step are given:

$$\Delta \epsilon_{cr} = C_1 \sigma^{C_2} \epsilon^{C_3} e^{\frac{-C_4}{T}} \Delta t \quad [\text{m/m}] \quad (3.1)$$

or

$$\Delta \epsilon_{cr} = C_1 \sigma^{C_2} t^{C_3} e^{\frac{-C_4}{T}} \Delta t \quad [\text{m/m}] \quad (3.2)$$

or

$$\Delta \epsilon_{cr} = C_1 \sigma^{C_2} r \cdot e^{-r \cdot t} \Delta t \quad [\text{m/m}] \quad (3.3)$$

Where:

$C_1 \dots C_5$ are constants (determined by experiment)

ϵ is the equivalent total deformation with the existing creep deformation included [m/m].

σ equivalent stress (von Mises) [N/m²]

T absolute temperature [K]

t time at the end of substep [s]

e base of natural logarithm and

$$r = C_5 \sigma^{C_3} e^{-\left(\frac{C_4}{T}\right)} \quad (3.4)$$

Precision of the calculation greatly depends on selected time increment t and certainly on constants C_1 to C_5 which are determined by experiment. For this purpose tests were conducted as shown in the figures in the previous chapter.

To prove the validity of these constants the tensile test was simulated by finite element method, a time consuming and tedious job with the first investigations.

It was shown that the expression (3.1) could apply to TStE 500 material in the area from 0,1% to 6% total deformation. The same holds good for the expression (3.3) which, with slight modifications, gives even better results.

To obtain easier the required values of the constants the diagram demonstrating the expression (3.3) (**figure 3.1**) was used with the following values of the constants:

$$C_1 = 1 \cdot 10^{-7}; \quad C_2 = C_3 = 0,64; \quad C_4 = 1000; \quad C_5 = 1,9 \cdot 10^{-7};$$

as a function of time t for different values T (here T is not considered as temperature but as a parameter dependent on the art of the stress state). Thus, depending on the point in the stress-strain diagram, the following is valid:

Equivalent stress [N/mm ²]	Parameter T
do 625	800
625 do 675	400
674 do 725	300

To simplify the determination of constants C_1 to C_5 the above given values were kept only while the constant C_1 was investigated, this constant being in fact the general multiplier in expression (3.3), while the results of the numerical calculation where $C_1 = 0,35 \cdot 10^{-7}$; are shown in **diagrams 3.2** and **3.3**.

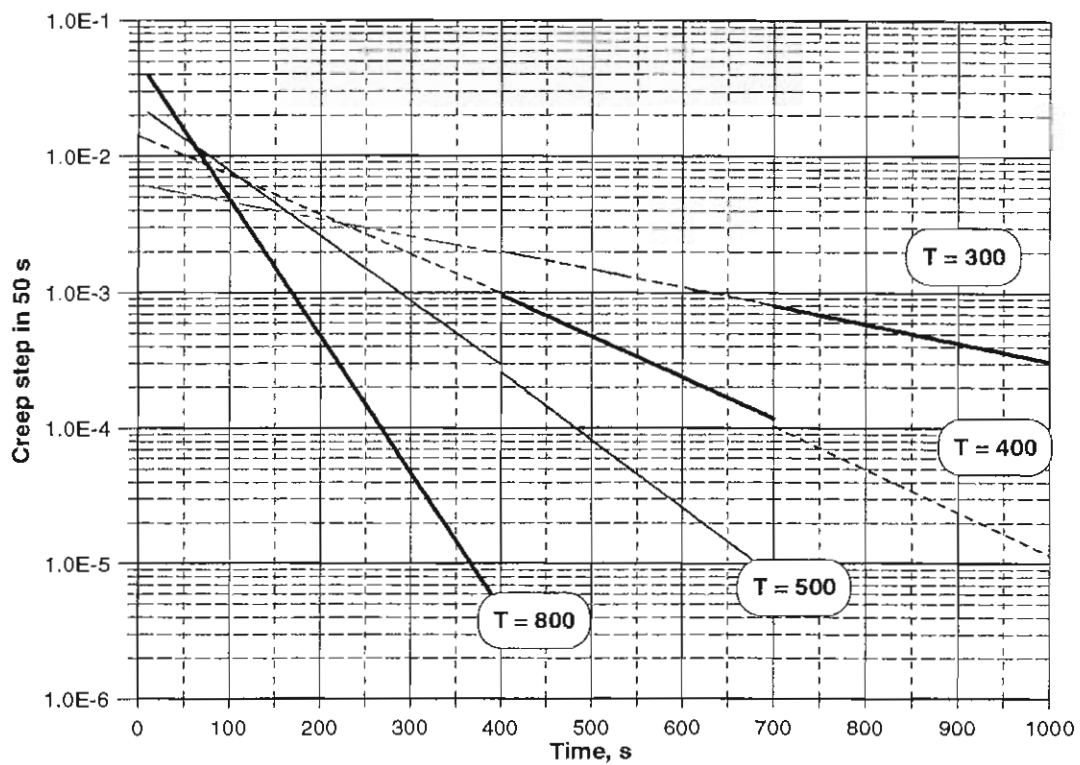


Figure 3.1 - Diagram of creep step within time increment 50 s

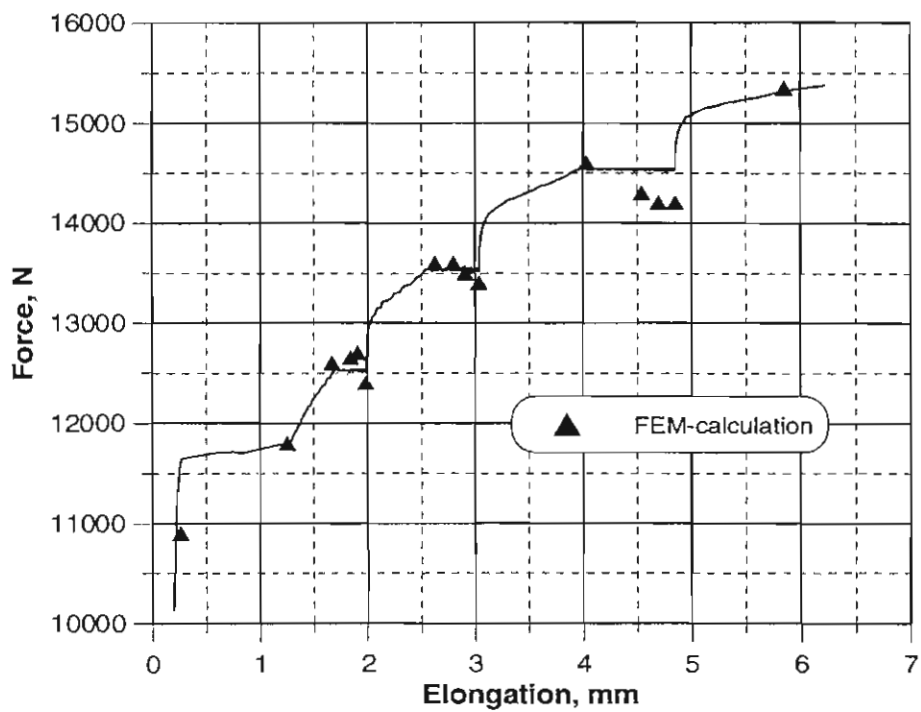


Figure 3.2 Force - elongation curve (FEM calculation under force control)

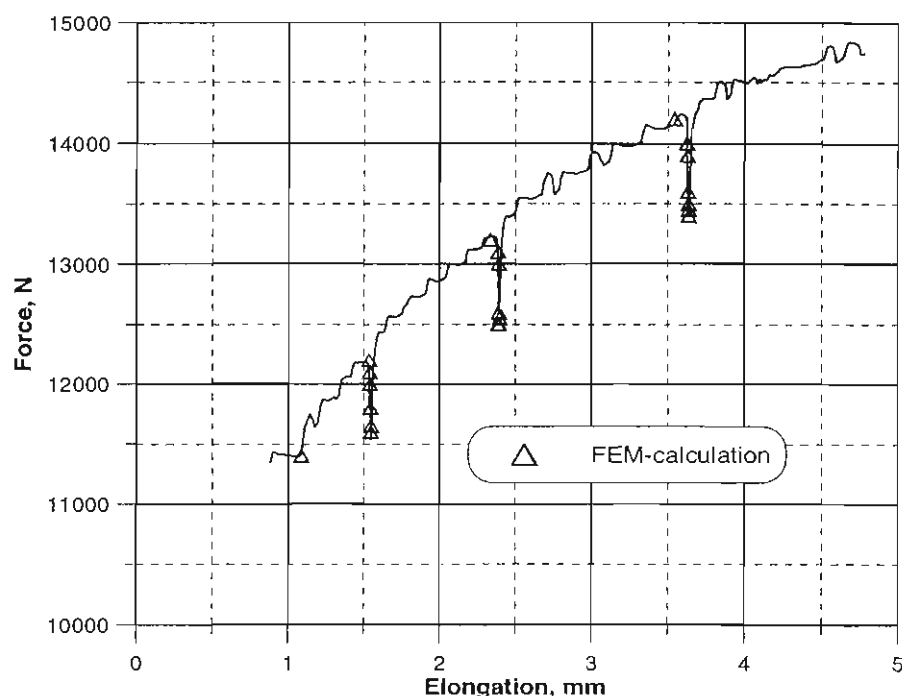


Figure 3.3 Force - elongation curve (FEM calculation under displacement control)

4 Discussion and conclusions

The FEM numerical calculation taking into account relaxation and true stress-strain curve shows satisfactory accordance with the measured values.

The deviations that appear in the diagrams are the result of the FEM precision on the one hand and of the included simplifications on the other hand.

The results obtained show that it is advisable to take the following into account when making similar calculations:

- as many true properties of materials as possible, determined by corresponding tests,
- use as complex a program as possible for the FEM calculation and
- use a powerful and fast computer as the calculations might have to be repeated many times.

Definition of the manufacturing parameters for plastic metal forming could be made considerably easier and faster by such calculations.

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INVERSE IDENTIFICATION OF THERMO-RHEOLOGY PARAMETERS

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SLOVENIA

The modern material science is largely engaged in processing of advanced materials. Since the behaviour of such materials is usually characterized by complex relationships an efficient material characterization is demanded, along with the material processing. Accordingly, special experimental techniques and material identification methods are developed. This paper considers the characterization of rheological properties of solid materials as determined by a numerical solution of the corresponding inverse problem. In particular, bending of an elastoplastic beam, heated initially in order to obtain a nonuniform temperature field in it, is considered.

1. INTRODUCTION

It has been long recognized that materials can be properly characterized only by adequate laboratory experiments. Often, due to complex material behaviour different specific experiments, limited usually to simple loading cases and specimen geometries, are needed for a proper characterization of the same material. Usually, the laboratory experiments used for the corresponding material characterization are designed in such a way, that only one or two material properties can be identified with one experiment. Consequently, several experiments of different kind have to be performed in order to obtain complete material characterization. Not always, however, the complete characterization is achieved. In contrast to this experimental experience, any computer based approach that is aiming at loosening of the severity of conditions imposed in an experiment is most welcome. Such approach would hopefully lead to possible simplification of the experimental set-up itself and to reduction of individual experiments needed for the complete material characterization.

In the evaluation of the experimental data the computer assistance has been present for a rather long time, its role in the material characterization being most significant. Nowadays, when considering recent advances in computational techniques for the solution of inverse problems, this assistance becomes even more firm and efficient. By introducing computer simulations to resimulate a real complex experiment, a powerful means for the material characterization is established. Namely, when performing a computer simulation under the same conditions, as met in a real experiment, it is possible, by imposing the equivalence of the computed and measured responses, to identify material properties.

2. PROBLEM DEFINITION

In technical applications we often meet problems that are characterized either by transient or steady state nonuniform temperature fields matched with severe plastic deformation fields. The temperature dependence of various mechanical properties is of particular interest in such problems. A method for the identification of physical properties that influence the

mechanical state in a deformed solid body will be presented in the sequel.

Let presume that the actual temperature field $T=T(x_k, t)$ in a solid, subjected to transient heat conduction, is known, x_k and t being the space and time coordinate, respectively. The evolution of the corresponding mechanical state characterized by the stress and strain tensor $\sigma_{ij}(x_k, t)$, $\epsilon_{ij}(x_k, t)$ and displacement vector $u_i(x_k, t)$ is given by the solution of the following system of differential equations

$$\sigma_{ji,j} + f_i = 0 \quad (1)$$

$$d\epsilon_{ij} = \frac{1}{2} (du_{i,j} + du_{j,i}) \quad , \quad x_k \in \Omega \quad (2)$$

$$d\sigma_{ij} = H_{ijkl} d\epsilon_{kl} - \frac{E\alpha}{1-2\nu} dT \delta_{ij} \quad (3)$$

taking the associated boundary conditions into account. Equations (1), (2) and (3) represent respectively the stress equilibrium, geometry of deformation and rheology of the problem. Here, f_i represents body forces, H_{ijkl} material properties tensor, E modulus of elasticity, α coefficient of thermal dilatation, ν Poisson's ratio and δ_{ij} Kronecker tensor. Along with the loading path dependence, which characterizes any nonlinear material response, the material physical properties, be mechanical or thermal, show also a pronounced temperature dependence. The number of independent material coefficients for an isotropic and elastoplastic material, necessary for describing tensor H_{ijkl} , reduces to Poisson's ratio $\nu=\nu(T)$, Young's modulus $E=E(T)$, initial yield stress $R_p=\sigma_v=\sigma_v(T)$ and instantaneous tangential modulus $E_T=d\sigma/d\epsilon=E_T(\epsilon, T)$, where all quantities refer to the uniaxial stress state. As a quantity which incorporates effects of the previous deformation history, quantitatively measured by equivalent plastic deformation ϵ , the tangential modulus can be determined only from temperature dependent yield curves $\sigma=\sigma(\epsilon, T)$.

A computer aided identification of the temperature dependence of any rheological property is based on an assumed functional structure $F=F(T)$ with temperature T as independent variable. Any mathematical form of a continuous function F can be assumed in principle, but the degree of agreement with the real material behaviour depends on the number of free parameters in it. In the considered case, the sectionally linear approximation of function F is assumed, and accordingly, the temperature domain of interest $[T_s, T_f]$ is divided into m number of intervals

$$[T_s, T_f] = [T_0, T_1] \cup (T_1, T_2] \cup \dots \cup (T_{m-1}, T_m] \quad , \quad T_s \equiv T_0 \quad , \quad T_f \equiv T_m \quad (4)$$

In accordance with the assumed function approximation the value of $F(T)$ at specific temperature $T \in [T_{i-1}, T_i]$ can be expressed as

$$F(T) = F^{i-1} + \frac{F^i - F^{i-1}}{T_i - T_{i-1}} (T - T_{i-1}) \quad , \quad 1 \leq i \leq m \quad (5)$$

where coefficients F^{i-1} and F^i are the corresponding values of function $F(T)$ at the borders of the i -th interval.

In the material characterization process the parameters specifying the respective function approximations of thermal dilatation $F(T) \equiv \alpha(T)$, Poisson's ratio $F(T) \equiv \nu(T)$, modulus of elasticity $F(T) \equiv E(T)$, as well as yield stress and yield curves $F(\epsilon, T) = \sigma(\epsilon, T)$, are material constants which are to be denoted, in the sense of equations (4) and (5), as $\alpha^0, \alpha^1, \dots, \alpha^a$,

$v^0, v^1, \dots, v^b, E^0, E^1, \dots, E^c$ and $\sigma^0, \sigma^1, \dots, \sigma^d$ with integers a, b, c and d representing the corresponding numbers of intervals for the respective function approximations. All function coefficients are assembled, for purposes of further mathematical manipulation, in a material parameters vector p

$$p^T = [p_1, \dots, p_n] = [\alpha^0, \dots, \alpha^a, v^0, \dots, v^b, E^0, \dots, E^c, \sigma^0, \dots, \sigma^d] \quad (6)$$

where n is the total number of coefficients to be identified.

3. METHOD OF PARAMETER ESTIMATION

A relationship that governs the response of a system, which is subjected to applied actions, can be in general described by operational equation of the form

$$Ax = f \quad (7)$$

When the structure of operator A is not known the problem of material characterization consists first of the identification of the governing rule, and afterwards, the structure of operator A being identified, it is followed by the estimation of material parameters that determine the system behaviour quantitatively. Since in our case the governing rule is adopted (3) the task is to estimate the unknown material parameters assembled in vector p , and the problem can be classified as an inverse problem of the second kind.

In the problem of material characterization a solution of the inverse problem is actually following solution strategies used in extremum problems [1]. In the identification / estimation problems responses of two systems are considered, one being related to a real physical model and the other being related to a mathematical model which is supposed to represent the behaviour of the physical model equivalently. The objective of any such problem is to enforce the response of the assumed mathematical model to be as close as possible to the physical one. Of course, this is achieved only by a proper adjustment of material parameters, which finally yields the sought material characterization. In the sequel, when referring to the considered two systems, the associated quantities will be addressed as "experimental" or "numerical".

The scalar objective function $S(p)$, which is to be minimized, is written as a matrix product

$$S(p) = [Y - U(p)]^T W [Y - U(p)] \quad (8)$$

where Y and $U(p)$ are vectors formed from a subset of the corresponding "experimental" and "numerical" quantities. The role of weighting matrix W is to introduce unequal weights due to measurement errors and due to physically different quantities. Superscript T indicates the matrix transpose. The computed response $U(p)$, which is usually obtained by the Finite Element Method analysis, is a function of material parameters vector p , assumed in the computational analysis. No regularization term, which includes vector p in an explicit manner in the objective function, is needed.

In this work, a method used for the minimization of equation (8) is the Gauss-Newton method of minimization [2]. Although more powerful methods of minimization exist, particularly when referring to the structural optimization domain [1], it seems that they are less efficient for identification purposes.

The necessary condition for $S(p)$ to be stationary is that the matrix derivative of $S(p)$ with

respect to p be equal to zero

$$\nabla_p S(p) = -2 X^T(p) W [Y - U(p)] = 0 \quad (9)$$

where the sensitivity matrix $X(p)$ is defined as

$$X(p) = [\nabla_p U^T(p)]^T \quad (10)$$

Vector p , that minimizes the considered objective function in accordance with equation (9), is obtained iteratively. With v being the iteration counter the relationship relating the responses U^v and U^{v+1} in two subsequent iteration computations can be approximated by the linearized Taylor series expansion of U in the parameter space

$$U(p^{v+1}) = U(p^v) + X(p^v)[p^{v+1} - p^v] \quad (11)$$

The substitution of this relationship in Eqn. (9) yields a set of equations

$$X^T(p^v) W X(p^v) p^{v+1} = X^T(p^v) W [X(p^v) p^v + [Y - U(p)]] \quad (12)$$

which is used as the iterative solution algorithm for obtaining a new estimate of vector p .

The search procedure for finding the extremum of objective function $S(p)$ is relying on the sensitivity analysis of each single parameter in vector p , establishing thus its influence on the calculated primary variables. Due to high implicit dependence and nonlinearity the solution is obtained iteratively. The sensitivity coefficients are calculated using finite differences, requiring thus in an n -parameter problem $(n+1)$ direct problem analyses for each iteration. The finite difference approach of the sensitivity analysis being very time consuming it should be replaced with analytical or semi-analytical methods, whenever possible.

4. NUMERICAL EXAMPLE

Bending of a beam is considered as our reference experiment, where some mechanical properties of a hypothetical material with steel-like properties are sought. The estimated properties are temperature dependent modulus of elasticity E and Poisson's ratio ν , as well as yield curves at various temperatures, preselected in a wide temperature range. Since the purpose of this paper is not to describe an experimental technique, but to point out the numerical aspects of the proposed identification method, this example has only an illustrative meaning.

4.1. PHYSICAL EXPERIMENT ARRANGEMENT

The experimental set-up (Fig. 1) consists of a 120mm long homogeneous beam sample with a uniform 5x10mm rectangular cross-section. The sample is loaded thermally and mechanically in such a way that the bending moment of constant magnitude is achieved in the part of the beam between the supports. The sensors, measuring longitudinal and transverse strains are located at positions denoted by A and B , while vertical displacement of the beam is measured in the centre of the span (point C in Fig. 1). Temperatures T_1 and T_2 of the top and bottom surfaces of the beam are measured as well, but locations of the temperature sensors are not shown in Fig. 1.

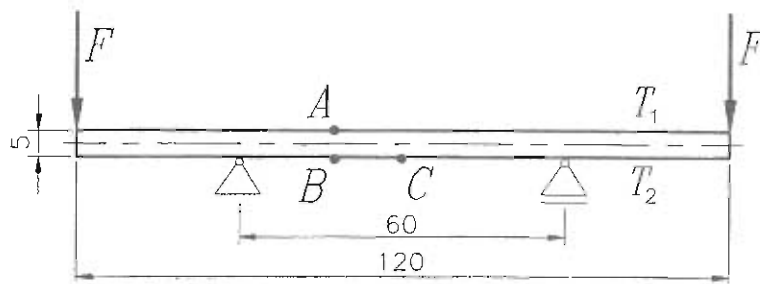


Figure 1: Experiment arrangement

The initial thermo-mechanical state at the beginning of the experiment is isothermal and stress-free. In the experiment the actions are applied in two steps. During the first step the beam is exposed to some heating until the nonuniform steady state temperature field is obtained, this steady state being assumed to not alter in the continuation of the experiment. Data, obtained from the deformation of the beam just due to thermal expansion, are used for the identification of temperature dependence of the coefficient of thermal expansion. In the second step of the experiment the beam is loaded mechanically by applying a gradually increasing load F at its' both ends, causing yielding of some parts of the beam.

4.2. EQUIVALENT NUMERICAL EXPERIMENT

The response of the physical system Y , i.e. the data acquired from the real physical experiment, represents a basis for the computer aided identification. For the investigation purposes only the material characterization, following the previously described approach, will be performed on an equivalent numerical experiment that will replace the physical one. Accordingly, the measured data in a physical experiment will be substituted by the computed "experimental" response, as provided by the direct FEM analysis of the observed system when assuming the material properties known. It is clear that, in comparison to the physical experiment, the corresponding numerical model is only an idealization, and consequently, it can not take deficiencies and imperfections that are characterizing any real experimentation into account.

The rheological properties, listed in Table 1 and plotted in Fig. 2 as thick curves, are a basis for the computed "experimental" response. At the same time they are the target values when performing the inverse identification. In the sequel they will be referred as "true" values in contrast to the "estimated" or "identified" values of material parameters that are resulting from the identification process.

TABLE 1. Rheological properties of material

T [°C]	E [10^5 MPa]	ν [/]	α [10^{-6} K $^{-1}$]
0.0	337	5.7	0.46
200.0	642	8.7	0.68
400.0	935	11.1	0.87
500.0	1216	13.0	1.10

The numerical experiment is conducted in two steps, following the same loading procedure as in the physical experiment (Fig. 1). In accordance with the imposed thermal boundary conditions, where the beam is first uniformly heated at the top surface ($T_1=500^\circ\text{C}$) and simultaneously cooled ($T_2=0^\circ\text{C}$) at the bottom surface, the linear variation of the temperatures through the beam height is supposed. In the second step with the temperature field kept stationary the two forces of magnitude $F=266\text{N}$ are applied.

From the computed "experimental" response results a subset of displacements and strains is formed to provide the "measured" data set, which is to be used subsequently in the material parameters estimation. Actually, we assume as "measured" the displacements and strains of surface points of the FE model which coincide with the sensors' locations. To thus obtained "experimental" data an artificial measurement error is added. Usually, conducting an experiment any known systematic effects are assumed to be removed, so that the remaining errors can be considered as random and their effect can be described statistically. The random error associated with strain measurements can be quantified as

$$\varepsilon_{err} = R \cdot \max[0.0003, 0.015 |\varepsilon|] \quad , \quad -1 \leq R \leq 1 \quad (13)$$

while the one associated with displacement measurement is

$$u_{err} = R \cdot \max[0.01 \text{ mm}, 0.0075 |u|] \quad , \quad -1 \leq R \leq 1 \quad (14)$$

where ε and u are the respective values of strain and displacement at the sensor location, and R is a Gaussian distributed random number.

4.3. RESULTS OF IDENTIFICATION

Rheological properties that are to be characterized are identified by solving the corresponding inverse problem, where in each iteration ν a set of equations (12) is to be solved. The computed response $U(p^\nu)$ is obtained by solving a direct problem, given by equations (1-3), with the material behaviour defined in accordance with the actual value of the material parameters vector p^ν . The numerical model is analysed by the same FEM mesh as it was used in the analysis that enabled the acquisition of a computed "experimental" response, though there is no special reason for it. In a real experiment, the mesh accuracy test should be performed in any case.

In the identification of the considered hypothetical material we exploit the nature of the mechanical response during the two step experiment. In the first step, where the resulting stresses are rather negligible, the identification of thermal dilatation coefficients α^d is performed, while in the second step the mechanical properties, including yield curves at two different temperatures, is estimated. According to this identification schedule, the material parameters vector p in the first step consists of 4 independent parameters

$$p^T = [p_1, \dots, p_4] = [\alpha^0, \dots, \alpha^3] \quad (15)$$

and in the second step of 17 independent parameters

$$p^T = [p_5, \dots, p_{22}] = [v^0, \dots, v^3, E^0, \dots, E^3, \sigma^0, \dots, \sigma^8] \quad (16)$$

The initial estimate for the parameters assumes that they are constant on the whole temperature range of interest (0-500°C)

$$\begin{aligned} E(T) &= 220\,000\text{MPa} \\ \sigma_y(T) &= 300\text{MPa} \end{aligned} \quad (17)$$

The initial yield curves at fixed temperatures of $T_1=500^\circ\text{C}$ and $T_2=0^\circ\text{C}$ are chosen such, that they express ideally plastic behaviour of material, i.e. they have constant values through the whole range of plastic deformation.

The results of the identification are tabulated in Table 2 and plotted in Fig. 2. In Table 2 the comparison of "true" and identified material properties is given. As it can be seen by comparing the obtained results with the "true" values the quality of the identified material parameters is pretty good.

Table 2: "True" versus identified function coefficients comparison of the steel

T [$^\circ\text{C}$]	E [10^5 MPa]		ν [/]		α [10^{-6} K $^{-1}$]	
	true	identif.	true	identif.	true	identif.
0.0	2.10	2.15	0.300	0.297	11.00	11.34
200.0	1.97	1.94	0.310	0.309	13.00	13.09
400.0	1.83	1.80	0.320	0.326	14.30	14.20
500.0	1.73	1.71	0.330	0.332	14.80	14.73

In Fig. 2 the "true" and estimated yield curves are compared. Agreement between two curves is satisfactory for the most part of them.

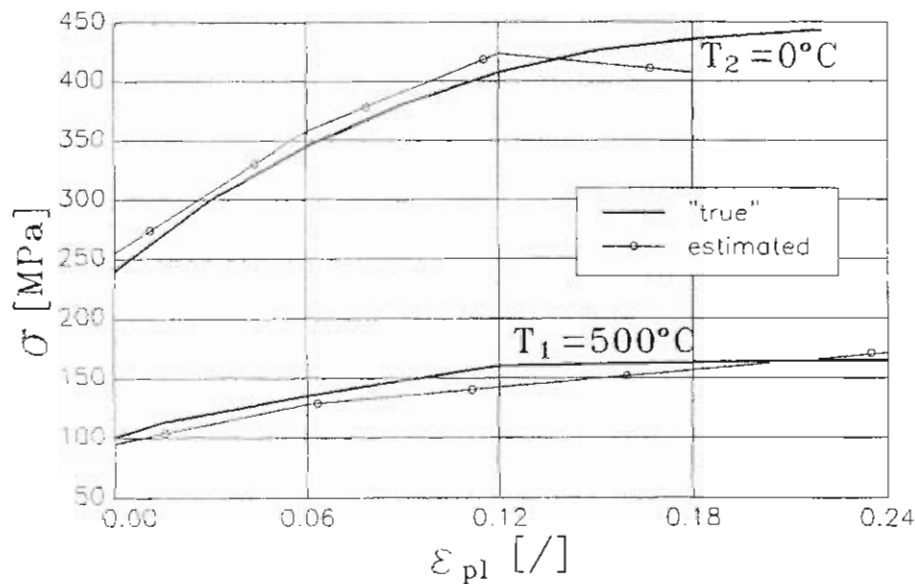


Figure 2: "True" and identified yield curves of the steel

6. CONCLUSION

An identification methodology used for the determination of temperature dependent rheological properties of materials that exhibit elastoplastic behaviour is presented in the paper. The method can be used generally for the estimation of an arbitrary number of parameters. However, because of the excessive time consuming identifications when estimating a larger number of material properties, the number of parameters is limited.

The considered numerical example has proven the efficiency and reliability of the inverse identification of material parameters. The number of iterations needed for successful identification depends on number of material parameters contained in the vector p and on good or ill posed kind of the inverse problem itself. The investigation of a heat transfer problem, where thermal properties of material are sought [3,4], and a mechanical one, which is presented in this paper, has shown that the thermal identification problem is definitely better posed than the mechanical one. In order to obtain a better conditioning of the problem the two step loading is applied, and material parameters vector p is splitted.

The method can be used as a tool for the optimal design of experiments. Relying on the sensitivity analysis, obtained from the consideration of the numerical model and the corresponding simulation of a real experiment, the points in time and space domain with the best data acquisition possibilities can be obtained.

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THE INFLUENCE OF THE TEMPERATURE ON THE FRACTURE TOUGHNESS, FOR X20CrMoV121 STEEL

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Abstract

The fracture toughness is a basic characteristic in order to appreciate the stability of a crack. Estimating the fracture toughness of a material is necessary and it is experimentally determined.

The fracture toughness is influenced by many factors. For a thermoresistant steel used in the steam pipes manufacturing from a Romanian thermo-electrical power station, the most important influences on the fracture toughness are: the temperature and the working-duration of the pipe.

In the paper it is presented the way for estimating the fracture toughness for a steam pipe material (X20CrMoV121 steel) from a Romanian thermo-electrical power station. As parameter it is considered, the temperature.

1. General considerations

The fracture toughness, was determined for the thermoresistant steel X20CrMoV121, which is used at manufacturing steam pipes in Romanian thermo-electrical power stations.

Steam pipes have functioned at the temperature $T=540^{\circ}\text{C}$ and at the internal pressure $p=14\text{ MPa}$.

Chemical composition of the steel which is analysed, prescribed by the producer and experimentally determined, is presented in Table 1.

Table 1. Chemical composition of X20CrMoV121 steel

Estimating procedure	Chemical composition [%]								
	C	Si	Mn	P	S	Ni	Cr	Mo	V
Prescribed	0,17	0,10	1,00	max.	max.	0,30	10,00	0,60	0,23
	0,23	0,50		0,03	0,03	0,80	12,50	1,20	0,35
Determined	0,19	0,43	0,67	0,028	0,006	0,48	11,7	0,95	0,25

It can be observed that even chemical composition was kept between prescribed limits mentioned by the producer.

2. Estimating fracture toughness

a) On traction compact specimen

At the beginning, in order to estimate fracture toughness K_{IC} there were used tension compact specimens (Fig.1), [1]. For this kind of

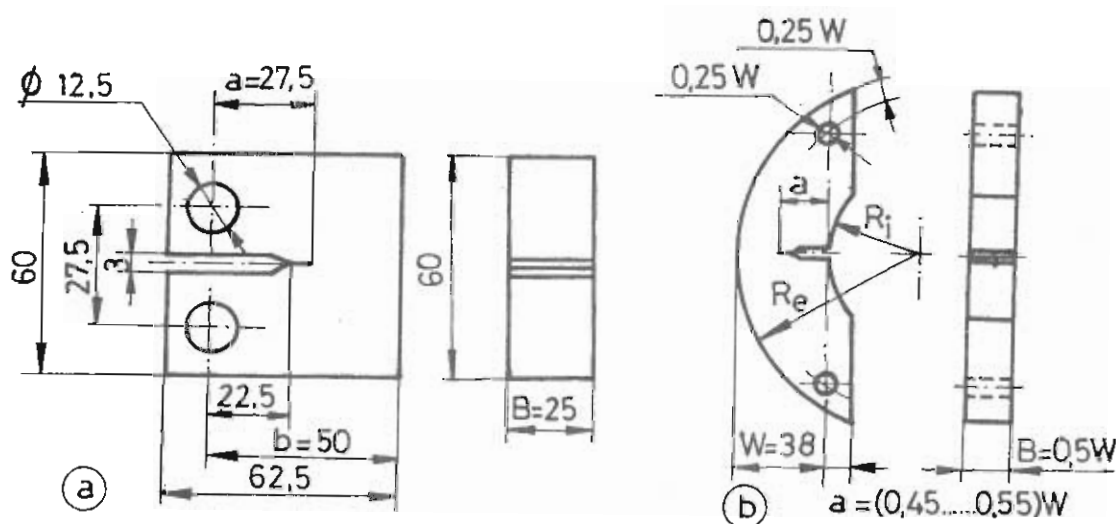


Fig.1 Tension specimens: a) Compact, b) Arc-shaped

specimens, it have to be satisfied the relations [1], [2]:

$$B, a > 2,5 \left(\frac{K_{IQ}}{\sigma_c} \right)^2 \quad (1)$$

where:

B -width of the specimen

a -length of crack

σ_c -yield strength of materials

K_{IQ} -stress intensity factor, estimated on the tension compact specimens.

For X20CrMoV121 steel, condition (1) was not satisfied and, so K_{IQ} can not be accepted as fracture toughness K_{IC} . In this situation, researches have to be done by using specimens with greater dimensions. But, it is impossible to achieve specimens having greater dimensions from the walls of the steam pipes. So, there were used Chevron specimens,

b) On Chevron specimens

There were used Chevron specimens, as that presented in Fig.2, [3].

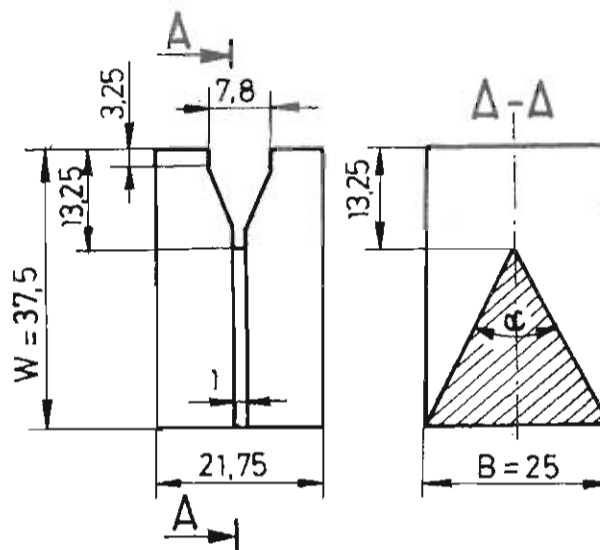


Fig.2 Chevron specimen used

In case of using Chevron specimens, it was determined the fracture toughness K_V , by using relation [4]:

$$K_V = \frac{22 F_{\max}}{B^{3/2}} \quad (2)$$

where:

$F_{\max}$ -maximum force on specimen taped during loading period
 B -width of the specimen.

The variation of the stress intensity factor K_{IQ} and fracture toughness K_V versus testing temperature, is presented in Fig.3

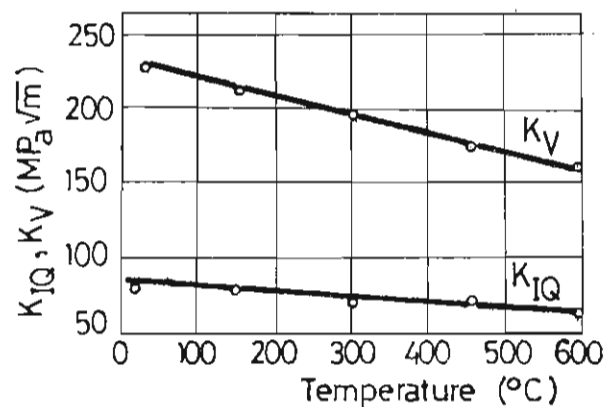


Fig.3 Variation of K_{IQ} and K_V versus temperature

3. Conclusions

Analysing diagrams of Fig.3, it results the following conclusions:

- Stress intensity factor K_{IQ} and fracture toughness K_V , diminish with the growing temperature of material.
- Between the values of K_{IQ} which was estimated on the tension compact specimens and the values of K_V which was estimated on Chevron specimens, there is an important difference. These difference decrease by increasing the temperature.

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MECHANISMS OF PLASTIC COLLAPSE OF CONTINUOUS BEAMS MADE OF ELASTIC-LINEAR STRAIN-HARDENING MATERIAL

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Abstract-This paper deals with analysing continuous beams on several supports loaded by concentrated forces and uniformly distributed continuous load. The dependence of bending moment on the beam curvature has been derived, assuming that the state of strain in the beam is elastic-plastic. Continuous beams of rectangular cross-section and I-cross-section, made of elastic-perfectly plastic and elastic-linear strain-hardening material, respectively, have been considered, with varying magnitude of the tangential elasto-plastic modulus. Expressions defining the dependence of the elastic range height on the bending moment have been derived. The spread of plastified regions in the beams has been monitored by means of those expressions. Finally, an analysis of the limit state of the structure has been carried out and plastic collapse mechanisms have been constructed. The limit load of the structure has been determined by using the virtual displacement method.

1. Introduction

Continuous beams on several supports loaded with concentrated forces and uniformly distributed continuous loading are being considered. It has been assumed that the load is increasing slowly and monotonously from zero to its final magnitude, with the state of strain in the beams changing from purely elastic through elasto-plastic to limit-plastic state. Such continuous beams are primarily used in civil engineering, but can also be found in naval architecture and mechanical engineering. We shall restrict ourselves to the analysis of metal structures, assuming that the stress-strain diagrams of the material can be represented schematically with the material in the first case being elastic-perfectly plastic, and in the second, having linear strain-hardening property. In the analysis, the magnitude of tangential elasto-plastic modulus has been varied by simulating different materials with differently pronounced strain-hardening property. The analysis has been carried out by analytic methods, fully exactly. The mathematical model offers a possibility of accurate determination of the elastic range height and of the plastic zone spread, respectively, in closed explicit form. The dependence of the bending moment $M_{y,pl}$ on the beam curvature $1/\rho$ has been derived as well.

By switching from elastic into plastic range in computation of the structure strength, i.e. allowing the occurrence of local plastic strains at still, static load, an increase of bearing capacity of the structure and a decrease of cross-section dimensions of the beam, respectively, has been

obtained for the same load of the beam, which resulted in lighter structure with less material expenditure and cheaper structure. Modern approach to the strength problems is based on the analysis of the limit state of the structure from which the permissible load of the structure can be determined, so that the analysis in the elastic state can be omitted. In the paper, plastic collapse mechanisms for some continuous beams have been constructed and the limit load has been determined by means of the virtual displacement method.

2. Dependence of bending moment $M_{y,pl}$ on beam curvature $1/\rho$ on assumption of elasto-plastic state of strain in a beam

a) rectangular cross-section of the beam

Figure 1 shows partly plastified cross-section of the beam made of elastic-linear strain-hardening material and adequate distribution of longitudinal strains and normal stresses. The bending moment in the cross-section is determined according to the expression:

$$M_{y,pl} = \int_A \sigma_x z dA = 2b \left\{ \int_0^{h_T/2} E \varepsilon_x z dz + \int_{h_T/2}^{h/2} [\sigma_T + E_T (\varepsilon_x - \varepsilon_T)] z dz \right\}. \quad (1)$$

Since $\varepsilon_x = z/\rho$ for $z = h_T/2$ is obtained $\varepsilon_T = h_T/2\rho$ and $\sigma_T = E\varepsilon_T = Eh_T/2\rho$ respectively, which inserted in the expression (1) gives:

$$M_{y,pl} = 2b \left(E \int_0^{h_T/2} \frac{z}{\rho} z dz + E \int_{h_T/2}^{h/2} \frac{h_T}{2\rho} z dz + E_T \int_{h_T/2}^{h/2} \frac{z}{\rho} z dz - E_T \int_{h_T/2}^{h/2} \frac{h_T}{2\rho} z dz \right). \quad (2)$$

After integration performed the result is:

$$M_{y,pl} = \frac{E}{\rho} \left[\frac{bh_T h^2}{8} - \frac{bh_T^3}{24} + \frac{E_T}{E} \left(\frac{bh^3}{12} + \frac{bh_T^3}{24} - \frac{bh_T h^2}{8} \right) \right]. \quad (3)$$

The height of elastic range depends on the beam curvature $1/\rho$. After certain transformations have been executed and this connection between h_T and ρ inserted in the expression (3), there follows:

$$\frac{M_{y,pl}}{M_{y,T}} = \frac{3}{2} - \frac{1}{2} \left(\frac{\rho}{\rho_T} \right)^2 + \frac{E_T}{E} \left[\frac{\rho_T}{\rho} + \frac{1}{2} \left(\frac{\rho}{\rho_T} \right)^2 - \frac{3}{2} \right]. \quad (4)$$

In the case of the beam material being elastic-perfectly plastic, the tangential elasto-plastic modulus $E_T = 0$, i.e. strain-hardening parameter $\lambda = 1$ so that the above formula assumes the form:

$$\frac{M_{y,pl}}{M_{y,T}} = \frac{3}{2} - \frac{1}{2} \left(\frac{\rho}{\rho_T} \right)^2. \quad (5)$$

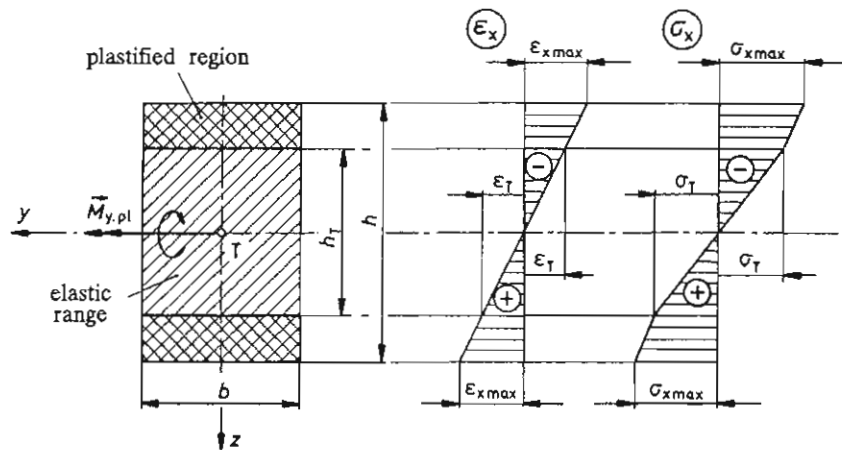


Figure 1. Rectangular cross-section of the beam. Distribution of strains and stresses along the cross-section height

According to expressions (4) and (5) the flexion diagrams $M_{y,pl} = f(1/\rho)$ have been computed and constructed for different values of the tangential elasto-plastic modulus E_T and shown on Fig. 2.

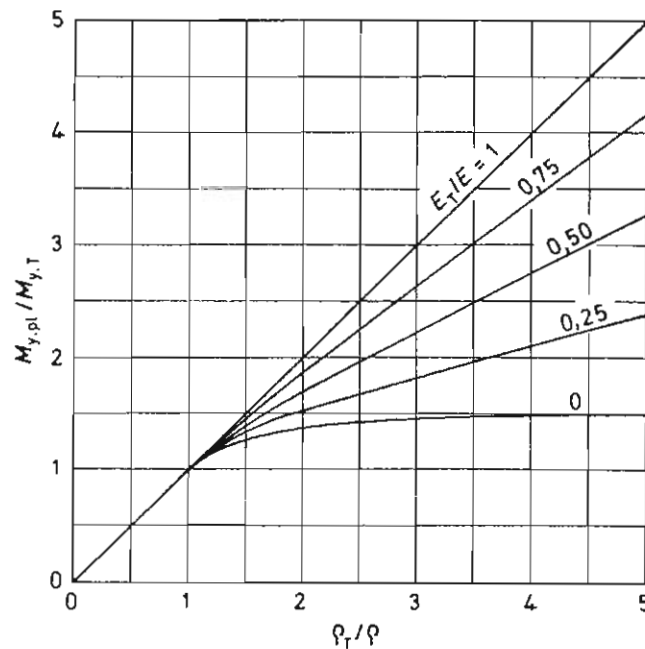


Figure 2. Dependence of the bending moment $M_{y,pl}$ on the beam curvature $1/\rho$

Dependence of the elastic range height h_T on the bending moment $M_{y,pl}$ is determined from the expression (3) and (4), respectively. This connection is necessary for the analysis of the plastic zones spread. If $\rho_T/\rho = h/h_T$ is inserted in the formulae (4) and (5) and arranged, according to (5) it is obtained:

$$\frac{h_T}{h} = \sqrt{3 - 2 \frac{M_{y,pl}}{M_{y,T}}}. \quad (6)$$

If the beam material is elastic-linear strain-hardening, then the formula (4) gives cubic equation for determining the height of elastic range:

$$\left(\frac{h_T}{h}\right)^3 + \left(\frac{2}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 3\right) \frac{h_T}{h} - 2 \frac{1-\lambda}{\lambda} = 0, \quad (7)$$

where $\lambda = 1 - \frac{E_T}{E}$ is strain-hardening parameter. One real root of this equation is determined by applying the Cardan formula and amounts to:

$$\frac{h_T}{h} = \sqrt[3]{\frac{1-\lambda}{\lambda} + \sqrt{\left(\frac{1-\lambda}{\lambda}\right)^2 + \left(\frac{2}{3\lambda} \frac{M_{y,pl}}{M_{y,T}} - 1\right)^3}} + \sqrt[3]{\frac{1-\lambda}{\lambda} - \sqrt{\left(\frac{1-\lambda}{\lambda}\right)^2 + \left(\frac{2}{3\lambda} \frac{M_{y,pl}}{M_{y,T}} - 1\right)^3}}. \quad (8)$$

b) I-cross-section of the beam

The analysis of the dependence of the bending moment on the beam curvature for the I-cross-section has been performed in two parts: the case when the flanges were partially plastified, and the rib was elastic (as shown in Fig. 3), was investigated first, and then it was assumed that the flanges were fully plastified, and the rib only partially as in Fig. 4.

- the flanges are partially plastified, and the rib is elastic

Here too the bending moment is defined fully by analogy to the expression (1). After performed integration and arrangement it is obtained:

$$M_{y,pl} = \frac{E}{\rho} \left[\frac{th^3}{12} - \frac{bh_T^3}{24} - \frac{bh^3}{12} + \frac{bh_T H^2}{8} + \frac{E_T}{E} \left(\frac{bH^3}{12} + \frac{bh_T^3}{24} - \frac{bh_T H^2}{8} \right) \right]. \quad (9)$$

After making slight transformations and taking into consideration that $H/h_T = \rho_T/\rho$, it follows:

$$\frac{M_{y,pl}}{M_{y,T}} \left(1 - \frac{h^3}{H^3} + \frac{t}{b} \frac{h^3}{H^3} \right) = \left(\frac{t}{b} \frac{h^3}{H^3} - \frac{h^3}{H^3} \right) \frac{\rho_T}{\rho} - \frac{1}{2} \left(\frac{\rho}{\rho_T} \right)^2 + \frac{3}{2} + \frac{E_T}{E} \left[\frac{\rho_T}{\rho} + \frac{1}{2} \left(\frac{\rho}{\rho_T} \right)^2 - \frac{3}{2} \right] \quad (10)$$

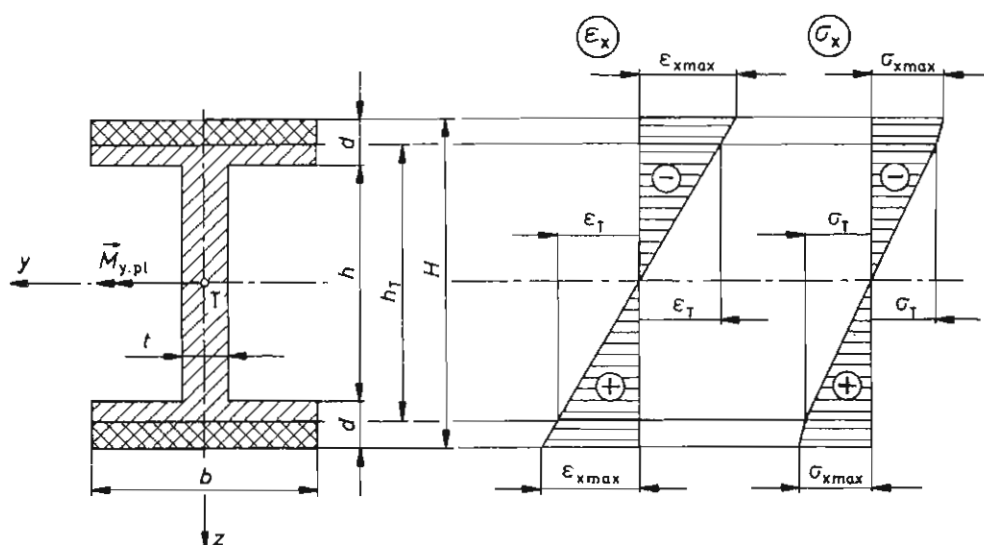


Figure 3. Partially plastified I-cross-section. Stress and strain distribution

The dependence of the elastic range height on the bending moment is determined from the equation (10) which is necessary for the analysis of the plastic zones spread. Since the analytical expression (10) is rather complex, it can be somewhat simplified if a concrete I-cross-section is chosen as a numerical example. Thus for example, by inserting the dimensions for the cross-section I24 ($H=240$ mm, $b=106$ mm, $t=8,7$ mm, $d=13,1$ mm and $h=H-2d=213,8$ mm) in the equation (10), it follows:

$$\frac{M_{y,pl}}{M_{y,T}} = 4,2726 - 1,8484 \frac{\rho_T}{\rho} - 1,4242 \left(\frac{\rho}{\rho_T} \right)^2 + 2,8484 \frac{E_T}{E} \left[\frac{\rho_T}{\rho} + \frac{1}{2} \left(\frac{\rho}{\rho_T} \right)^2 - \frac{3}{2} \right]. \quad (11)$$

Taking into consideration that $\rho_T/\rho = H/h_T$ and by making appropriate arrangements, the cubic equation is obtained for determination of the elastic range height in the form of:

$$\left(\frac{h_T}{H} \right)^3 + \left(\frac{0,702}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 3 \right) \frac{h_T}{H} + \frac{2\lambda - 0,702}{\lambda} = 0. \quad (12)$$

One real root of the equation (12) determines the height of the elastic range and equals to:

$$\frac{h_T}{H} = 2 \left(\frac{0,234}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 1 \right)^{\frac{1}{2}} \cdot \cos \left[\frac{\pi}{3} - \frac{1}{3} \arccos \frac{\lambda - 0,35}{\lambda \left(\frac{0,234}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 1 \right)^{3/2}} \right] \quad (13)$$

In the case of the beam material being elastic-perfectly plastic, it is necessary to insert $E_T = 0$ and $\lambda = 1$ respectively in the above formula, thus obtaining:

$$\frac{h_T}{H} = 2 \left(\left[0,234 \frac{M_{y,pl}}{M_{y,T}} - 1 \right] \right)^{\frac{1}{2}} \cdot \cos \left[\frac{\pi}{3} - \frac{1}{3} \arccos \frac{0,65}{\left(\left[0,234 \cdot \frac{M_{y,pl}}{M_{y,T}} - 1 \right] \right)^{3/2}} \right] \quad (14)$$

- flanges are fully plastified, and the rib only partially

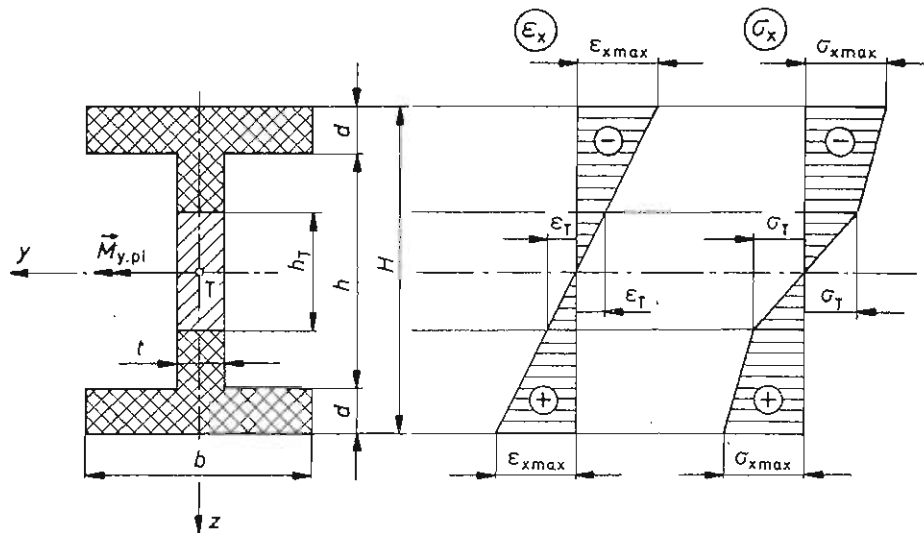


Figure 4. Partially plastified I-cross-section. Strain and stress distribution

$$M_{y,pl} = \frac{E}{\rho} \left[\frac{th_T h^2}{8} - \frac{th_T^3}{24} + \frac{bh_T H^2}{8} - \frac{bh_T h^2}{8} + \right. \\ \left. + \frac{E_T}{E} \left(\frac{th^3}{12} + \frac{th_T^3}{24} - \frac{th_T h^2}{8} + \frac{bH^3}{12} - \frac{bh^3}{12} - \frac{bh_T H^2}{8} + \frac{bh_T h^2}{8} \right) \right] \quad (15)$$

$$\frac{M_{y,pl}}{M_{y,T}} \left[1 - \frac{h^3}{H^3} \left(1 - \frac{t}{b} \right) \right] = \frac{\rho_T}{\rho} \left\{ \frac{3}{2} \left(\frac{t}{b} \frac{h^2}{H^2} + 1 - \frac{h^2}{H^2} \right) \frac{h_T}{H} - \frac{1}{2} \frac{t}{b} \frac{h_T^3}{H^3} + \right. \\ \left. + \frac{E_T}{E} \left[1 - \frac{h^3}{H^3} \left(1 - \frac{t}{b} \right) - \frac{3}{2} \left(1 + \frac{t}{b} \frac{h^2}{H^2} - \frac{h^2}{H^2} \right) \frac{h_T}{H} + \frac{1}{2} \frac{t}{b} \frac{h_T^3}{H^3} \right] \right\} \quad (16)$$

$$\frac{M_{y,pl}}{M_{y,T}} \left[1 - \frac{h^3}{H^3} \left(1 - \frac{t}{b} \right) \right] = \frac{3}{2} \left(\frac{t}{b} \frac{h^2}{H^2} + 1 - \frac{h^2}{H^2} \right) - \frac{1}{2} \frac{t}{b} \left(\frac{\rho}{\rho_T} \right)^2 + \\ + \frac{E_T}{E} \left\{ \left[1 - \frac{h^3}{H^3} \left(1 - \frac{t}{b} \right) \right] \frac{\rho_T}{\rho} - \frac{3}{2} \left(1 + \frac{t}{b} \frac{h^2}{H^2} - \frac{h^2}{H^2} \right) + \frac{1}{2} \frac{t}{b} \left(\frac{\rho}{\rho_T} \right)^2 \right\} \quad (17)$$

On inserting dimensions for the cross-section I24 in the equation (17) it follows:

$$\frac{M_{y,pl}}{M_{y,T}} = 1,16 - 0,117 \left(\frac{\rho}{\rho_T} \right)^2 + \frac{E_T}{E} \left[\frac{\rho_T}{\rho} - 1,16 + 0,117 \left(\frac{\rho}{\rho_T} \right)^2 \right], \quad (18)$$

$$\left(\frac{h_T}{H} \right)^3 + \left(\frac{8,555}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 9,9258 \right) \frac{h_T}{H} - 8,555 \cdot \frac{1-\lambda}{\lambda} = 0. \quad (19)$$

One real root of this equation is determined by applying the Cardan formula and it makes:

$$\begin{aligned} \frac{h_T}{H} = & \sqrt[3]{4,2775 \cdot \frac{1-\lambda}{\lambda} + \sqrt{\left(4,2775 \cdot \frac{1-\lambda}{\lambda}\right)^2 + \left(\frac{2,852}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 3,3086\right)^3}} + \\ & + \sqrt[3]{4,2775 \cdot \frac{1-\lambda}{\lambda} - \sqrt{\left(4,2775 \cdot \frac{1-\lambda}{\lambda}\right)^2 + \left(\frac{2,852}{\lambda} \frac{M_{y,pl}}{M_{y,T}} - 3,3086\right)^3}} \end{aligned} \quad (20)$$

According to the formulae (10), (11), (17) and (18) the flexion diagrams $M_{y,pl} = f(1/\rho)$ are computed and constructed for different values of the tangential elasto-plastic modulus E_T and are shown on Fig. 5.

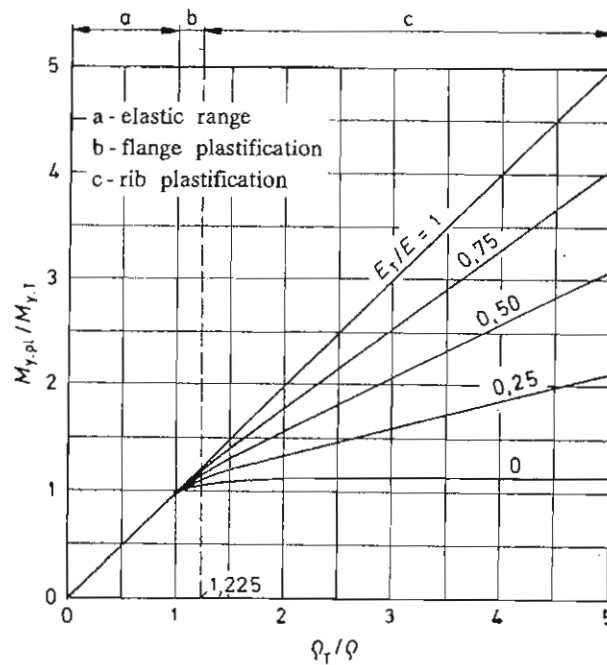


Figure 5. Dependence diagram of the bending moment $M_{y,pl}$ on the beam curvature $1/\rho$ for the I24-cross-section

3. Analysis of occurrence and spreading of plastic zones in beams

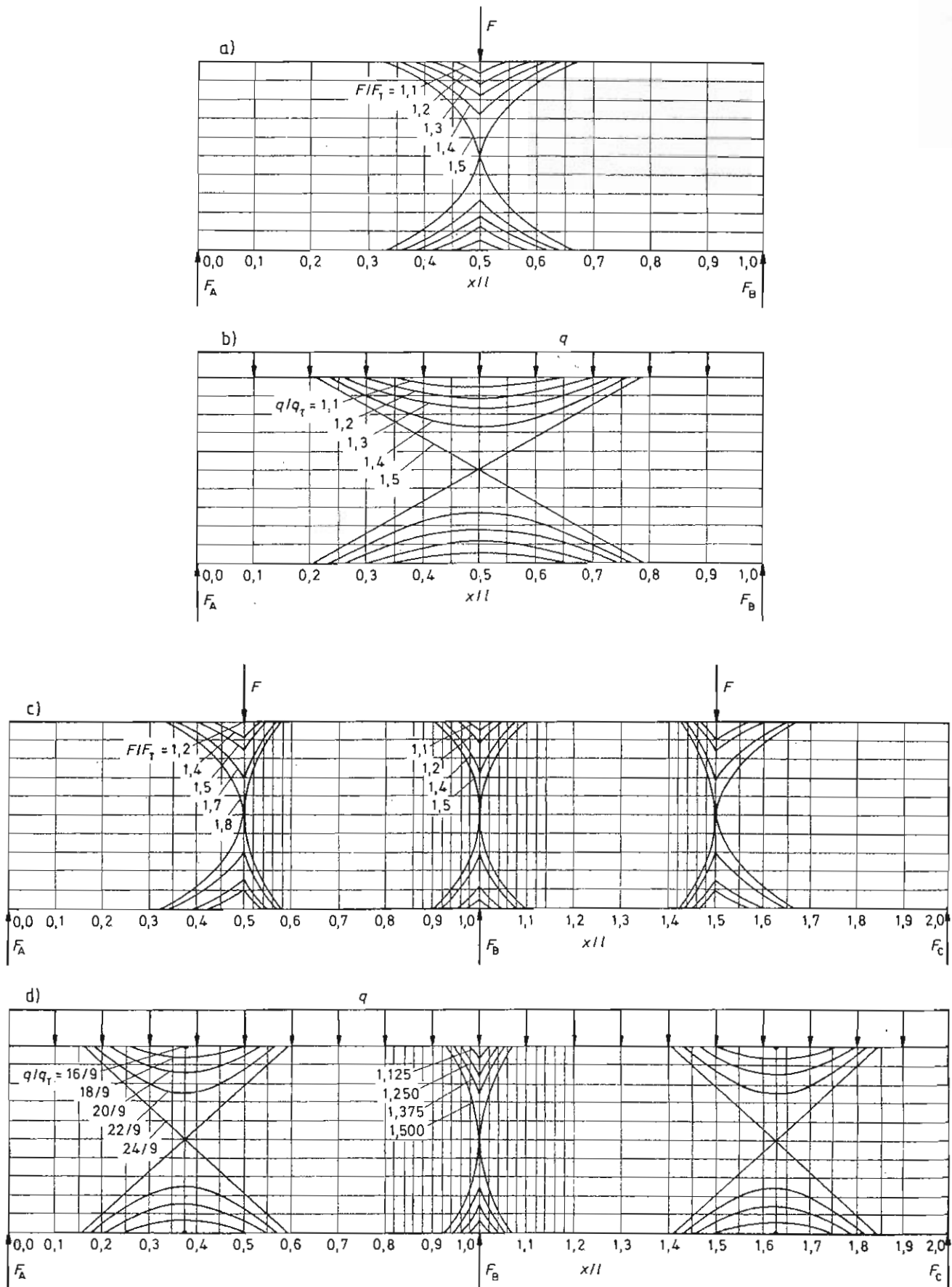


Figure 6. Spreading of plastic zones in the beams of rectangular cross-section; the beam material is elastic-perfectly plastic

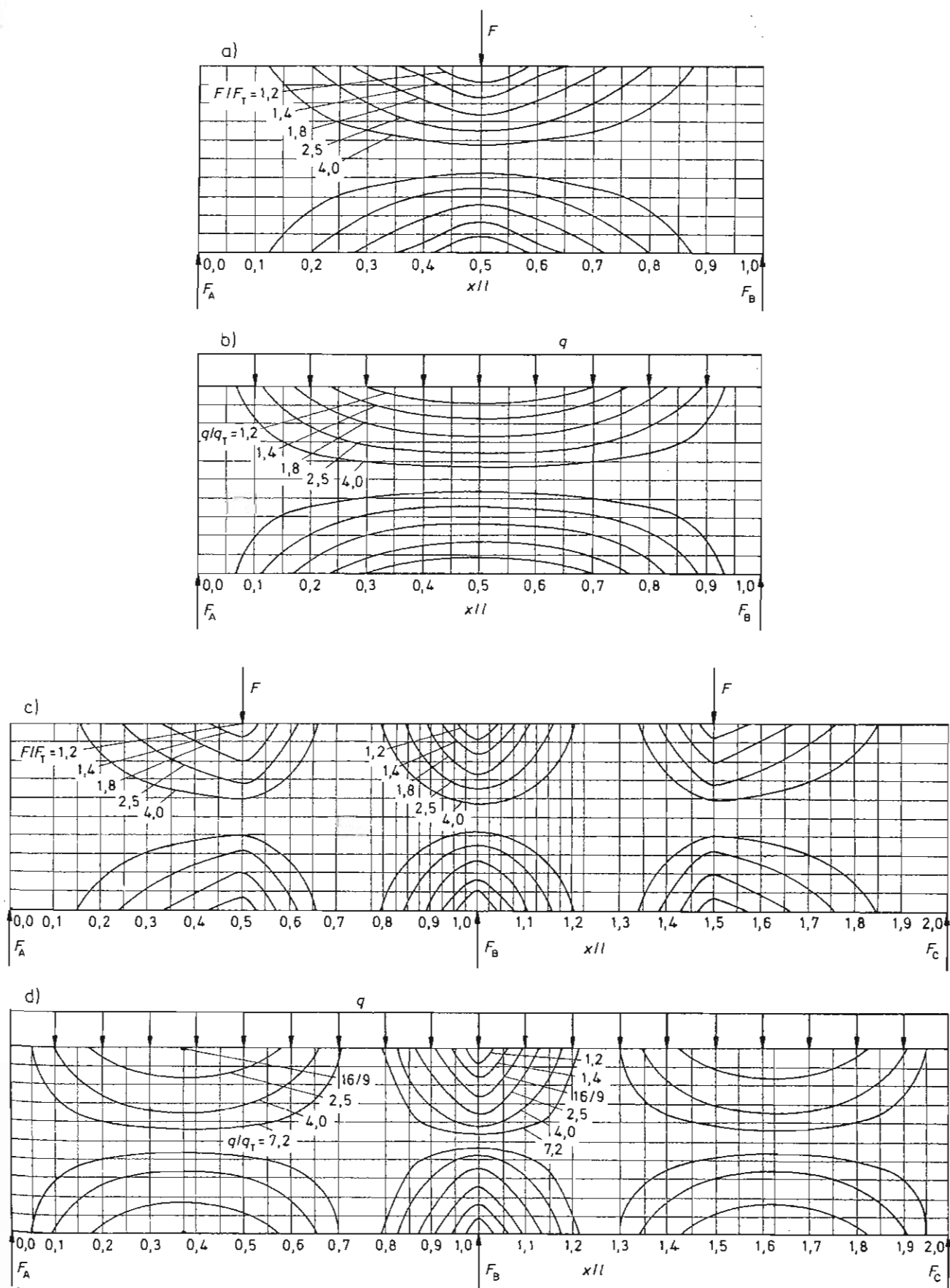


Figure 7. Spreading of plastic zones in the beams of rectangular cross-section; the beam material is elastic-linear strain-hardening

4. Conclusion

In the paper the dependence of the bending moment $M_{y,pl}$ on the beam curvature $1/\rho$ has been derived, on the assumption of elasto-plastic state of strain in the beam. At bending in the plastic range this connection becomes non-linear and depends markedly on the magnitude of the tangential elasto-plastic modulus. The investigation has been done for rectangular cross-section and for I-cross-section. In the former case the material is elastic-perfectly plastic, and in the latter, elastic-linear strain-hardening. As shown on Fig. 5, even an insignificant increase of bending moment in relation to $M_{y,T}$ is enough to cause plastification of the flange, whereas for plastification of the rib it is necessary to increase considerably the bending moment, all the more so as the material has more strongly pronounced property of strain-hardening one.

Further, the expressions defining the dependence of the elastic range height on the bending moment have been derived. By means of these expressions, the spread of plastified regions in the beams has been monitored. By comparing the results on Figures 6 and 7 it can be generally concluded that in the case of elastic-perfectly plastic material of the beams the plastified regions are concentrated on the narrow region around the cross-section in which external load is at work, while in the case of elastic-linear strain-hardening material the plastic zones extend over much wider region of the beam.

At the end of the paper an analysis of the limit state has been carried out on the assumption that the material is elastic-perfectly plastic. The mechanisms of plastic collapse have been constructed and by means of virtual displacements method the limit load of the structure has been determined. In the case of elastic-linear strain-hardening material it would be necessary to define the concept of the limit state and of the limit bending moment, respectively. Namely, in such material the fracture will begin by rupture of fibres at the upper and lower boundary of the beam. The fracture occurs when maximum longitudinal strain $\varepsilon_{x,max}$, in the critical cross-section of the beam becomes equal to the strain which is adequate to the tensile strength and to the compressive strength of the material, respectively.

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THERMOELASTIC ANALYSIS OF FOUNDATIONS BY COMPLEX VARIABLE METHOD

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SUMMARY

A centrically loaded concrete strip foundation exposed to thermal action is studied in the paper. It is assumed that the pressure distribution in the contact zone between foundations and the soil is regulated according to the law of square parabola. The analysis of stress and displacement in the concrete strip foundation was conducted using the method of complex variable. For a given load and a stationary temperature range, complex potentials were defined from boundary conditions using the least-squares method. The field of displacement and stress in the strip foundation was obtained from complex potentials.

1. INTRODUCTION

A centrically loaded concrete strip foundation exposed to temperature variations is considered. The real pressure distribution in the zone of contact between foundations and the soil depends on mechanical properties of soil, as well as on the material and rigidity of foundations. In the case under study it is assumed that pressure distribution in the contact zone between foundations and soil is regulated according to the law of square parabola (Figure 1).

In the initial state, the temperature $T_0 = \text{const.}$, while the temperature range in the final state is defined by the expression :

$$T(x, y) = T_0(1 + n e^{\lambda x} \cos \lambda y) \quad (1)$$

The temperature increase with respect to the initial state T_0 is :

$$\theta(x, y) = T - T_0 = T_0 n e^{\lambda x} \cos \lambda y \quad (2)$$

The temperature increase in the origin of coordinates is defined by the selection of the constant n , and in contour points by the selection of constant λ .

2. STATE OF STRESS AND DISPLACEMENT IN THE STRIP FOUNDATION

The state of stress and displacement in the strip foundation will be defined by the method of complex variable. It can easily be demonstrated that the function θ defined in the expression (2) is a harmonic function, and that it represents a stationary temperature field without a source of heat. The complex form for the studied plane strain problem is :

- for boundary conditions

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = i \int_0^s (\rho_x + i\rho_y) ds + \text{const.} = f(t) + \text{const.} \quad (3)$$

- for stress components

$$\begin{aligned} \sigma_x + \sigma_y &= 2[\varphi'(z) + \overline{\varphi'(z)}] \\ \sigma_y - \sigma_x + 2i\sigma_{xy} &= 2[\overline{z}\varphi''(z) + \psi'(z)] \end{aligned} \quad (4)$$

- for displacement components

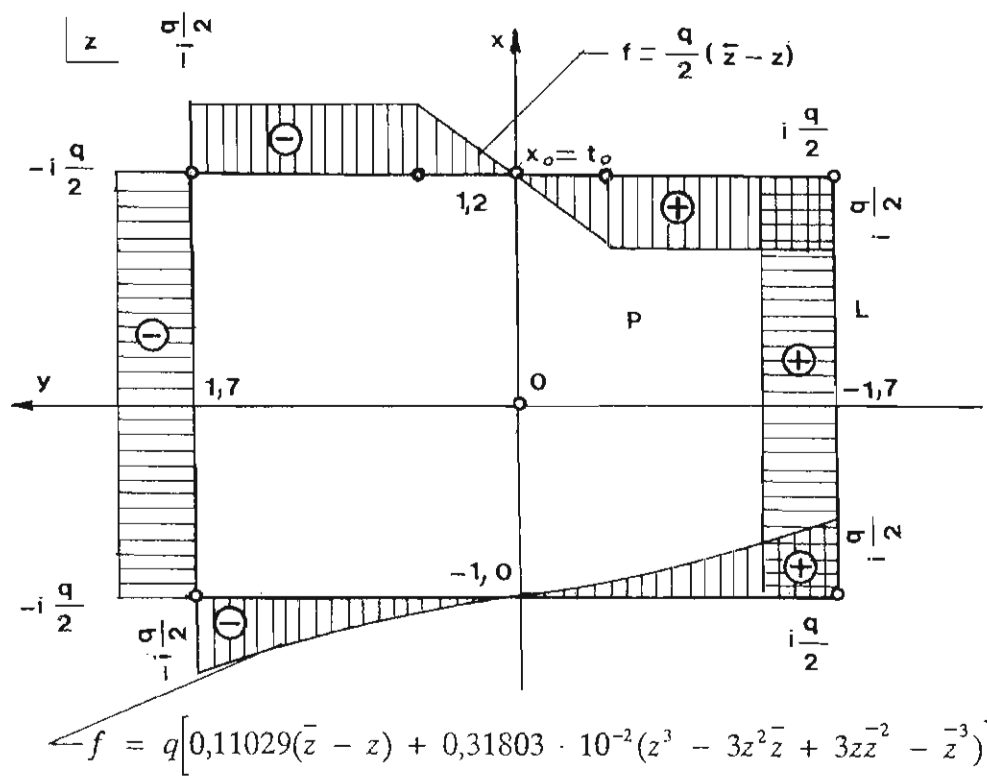
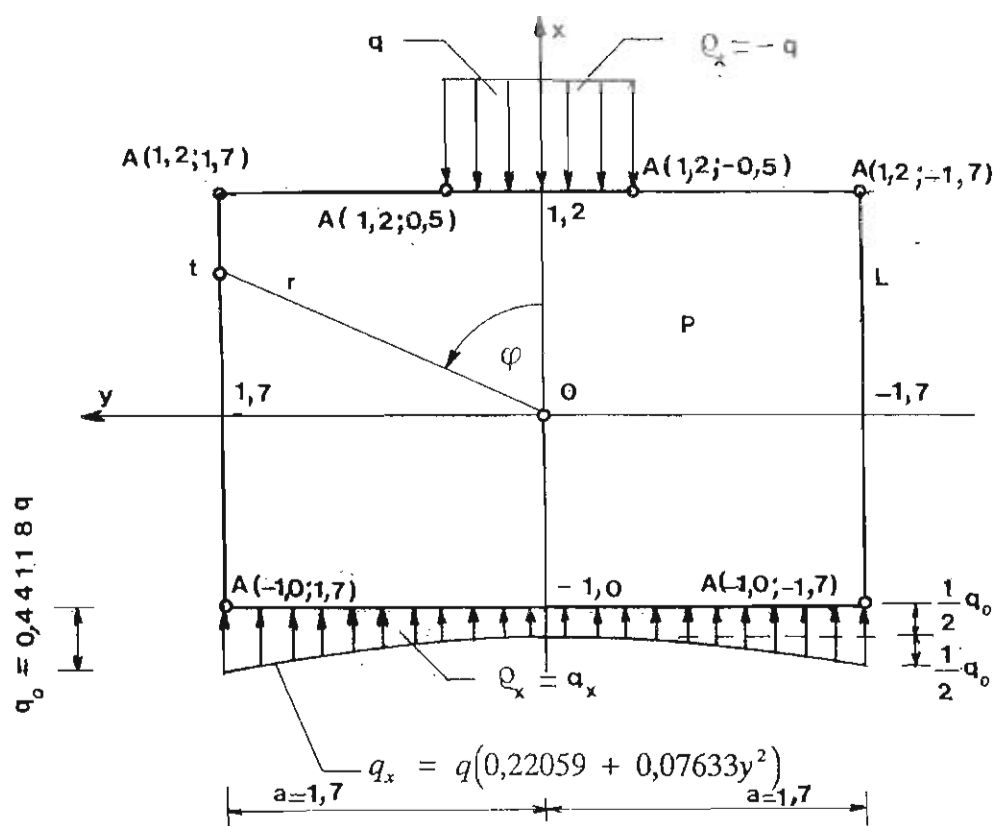
$$u + iv = \frac{1 + \nu}{E} \left[(3 - 4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \right] + \alpha_T(1 + \nu) \int f_p(z) dz \quad (5)$$

In the expression (5) $f_p(z)$ is the analytic function of the complex variable and is described by the following expression :

$$f_p(z) = \theta + i Q_p \quad (6)$$

and

$$\varphi_p(z) = p_p + iq_p = \int f_p(z) dz \quad (7)$$



where Q_p is the harmonic function conjugated by the function θ , while p_p and q_p are harmonic functions for which the following relations are valid :

$$\frac{\partial p_p}{\partial x} = \frac{\partial q_p}{\partial y} = \theta, \quad \frac{\partial p_p}{\partial y} = -\frac{\partial q_p}{\partial x} = -Q_p \quad (8)$$

For the harmonic function θ set in the expression (2), the following conjugated function is defined:

$$Q_p = T_0 n e^{\lambda x} \sin \lambda y + C_1 \quad (9)$$

a then the expression (6) was used to obtain the following function

$$f_p(z) = T_0 n e^{\lambda x} (\cos \lambda y + i \sin \lambda y) + iC_1 \quad (10)$$

Functions p_p and q_p are defined by means of expressions (2), (8) and (9) and can be presented as follows :

$$\begin{aligned} p_p &= T_0 \frac{n}{\lambda} e^{\lambda x} \cos \lambda y - C_1 y + C_2 \\ q_p &= T_0 \frac{n}{\lambda} e^{\lambda x} \sin \lambda y + C_1 x + C_3 \end{aligned} \quad (11)$$

By means of expressions (7) and (11) we obtain :

$$\begin{aligned} \int f_p(z) dz &= p_p + iq_p = T_0 \frac{n}{\lambda} e^{\lambda x} \cos \lambda y - C_1 y + C_2 + \\ &+ i(T_0 \frac{n}{\lambda} e^{\lambda x} \sin \lambda y + C_1 x + C_3) \end{aligned} \quad (12)$$

where C_1, C_2, C_3 are arbitrary constants expressing displacement of rigid body in its plane. According to expressions (5) and (12), components of displacement due to increase in temperature are :

$$\begin{aligned} u_T &= \alpha_T T_0 \frac{n}{\lambda} e^{\lambda x} \cos \lambda y - \alpha_T C_1 y + \alpha_T C_2 \\ v_T &= \alpha_T T_0 \frac{n}{\lambda} e^{\lambda x} \sin \lambda y + \alpha_T C_1 x + \alpha_T C_3 \end{aligned} \quad (13)$$

Based on predefined conditions according to which, due to temperature influence in point A (-1.0;0), the displacement components are: $u_T(-1.0;0)=0$, $v_T(-1.0;0)=0$ and the rotation is

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0$$

the constants have been defined: $C_1 = C_3 = 0$, $C_2 = -T_0 \frac{n}{\lambda} e^{-\lambda}$, so that the equation (12) assumes the following form :

$$\int f_p(z)dz = p_p + iq_p = T_0 \frac{n}{\lambda} \left(e^{\lambda x} \cos \lambda y - \bar{e}^{\lambda} + i e^{\lambda x} \sin \lambda y \right) \quad (14)$$

If the expression (14) is included in expression (5) we obtain a complex form for displacement components :

$$u + iv = \frac{1 + \nu}{E} \left[(3 - 4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \right] + \alpha_T(1 + \nu) T_0 \frac{n}{\lambda} \left(e^{\lambda x} \cos \lambda y - e^{-\lambda} + i e^{\lambda x} \sin \lambda y \right) \quad (15)$$

The load function $f(t)$ was defined on the contour L by means of the expression (3). This function is presented graphically in Figure 2.

As it is assumed that the constant from expression (3) is equal to zero, additional conditions for unambiguous determination of functions $\varphi(z)$ and $\psi(z)$ are :

$$\varphi(0) = 0 \quad \text{or} \quad \psi(0) = 0, \quad \text{Im } \varphi'(0) = 0 \quad (16)$$

Approximate values of functions $\varphi(z)$ and $\psi(z)$ should be in the form of the following power series :

$$\varphi_n(z) = \sum_{k=1}^n \alpha_k z^k, \quad \psi_n(z) = \sum_{k=0}^n \beta_k z^k \quad (17)$$

Boundary conditions (3) are reduced to the following form :

$$\varphi_n(t) + t\overline{\varphi'_n(t)} + \overline{\psi_n(t)} = f_n(t) \quad (18)$$

Based on the stress-field symmetry, it may be concluded that coefficients α_k and β_k are also realistic in the expression (17). If new symbols are introduced in the expression (17) :

$$\alpha_k = a_{2k-1}, \quad k = 1, 2, 3, \dots, n; \quad \beta_k = a_{2k}, \quad k = 0, 1, 2, 3, \dots, n \quad (19)$$

$$\varphi_{(2k-1)}(z) = z^k, \quad k = 1, 2, 3, \dots, n; \quad \varphi_{(2k)}(z) = 0, \quad k = 0, 1, 2, 3, \dots, n \quad (20)$$

$$\psi_{(2k)}(z) = z^k, \quad k = 0, 1, 2, 3, \dots, n; \quad \psi_{(2k-1)}(z) = 0, \quad k = 1, 2, 3, \dots, n \quad (21)$$

then the expression (17) may assume the following form :

$$\varphi_n(z) = \sum_{k=0}^{2n} a_k \varphi_{(k)}(z) \quad ; \quad \psi_n(z) = \sum_{k=0}^{2n} a_k \psi_{(k)}(z) \quad (22)$$

If the expression (22) is included in the expression (18), we obtain

$$f_n(t) = \sum_{k=0}^{2n} a_k f_{(k)}(t) \quad (23)$$

where

$$f_{(k)}(t) = \varphi_{(k)}(t) + t \overline{\varphi'_{(k)}(t)} + \overline{\psi_{(k)}(t)} \quad (24)$$

Coefficients a_k of the polynomial (22) will be determined from the condition according to which the integral of the square of the difference between the required function $f_n(t)$ and the defined load function $f(t)$ along the closed contour L has the lowest value :

$$\int_L |f_n(t) - f(t)|^2 ds = \min. \quad (25)$$

If the scalar product of two functions $\varphi_m(t)$ and $\varphi_n(t)$ is defined :

$$(\varphi_m, \varphi_n) = \int_L \varphi_m(t) \overline{\varphi_n(t)} ds ; \quad ds = |dt| \quad (26)$$

then the expression (25) can be presented as follows :

$$\int_L |f_n(t) - f(t)|^2 ds = \left\| \sum_{k=0}^{2n} a_k f_{(k)}(t) - f(t) \right\|^2 = \min \quad (27)$$

As coefficients a_k are realistic, we obtain :

$$\sum_{j,k=0}^{2n} a_j a_k (f_{(j)}, f_{(k)}) - 2 \sum_{k=0}^{2n} a_k \operatorname{Re}(f, f_{(k)}) + (f, f) = \min \quad (28)$$

In order to apply minimum values to the expression (28), its derivations according to a_j and a_k must be equal to zero. In this way we obtain the following system of linear algebraic equations with unknowns a_k :

$$\sum_{k=0}^{2n} a_k \operatorname{Re}(f_{(k)}, f_{(j)}) = \operatorname{Re}(f, f_{(j)}), \quad j = 0, 1, 2, 3, \dots, 2n \quad (29)$$

The equation system (29) was solved for $n=15$. Thus obtained potential functions $\varphi(z)$ and $\psi(z)$ are :

$$\begin{aligned} \varphi(z) = & q (-0,87080 \cdot 10^{-1}z - 0,64570 \cdot 10^{-1}z^2 - 0,14392 \cdot 10^{-1}z^3 - \\ & -0,15025 \cdot 10^{-1}z^4 - 0,48835 \cdot 10^{-2}z^5 - 0,29327 \cdot 10^{-2}z^6 - \\ & -0,14115 \cdot 10^{-2}z^7 - 0,58329 \cdot 10^{-3}z^8 - 0,30713 \cdot 10^{-3}z^9 - \\ & -0,77899 \cdot 10^{-4}z^{10} - 0,48379 \cdot 10^{-4}z^{11} - 0,69660 \cdot 10^{-5}z^{12} - \\ & -0,50452 \cdot 10^{-5}z^{13} - 0,38640 \cdot 10^{-6}z^{14} - 0,13299 \cdot 10^{-6}z^{15}) \end{aligned} \quad (30)$$

$$\begin{aligned} \psi(z) = & q (0,20320 + 0,26345 z + 0,12548 z^2 + 0,60724 \cdot 10^{-1}z^3 + \\ & + 0,32329 \cdot 10^{-1}z^4 + 0,20311 \cdot 10^{-1}z^5 + 0,78213 \cdot 10^{-2}z^6 + \\ & + 0,43564 \cdot 10^{-2}z^7 + 0,10464 \cdot 10^{-2}z^8 + 0,42251 \cdot 10^{-3}z^9 + \\ & + 0,52475 \cdot 10^{-4}z^{10} - 0,55253 \cdot 10^{-4}z^{11} - 0,11331 \cdot 10^{-5}z^{12} - \\ & - 0,30764 \cdot 10^{-4}z^{13} - 0,68237 \cdot 10^{-6}z^{14} - 0,29165 \cdot 10^{-5}z^{15}) \end{aligned} \quad (31)$$

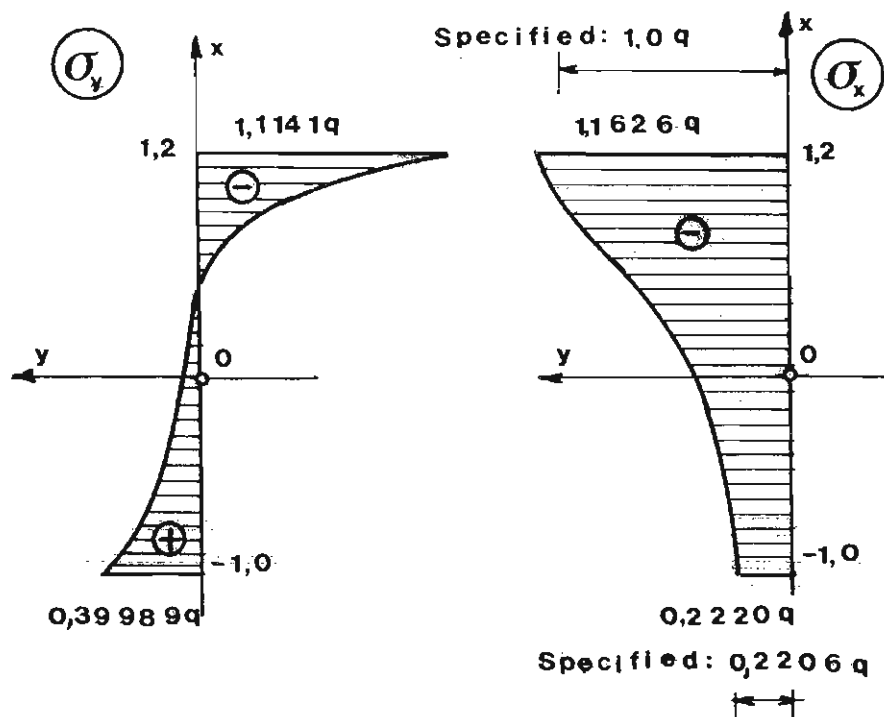


Figure 3 Stress diagrams in section $y = 0$

Scale of displacement
 $1 \text{ cm} :: 4 \cdot 10^{-5}$

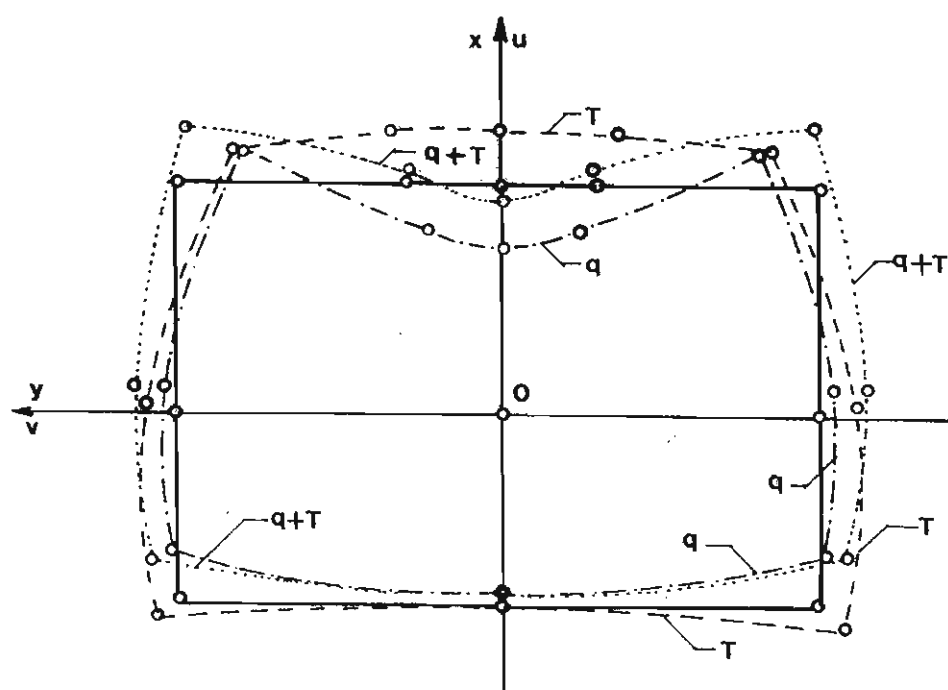


Figure 4 Displacements of contour points

Table 1 Displacement components of contour points

Point	Displacements for q = 1 MPa		Displacements for temperature increase $\theta(0,0) = T_0 n = 1K$		Total displacement due to loading and temperature increase	
	$[10^{-5}]$					
	u	v	u_T	v_T	u	v
A(1,2; 0)	-3.8342	0	2.8854	0	-0.9488	0
A(1,2; 0.5)	-2.1375	-1.0555	2.7653	1.0353	0.6278	-0.0202
A(1,2; 1.7)	1.3473	-3.3426	1.5610	3.1974	2.9083	-0.1452
A(0; 1.7)	1.3044	0.5847	0.2002	1.8366	1.5046	2.4213
A(-1.0; 1.7)	2.6219	0.1375	-0.4793	1.1571	2.1426	1.2946
A(-1.0; 0)	0.3307	0	0	0	0.3307	0

Expressions (30), (31) and (14) are the solution to the quasi-static problem of thermoelasticity. Diagrams of cross sectional stress along the real axis are presented in Figure 3. In this cross section the shear stresses are $\sigma_{xy} = 0$. The relative error under balanced conditions in this cross section amounts to less than 2.5%. It can be seen from the σ_x diagram that boundary conditions have been fulfilled with sufficient accuracy. The numerical realization of the displacement field was conducted for concrete strip foundations with the elasticity modulus of $E_b = 28.5 \cdot 10^3 \text{ MPa}$, the Poisson's ratio of $\nu = 0.20$ and the linear thermal elongation ratio of $\alpha_T = 10^{-5}/K$. The λ constant was selected so that $\lambda \cdot 2a = \pi/2$. On this basis, the value of $\lambda = 0.462$ was obtained. Displacements of typical contour points were determined for load $q = 1 \text{ MPa}$ and for temperature increase of $\theta(0,0) = T_0 n = 1K$. The corresponding displacement components are presented in Table 1, while their graphic representation is given in Figure 4.

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NONLINEAR BENDING OF BEAMS HAVING CROSS SECTIONS WITH ONE AXIS OF SYMMETRY

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ABSTRACT: An expression for the mathematical description of the stress-strain diagram in the form of odd polynomials of the n -th order has been proposed. Particularly, application of polynomial of the third order was successful. Nonlinear bending of the beams having cross sections with one axis of symmetry is analyzed in view of the new expression. New concepts of the moments of inertia of the third and fourth order as well as section modulus of the fourth order are introduced.

1. Introduction

Many engineering materials such as copper and copper alloys, aluminum and aluminum alloys, different polymers and some organic materials exhibit pronounced nonlinearity in the stress-strain diagram. In that case Hooke's law cannot be applied and therefore formulae of ordinary (linear) strength of materials are not valid any more. An adequate and simple engineering theory to deal with such problems has not been yet developed. Recently there have been some attempts to develop simple solution for pure bending of straight beams [6], [11]. However that solution is limited to the beams having cross sections with two axes of symmetry and bending moments acting around one axis of symmetry.

The main aim of this contribution is to extend the proposed theory to the straight beams having cross section with one axis of symmetry. Bending moment can act about the axis of symmetry or perpendicularly to it. Figure 1 shows some typical cross sections to which this theory will apply.

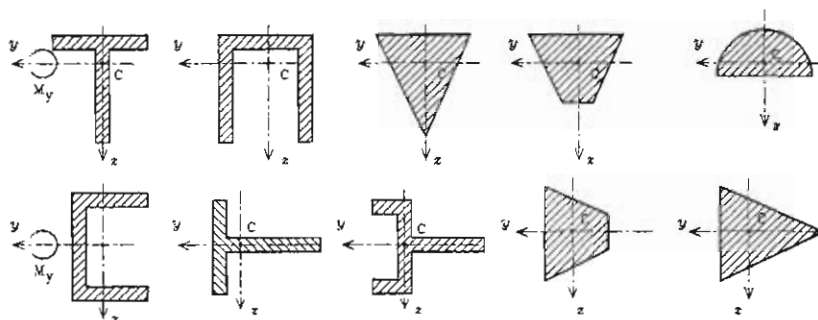


Figure 1 Cross sections with one axis of symmetry

2. Mathematical description of the stress-strain diagram

Only materials that exhibit the same tensile and compressive properties will be considered. The stress strain diagram of such two materials is shown in Figure 2. It is apparent that stress is an odd function of strain. Therefore the stress strain diagram can be represented by odd polynomials [6], [8]. Particularly, applications of the polynomial

of the third order (cubic parabola) appeared to be very successful. In that case mathematical description of the stress strain diagram takes the form:

$$\sigma = E\varepsilon - H\varepsilon^n \quad (1)$$

where E , H , n are material constants. In the case of cubic parabola $n = 3$.

In the case of *material M* (Fig. 2a) it is sufficient to measure only maximum (ultimate) stress σ_M and the corresponding strain ε_M . The second information is obtained from the condition that for $\varepsilon = \varepsilon_M$ the function $\sigma = \sigma(\varepsilon)$ reaches its maximum. This condition is

$$\frac{d\sigma}{d\varepsilon} = 0. \quad (2)$$

In this case the stress can be easily expressed as a function of strain, i.e.

$$\sigma = \frac{\sigma_M}{2} \left[3 \left(\frac{\varepsilon}{\varepsilon_M} \right) - \left(\frac{\varepsilon}{\varepsilon_M} \right)^3 \right]. \quad (3)$$

In this case expressions for elastic constants are given by

$$E = \frac{3}{2} \frac{\sigma_M}{\varepsilon_M}, \quad H = \frac{1}{2} \frac{\sigma_M}{\varepsilon_M^3}. \quad (4)$$

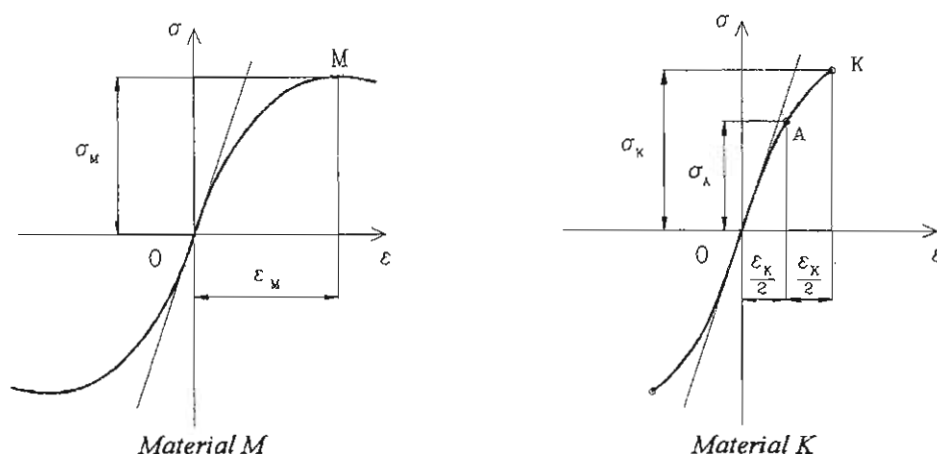


Figure 2 Representation of the stress strain diagram by cubic parabola

Mathematical description of the stress strain diagram for *material K* two pairs of the stress and strain have to be measured. One pair is rupture stress σ_K and corresponding strain ε_K . The other pair is σ_A , ε_A as shown in Figure 2b. It is convenient to select strain ε_A as one half of the strain ε_K , i.e.

$$\varepsilon_A = \varepsilon_K / 2. \quad (5)$$

Now, the stress expressed as a function of strain is given by

$$\sigma = \frac{1}{3} \left[(8\sigma_A - \sigma_K) \frac{\varepsilon}{\varepsilon_K} - (8\sigma_A - 4\sigma_K) \left(\frac{\varepsilon}{\varepsilon_K} \right)^3 \right], \quad (6)$$

or

$$\sigma = \frac{\sigma_K}{3} \left[(8\xi - 1) \frac{\varepsilon}{\varepsilon_K} - (8\xi - 4) \left(\frac{\varepsilon}{\varepsilon_K} \right)^3 \right], \quad (7)$$

where

$$\xi = \frac{\sigma_A}{\sigma_K}, \quad (8)$$

and

$$0,5 < \xi < 0,6875 = 11/16, \quad (9)$$

If $\xi = 0,5$, the material is linearly elastic and Hooke's law can be applied. However, if $\xi = 0,6875$, equation (7) coincides with (3).

3. Analysis of pure bending of straight beams

3.1 Determination of radius of curvature of the deflection curve

A straight beam in pure bending is shown in Figure 3. Strain analysis will be performed using Bernoulli-Euler kinematic assumptions. These assumptions imply that normal strain ε_x varies linearly over the cross section of a beam. Thus

$$\varepsilon_x = \frac{z}{\rho}, \quad (10)$$

where ρ is radius of curvature of the deflection curve. In the case of pure bending of a homogenous prismatic beam it is a circular arc. By substituting (10) into (7) gives

$$\sigma_x = \frac{\sigma_K}{3} \left[\frac{8\xi - 1}{\varepsilon_K} \frac{z}{\rho} - \frac{8\xi - 4}{\varepsilon_K^3} \frac{z^3}{\rho^3} \right]. \quad (11)$$

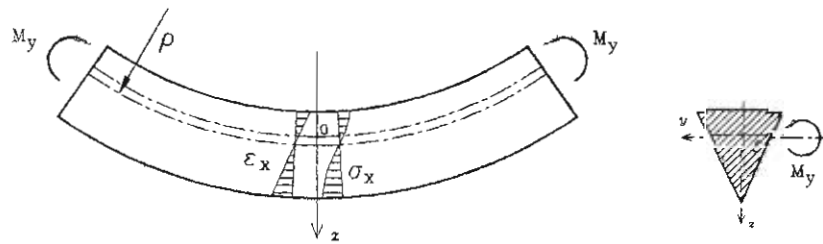


Figure 3 Pure bending of a prismatic beam

Since there is no axial load ($N = 0$) and no bending around the z axis ($M_z = 0$), the equilibrium equations are

$$N = \int_A \sigma_x dA = 0, \quad (12)$$

$$M_y = \int_A \sigma_x z dA, \quad (13)$$

$$M_z = -\int_A \sigma_x y dA = 0. \quad (14)$$

Substituting (7) into (12) yields

$$M_y = \frac{\sigma_K}{3} \left[\frac{1}{\rho} \frac{8\xi-1}{\varepsilon_K} \int_A z^2 dA - \frac{1}{\rho^3} \frac{8\xi-4}{\varepsilon_K^3} \int_A z^4 dA \right]. \quad (15)$$

Integrals on the right hand side of the above equation represent moment of inertia of the second and fourth order respectively [3,4,5], i.e.

$$I_{yy} = \int_A z^2 dA, \quad I_{yyyy} = \int_A z^4 dA. \quad (16)$$

Now (15) can be rewritten as

$$M_y = \frac{\sigma_K}{3} \left[\frac{1}{\rho} \frac{8\xi-1}{\varepsilon_K} I_{yy} - \frac{1}{\rho^3} \frac{8\xi-4}{\varepsilon_K^3} I_{yyyy} \right]. \quad (17)$$

If radius of curvature ρ is known, bending moment M_y , stress σ_x and strain ε_x can be calculated directly using (17), (11) and (10). However, if ρ is unknown and bending moment M_y is given instead, first the cubic equation (17) has to be solved for $1/\rho$. When this is done, stress and strain can be determined by substituting $1/\rho$ into (10) and (11).

3.2 Location of neutral axis

After substitution of (11) into (12) an equation locating the neutral axis can be found. This yields

$$M_y = -\frac{\sigma_K}{3} \left[\frac{1}{\rho} \frac{8\xi-1}{\varepsilon_K} \int_A z dA - \frac{1}{\rho^3} \frac{8\xi-4}{\varepsilon_K^3} \int_A z^3 dA \right] = 0.$$

This equation can be transformed into

$$\frac{8\xi-1}{\rho \varepsilon_K} S_y - \frac{8\xi-4}{(\rho \varepsilon_K)^3} I_{yyy} = 0, \quad (18)$$

where

$$S_y = \int_A z dA, \quad I_{yyy} = \int_A z^3 dA, \quad (19)$$

are the first and third moments of area respectively [10]. If the y axis passes through the centroid of the cross section, the equation will not be satisfied. Namely, in that case $S_y = 0$. However, as may be expected, I_{yyy} is different from zero. The neutral axis y' will be located at a distance e from the centroidal axis y as shown in Figure 4.

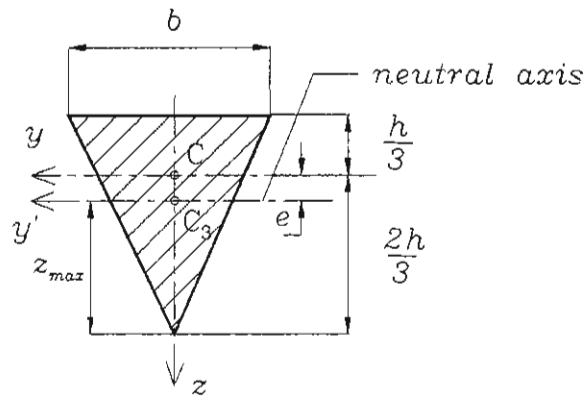


Figure 4 Location of the neutral axis

The first and the third moment of an area for the y' axis are given by [10]

$$S'_y = e A, \quad I'_{yyy} = I_{yyy} - e I_{yy} - e^3 A. \quad (20)$$

Substituting (20) into (18) yields

$$(\rho \varepsilon_K)^2 (8\xi - 1) e A - (8\xi - 4) (I_{yyy} - e I_{yy} - e^3 A) = 0,$$

or

$$(8\xi - 4) A e^3 + \left[(\rho \varepsilon_K)^2 (8\xi - 1) + (8\xi - 4) I_{yy} \right] e - (8\xi - 4) I_{yyy} = 0. \quad (21)$$

This is a cubic equation in e . Solving this equation for e , the location of the neutral axis is determined.

The origin of coordinate system lies in the neutral axis. It may be inconvenient for practical computations since the location of the neutral axis is not known in advance. If the origin coincides with the centroid of the cross section, $z - e$ instead of z should be used in formulae (10) and (11). Thus

$$\varepsilon_x = \frac{z - e}{\rho}, \quad (22)$$

$$\sigma_x = \frac{\sigma_K}{3} \left[\frac{8\xi - 1}{\varepsilon_K} \frac{z - e}{\rho} - \frac{8\xi - 4}{\varepsilon_K^3} \left(\frac{z - e}{\rho} \right)^3 \right]. \quad (23)$$

3.3 Position of the bending plane

Substituting (11) into (14) an equation locating the bending plane can be obtained. Thus

$$-\frac{8\xi - 1}{\rho \varepsilon_K} \int_A y z dA + \frac{8\xi - 4}{(\rho \varepsilon_K)^3} \int_A y z^3 dA = 0,$$

or

$$-\frac{8\xi - 1}{\rho \varepsilon_K} I_{yz} + \frac{8\xi - 4}{(\rho \varepsilon_K)^3} I_{yyy} = 0. \quad (24)$$

If the above equation is satisfied, the neutral axis and the axis around which the applied moment acts coincide, and consequently the bending plane is perpendicular to the y axis. Since only the beams with symmetric cross sections are treated, the condition

$$I_{yz} = 0, \quad I_{yyz} = 0, \quad (25)$$

is always fulfilled. Therefore, the bending plane is perpendicular to the y axis.

4. Resisting bending moment

The maximum stress and strain occur at a point most remote from the neutral axis. That point lies at the periphery of the cross section at a distance $z_{\max}$ from the neutral axis y' as shown in Figure 4. By increasing the bending moment the maximum stress and strains are increased, too. When the stress reaches the ultimate value σ_K , the bending moment reaches its maximum value M_p called the resisting moment. Then the curvature of the neutral surface attains its maximum value $(1/\rho)_{\max}$. Consequently, the radius of curvature reaches its minimum $\rho_{\min}$. Substituting $\varepsilon_x = \varepsilon_K$ into (10) yields

$$\frac{1}{\rho_{\min}} = \frac{\varepsilon_K}{z_{\max}}, \quad (26)$$

If (26) is substituted into (17), the following equations are obtained

$$M_p = M_{y,\max} = \frac{\sigma_K}{3} \left[(8\xi - 1) \frac{I'_{yy}}{z_{\max}} - (8\xi - 4) \frac{I'_{yyyy}}{z_{\max}^3} \right], \quad (27)$$

where I'_{yy} and I'_{yyyy} are moments of inertia with respect to the neutral axis y' . They may be calculated from moments of inertia about the centroidal axis using parallel axis theorem [10]. Thus

$$I'_{yy} = I_{yy} + e^2 A, \quad I'_{yyyy} = I_{yyyy} + 6e^2 I_{yy} + e^4 A. \quad (28)$$

Equation (27) may be written more compactly, i.e.

$$M_p = M_{y,\max} = \frac{\sigma_K}{3} \left[(8\xi - 1) W'_{yy} - (8\xi - 4) W'_{yyyy} \right], \quad (29)$$

where

$$W'_{yy} = \frac{I'_{yy}}{z_{\max}}, \quad W'_{yyyy} = \frac{I'_{yyyy}}{z_{\max}^3}. \quad (30)$$

are section moduli with respect to the neutral axis.

5. Some applications of the theory

5.1 Bending of a beam with rectangular cross section

The developed theory was utilized for solution of two practical problems. In the first one a beam with rectangular cross section made of the material M was subjected to pure bending. The material properties are $\sigma_M = 100 \text{ MPa}$, $\varepsilon_M = 0.004$. The rectangular

cross section has width b and height $h = 2b$. Figure 5 shows the stress distribution over the cross section for different bending moments in dimensionless form.

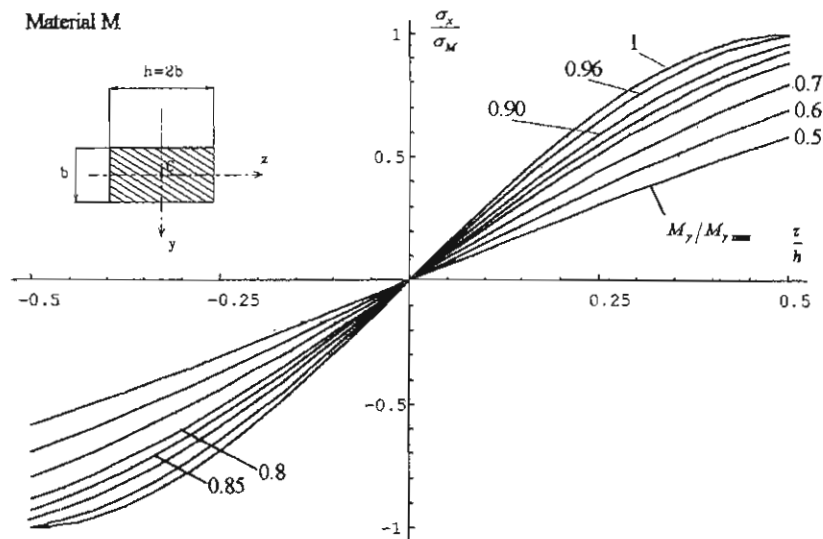


Figure 5 Dimensionless stress distribution over cross section for different moment ratios $M_y / M_{y,max}$

If it is assumed that equation (17) holds even if σ_x exceeds the value σ_M , the bending moment M_y can slightly exceed the value M_p . In this example it is 7.4 percent.

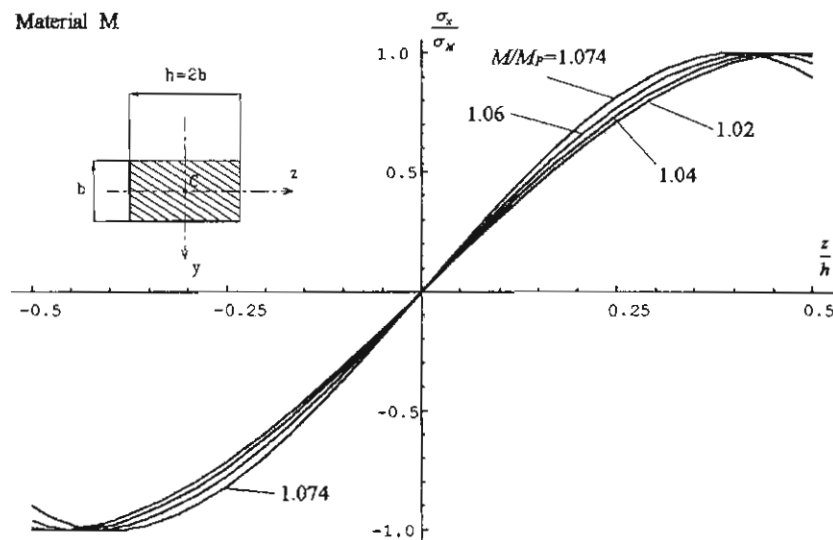


Figure 6 Dimensionless stress distribution when bending moment M_y is greater than M_p

5.2 Bending of a beam with triangular cross section

The beam is made of the material K with the following properties: $\sigma_K = 100 \text{ MPa}$, $\varepsilon_K = 0.004$, $\xi = 0.65$. The cross section of the beam is an isosceles triangle with width b and height $h = 2b$. The bending moment equals the resisting moment M_p .

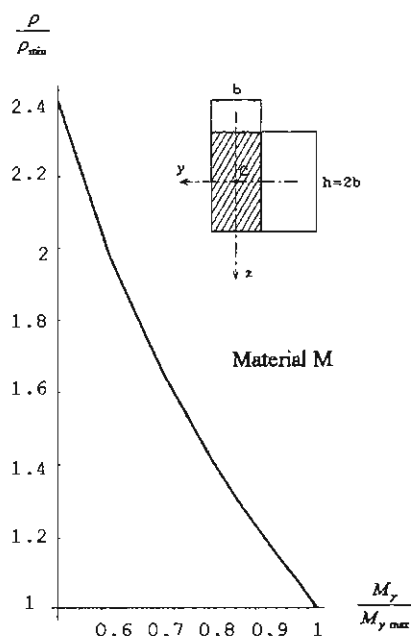


Figure 7 Radius of curvature as a function of bending moment

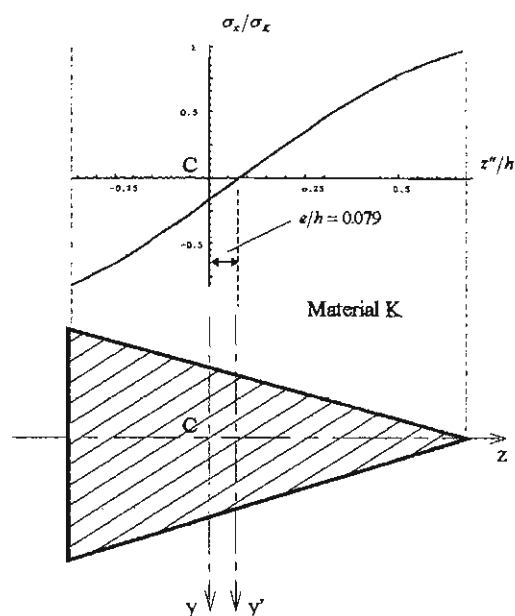


Figure 8 Dimensionless stress distribution over the cross section

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Flexible Mechanisms Modelling using Modal Approach

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Abstract

The scope of the paper is mathematical modelling of flexible mechanisms using modal approach, and a comparison of the given method with the finite element method (FEM) approach considering numerical and computational issues. Although the rigid body dynamics and the vibrations of fixed structures are already well researched, the flexible robot manipulators and similar flexible mechanism systems with low stiffness links still need better and more efficient algorithms regarding the automatic control requirements. Advanced model based control strategies require models of relatively low order, but still able to describe system dynamics well enough. In order to develop the mathematical model Lagrangian formulation is employed. The small deformation of the flexible links is assumed. The eigenvectors are used as the space approximation functions and Galerkin's method is applied to reach partial formulation. The time solution procedure is step by step based algorithm. The results are compared with those obtained by the finite element procedure and with the experimental results obtained on the laboratory model of the flexible robot manipulator. The numerical and computational efficiency of applied method is discussed.

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1 Introduction

This work has arise from our research on control of flexible robot manipulators. Lightweight robot arms with flexible links have stimulated great interest of researchers from the field of robotics, automatic control and applied mechanics. Motives for that interest are due to application of lighter materials and making less rigid construction of the fast tracking systems like industrial robots, space and earth based telescopes, space manipulators and other mechanical systems. Flexible robots compared with their rigid and much more common counterparts should be more productive, more economical and cheaper, and more applicable in space operations due to their lighter construction, faster motion and smaller actuator requirement. However, very important characteristics of precise positioning, repeatability and precise trajectory tracking are deteriorated due to significant vibrations of their links. A lot of research efforts have been given in order to develop advanced controllers that can be able to overcome the problems originated from lighter and less robust constructions of the mechanisms. We can say the lack of "muscles" is compensated for by more "brain". Such advanced controllers are, in principle, model based. They require mathematical model of the system to control it. The question is how accurate that model should be? It is particularly emphasised in flexible mechanisms where the original model is of infinite order. More knowledge we have about the controlled system, or more accurate model we have, the better performances of the controlled system we can get. However, due to control application requirements, the mathematical model should be as small as possible. The goal of our research, partially described in this paper, is to give the answers about approach to model flexible mechanisms, particularly regarding control requirements. The finite elements are well-known method for the discretization of continuous system. The idea behind the finite element method (FEM) is to provide formulation that can exploit computer automation for the analysis of irregular systems. That makes the method particularly favourable for complex structures. However, it is somehow disregarded in the modelling for control. That is due to enlargement of the models obtained by this technique. The modal expansion methods give smaller models for the same accuracy. Still, the principal drawback of the modal approach is that the application is difficult for complex structures. In the paper mathematical mod-

elling using modal approach is presented, and the comparison with the FEM considering numerical and computational issues is given using example of the one-link flexible robot manipulator. Experimental results are given and commented, as well. The paper is structured as follows: Chapter 2 briefly explains mathematical model of the mechanisms elastic link motion. Chapter 3 introduces modal approach to the mechanism motion equation. Numerical example is given in Chapter 4, while the comparisons between numerical and experimental results are in Chapter 5. Chapter 6 is conclusion.

2 Mathematical model of the mechanism's elastic link motion

Mathematical model of the elastic beam under "pin-free" boundary conditions, showed in Figure 1 is considered. The following data, in accordance with real model, are used:

$$\begin{array}{lll} L = 1m & h = 0.1m & b = 0.0015m \\ A = 1.5 \cdot 10^{-4}m^2 & I_y = 2.8 \cdot 10^{-11}m^4 & E = 200 \cdot 10^9 N/m^2 \\ \rho = 7850 kg/m^3 & t_o = 0.1s & T_o = 3.7272 Nm \\ \omega_o = 6.9114s^{-1} & & \end{array}$$

$$T = \begin{cases} T_o \sin[\omega_o(t - t_o)] & t_o \leq t \leq t_o + 2\pi/\omega_o \\ 0 & \text{otherwise} \end{cases}$$

The left bearing is supposed to be fixed. Governing equation is elementary Euler-Bernoulli partial differential equation:

$$EI \frac{d^4 y}{dx^4} + \rho A \frac{d^2 y}{dt^2} = p \quad (1)$$

where $y(x, t)$ is transverse displacement, $EI(x)$ is the flexural rigidity, $\rho A(x)$ is uniform mass of the beam and p is transversal load.

After discretization, either by finite element method or modal expansions, and after describing the motion either by elements of vectorial mechanics or by analytical mechanics (we have used Lagrangian formulation), the final form of the motion is obtained:

$$M\ddot{q} + C\dot{q} + Kq = Q + f \quad (2)$$

where $\mathbf{M}$ is inertia matrix, $\mathbf{K}$ is stiffness matrix, $\mathbf{C}$ is viscous damping matrix, $\mathbf{Q}$ is vector of external forces and torque, $\mathbf{f}$ is vector of nonlinear effects and $\mathbf{q}$ is generalized variables vector.

3 Modal approach to the solution of mechanism link motion equation

Governing equations (1) and (2) can be solved directly or after transformation. Both approaches have some advantages. Here, solution procedure involving modal transformation is presented. A new variable $\mathbf{r}$ is introduced in order to reduce complexity of governing equations:

$$\mathbf{q} = \mathbf{V}\mathbf{r} \quad (3)$$

where orthogonal matrix $\mathbf{V}$ represents eigenvectors of the eigenproblem

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{v} = 0 \quad (4)$$

normalised to the mass matrix $\mathbf{M}$. Introducing new variable $\mathbf{r}$ in governing equation it becomes a set of decoupled equations:

$$\ddot{\mathbf{r}}_i - \omega_i^2 \mathbf{r}_i = \mathbf{f}_i \quad (5)$$

where $i = 1, 2, \dots, n \leq N$ and

$$\begin{aligned} \mathbf{V}^T \mathbf{M} \mathbf{V} &= \mathbf{I} \\ \mathbf{V}^T \mathbf{K} \mathbf{V} &= \Omega^2 \\ \mathbf{f}_i &= \mathbf{v}_i \mathbf{Q}^T \end{aligned} \quad (6)$$

The transformed equation (5) is usually solved by means of the step by step numerical methods. The obvious advantages of modal approach are lower numerical effort in numerical calculations in each step. The disadvantages are eigenvalue problem solution and transformations of load in each step and necessary transformations variable $\mathbf{r}$ to calculate displacement $\mathbf{q}$. Modal solution of non-linear equation of motion is already used [1-3] but not in solving coupled rigid body motions and elastic displacement.

4 Numerical example

A problem according to the Figure 1 is solved analytically and numerically. The eigenvectors of governing equation read,[4-5]:

$$y_i(x) = D_i((\sin(\beta_i x) + \sinh(\beta_i x)) + (\sin(\beta_i x) - \sinh(\beta_i x))) \quad (7)$$

where β_i , and D_i are listed in Table 1.

β_i	D_i
0.00583	176569
3.9266	0.945762
7.06858	1.00241
10.2102	0.999896
13.3518	1.0

Table 1: Eigenfunctions coefficients

The graph of eigenvectors are shown in Figure 2. The eigenvalues and eigenvectors for five finite element model, Figure 3, are listed in Table 2 and Figure 4.

<i>Analytical</i>	<i>Numerical</i>	<i>Experimental</i>
0.	0.	-
33.6838	33.6922	30.5
109.157	109.419	85.5
227.748	229.851	158.3
389.462	397.448	-

Table 2: Analytical, Numerical (FEM) and Experimental Frequencies

The obvious difference between analytical model eigenfunctions and finite element model vectors stems from the way in which rotation is decoupled from elastic displacement in finite element model [5]. In the finite element model, the rigid body rotation of the link is represented by the first equation i.e. by the degree of the freedom related to the pole, while the linear function describing this motion is included in

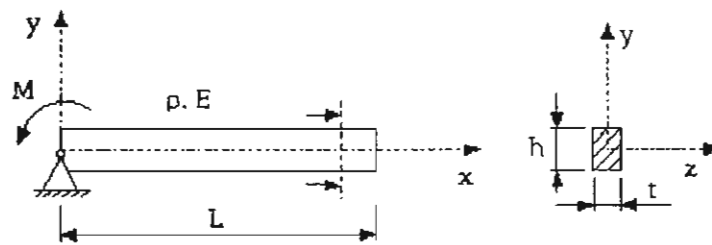


Figure1. Beam with "pin-free" boundary conditions

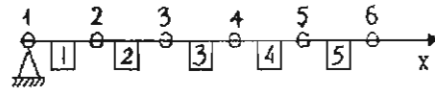


Figure3 Finite element model for pin-free beam

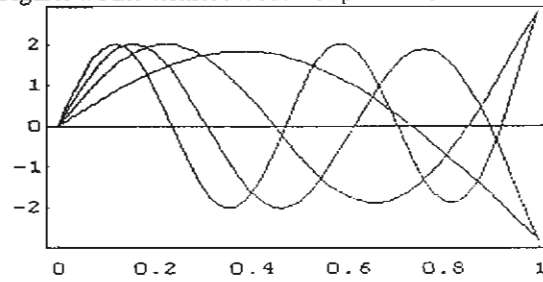


Figure 2. Eigenfunctions of governing equation

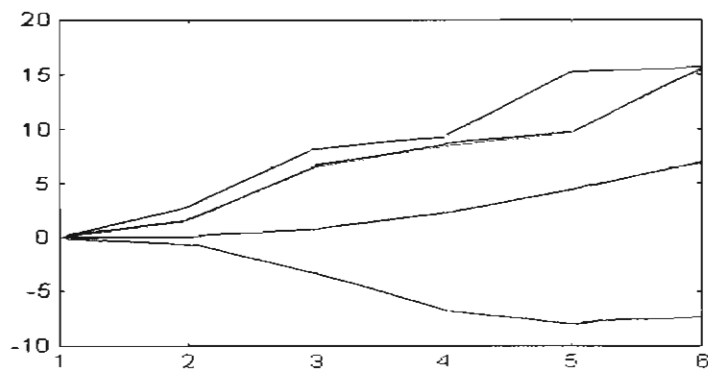


Figure 4 Eigen modes for beam finite element model

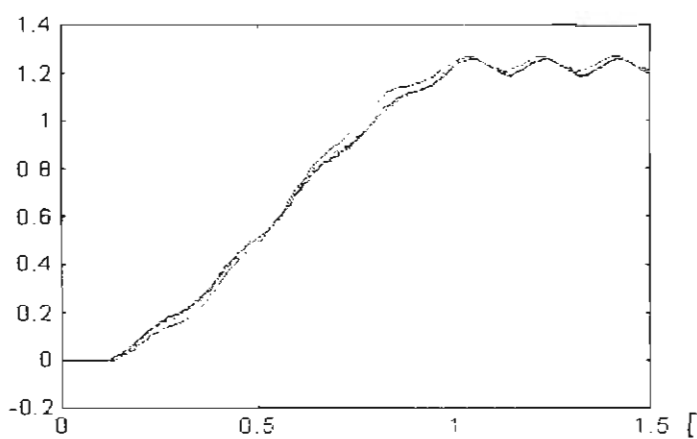


Figure 5. The angle of rotation of the beam pin support

the elastic modes. Integration of the modal equations is performed by Runge-Kutta method using adaptable step size algorithm to provide the numerical accuracy. The results are presented in Figure 5.

The dot solution in Figure 5 represent motion for rigid body. The further solutions include always one elastic mode more.

5 A comparison of numerical solutions with the experimental results

Comparative analysis of experimental and simulation results has been performed in order to verify different approaches in modelling and control of flexible mechanisms. For those purposes, the laboratory model of the flexible robot arm is built, and it serves as a test-bed. The arm is driven by the permanent- magnet synchronous motor. Resolver is employed for the joint angle measurement, and two pairs of strain gauges connected in the half-bridge are employed for the measurement of bending moments of the link. Description of the laboratory model of the flexible arm model and belonging equipment is given in [6-7]. The significance of the hard nonlinearities induced by the actuator (the stick-slip friction in the joint and the effect of "valleys" and "peaks" of permanent magnet motor) requires their inclusion in the numerical model. Such fully analysis of the experiments and numerical results has been performed only for the finite element modelling approach so far, and it is described in the [5]. The numerical results there show good agreement with experiments, both in frequency domain analysis and in time domain analysis. The resonant frequencies obtained experimentally and those from finite element mathematical models (discretized by 5 elements) and from modal mathematical model are given in Table 2.

6 Conclusion

A modal approach to the flexible robot link is used in order to describe nonlinear transient motion of mechanism. Results are compared with direct solution and experimental data. Modal approach is shown as efficient, accurate and reliable procedure. In further development a modal synthesis will be used to build up to simulate more complex

systems.

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A System Approach to Modelling and Simulation of Multibody Systems Using Acausal Bond Graphs

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Abstract:

An approach to simulation of mechanical systems using bond graphs is presented. The mechanical system may consist of several rigid bodies connected by joints. Modelling of such a system can be developed by system decomposition. At each level of decomposition processes considered were described using bond graphs. In that way model of the system has been represented as a tree of models. Models in the tree represent objects that contain all information on its functionality. The bond graph used are acausal not implying any specific input-output relationships. Description of the system motion appears as a system of the sparse nonlinear differential and algebraic equation. The equations are solved using methods based on backward differentiation formula. The approach has been applied to simulation of the motion of the reciprocating mechanism.

1. Introduction

Many mechanical systems consist of interconnected bodies. Kinematic relationships describing such systems are typically nonlinear. The dynamic processes in the systems are complex too so that numerical simulation of such systems is not an easy task, see [1-4]. In the case of tree like (open) systems, governing equations can be generated in the state space form. Unfortunately, often that is not feasible because of the closed loops in the system. Modelling of such systems were usually done using the Lagrange multiplier approach. The resulting equations have the form of the differential-algebraic equations (DAEs). The common approaches to numerical solution of such equations are based on using some form of transformations to the ordinary differential equations form [1,3]. With development of differential algebraic equation methods direct solutions of equations are possible too [2,5]. For review of various approaches to the multibody dynamic see [4].

The bond graphs [6] attract fairly much attention as the possible modelling tools for multibody systems too. Classical multibody approaches place attention to system equations and ways to satisfy them. The bond graphs on the other hand take physical point of view on multibody dynamics and are more engineering oriented. Using bond graphs the interaction in the system can be more easily understood. In addition they enable relatively easy modification of models, such as modelling of deformations, modelling of nonmechanical parts of the systems, etc.. At first glance there are differences in the approaches, but in essence there are many similarities too ([7]).

The classical bond graph approach to multibody dynamics is based on causality relationships in the systems. The systems are usually described in the state space form. In the case of algebraic loops or differential causalities, some form of bond graph transformations, or transformation at equation level, are necessary. Acausal form of bond graphs are more general and are leading to the DAEs that can be solved using

suitable methods [5] mentioned above. In the following the short description of modelling approach and methods used are given.

The processes in a typical multibody system are fairly complex so it is essential to represent bond graph models in an easily understandable form. The approach that has been taken was based on the system decomposition. Starting from the system level, at each level of decomposition the system are seen as set of simpler objects that interact one with other through its ports. The structure of the system and the interactions are depicted as word models connected by bonds. The decomposition process ends up at the level of detail chosen by modeler. In the case of multibody systems that were usually the basic kinematics and dynamics of bodies the system consist of. The resulting model were created in the form of tree of objects connected through its ports [8].

In essence the models of objects are created as objects presenting themselves externally as bond graphs and containing internally all information describing its functionality, i.e. its parameters and relations. The same is true for interconnections between the objects. Every port and bond are represented as array of simpler ports and bonds respectively. Looking from multibody perspective the developed approach supports usual vector, matrix or tensor representation as well. In this way models can be presented in the form that helps multibody system dynamics understanding by pushing details to the lower levels. The model of the complete system or its parts can be held in libraries and used later on for building up some other systems as well.

Based on the system representation described above the mathematical model created are in the form of semi-explicit DAEs and are to be solved using suitable DAEs solvers. During the system model build up, matrix of partial differentials are created using procedures for symbolic differentiation.

Solving the system DAEs are performed using a method based on backward differentiation formula (BDF) described in [5]. The method has been designed in a such way to enable evaluation of the initial conditions for all variables used. To enable solving of the multibody dynamics models it was chosen to control the differentiated variables only at user chosen level of relative and absolute error. The method developed can handle successfully the semi-explicit systems of model equations of index up to two.

The approach and methods for modelling and simulation of engineering systems outlined above have been implemented as Bond Graph Simulations System (BondSym). The developed system runs on PCs under Windows 95 and NT operating systems [8]. The method has been applied to the problem of motion of reciprocating mechanism.

2. Description of the body motion

In this section governing equations of multibody system in plane are developed. Method can be readily applied to three dimension too. A typical body with forces applied at two points A and B and body centre of mass C is shown in Figure 1. Two frames of reference are defined, an inertial Oxy and body fixed frame $Cx'y'$. The positions of points are given by

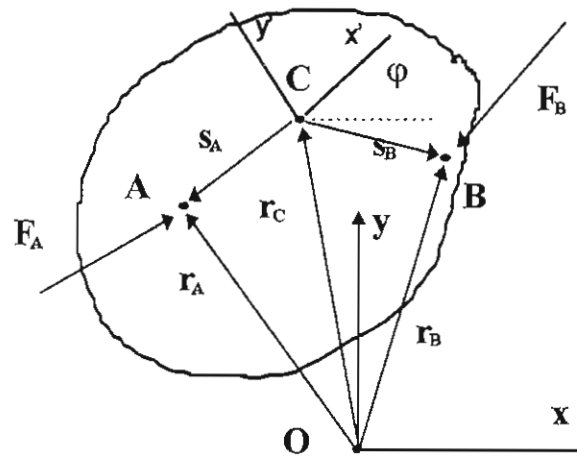


Figure 1

$$r_A = r_C + A s_A \quad (1)$$

$$r_B = r_C + A s_B \quad (2)$$

where A is coordinate transformation matrix. After differentiation, the corresponding equations for velocities are

$$v_A = v_C + R A s_A \omega \quad (3)$$

$$v_B = v_C + R A s_B \omega \quad (4)$$

where R presents rotation matrix, v_A , v_B and v_C corresponding velocities and ω body angular velocity.

Using familiar laws from Engineering Mechanics dynamic equations of body motion are given by

$$\frac{dp}{dt} = F_A + F_B \quad (5)$$

$$p = M v_C \quad (6)$$

$$\frac{dK_C}{dt} = (R A s_A)^T F_A + (R A s_B)^T F_B \quad (7)$$

$$K_C = I_C \omega \quad (8)$$

where p presents body momentum, M mass matrix, v_C velocity of body center of mass, K_C moment of momentum for center of mass and I_C mass moment of inertia with respect to center of mass. If A and B are joints, corresponding forces represent reaction forces. There can be other forces acting on the body too.

The joints impose constraints on motion of connected bodies i and j . In the case of revolute joints, constraints require that points A on body i and point B on body j are coincident

$$(r_C + A s_A)_i = (r_C + A s_B)_j \quad (9)$$

Corresponding relationship in terms of velocities are

$$(v_C + R A s_A \omega)_i = (v_C + A s_B \omega)_j \quad (10)$$

Similar relationships hold for translational joints restricting the motion in single direction only. Assembling all equations, governing equations of motion of the system can be put in the form

$$\frac{dP}{dt} = G^T F + F_e \quad (11)$$

$$P = Mv \quad (12)$$

$$\Phi(q,t) = 0 \quad (13)$$

where P presents the vector of momentum and moment of momentum of bodies, q the vector of body generalized coordinates, v corresponding velocities, F joint forces, F_e other forces and moments on the system, M generalized mass and moment of inertia matrix. The last equation presents the position constraints imposed by the joints, and G is corresponding Jacobian matrix.

The equations (11)-(13) present a system of differential-algebraic equations that correspond to usual formulation in multibody dynamics based on the Lagrange multipliers method. The system has index 3 and are hard to solve at present state of DAEs methods [6]. The problem can be solved by lowering the index of DAEs through differentiation of constraint equation.

The relationship corresponding to the velocity constraints given by (10) in assembled form is

$$Gv = 0 \quad (14)$$

The equations (11), (12) and (14) are DAEs of index 2. They can be solved using for instance BDF methods with some modifications. The problem that remains is how to satisfy the positional constraint (13) too. With relatively tight error control it can be expected that corresponding drift in joints position can be tolerable. If not there are various penalty functions and constraints stabilization techniques that can be used to improve the solution ([5]). The last form of the system governing equations are well suited for bond graph representation.

3. Modelling of the reciprocating mechanism

Bond graph method developed enables systematic development of an engineering systems models. The approach has been applied to modelling of the reciprocating mechanism shown in Figure 2. Such a mechanism presents a vital part of many engines and pumps so that understanding of its behaviour is of great practical importance. It is relatively simple, nonlinear, closed system and was well suited for testing modelling method and simulations results.

General structure of the mechanism is given in Figure 3. The system consists of a crank, conrod and piston connected by bearings. The piston slides inside the cylinder. It was assumed that the system was driven by suitable shaft rate controlled torque generator (source). The monitoring of the system behaviour is represented using objects named after parameters observed. These objects have been implemented as X-T or X-Y plotters.

The crank and conrod present the rigid bodies that can be mathematically described by the equations (3-8). In terms of bond graphs dynamic structure of the crank can be represented as in Figure 4. Practically the same representation was used for conrod. Differences are in the direction of assumed power flows.

The Rotation object represents rotational dynamics and was modelled using simple inertial element. Similarly the Translation represents translational dynamics of the body and were modelled by the pair of inertial elements for two axes of motion. The object LinRot presents an array of transformers describing relationships

between linear velocities that are caused by the body rotation with corresponding angular velocity. Simultaneously the object describes corresponding relationship between applied force forces and its moment for other point, the center of mass in case shown.

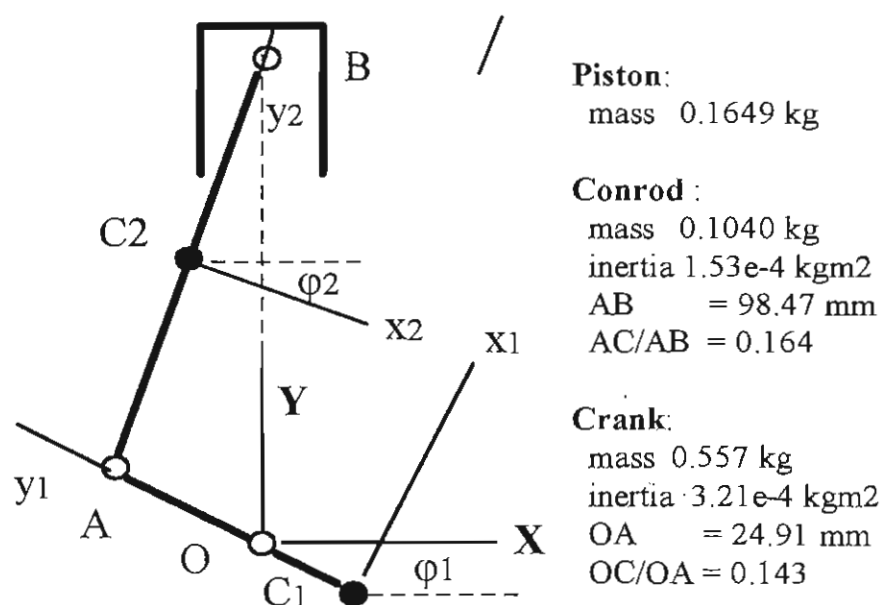


Figure 2

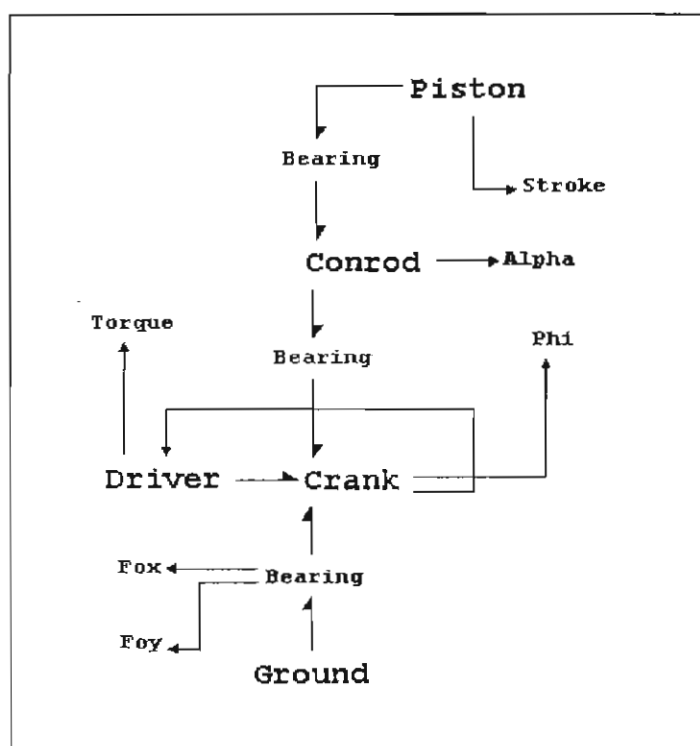


Figure 3

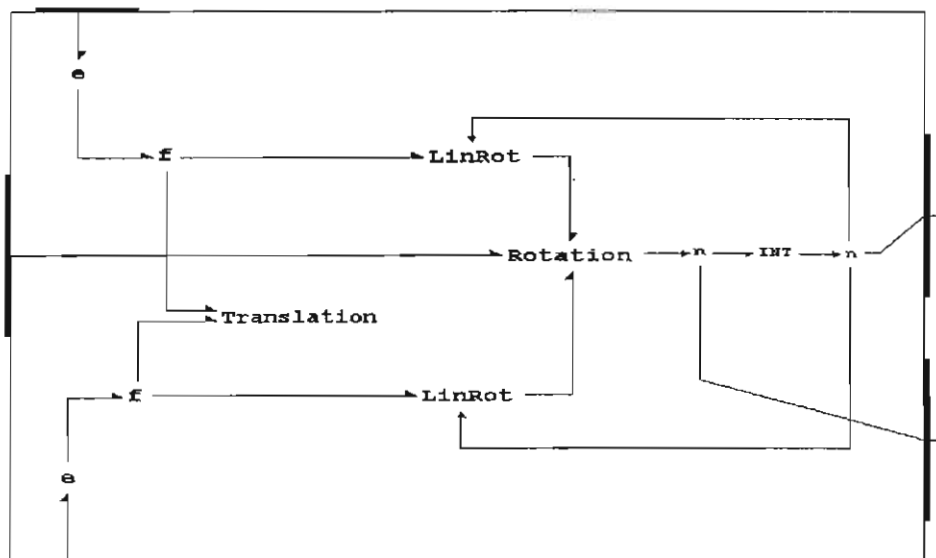


Figure 4

The bearing used for connection of crank and conrod etc. can be modelled very simply using the fact that corresponding connected parts have equal velocities. Piston can move in y directions only. Constraining the motion in x direction is represented by zero velocity source. Similarly the Ground imposes fixation of system and was represented by pair of zero velocity sources for x and y directions.

The model of reciprocating mechanism was presented in idealized form, neglecting any deformation or friction. Using bond graphs it is easy matter to add on such characteristics, but this "naked" model can serve as fairly tough test problem. The parameters of models are taken from [10] where it has been used for analysis of vibrations using more classical (non bond graph) approach.

3. The results of simulations

Based on model developed, governing equations has been generated automatically in descriptor form.. That means that not only generalized coordinates are used, but also most of effort (forces, moments) and flows (velocities) variables. Resulting systems had the form of DAEs of very sparse structure (two or three variables per equations only). Simulation was made with relative and absolute error tolerance of 10^{-6} . Some simulation results are shown in Figure 5 and 6.

To estimate error of results obtained the simple formulas relating the crank and conrod angles and piston stroke has been used

$$\sin \varphi_2 = -\frac{OA}{AB} \sin \varphi_1$$

$$y_p = -[OA \cdot (1 - \cos \varphi_1) + AB \cdot (1 - \cos \varphi_1)]$$

The error in the piston stroke has been found of order 10^{-5} and this correspond to the range of errors in algebraic variables expecting of BDF methods (see [5]). The error in conrod angle was much larger (about $7 \cdot 10^{-4}$). Simulations has been repeated with errors tolerances of 10^{-8} and the new errors found was 10^{-6} and 10^{-4} respectively.

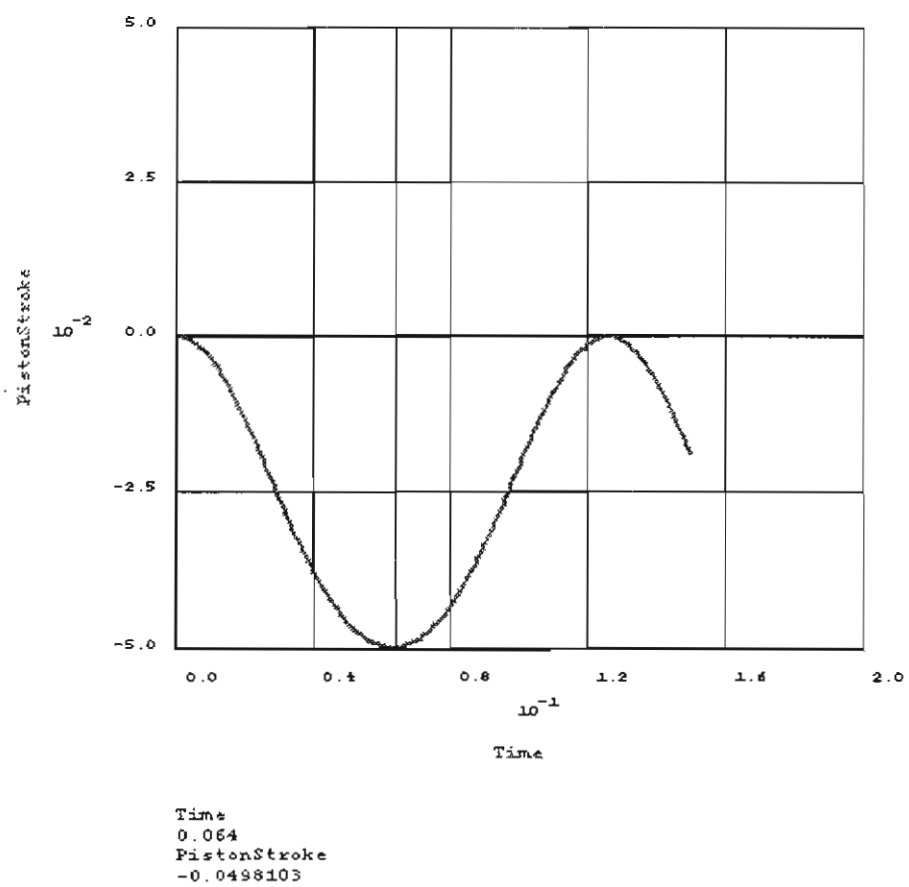


Figure 5

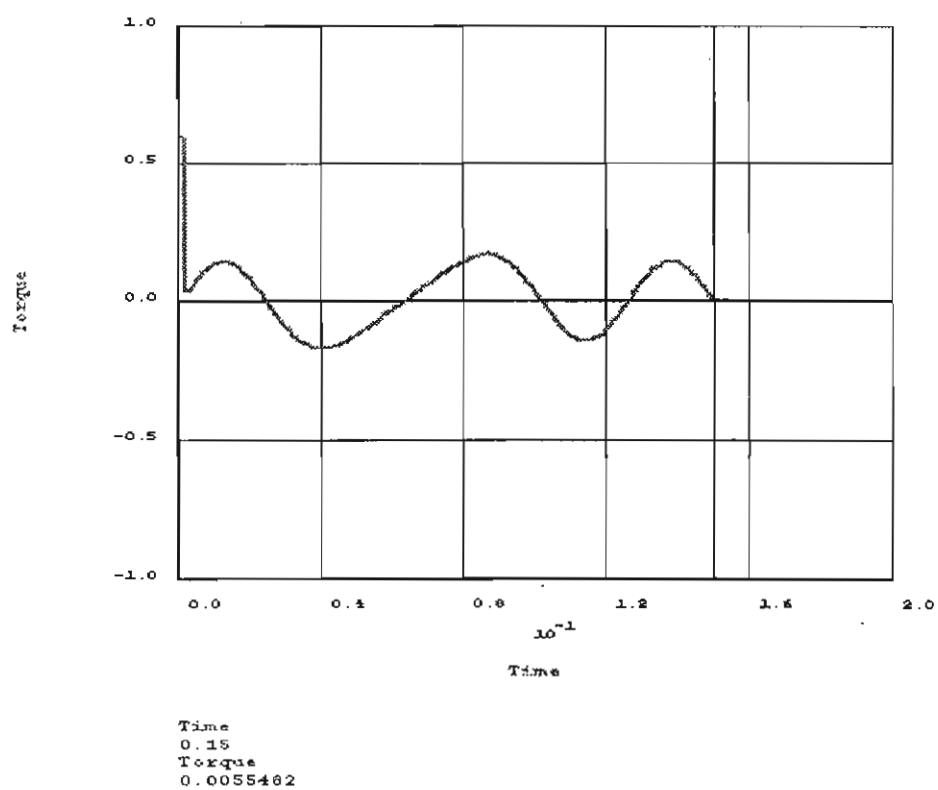


Figure 6

The bond graph models implement velocity constraints given by (14) only. The position constraints are in general not satisfied. In the case considered constraints violation didn't seem dramatical. Some authors suggest implementing spring and damper elements in the joints in order to force approximate fulfillment of constraints requirements (see [9]). That was found ineffective leading to very stiff systems.

Among various shemas used for satisfying position constraints from point of view of bond graphs, an interesting approach was suggested in [5] (see also [2]). Instead of simple integration, generalized coordinates are given by

$$\frac{dq}{dt} = v + G^T u \quad (15)$$

The equations (13) and (15) together with equations (11), (12) and (14) now constitute system of DAEs describing the systems motion. Such an approach is usually termed constraint stabilization. The variable u can be viewed as corrections in direction of gradient to constraint space (see ref. [4]) for more general approach to geometry of motion of constrained mechanical systems).

5. Conclusion

The bond graph modelling approach can be successfully applied to modelling of multibody systems. As general modelling approach it is not specially designed for multibody systems. Nevertheless, it offers systematic approach to simulation model building. It is very similar in form to the real equipment, and offers better understanding of processes in the system. Results achieved are quite satisfactory and can be further improved.

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INCREMENTAL HARMONIC BALANCE ANALYSIS OF FORCED AND AUTONOMOUS COUPLED NONLINEAR OSCILLATORS

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Abstract

The paper presents the incremental harmonic balance analysis of Van der Pol- Duffing oscillators with finite degrees of freedom exhibiting self sustained oscillations (autonomous systems) or performing oscillations under periodic excitation. The proposed method is used for calculation of stationary time responses, limit cycles, stable and unstable branches of the resonance curves and for construction of solution diagrams in dependence of system parameters. The stability of the calculated solutions is proved by Floquet theory. Internal resonance of a nonlinear mechanical system consisting of two masses and two springs with a linear damper is analyzed, where one of the springs is linear and the other has a cubic nonlinear characteristic. As example of autonomous system, the coupled nonlinear oscillator with two degrees of freedom is calculated, where the solution diagram is shown

1. Introduction

The paper presents computation of nonlinear periodic solutions in coupled Van der Pol – Duffing oscillators by using incremental harmonic balance method (IHB method). These solutions can be obtained also by numerical time-integration methods and various perturbation methods. Methods of numerical integration cannot be computationally efficient, because we look for steady-state solutions, which can be computed after transient period. Perturbation methods [1] are efficient and make global stability analysis possible, but they are limited on dynamical systems with small nonlinearity.

By using IHB method the frequency response of the resonance curve of periodically excited nonlinear oscillators can be easily computed by incrementing the frequency of the exciting signal. In principle stable as well as unstable branches of resonance curve can be computed. If an additionally parameter is varied inside the augmentation process, the complete solution diagram with bifurcation points is constructed. In the process of frequency incrementation as well as in the augmentation of the selected system parameter we proceed from previous or assumed solution and then introduce Newton-Raphson iterative procedure until convergent solution is obtained.

2. Incremental Harmonic Balance (IHB) Method

Oscillations of nonlinear dynamical systems with N degrees of freedom, subjected to periodic excitation can be described by system of nonlinear ordinary differential equations:

$$\omega^2 \ddot{x}_i + f_i(x_1, \dots, x_N; \dot{x}_1, \dots, \dot{x}_N; \omega, \lambda) = g_i \cdot \cos[(2m - l)\tau] \quad (i = 1, \dots, N). \quad (1)$$

where dimensionless time $\tau = \omega t$ is introduced as independent variable and where $f_i(x_1, \dots, x_N; \dot{x}_1, \dots, \dot{x}_N; \omega, \lambda)$ denote nonlinear functions of dependent variables x_i and corresponding first derivatives $\dot{x}_i = \frac{1}{\omega} \frac{dx_i}{dt}$, ($i = 1, \dots, N$) with exciting frequency ω and arbitrary changeable parameter λ . Assuming harmonic excitation, g_i are amplitudes of exciting signals and m is an appropriate integer. The Newton–Raphson iterative procedure starts with an initial or guessed solution $x_{i0}, \dot{x}_{i0}, \omega_0, \lambda_0, g_{i0}$, ($i = 1, \dots, N$) and computes the neighbouring solution:

$$\begin{aligned} x_i &= x_{i0} + \Delta x_i, \dot{x}_i = \dot{x}_{i0} + \Delta \dot{x}_i, \omega = \omega_0 + \Delta \omega, \lambda = \lambda_0 + \Delta \lambda; \\ g_i &= g_{i0} + \Delta g_i, (i = 1, \dots, N) \end{aligned} \quad (2)$$

with adding increments $\Delta x_i, \Delta \dot{x}_i, \Delta \omega, \Delta \lambda, \Delta g_i$ to initial solution. Putting expressions (2) into equations (1) and developing equations (1) in Taylor's series about initial solution, linearized incremental differential equations are obtained:

$$\begin{aligned} \omega_0^2 \Delta \ddot{x}_i + \sum_{j=1}^N \left(\frac{\partial f_i}{\partial \dot{x}_j} \right)_0 \Delta \dot{x}_j + \sum_{j=1}^N \left(\frac{\partial f_i}{\partial x_j} \right)_0 \Delta x_j &= \\ = [g_{i0} \cos[(2m-1)\tau] - \omega_0^2 \dot{x}_{i0} - f_{i0}] - \left[\left(\frac{\partial f_i}{\partial \omega} \right)_0 + 2\omega_0 \dot{x}_{i0} \right] \Delta \omega - \left(\frac{\partial f_i}{\partial \lambda} \right)_0 \Delta \lambda + \cos[(2m-1)\tau] \cdot \Delta g_i \end{aligned} \quad (3)$$

where all incremental expressions of higher order are neglected. If initial values and corresponding increments are combined in following vectors:

$$\{X_0\} = [x_{i0}, \dots, x_{N0}]^T, \{\dot{X}_0\} = [\dot{x}_{i0}, \dots, \dot{x}_{N0}]^T, \{G_0\} = [g_{i0}, \dots, g_{N0}]^T, \{\Delta G\} = [\Delta g_1, \dots, \Delta g_N]^T, \{F_0\} = [f_{i0}, \dots, f_{N0}]^T \quad (4)$$

and matrices $[C]$ and $[K]$ as well as vectors $\{Q\}$, $\{P\}$ and $\{R\}$ are defined with elements:

$$C_{ij} = \left(\frac{\partial f_i}{\partial \dot{x}_j} \right)_0, \quad K_{ij} = \left(\frac{\partial f_i}{\partial x_j} \right)_0, \quad Q_i = \left(\frac{\partial f_i}{\partial \omega} \right)_0, \quad P_i = \left(\frac{\partial f_i}{\partial \lambda} \right)_0, \quad R_i = g_{i0} \cos[(2m-1)\tau] - (\omega_0^2 \dot{x}_{i0} + f_{i0}) \quad (5)$$

then linearized incremental differential equations can be written in compact matrix form as:

$$\omega_0^2 \{\Delta \ddot{X}\} + [C] \cdot \{\Delta \dot{X}\} + [K] \cdot \{\Delta X\} = \{R\} - (2\omega_0 \{\dot{X}_0\} + \{Q\}) \Delta \omega - \{P\} \cdot \Delta \lambda + \cos[(2m-1)\tau] \cdot \{\Delta G\} \quad (6)$$

Matrix differential equation (6) is linear with periodic coefficients. Solution of this matrix equation can be obtained by Galerkin procedure, which presents the second step in the IHB method. Because of periodic excitation, we can expect, that steady-state solution is also periodic:

$$\begin{aligned} x_{i0} &= \sum_{n=1}^K (a_{i,n} \cos[(2n-1)\tau] + b_{i,n} \sin[(2n-1)\tau]) = [T] \{a_i\}, \\ \Delta x_i &= \sum_{n=1}^K (\Delta a_{i,n} \cos[(2n-1)\tau] + \Delta b_{i,n} \sin[(2n-1)\tau]) = [T] \{\Delta a_i\}; (i = 1, \dots, N) \end{aligned} \quad (7)$$

where:

$$\begin{aligned} [T] &= [\cos \tau, \cos 3\tau, \dots, \cos[(2K-1)\tau]; \sin \tau, \sin 3\tau, \dots, \sin[(2K-1)\tau]] \\ \{a_i\} &= [a_{i,1}, a_{i,2}, \dots, a_{i,K}; b_{i,1}, b_{i,2}, \dots, b_{i,K}]^T \\ \{\Delta a_i\} &= [\Delta a_{i,1}, \Delta a_{i,2}, \dots, \Delta a_{i,K}; \Delta b_{i,1}, \Delta b_{i,2}, \dots, \Delta b_{i,K}]^T \end{aligned} \quad (8.a,b,c)$$

By introducing matrix $[Y]$ and vectors $\{A_0\}, \{\Delta A\}$:

$$[Y] = \begin{bmatrix} [T] & [0] & \dots & [0] \\ [0] & [T] & \dots & [0] \\ \vdots & \vdots & \ddots & \vdots \\ [0] & [0] & \dots & [T] \end{bmatrix}, \quad \{A_0\} = \begin{bmatrix} \{a_1\} \\ \{a_2\} \\ \vdots \\ \{a_N\} \end{bmatrix}, \quad \{\Delta A\} = \begin{bmatrix} \{\Delta a_1\} \\ \{\Delta a_2\} \\ \vdots \\ \{\Delta a_N\} \end{bmatrix} \quad (9)$$

the vectors $\{X_0\}, \{\Delta X\}$ can be expressed as follows:

$$\{X_0\} = [Y] \cdot \{A_0\}, \quad \{\Delta X\} = [Y] \cdot \{\Delta A\} \quad (10)$$

Performing Galerkin procedure for one period of dimensionless time $\tau = \omega t$, that is on the interval $(0, 2\pi)$:

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} \delta\{\Delta A\}^T \left\{ [Y]^T \left[\omega_0^2 [\ddot{Y}] + [C][\dot{Y}] + [K][Y] \right] \{\Delta A\} \right\} d\tau = \\ \frac{1}{\pi} \int_0^{2\pi} \delta\{\Delta A\}^T \left\{ [Y]^T \left[\{R\} - (\{Q\} + 2\omega_0 [\ddot{Y}]\{A_0\})\Delta\omega - \{P\}\Delta\lambda + \{\Delta G\} \cos[(2m-l)\tau] \right] \right\} d\tau \end{aligned} \quad (11)$$

where $\delta\{\Delta A\}^T$ presents arbitrary variations of harmonic increments vector, one obtains the linear matrix equation for unknown harmonic increments vector $\{\Delta A\}$:

$$[k]\{\Delta A\} = \{r\} + \{q\}\Delta\omega + \{p\}\Delta\lambda + [s]\{\Delta G\} \quad (12)$$

where $[k]$ and $[s]$ are matrices and $\{r\}, \{q\}, \{p\}$ are following vectors:

$$[k] = \frac{1}{\pi} \int_0^{2\pi} [Y]^T \left[\omega_0^2 [\ddot{Y}] + [C][\dot{Y}] + [K][Y] \right] d\tau, \quad [s] = \frac{1}{\pi} \int_0^{2\pi} [Y]^T \cos[(2m-l)\tau] d\tau. \quad (13.a,b)$$

$$\{r\} = \frac{1}{\pi} \int_0^{2\pi} [Y]^T \left(\{G_0\} \cos[(2m-l)\tau] - \omega_0^2 [\ddot{Y}]\{A_0\} - \{F_0\} \right) d\tau, \quad (14)$$

$$\{q\} = -\frac{1}{\pi} \int_0^{2\pi} [Y]^T (\{Q\} + 2\omega_0 [\ddot{Y}]\{A_0\}) d\tau, \quad (15)$$

$$\{p\} = -\frac{1}{\pi} \int_0^{2\pi} [Y]^T \{P\} d\tau \quad (16)$$

The matrix equation (12) is solved in each step of Newton-Raphson iterative procedure until convergent solution is obtained. Construction of resonance curves provides an constant amplitude of excitation and fixed value of parameter λ , so that $\Delta\lambda = 0, \{\Delta G\} = \{0\}$. Equation (12) is reduced to $[k]\{\Delta A\} = \{r\} + \{q\}\Delta\omega$ and then solved by incrementing frequency ω or by incrementing one component of vector $\{A\}$. If $\Delta\omega$ is prescribed as control increment, then through Newton iteration ω remains constant ($\Delta\omega = 0$) and increments $\{\Delta A\}$ can be computed from equation $[k]\{\Delta A\} = \{r\}$. In the next iteration, the updated vector $\{A\} + \{\Delta A\}$ is computed and Newton - Raphson process is repeated until the magnitude of residual vector $\{r\}$ becomes sufficiently small. Using $\Delta\lambda$ as control increment, IHB method can be used for construction of solution diagram.

When steady-state solution by using IHB method is computed, usually stability of obtained solution is of great interest. Supposing, that $\{X_0\}$ means steady-state solution of equation (6), which implies, that $\{R\} = \{0\}$. If exciting frequency, as well as amplitude of exciting signal and value of parameter λ are not changed, small changes of solution $\{X\} = \{X_0\} + \{\Delta X\}$ will satisfy the perturbed equation:

$$\omega_0^2 \{\Delta \ddot{X}\} + [C] \cdot \{\Delta \dot{X}\} + [K] \cdot \{\Delta X\} = \{0\} \quad (17)$$

which is matrix ordinary linear differential equation with periodic coefficients in $[C]$ or $[K]$ or both. Equation (17) can be presented by first order system introducing combined vectors:

$$\frac{d}{d\tau} \begin{Bmatrix} \{\Delta X\} \\ \{\Delta \dot{X}\} \end{Bmatrix} = -\frac{1}{\omega_0^2} \begin{bmatrix} [0] & [I] \\ [K] & [C] \end{bmatrix} \begin{Bmatrix} \{\Delta X\} \\ \{\Delta \dot{X}\} \end{Bmatrix}, \{Z\} = \begin{Bmatrix} \{\Delta X\} \\ \{\Delta \dot{X}\} \end{Bmatrix}, [A] = -\frac{1}{\omega_0^2} \begin{bmatrix} [0] & [I] \\ [K] & [C] \end{bmatrix}, \{\dot{Z}\} = [A]\{Z\} \quad (18)$$

With this transformation, stability of steady-state solution corresponds to the stability of solutions of the system $\{\dot{Z}\} = [A]\{Z\}$. Assuming, that $[\Phi(\tau)]$ denotes the transition matrix, which is used to transform the state vector $\{Z(\tau)\}$ at the time $\tau = 0$ in the state vector $\{Z(\tau + 2\pi)\}$ at the end of one period. In accordance with Floquet theory [2], solutions of the system (18) are stable, if all moduli of the eigenvalues of the transition matrix at the end of one period $[\Phi(2\pi)]$ are less than unity, otherwise solutions are unstable. For this purpose, each period 2π is divided into N_k intervals (τ_{k-1}, τ_k) , where $\Delta_k = \tau_k - \tau_{k-1}$ means the length of the several intervals. In each interval the periodic matrix of coefficients $[A(\tau)]$ is replaced by constant matrix:

$$[A_k] = \frac{1}{\Delta_k} \int_{\tau_{k-1}}^{\tau_k} [A(\zeta)] d\zeta \quad (19)$$

The transition matrix at the end of one period is given by :

$$[\Phi(2\pi)] = \prod_{k=1}^{N_k} \exp(\Delta_k [A_k]) \quad (20)$$

3. Application of IHB Method to coupled Van der Pol – Duffing oscillator with two degrees of freedom

In this section, the proposed method is applied on coupled Van der Pol – Duffing oscillator with two degrees of freedom ($N=2$), where the following equations are presumed:

$$\begin{aligned} \omega^2 \ddot{x}_1 - \varepsilon_1 (\lambda - \mu_1 x_1^2) \omega (\dot{x}_1 - \dot{x}_2) + \nu_1 (x_1 - x_2) + \kappa_1 x_1 + \alpha_{11} x_1^3 + \alpha_{12} x_1^2 x_2 + \alpha_{13} x_1 x_2^2 + \alpha_{14} x_2^3 &= g_1 \cos[(2m-1)\tau] \\ \omega^2 \ddot{x}_2 - \varepsilon_2 (\lambda - \mu_2 x_2^2) \omega (\dot{x}_2 - \dot{x}_1) + \nu_2 (x_2 - x_1) + \kappa_2 x_2 + \alpha_{21} x_1^3 + \alpha_{22} x_1^2 x_2 + \alpha_{23} x_1 x_2^2 + \alpha_{24} x_2^3 &= 0 \end{aligned} \quad (21a,b)$$

By introducing the vectors:

$$\{X_0\} = [x_{10}, x_{20}]^T, \{\ddot{X}_0\} = [\ddot{x}_{10}, \ddot{x}_{20}]^T, \{\Delta X\} = [\Delta x_1, \Delta x_2]^T, \{\Delta \dot{X}\} = [\Delta \dot{x}_1, \Delta \dot{x}_2]^T, \{\Delta \ddot{X}\} = [\Delta \ddot{x}_1, \Delta \ddot{x}_2]^T \quad (22)$$

as well as amplitude vectors and vectors of corresponding increments:

$$\begin{aligned} \{a_1\} &= [a_{11}, a_{12}, \dots, a_{1K}; b_{11}, \dots, b_{1K}]^T, \{\Delta a_1\} = [\Delta a_{11}, \Delta a_{12}, \dots, \Delta a_{1K}; \Delta b_{11}, \dots, \Delta b_{1K}]^T \\ \{a_2\} &= [a_{21}, a_{22}, \dots, a_{2K}; b_{21}, \dots, b_{2K}]^T, \{\Delta a_2\} = [\Delta a_{21}, \Delta a_{22}, \dots, \Delta a_{2K}; \Delta b_{21}, \dots, \Delta b_{2K}]^T \end{aligned} \quad (23.a,b)$$

one obtain the matrix $[k]$ by using the equation (13.a):

$$[k] = \begin{bmatrix} (\nu_1 + \kappa_1)[I] - \omega_0^2[M] & [0] \\ [0] & (\nu_2 + \kappa_2)[I] - \omega_0^2[M] \end{bmatrix} + \omega_0 \begin{bmatrix} -\varepsilon_1(\lambda_0[C_L] - \mu_1[C_N(\{a_1\})]) & \varepsilon_1(\lambda_0[C_L] - \mu_1[C_N(\{a_1\})]) \\ \varepsilon_2(\lambda_0[C_L] - \mu_2[C_N(\{a_2\})]) & -\varepsilon_2(\lambda_0[C_L] - \mu_2[C_N(\{a_2\})]) \end{bmatrix} + 2\omega_0 \begin{bmatrix} \varepsilon_1\mu_1([K_N^1(\{a_1\})] - [K_N^1(\{a_1\}, \{a_2\})]) & [0] \\ [0] & -\varepsilon_2\mu_2([K_N^1(\{a_2\}, \{a_1\})] - [K_N^1(\{a_2\})]) \end{bmatrix} + \begin{bmatrix} 3\alpha_{11}[K_N^{(2)}(\{a_1\})] + 2\alpha_{12}[K_N^{(2)}(\{a_1\}, \{a_2\})] + \alpha_{13}[K_N^{(2)}(\{a_2\})] & \alpha_{12}[K_N^{(2)}(\{a_1\})] + 2\alpha_{13}[K_N^{(2)}(\{a_1\}, \{a_2\})] + 3\alpha_{14}[K_N^{(2)}(\{a_2\})] \\ 3\alpha_{21}[K_N^{(2)}(\{a_1\})] + 2\alpha_{22}[K_N^{(2)}(\{a_1\}, \{a_2\})] + \alpha_{23}[K_N^{(2)}(\{a_2\})] & \alpha_{22}[K_N^{(2)}(\{a_1\})] + 2\alpha_{23}[K_N^{(2)}(\{a_1\}, \{a_2\})] + 3\alpha_{24}[K_N^{(2)}(\{a_2\})] \end{bmatrix} \quad (24)$$

where matrices $[M]$, $[C_L]$, $[C_N(\{a\})]$, $[K_N^{(1)}(\{a\})]$, $[K_N^{(2)}(\{a\})]$, $[K_N^{(2)}(\{a\}, \{b\})]$ are defined as:

$$[M] = \begin{bmatrix} \text{diag}(2n-1)^2 & [0] \\ [0] & \text{diag}(2n-1)^2 \end{bmatrix}, [C_L] = \begin{bmatrix} [0] & \text{diag}(2n-1) \\ -\text{diag}(2n-1) & [0] \end{bmatrix}, n=1,2,\dots,K \quad (25.a,b)$$

$$[C_N(\{a\})] = \frac{1}{\pi} \int_0^{2\pi} \{a\}^T [T]^T [T] \{a\} [T]^T [T] d\tau, [K_N^{(1)}(\{a\})] = \frac{1}{\pi} \int_0^{2\pi} \{a\}^T [T]^T [T] \{a\} [T]^T [T] d\tau, \quad (26.a,b)$$

$$[K_N^{(2)}(\{a\})] = \frac{1}{\pi} \int_0^{2\pi} \{a\}^T [T]^T [T] \{a\} [T]^T [T] d\tau, [K_N^{(2)}(\{a\}, \{b\})] = \frac{1}{\pi} \int_0^{2\pi} \{a\}^T [T]^T [T] \{b\} [T]^T [T] d\tau.$$

On analogous way one obtain the vectors $\{r\}$, $\{q\}$, $\{p\}$ in equations (14, 15, 16):

$$\{r\} = \begin{Bmatrix} g_1 \{e_{2m-1}\} \\ \{0\} \end{Bmatrix} - \left[\begin{bmatrix} ((\nu_1 + \kappa_1)[I] - \omega_0^2[M]) & -\nu_1[I] \\ -\nu_2[I] & ((\nu_2 + \kappa_2)[I] - \omega_0^2[M]) \end{bmatrix} - \omega_0 \begin{bmatrix} \varepsilon_1(\lambda_0[C_L] - \mu_1[C_N(\{a_1\})]) & -\varepsilon_1(\lambda_0[C_L] - \mu_1[C_N(\{a_1\})]) \\ -\varepsilon_2(\lambda_0[C_L] - \mu_2[C_N(\{a_2\})]) & \varepsilon_2(\lambda_0[C_L] - \mu_2[C_N(\{a_2\})]) \end{bmatrix} + \begin{bmatrix} \alpha_{11}[K_N^{(2)}(\{a_1\})] & \alpha_{12}[K_N^{(2)}(\{a_1\})] + \alpha_{13}[K_N^{(2)}(\{a_1\}, \{a_2\})] + \alpha_{14}[K_N^{(2)}(\{a_2\})] \\ \alpha_{21}[K_N^{(2)}(\{a_1\})] & \alpha_{22}[K_N^{(2)}(\{a_1\})] + \alpha_{23}[K_N^{(2)}(\{a_1\}, \{a_2\})] + \alpha_{24}[K_N^{(2)}(\{a_2\})] \end{bmatrix} \right] \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} \quad (27)$$

$$\{q\} = \left(2\omega_0 \begin{bmatrix} [M] & [0] \\ [0] & [M] \end{bmatrix} + \begin{bmatrix} \varepsilon_1(\lambda_0[C_L] - \mu_1[C_N(\{a_1\})]) & -\varepsilon_1(\lambda_0[C_L] - \mu_1[C_N(\{a_1\})]) \\ -\varepsilon_2(\lambda_0[C_L] - \mu_2[C_N(\{a_1\})]) & \varepsilon_2(\lambda_0[C_L] - \mu_2[C_N(\{a_1\})]) \end{bmatrix} \right) \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} \quad (28)$$

$$\{p\} = \omega_0 \begin{bmatrix} \varepsilon_1[C_L] & -\varepsilon_1[C_L] \\ -\varepsilon_2[C_L] & \varepsilon_2[C_L] \end{bmatrix} \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix} \quad (29)$$

The matrix $[s]$ in equation (12) can be computed by using the formula:

$$[s] = \frac{1}{\pi} \int_0^{2\pi} [Y]^T \cos[(2m-1)\tau] \cdot d\tau = \begin{bmatrix} \{e_{2m-1}\} & \{0\} \\ \{0\} & \{e_{2m-1}\} \end{bmatrix} \quad (30)$$

where $\{e_{2m-1}\}$ denotes the unity vector with $(2m-1)^{\text{th}}$ component equal to 1.

4. Numerical examples

4.1. Coupled nonlinear oscillator with cubic nonlinearity and two degrees of freedom

In the first example, the internal resonance of a nonlinear mechanical system in Fig. 1, consisting of two point masses and two springs with a linear damper is analyzed. First of

the springs is linear, whereas the other has a cubic nonlinear characteristic. The restoring force of the nonlinear spring is defined by:

$$F_2 = k_2(x_1 - x_2) + \mu(x_1 - x_2)^3 \quad (31)$$

Nonlinear oscillations of the mechanical system, can be described by equations:

$$\begin{aligned} \omega^2 \ddot{x}_1 + \frac{\mu l}{m_1} \omega(\dot{x}_1 - \dot{x}_2) + \frac{k_1}{m_1} x_1 + \frac{k_2}{m_1} (x_1 - x_2) + \frac{\mu}{m_1} (x_1 - x_2)^3 &= \frac{g}{m_1} \cos[(2m-1)\tau] \\ \omega^2 \ddot{x}_2 - \frac{\mu l}{m_2} \omega(\dot{x}_1 - \dot{x}_2) + \frac{k_2}{m_2} (x_2 - x_1) - \frac{\mu}{m_2} (x_1 - x_2)^3 &= 0; \end{aligned} \quad (32.a,b)$$

Equations (32. a,b) are equivalent to equations (21. a,b) if the following choice of parameters is applied:

$$\begin{aligned} g_1 &= \frac{g}{m_1}; \quad g_2 = 0; \\ \nu_1 &= \frac{k_2}{m_1}; \quad \kappa_1 = \frac{k_1}{m_1}; \quad \varepsilon_1 = -\frac{\mu l}{m_1}; \quad \lambda = 1; \quad \mu_1 = 0; \quad g_1 = \frac{g}{m_1}; \quad \alpha_{11} = \frac{\mu}{m_1}; \quad \alpha_{12} = -3\frac{\mu}{m_1}; \quad \alpha_{13} = 3\frac{\mu}{m_1}; \quad \alpha_{14} = -\frac{\mu}{m_1}; \\ \nu_2 &= \frac{k_2}{m_2}; \quad \kappa_2 = 0; \quad \varepsilon_2 = -\frac{\mu l}{m_2}; \quad \lambda = 1; \quad \mu_2 = 0; \quad g_2 = 0; \quad \alpha_{21} = -\frac{\mu}{m_2}; \quad \alpha_{22} = 3\frac{\mu}{m_2}; \quad \alpha_{23} = -3\frac{\mu}{m_2}; \quad \alpha_{24} = \frac{\mu}{m_2} \end{aligned} \quad (33)$$

The internal resonance curves are computed for values of parameters:

$$\nu_1 = 1.582; \quad \kappa_1 = 1.5; \quad \varepsilon_1 = -0.03164; \quad g_1 = 2;$$

$$\alpha_{11} = 0.01582; \quad \alpha_{12} = -0.04746; \quad \alpha_{13} = 0.04746; \quad \alpha_{14} = -0.01582;$$

$$\nu_2 = 1; \quad \kappa_2 = 0; \quad \varepsilon_2 = -0.02; \quad g_2 = 0;$$

$$\alpha_{21} = -0.01; \quad \alpha_{22} = 0.03; \quad \alpha_{23} = -0.03; \quad \alpha_{24} = 0.01;$$

where the value $m = 2$ in excitation signal $g_1 \cdot \cos[(2m-1)\tau]$ is chosen. Computed results, that show the course of magnitudes A_{12} and A_{22} :

$$A_{12} = \sqrt{a_{12}^2 + b_{12}^2}, \quad A_{22} = \sqrt{a_{22}^2 + b_{22}^2}, \quad (34)$$

in dependence on the frequency ω , are plotted on the Fig. 1. Magnitudes A_{11} and A_{21} are both equal zero in this type of resonance. Fig. 1 shows, that both stable as well as unstable branches of resonance curves can be computed by using the IHB method.

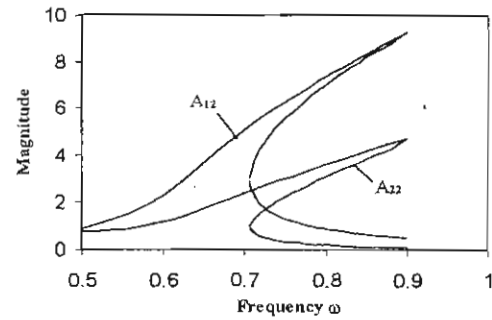
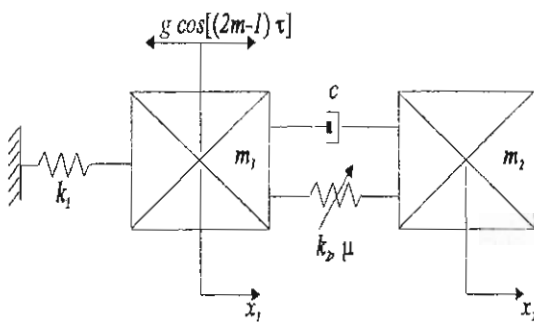


Fig. 1. Internal resonance of nonlinear mechanical system

4.2. Autonomous nonlinear oscillator with two degrees of freedom

As second example the autonomous coupled nonlinear oscillator with two degrees of freedom is analyzed. The oscillator is described by means of equations:

$$\begin{aligned} \omega^2 \ddot{x}_1 - \varepsilon_1 (\lambda - \mu_1 x_1^2) \omega \dot{x}_1 + \kappa_1 x_1 + \alpha_1 x_1^3 + \alpha_{12} x_1^2 x_2 + \alpha_{13} x_1 x_2^2 + \alpha_{14} x_2^3 &= 0 \\ \omega^2 \ddot{x}_2 - \varepsilon_2 (\lambda - \mu_2 x_2^2) \omega \dot{x}_2 + \kappa_2 x_2 + \alpha_2 x_2^3 + \alpha_{22} x_1^2 x_2 + \alpha_{23} x_1 x_2^2 + \alpha_{24} x_2^3 &= 0 \end{aligned} \quad (35)$$

which are somewhat simplified equations (21. a,b). Because of this, matrix $[k]$ in the equation (24) as well as the vectors $\{r\}, \{q\}, \{p\}$ in equations (27, 28, 29) are also simplified. Other values of parameters are chosen as follows:

$$\lambda = 0.08, \varepsilon_1 = 0.2, \varepsilon_2 = -1, \mu_1 = \mu_2 = 0, \nu_1 = \nu_2 = 0, \kappa_1 = 1, \kappa_2 = 9,$$

$$\alpha_{11} = 0, \alpha_{12} = -0.008, \alpha_{13} = \alpha_{14} = 0, \alpha_{21} = -0.0064, \alpha_{22} = \alpha_{23} = \alpha_{24} = 0$$

At this parameter values the above coupled oscillator can exhibit self excited oscillations, where frequency of oscillations is not known in advance and must be included among unknowns. Therefore, we have $2 \times N \times K = 4K$ equations in the matrix system (12) and $4K + 1$ unknowns, including unknown frequency ω . To solve the equation (12), one harmonic increment, e.g. $\Delta b_{1i} (i = 1)$ is set to zero, which implies, that b_{1i} has constant value during the Newton-Raphson iterative procedure. Matrix $[k]$ in the equation (12) is arranged in such a way, that the $(K + 1)$ column of the matrix $[k]$ is interchanged with the vector $\{q\}$, appearing on the right hand side of the equation (12) and harmonic increment Δb_{1i} is replaced by increment $\Delta \omega$. This modification is permitted because the system is autonomous and assumption $\Delta b_{1i} = 0$ will cause only the phase shift without changes in the frequency spectrum. From analysis for parameter $\lambda = 0.08$ follows, that two kinds of oscillations exist, the first one with frequency $\omega = 0.978$ and the second one with frequency $\omega = 0.9926$. The time responses of amplitudes $x_1(\tau)$ and $x_2(\tau)$ over two periods of dimensionless time τ and corresponding limit cycles at first mode oscillation shows Figure 2. a,b. and for second mode oscillation the Figure 3. a,b.

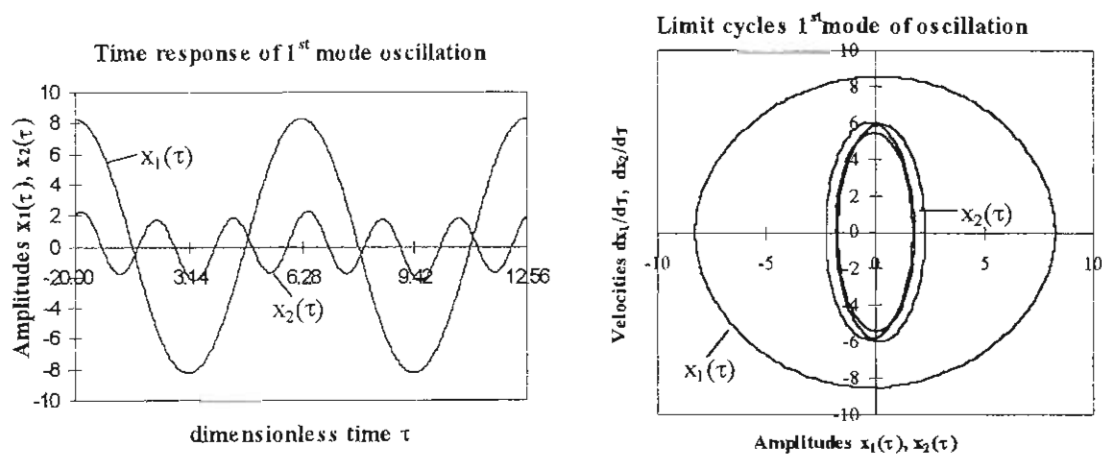


Fig. 2. a) Time response of 1st mode oscillation, b) corresponding limit cycles at $\lambda = 0.08$

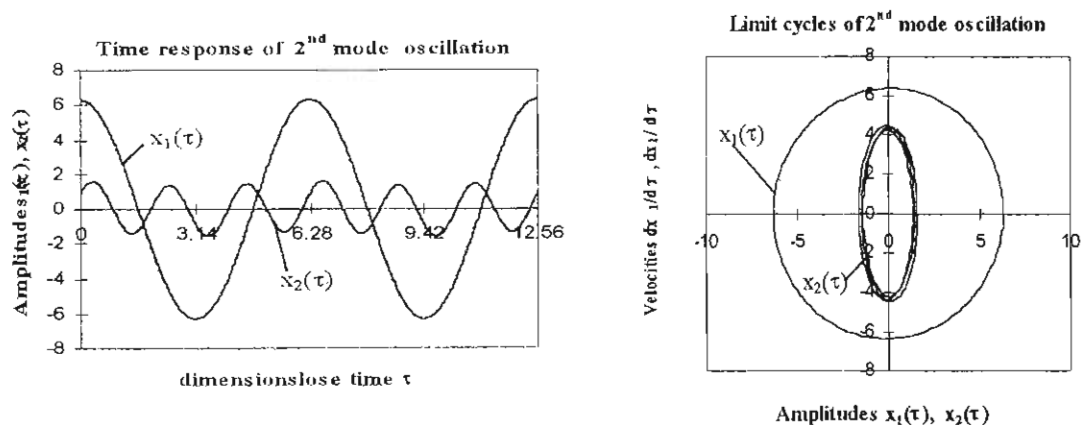


Fig. 3. a) Time response of 2nd mode oscillation, b) corresponding limit cycles at $\lambda = 0.08$

By using augmentation of parameter λ , the complete solution diagram is constructed and plotted on the Figure 4. In the solution diagram, both kinds of self excited periodic oscillations for value of parameter $\lambda=0,08$ are indicated as well as the value of parameter $\lambda=0,092$, where only one mode of oscillation is obtained. From the solution diagram follows, that at values $\lambda>0,092$ periodic oscillations do not exist anymore.

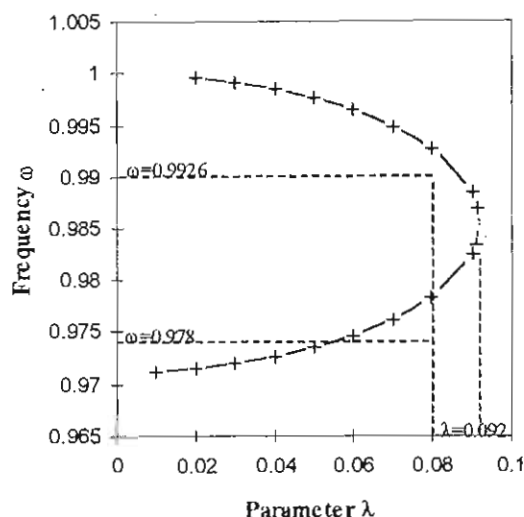


Fig. 4. Solution diagram

5. Conclusion

In this paper the incremental harmonic balance method is presented. The method is especially suitable for analysing periodically excited nonlinear dynamical systems with high nonlinearity and enables computation of stable as well as unstable branches of resonance curves. Moreover, the analysis of autonomous systems is also carried out. Stability of computed solutions is proved by Floquet theory. Internal resonance of nonlinear mechanical system with two degree of freedom is analyzed. Self excited oscillations of nonlinear coupled autonomous oscillator with two degree of freedom are computed together with corresponding solution diagram.

6. References

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TRANSIENT VIBRATIONS OF THE MECHANICAL CONSTRUCTIONS

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Abstract: In the paper the mathematical model of 3-D mechanical construction which is in the vibration state of the forced vibrations with damping had been shown. Due the impact impulse on the construction the transient vibrations has been occurred. The mathematical model has been solved by using the finite elements method in the program package I-DEAS. In the experimental part of work the method of excitation with an impact hammer has been used. Both numerical and experimental results had been given in the form of diagrams.

1. INTRODUCTION

In the theory of forced vibrations two types of excitation, deterministic and random, are known. Deterministic type of excitation can be defined as a time or frequency function. Transient vibrations are occurred if the aperiodic time-varying excitation acting on the vibrational object. This paper is dealing with numerical and experimental ordering of the transient response of the mechanical structures which are excited by an impact impulse.

2. THEORY OF THE TRANSIENT VIBRATIONS OF THE VIBRATIONAL OBJECTS

2.1. The vibrational object equation

The problem of the transient vibrations of the vibrational objects will be formulated with finite elements method. On the basis of principle of the virtual work formulation of the vibrational object as a system with the n degrees of freedom, which is:

$$\{\delta U\}^T ([M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} - \{F\}) = 0 \quad (1)$$

where are:

$[M], [C], [K]$	mass, damping and stiffness matrix,
$\{\ddot{U}\}, \{\dot{U}\}, \{U\}$	acceleration, velocity and displacement vector
$\{\delta U\}$	virtual displacement vector,
$\{F\}$	excitation force vector,

its elementary equation (2) can be get:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{F\} \quad (2)$$

2.2. The natural frequencies and natural forms of vibrations determination

The natural frequencies and forms of vibrations of the vibrational system defined by equation (2), which is in the vibrational state of the forced vibrations with damping, is possible to get from the system of the free undamped vibrations differential equations, which is:

$$[M]\{\ddot{U}\} + [K]\{U\} = 0 \quad (3)$$

The differential equation system (3) solution has the form:

$$\{U\} = \{q\}e^{i\omega t} \quad (4)$$

where $\{q\}$ is amplitude vector, then is:

$$\{\ddot{U}\} = -\{q\}\omega^2 e^{i\omega t} \quad (5)$$

and by inserting of this statements in (3) follows:

$$(-\omega^2[M] + [K])\{q\} = 0 \quad (6)$$

Equation (6) represents the homogeneous system of linear equations, in which the amplitude vector $\{q\}$ and natural frequency ω are unknown, and will have the nontrivial solutions if the following condition is satisfied:

$$|-\omega^2[M] + [K]| = 0 \quad (7)$$

Development of this determinant will give the polynomial of ω^2 , roots of which represents vibrational system natural frequencies ω_i . For each ω_i it is possible to calculate the amplitude vector $\{q_i\}$ which is called the vibrational normal mode.

The amplitude vector $\{q_i\}$ can be obtained in different ways, but with using the matrix methods is the most convenient treatment by which the normalized modal vector $\{\Phi_i\}$ is determined. Normalized modal vector is dimensionless vector whose n components describe relative amplitudes of vibrational object nodal points displacements in the case of free vibrations in i -th natural mode (as well as amplitude vector $\{q_i\}$), for whom is:

$$\{\Phi_i\} = c_i \{q_i\} \quad (8)$$

where c_i is such constant that is:

$$\{\Phi_i\}^T [M] \{\Phi_i\} = \bar{m} \quad (9)$$

Value of constant $\bar{m}$, which because of normalized modal vectors dimensionless has a mass dimension, can be free chosen for each problem. Normalized modal vectors are collected in a single square matrix of order n , by which the modal matrix $[\Phi]$ is got.:

$$[\Phi] = [\{\Phi_1\} \ \{\Phi_2\} \ \dots \ \{\Phi_n\}] \quad (10)$$

The most important characteristic of modal matrix is that by its using the matrix $[M]$ i $[K]$ diagonalization can be executed, so it is:

$$\begin{aligned} [\Phi]^T [M] [\Phi] &= \bar{m} [I] \\ [\Phi]^T [K] [\Phi] &= \bar{m} [\omega^2] \end{aligned} \quad (11)$$

when $[I]$ is unit matrix, while $[\omega^2]$ is a diagonal matrix whose elements are squares of the natural frequencies.

2.3. Determining of vibrational object response

By equation (2) the system of coupling differential equations of motion is given. By using a vector of generalized coordinates $\{p\}$ it is possible to uncoupled it, such that:

$$\{U\} = [\Phi] \{p\} \quad (12)$$

where with the statement (13) is obtained:

$$[\Phi]^T [M] [\Phi] \{\ddot{p}\} + [\Phi]^T [C] [\Phi] \{\dot{p}\} + [\Phi]^T [K] [\Phi] \{p\} = [\Phi]^T \{F\} \quad (13)$$

which represents the system of n uncoupled differential equations only in case that the damping is modelled as a proportional Rayleigh's damping, where the damping matrix $[C]$ is represented as a linear superposition of mass matrix $[M]$ and stiffness matrix $[K]$ on the next manner [1]:

$$[C] = \alpha_c [M] + \beta_c [K] \quad (14)$$

where α_c and β_c are Rayleigh's damping coefficients by which the i -th damping ratio ξ_i is defined, for whom is:

$$\xi_i = \frac{C_i}{C_{ikr}} = \frac{1}{2} \left(\frac{\alpha_c}{\omega_i} + \beta_c \omega_i \right), \quad \forall i = 1, \dots, n \quad (15)$$

The i -th uncoupled differential equation is now possible to write as:

$$\ddot{p}_i + (\alpha_c + \beta_c \omega_i^2) \dot{p}_i + \omega_i^2 p_i = \frac{\{\Phi_i\}^T \{F\}}{\bar{m}} = \frac{Q_i}{\bar{m}} = F_{gi}, \quad \forall i = 1, \dots, n \quad (16)$$

where the scalar product of transposed normalized modal vector $\{\Phi_i\}$ and excitation force vector $\{F\}$ is called generalized force Q_i .

On this way the problem is deduced to the problem of the vibrational system with one degree of freedom which can be modeled by mass m , spring with stiffness k and damper with damping coefficient c (fig. 1.). In this paper the problem of transient vibrations will be worked out, under which the unsteady vibrations which are resulted under the activity of nonperiodic force $F(t)$ are considered. If, in the moment $t_0 = \tau$, to the considered vibrational object acts infinitesimal impulse $dG = F(\tau)d\tau$, then it makes a velocity increase (Fig. 2.):

$$d\dot{x} = \frac{dG}{m} = \frac{F(\tau)d\tau}{m} \quad (17)$$

This velocity increase, which will be continuously produced through all the time at which excitation force $F(t)$ will be acted, with regard to the damped vibrations, will give displacement for the case $x_0 = \dot{x}_0 = 0$:

$$x(t) = \frac{F(\tau)}{m\nu} \int_0^t e^{-\delta(t-\tau)} \sin \nu(t-\tau) d\tau \quad (18)$$

where is:

$$\delta = \frac{c}{2m} \quad \text{and} \quad \nu^2 = \frac{k}{m} - \left(\frac{c}{2m} \right)^2 = \omega^2 - \delta^2 \quad (19)$$

with meaning:

$F(\tau)$ excitation force (N)
 t time (s)
 ω natural frequency (s^{-1})

For the elementarily excitation force types, it is possible on the relatively elementary manner to derive the expression for the transient vibrations response displacement.

For example, for the triangular excitation in which case the excitation force is fully defined in accordance with fig. 3., if the damping is neglected, it is possible to obtain analytical form of response displacement:

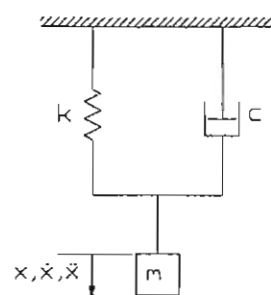
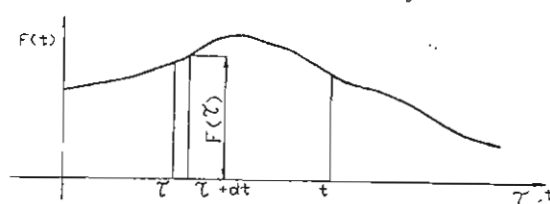


Fig. 1. One degree of freedom vibrational object



Sl. 2. Arbitrary form of the transient excitation

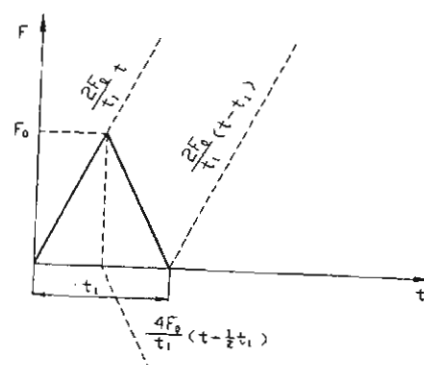


Fig. 3. Triangular excitation

$$\begin{aligned}
 x &= \frac{2F_0}{k} \left(\frac{t}{t_1} - \frac{\sin(\omega t)}{\omega t_1} \right), \quad 0 < t < \frac{t_1}{2} \\
 x &= \frac{2F_0}{k} \left(1 - \frac{t}{t_1} - \frac{\sin(\omega t)}{\omega t_1} + \frac{2\sin\left(\omega\left(t - \frac{t_1}{2}\right)\right)}{\omega t_1} \right), \quad \frac{1}{2}t_1 < t < t_1 \\
 x &= \frac{2F_0}{k} \left(\frac{1}{\omega t_1} \left(2\sin\left(\omega\left(t - \frac{t_1}{2}\right)\right) - \sin(\omega(t - t_1)) - \sin(\omega t) \right) \right), \quad t > t_1
 \end{aligned} \tag{20}$$

The identical expressions are given in the literature [2].

In the case of equation (16) the displacement, written in the generalized coordinates, with condition $p_{i0} = \dot{p}_{i0} = 0$, will be:

$$p_i(t) = \frac{1}{v_i} \int_0^t F_{gi}(\tau) e^{-\delta_i(t-\tau)} \sin v_i(t-\tau) d\tau \tag{21}$$

Now, it is possible to write the solution vector in generalized coordinates, in matrix form, as:

$$\{p\} = \frac{[v]^{-1}}{\bar{m}} \int_0^t [E_\tau] [Sin_\tau] [\Phi]^T \{F\} d\tau \tag{22}$$

where are:

$$\left. \begin{aligned}
 [v] &= \text{diag } v_i \\
 [E_\tau] &= \text{diag} [e^{-\delta_i(t-\tau)}] \\
 [Sin_\tau] &= \text{diag} [\sin v_i(t-\tau)]
 \end{aligned} \right\} \tag{23}$$

Finally, by means of transformation from generalized in the real coordinates, follows[3]:

$$\{U\} = \frac{[\Phi]}{\bar{m}} [v]^{-1} \int_0^t [E_\tau] [Sin_\tau] [\Phi]^T \{F\} d\tau \tag{24}$$

3. EXCITATION AND MEASUREMENT THE TRANSIENT VIBRATIONS

For the purpose of excitation and measuring transient vibrations the following measuring equipment has been used:

1. two-channels measuring device SCHENCK Vibroport 41,
2. impact hammer KISTLER model 9726120000 typ 9928,
3. measuring sensor AS-020 SCHENCK with sensivity 100 (mV/g),

Method of impact hammer is one of the most popular methods for vibration exciting on the vibrational objects. It serves of production of the transient, short duration mechanical energy transfer event which can be explained by the shock theory. The force obtained by the shock has a high intensity and it's time variation consists of the compresional (crescending) and restitutional (descending) part, fig.4. The frequency spectrum has a periodic structure with maximal amplitude on the frequency of 0(Hz) and with the zero values on the

frequencies of $\frac{n}{d\tau}$ where $d\tau$ is the effective duration of impulse while with n are marked the

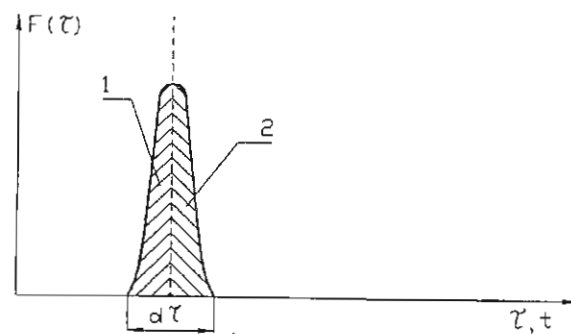


Fig. 4. Impact impulse: compresional (1) and restitutional (2) part

integer numbers. The time of impulse duration and consequently the shape of frequency spectrum depends of the hammers changeable tip type which is, during the impact, in the contact with vibrational object. The softer hammers tip materials (for example rubber) shall give the impact impulse of longer duration that is useful impulse frequency will be lower. For exciting a low vibrational objects natural frequencies (for example below 100(Hz) the hammers tip from very soft rubber is used) [4].

In the task of complex transfer function measuring, it is possible to use the force transducer. By using the force transducer the excitation force amplitude and its phase angle with regard to measured vibrational displacement can be determined. Unfortunately, in the task of transient vibrations measuring SCHENCK's equipment is not foreseeing the using of force transducer. Therefore on the opposite end of impact hammer, related to hammer's changeable tip by which the impact is produced, the vibration sensor SCHENCK AS-020 is attached. By this sensor the hammer's vibrational acceleration on the occasion of impact is measured. The hammers cross section as well as the place for acceleration sensor are shown on the fig. 5, [5]. On the basis of hammer's measured vibrational acceleration it is possible to calculate impact force in accordance with:

$$F = m a \quad (25)$$

where is:

- F - impact force
- m - mass of hammer with sensor
- a - acceleration of vibration sensor on the impact hammer

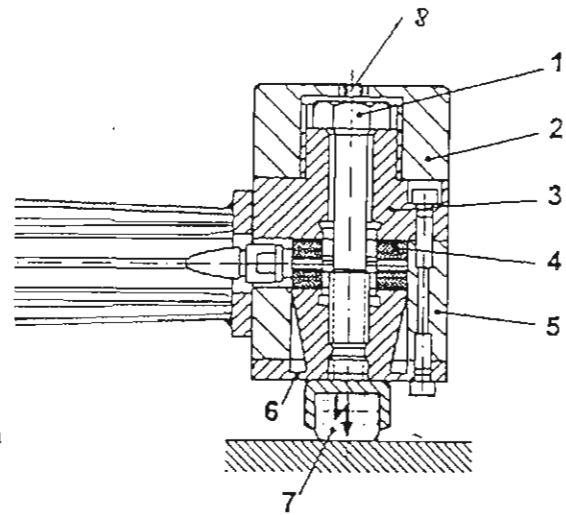


Fig. 5. Impact hammers cross section:

- 1- Extension bolt, 2- Additional mass,
- 3- Hammer mass, 4- Quartz element
- 5- Housing, 6- Tubular element
- 7- Tip element, 8- Place for the acceleration sensor

On the upper part of the diagram 4 the typical shape of impact hammer acceleration, during the impact, is shown. In comparison with fig. 4. is visible that besides the acceleration shape, and by this the excitation force, which corresponds to impact, in case of real measurement the parts which corresponds to swing acceleration before the impact as well as acceleration of hammers moving off from vibrational object after the impact appeared. Such parts of excitation force are not given over to vibrational object so they will not have influence to the vibration displacement values which the vibration sensor on the vibration object will measure.

4. NUMERICAL MODELLING OF TRANSIENT VIBRATIONS

For the purpose of control of the results obtained experimentally the world well known program package I-DEAS is used [6]. As a method of solving, among other the vibrational problems, this program package uses the finite elements method.

The classic methods of solving the continuous system problems are based on the solving of differential equations by which the state in the continuum is described. In difference to them the finite elements method uses the algebraic equations because instead of element with differentially small dimensions take into considerations the part of continuum with finite dimensions, so called finite element. The displacement method is used, where the unknown displacements in the finite element nodes with the known forces are bound with the stiffness matrix.

The vibrational object is modelled as a 3-D line construction composed of the beam finite elements. Stiffness matrix, mass matrix and generalized transformation matrix in the case of 3-D line construction are given in [7]. Construction stiffness matrix as well as construction mass matrix are obtained by inserting the transformed submatrix of finite elements stiffness

matrix and mass matrix, where they assume the final form by elimination of rows and columns which are referred to the bounded components of nodal displacement vector. It is not necessary to know the construction damping matrix because the damping is usually taken as Rayleigh's damping, related to the natural modes of vibrating by means of damping ratio (15).

The construction, on which the experiment of obtaining the transient vibrations is carried out, is modelled in the modul SIMULATION which is the part of the program package I-DEAS in which the calculation of the natural frequencies and modes of vibrations is carried out. The construction geometry is defined in the submodul MASTER MODELER. In the submodul BEAM SECTION cross sections geometry is defined. Submodul MESHING is used for generating the finite elements mesh and for generating the concentrated masses in the finite element's nodes. Submodul BOUNDARY CONDITIONS deals with the definition of boundary conditions and submodul MODEL SOLUTION with the definition of elementary parameters (number of vector iterations, precision...) of the numerical method SVI, by means of which the system of algebraic equations (6) is solved, as well as with definition of the number of natural frequencies and modes of vibrations which has to be obtained by calculation. By solving the system the natural frequencies and modes of vibrations are obtained and its graphical representation is given in the submodul POST PROCESSING.

Numerical solving of the problem of transient vibrations is carried out in the supplementary modul MODEL RESPONSE which works as an annex of the program package I-DEAS. It consists of two submoduls EXCITATION DEFINITION and RESPONSE DEFINITION. In first submodul the excitation force of the vibrational object is defined. This excitation force is calculated earlier in accordance to equation (25). The excitation force is placed to the excitation set where is possible to put more excitation forces and momentums which affect the vibrational object. It is possible that more complicated excitations, which require relatively numerous points for its definition, can be inserted into the package by using of UNV file which gives considerable saving of time. The correctness of the data which are carried out by excitation definition can be controlled since its graphical representation in the time domain is enabled. Submodul RESPONSE DEFINITION enables us to choose the wanted time response form (vibrating displacement, velocity and acceleration) and its graphical representation in the time domain.

5. EXAMPLE

For the purpose of comparison of numerically and experimentally obtained results the vibrating displacements of iron construction whose discretizing model is shown on fig.6. are determined.

In program package I-DEAS the mathematical model of construction is done. The construction is discretized by 36 nodes and 17 finite elements. In characteristic point 1 of construction the boundary conditions are defined as 6 bounded degrees of freedom. Concentrated mass $m = 0.373 \text{ (kg)}$, by which the vibration sensor is modelled, is situated at characteristic point 4. Mass moments of inertia of vibration sensor with respect to axes of local co-ordinate system are neglected.

By solving the eigenvalues problem by using the SVI method, in the program package I-DEAS, the natural frequencies and modes of vibrations, are obtained. First 6 natural frequencies and modes of vibrations are shown in table 1.

Ordinal number of natural frequency	Natural frequency (Hz)	Explanation
1.	3.88	bending about axe Z
2.	4.02	bending about axe X
3.	10.25	bending about X + torsion about Y
4.	11.38	bending about axe Z
5.	12.55	bending about X + torsion about Y
6.	37.53	bending about axe Z

Table 1. To the example

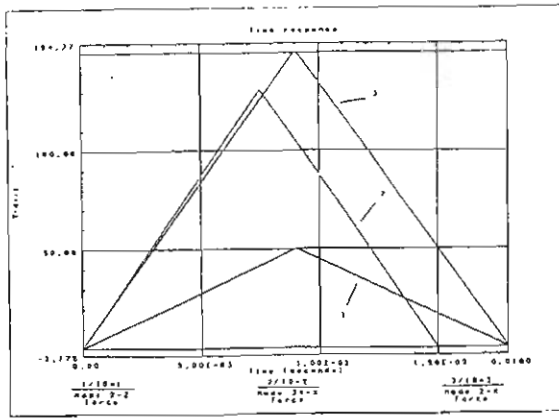


Diagram 1. Values of excitation forces

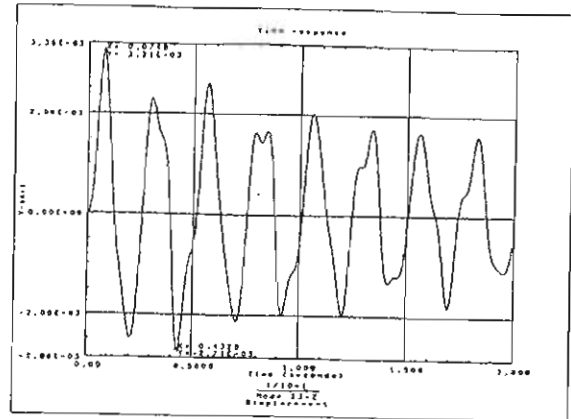


Diagram 2. To the example - calculation 1

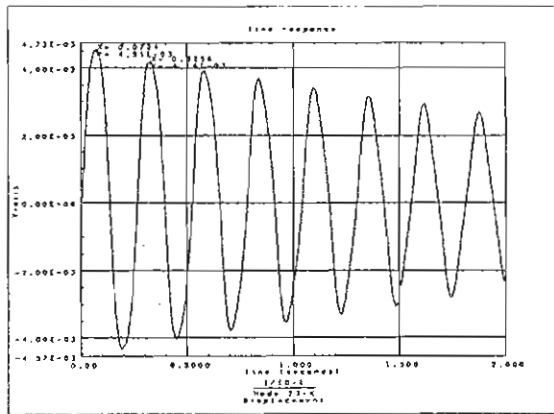


Diagram 3. To the example - calculation 2

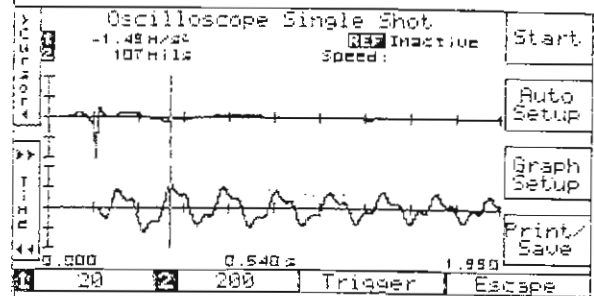


Diagram 4. To the example - experiment 1

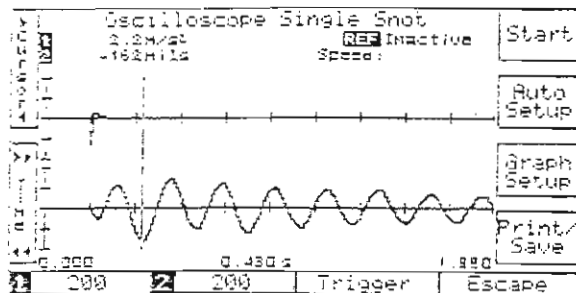


Diagram 5. To the example - experiment 2

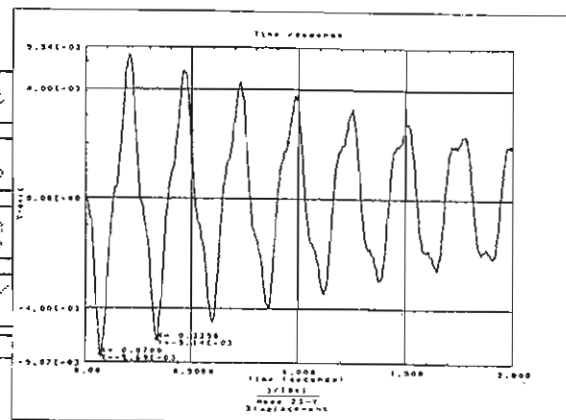


Diagram 6. To the example - calculation 3

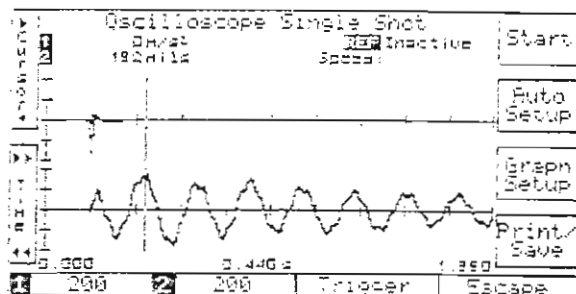


Diagram 7. To the example - experiment 3

Three experiments were carried out. In the first experiment construction was excited to vibrate with bending about axe X and torsion about axe Y as a result of an impact impulse acting in point 2 on the straight line which is paralell to axe Z of global coordinate system and measuring was done at point 4 on the same straight line. The second experiment excited the bending about axe Z because the construction was striken at point 3 on the straight line paralell to axe X. Vibrational displacement is measured at point 4 on the same straight line. The last experiment which excited the bending in the same plane is performed by striking the construction at the point 2 on the straight line paralell to axe X and by measuring at point 4 on the straight line paralell to axe Y. The obtained vibrating displacements were shown at the diagrams 4,5 and 7. During the measurements when the measuring is carried out, the measuring unit *mils* for whom is $1 \text{ mils} = 25.4 \cdot 10^{-6} \text{ (m)}$ was used, because of practical reasons.

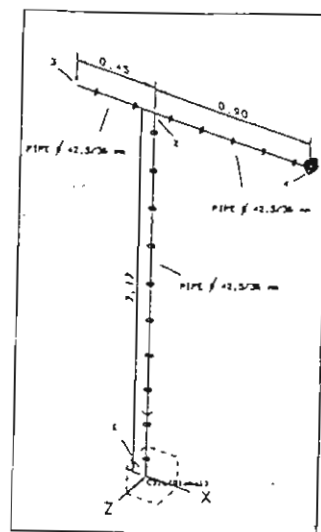


Fig. 6. To the example - diskretized model

On the basis of measured hammer's tip acceleration on the occasion of impact, the exciting forces are calculated, according to expression (25). The hammer mass was $m = 0.9 \text{ (kg)}$. By this calculation the entering data for the vibrating displacement estimation, on the program package I-DEAS were obtained. For the comparison, the examples which responds to experiments 1-3 was calculated. Values of exciting force are given on a diagram 1 while the results of the numerically part of estimation are shown on diagrams 2,3 and 6.

6. CONCLUSION

In accordance to the matter which is brought in this paper it is possible to conclude that the transient vibrations, which on mechanical constructions can be excited by striking with impact hammer, can be relatively exactly modelled on the program package I-DEAS if the time variation of the impact load is idealized with triangular profile. Triangular profile of impact load is chosen because the impact is approximated with the first degree polynomial which, related to the more complicated polynomials considerably facilitates integration in expression (24). Since the incorrectness in measurements has appeared only inside the first period of response displacement, where the lower displacements then expected are measured, it is possible to conclude that measuring equipment which was used had serious problems with displacement measuring in the very short period after the measuring device is turned on.

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VIBRATIONS OF A MECHANICAL SYSTEM WITH CLEARANCE SUBJECTED TO A MULTI-HARMONIC PERIODIC EXCITATION

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ABSTRACT

Forced steady state vibrations of a mechanical system with a clearance subjected to a multi-harmonic periodic excitation are considered. A clearance is modeled by a symmetric piecewise-linear force-displacement relationship. Equations for calculating non-linear frequency response characteristics are derived in the frequency domain by the incremental harmonic balance method. The results are valuated by a piecewise-linear technique. By comparing the results (very good agreement is obtained) it is shown, that the incremental harmonic balance method can be successfully applied to such systems.

1. INTRODUCTION

Clearances exist in many mechanical systems either by design or due to manufacturing errors and wear. Forced vibration of a system with clearances can result in relative motion across the clearance space and impacting between the components, what very often lead to noise and large dynamic loads. The solutions of these problems are usually obtained by using a piecewise-linear technique and digital simulation. The obtained response is the actual displacement time history and it can be very complicated and may include higher harmonics and chaotic motion. These methods can give transient and steady state responses for given initial conditions. Parametric studies and obtaining the wide-range frequency response by using these methods are very often cumbersome and time-consuming, particularly for the frequency regions of multiple-valued response, and for lightly damped systems. But, they are very valuable as a tool to validate other approximate methods.

For many engineering applications more meaningful description of the vibro-impact problem is given by the r.m.s. values than by the actual displacement time history. A detailed study of the frequency response functions of the systems with clearances are carried out by using the method of harmonic balance [1]. It is a spectral method in which the solution is assumed to have one dominant frequency component [2,3]. In cases where it can be applied (for systems with a single-harmonic excitation where a single harmonic description of the non-linearity is adequate), the harmonic balance method is by far the most

computationally efficient method for obtaining steady-state responses of non-linear vibration problems [4]. But, it becomes increasingly inaccurate as higher harmonics (superharmonics and subharmonics) in the response become significant.

The incremental harmonic balance method provides (also in the frequency domain) the study of effects of superharmonics and subharmonics to frequency response characteristic. In this method, non-linear differential equations are transformed into the set of linearized incremental algebraic equations in terms of the Fourier coefficients. So, only linear equations have to be formed and solved in each incremental step. The method is very good suited to parametric studies, because the new solution can be sought iteratively with the previous solution used as an approximation. The incremental harmonic balance method was introduced by Lau et al. [5] for treating continuous non-linearities. Later, the method was extended by Pierre et al. [6] to analyze systems with dry friction, and by Wong et al. [7] to analyze systems with unsymmetrical piecewise-linear force-displacement relationship.

In this paper a piecewise-linear technique and the incremental harmonic balance method are used for determining frequency response characteristics of an impact-pair with symmetric force-displacement relationship, which was subjected to a multi-harmonic excitation.

2. MODEL OF MECHANICAL SYSTEM WITH A CLEARANCE

Model of a mechanical system with clearance shown in Figure 1. consists of an inertia element m , an energy dissipation element k (viscous damping), and a non-linear elastic element $cg(x)$. The non-linear elastic element is defined by the function $g(x)$ shown in Figure 2.

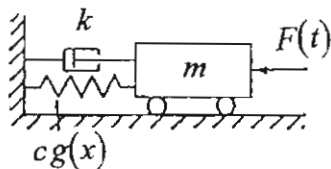


Figure 1. Model of a mechanical system.

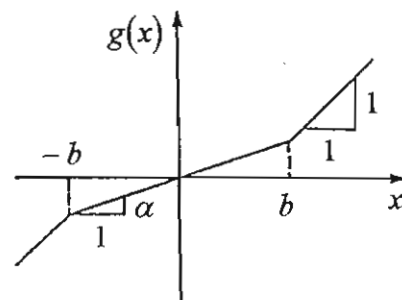


Figure 2. Non-linear force-displacement relationship.

b is one half of the clearance space and α is the ratio of the second stage stiffness and the first stage stiffness. When the system is excited by a periodic force $F(t)$ equation of motion is:

$$m\ddot{x} + k\dot{x} + cg(x) = F(t) = \sum_{n=0}^M (f_n \cos n\Omega t + g_n \sin n\Omega t), \quad (1)$$

where f_n and g_n ($n=0,1,\dots,M$) are the force component amplitudes of the corresponding harmonic terms, Ω is a fundamental frequency of the excitation, and

$$g(x) = \begin{cases} x - (1-\alpha)b, & x > b \\ \alpha x, & -b \leq x \leq b \\ x + (1-\alpha)b, & -b > x \end{cases}. \quad (2)$$

2. INCREMENTAL HARMONIC BALANCE METHOD

By introducing the non-dimensional time θ as a new independent variable, the period of the response (with ν subharmonics taken into consideration) is always 2π . The differential equation (1) can be written in a non-dimensional form:

$$\frac{\eta^2}{\nu^2} \frac{d^2 \bar{x}}{d\theta^2} + \frac{2\zeta\eta}{\nu} \frac{d\bar{x}}{d\theta} + g(\bar{x}) = \bar{F}(\theta) = \sum_{n=0}^M (\bar{f}_n \cos n(\nu\theta) + \bar{g}_n \sin n(\nu\theta)), \quad (3)$$

$$\text{where } \bar{x} = \frac{x}{l}, \bar{b} = \frac{b}{l}, \omega_0 = \sqrt{\frac{c}{m}}, \zeta = \frac{k}{2m\omega_0}, \eta = \frac{\Omega}{\omega_0}, \quad (4a-e)$$

$$\tau = \omega_0 \cdot t, \Omega t = \eta \tau = \nu \theta, \bar{f}_n = \frac{f_n}{ml\omega_0^2}, \bar{g}_n = \frac{g_n}{ml\omega_0^2}. \quad (4f-i)$$

l is any characteristic dimension of the system, usually $l=b$. A non-linear force-displacement relationship $g(x)$ can be expressed as:

$$g(\bar{x}) = \alpha \bar{x} + h^* (1-\alpha)(\bar{x} - \bar{b}^*), \quad (5)$$

where

$$h^* = \begin{cases} 1, & \bar{b} < \bar{x} \\ 0, & -\bar{b} \leq \bar{x} \leq \bar{b} \\ 1, & \bar{x} < -\bar{b} \end{cases}, \quad b^* = \begin{cases} \bar{b}, & \bar{b} < \bar{x} \\ 0, & -\bar{b} \leq \bar{x} \leq \bar{b} \\ -\bar{b}, & \bar{x} < -\bar{b} \end{cases}. \quad (6)$$

If a current state of vibration (correspondence to the excitation parameters η and $\bar{F}(\theta)$) is denoted by $\bar{x}_0(\theta)$, and a neighboring state can be expressed by adding an increment $\Delta \bar{x}(\theta)$:

$$\bar{x}(\theta) = \bar{x}_0(\theta) + \Delta \bar{x}(\theta). \quad (7)$$

By substituting equation (7) into (3) an incremental equation is obtained:

$$\frac{\eta^2}{\nu^2} \frac{d^2 \Delta \bar{x}}{d\theta^2} + \frac{2\zeta\eta}{\nu} \frac{d\Delta \bar{x}}{d\theta} + [\alpha + h^*(1-\alpha)] \Delta \bar{x} = R, \quad (8)$$

where

$$R = - \left[\frac{\eta^2}{\nu^2} \frac{d^2 \bar{x}_0}{d\theta^2} + \frac{2\zeta\eta}{\nu} \frac{d\bar{x}_0}{d\theta} + \alpha \bar{x}_0 + h^*(1-\alpha)(\bar{x}_0 - \bar{b}^*) - \bar{F}_0(\theta) \right]. \quad (9)$$

R is the correction term that vanishes in the case when $\bar{x}_0(\theta)$ is an exact solution of the equation (3). Equation (3) is a linear differential equation

with variable coefficients that can be approximately solved by using Galerkin's procedure. For this purpose the periodic solution and its increment are written as:

$$\bar{x}_0(\theta) = \sum_{i=0}^N (a_i \cos i\theta + b_i \sin i\theta), \quad \Delta \bar{x}(\theta) = \sum_{i=0}^N (\Delta a_i \cos i\theta + \Delta b_i \sin i\theta), \quad N = \nu K. \quad (10)$$

N is the number of all harmonic terms taken into consideration, ν is the number of subharmonics and K is the number of superharmonics. By averaging equation (8) inside the interval $[0, 2\pi]$, one obtains:

$$\int_0^{2\pi} \left\{ \frac{\eta^2}{\nu^2} \frac{d^2 \Delta \bar{x}}{d\theta^2} + \frac{2\zeta \eta}{\nu} \frac{d \Delta \bar{x}}{d\theta} + [\alpha + h^*(1-\alpha)] \Delta \bar{x} \right\} \delta(\Delta \bar{x}) d\theta = \int_0^{2\pi} (R) \delta(\Delta \bar{x}) d\theta, \quad (11a)$$

or

$$\int_0^{2\pi} \left\{ \frac{\eta^2}{\nu^2} \frac{d^2 \Delta \bar{x}}{d\theta^2} + \frac{2\zeta \eta}{\nu} \frac{d \Delta \bar{x}}{d\theta} + [\alpha + h^*(1-\alpha)] \Delta \bar{x} - R \right\} \left(\sum_{j=0}^N \cos j\theta + \sin j\theta \right) d\theta = 0. \quad (11b)$$

After substitution (10) into (11b) and integrating from 0 to 2π one obtains $2N+1$ linear equations with Δa_i and Δb_i as $2N+1$ unknowns ($i = 0, 1, \dots, N$, with $\Delta b_0 = 0$):

$$\sum_{i=0}^N \Delta a_i C_{aaij} + \Delta b_i C_{abij} = R_{aj}, \quad (j = 0, 1, \dots, N), \quad (12a)$$

$$\sum_{i=0}^N \Delta a_i C_{baaj} + \Delta b_i C_{bbij} = R_{bj}, \quad (j = 1, 2, \dots, N), \quad (12b)$$

where

$$C_{aaij} = \delta_{ij} \left(\alpha - \frac{\eta_0^2}{\nu^2} i^2 \right) \pi + C_{aaij}^N, \quad C_{abij} = \delta_{ij} \frac{2\zeta \eta_0}{\nu} i \pi + C_{abij}^N, \quad (13a,b)$$

$$C_{baaj} = -\delta_{ij} \frac{2\zeta \eta_0}{\nu} i \pi + C_{baaj}^N, \quad C_{bbij} = \delta_{ij} \left(\alpha - \frac{\eta_0^2}{\nu^2} i^2 \right) \pi + C_{bbij}^N, \quad (13c,d)$$

$$R_{aj} = - \left[\left(\alpha - \frac{\eta_0^2}{\nu^2} j^2 \right) a_j + \frac{2\zeta \eta_0}{\nu} j b_j - \bar{f}_j \right] \pi - R_{aj}^N, \quad (13e,f)$$

$$R_{bj} = - \left[\left(\alpha - \frac{\eta_0^2}{\nu^2} j^2 \right) b_j - \frac{2\zeta \eta_0}{\nu} j a_j - \bar{g}_j \right] \pi - R_{bj}^N, \quad (13g,h)$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

$$C_{aaij}^N = (1-\alpha) \int_0^{2\pi} h^* \cos i\theta \cos j\theta d\theta, \quad C_{abij}^N = (1-\alpha) \int_0^{2\pi} h^* \sin i\theta \cos j\theta d\theta, \quad (13i,j)$$

$$C_{baaj}^N = (1-\alpha) \int_0^{2\pi} h^* \cos i\theta \sin j\theta d\theta, \quad C_{bbij}^N = (1-\alpha) \int_0^{2\pi} h^* \sin i\theta \sin j\theta d\theta, \quad (13k,l)$$

$$R_{aj}^N = \int_0^{2\pi} h^* (1-\alpha) (\bar{x}_0 - \bar{b}^*) \cos j\theta d\theta, \quad R_{bj}^N = \int_0^{2\pi} h^* (1-\alpha) (\bar{x}_0 - \bar{b}^*) \sin j\theta d\theta. \quad (13m,n)$$

To evaluate expressions (13) one must know the zeros of the equations:

$$\bar{x}_0(\theta) = \bar{b} \quad \text{and} \quad \bar{x}_0(\theta) = -\bar{b}. \quad (14)$$

L solutions of trigonometric equations (14) denoted by $\theta_1, \theta_2, \dots, \theta_L$, with $\theta_0 = 0$ and $\theta_{L+1} = 2\pi$ ($\theta_1 < \theta_2 < \dots < \theta_L$), can be obtained numerically. h^* and b^* one determines inside each interval $[\theta_0, \theta_1], [\theta_1, \theta_2], \dots, [\theta_L, \theta_{L+1}]$ from equation (6). Solving the equations (12) starts from initial vibration state. One performs iterative solving until the norm of the corrective vector $\| \{R\} \| = \| [R_{a0}, R_{a1}, \dots, R_{aN}, R_{b1}, R_{b2}, \dots, R_{bN}]^T \|$ becomes smaller than a certain convergence parameter.

When the steady state solution is obtained, the stability of the given solution can be determined by examining the perturbed solution:

$$\bar{x}_0^* = \bar{x}_0 + \Delta \bar{x}_0^*, \quad (15)$$

where $\Delta \bar{x}_0^*$ is a small perturbation of $\bar{x}_0$. If perturbed motion decays with time, the steady state solution is dynamically stable, and if perturbed motion grows, it is dynamically unstable. Substitution of equation (15) into (3) gives linear differential equation with periodic coefficients:

$$\frac{\eta^2}{\nu^2} \frac{d^2 \Delta \bar{x}_0^*}{d\theta^2} + \frac{2\zeta\eta}{\nu} \frac{d\Delta \bar{x}_0^*}{d\theta} + [\alpha + h^*(1-\alpha)] \Delta \bar{x}_0^* = 0. \quad (16)$$

Whether the perturbed solution decays or grows with time for a certain $\bar{x}_0$ (and consequently known h^*) can be directly determined by solving parametrically excited system (16) by using the method of piecing exact solutions. A more efficient method for determining the stability of the steady state solution based on Floquet-Liapounov theorem [8] is used in this work. For that purpose equation (16) can be rewritten in state variable form as:

$$\{\dot{\Delta \bar{X}_0^*}\} = [A(\theta)] \{\Delta \bar{X}_0^*\}, \quad (17)$$

where

$$\{\Delta \bar{X}_0^*\} = \begin{Bmatrix} \Delta \bar{x}_0^* \\ \frac{d\Delta \bar{x}_0^*}{d\theta} \end{Bmatrix}, \quad \{\dot{\Delta \bar{X}_0^*}\} = \begin{Bmatrix} \frac{d\Delta \bar{x}_0^*}{d\theta} \\ \frac{d^2 \Delta \bar{x}_0^*}{d\theta^2} \end{Bmatrix}, \quad [A(\theta)] = \begin{bmatrix} 0 & 1 \\ -\frac{\nu^2}{\eta^2}(\alpha + h^*(1-\alpha)) & -\frac{2\nu\zeta}{\eta} \end{bmatrix}. \quad (18)$$

Since coefficient matrix $[A(\theta)]$ is periodic function of θ with period 2π , the stability criteria are related to the eigenvalues of the transition matrix at the end of one period $[\Phi(2\pi, 0)]$, where $[\Phi(\theta, \theta_0)]$ denotes the transition matrix at θ for initial conditions given at θ_0 . According to Floquet-Liapounov theorem: the solution is stable if all the moduli of the eigenvalues of $[\Phi(2\pi, 0)]$ are less than unity. Otherwise the solution is unstable. Inside each of interval $[\theta_i, \theta_{i+1}]$ h^* is a constant and consequently

$[A(\theta)]$ is also a constant matrix inside that interval. According to [9], for constant $[A(\theta)]$ (inside the interval $[\theta_i, \theta_{i+1}]$) transition matrix $[\Phi(\theta_{i+1}, \theta_i)]$ can be expressed as:

$$[\Phi(\theta_{i+1}, \theta_i)] = e^{[A(\theta)](\theta_{i+1} - \theta_i)}. \quad (19)$$

For the whole interval $[0, 2\pi]$ one obtains:

$$[\Phi(2\pi, 0)] = \prod_{i=0}^L \left(e^{[A(\theta)](\theta_{i+1} - \theta_i)} \right). \quad (20)$$

3. A PIECEWISE-LINEAR TECHNIQUE

Force-displacement relationship $g(\bar{x})$ (Figure 2.) is piecewise-linear. Local solutions of differential equation (1) are known explicitly inside each of the stage stiffness. Local solutions can be repeatedly matched at $\bar{x} = \bar{b}$ and $\bar{x} = -\bar{b}$, to obtain a global solution of (1). Piecing together of these local solutions is not directly possible, because the times of flight in each stage stiffness region cannot be found in closed form. But, the matching of local solutions can be done easily numerically. Only approximation done by applying of this procedure is in numerical determination of times in which the system changes stage stiffness region ($\bar{x} = \bar{b}$, $\bar{x} = -\bar{b}$).

4. NUMERICAL EXAMPLE

A system shown in Figure 1. with symmetric piecewise-linear force displacement relationship (Figure 2.) subjected to periodic excitation is considered. The excitation consists of a constant load ($\bar{f}_0 = 0.25$) and three harmonic terms ($\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0.125$). Equivalent r.m.s. amplitudes ($\bar{x}_p$) of the time domain responses obtained by a piecewise linear technique are shown in Figure 3a. Periodic responses are denoted by "o", and unperiodic by "+". The numerical precision used for obtaining the times in which the system changes stage stiffness region ($\bar{x} = \bar{b}$, $\bar{x} = -\bar{b}$) is $\varepsilon = 2.2204e-016$, what is the maximum numerical precision of the computer. Up to 800 excitation cycles were considered for examining a periodicity of the response. A response is considered as a periodic, if equations (21) are satisfied

$$\bar{x}(t = jT) = \bar{x}(t = (j + \nu)T), \quad \frac{d\bar{x}}{dt}(t = jT) = \frac{d\bar{x}}{dt}(t = (j + \nu)T), \quad (21)$$

where T denotes the period of the excitation. In Figure 3b, the numerical results obtained by the incremental harmonic balance method are compared with those obtained by a piecewise-linear technique. Figure 3b. shows excellent agreement between the results obtained by these methods. Unperiodic responses obtained by a piecewise linear technique are not found by the incremental harmonic balance method, because this method is

limited only to consideration of periodic vibrations. Also, frequency response characteristics in Figure 3a are incomplete, because the results of a piecewise-linear technique depend on given initial conditions, making it difficult to find all possible solutions.

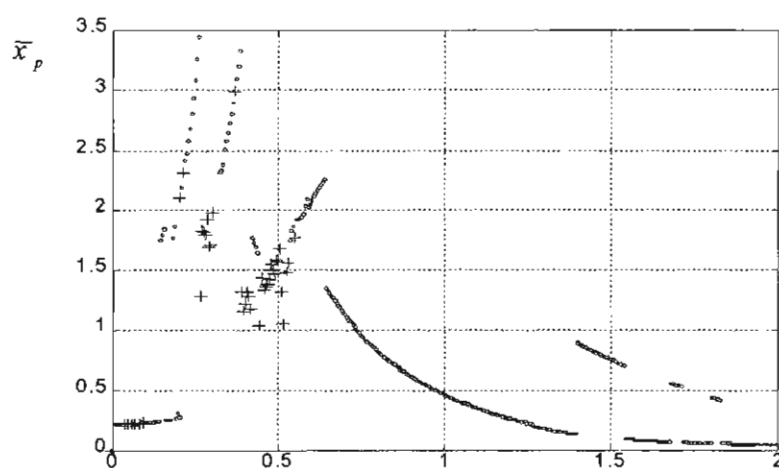


Figure 3a.

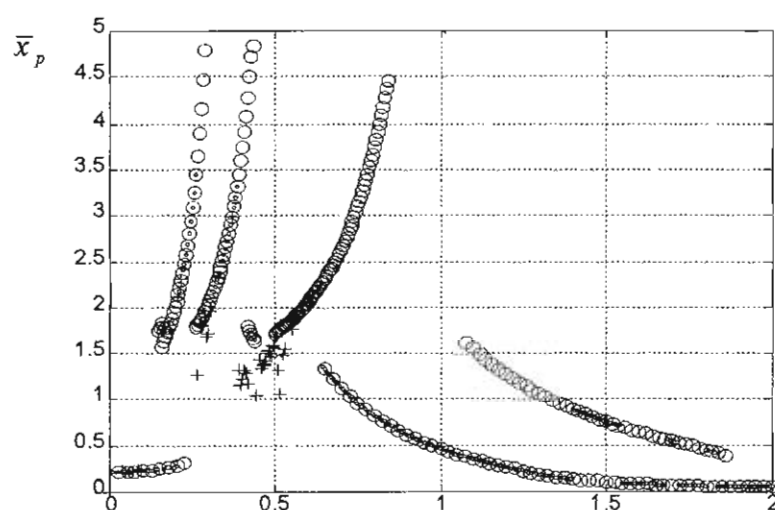


Figure 3b.

- ° , periodic response obtained by a piecewise-linear technique,
- + , unperiodic response obtained by a piecewise-linear technique,
- , response obtained by the incremental harmonic balance method.

Figure 3a,b. Frequency response $\bar{x}_p = \bar{x}_p(\eta)$ for $\bar{b} = 1$, $\alpha = 0$,
 $\bar{f}_0 = 0.25$, $\bar{f}_1 = \bar{f}_2 = \bar{f}_3 = 0.125$, $\bar{g}_1 = \bar{g}_2 = \bar{g}_3 = 0$,
 $\zeta = 0.015$.

5. CONCLUSION

The incremental harmonic balance method (in the frequency domain) and a piecewise-linear technique (in the time domain) are used in this

paper to obtain the steady state response of a mechanical system with a clearance subjected to a multi-harmonic excitation. An excellent agreement of the results obtained by these methods is achieved. So, it is shown, that the incremental harmonic balance method can be successfully applied for obtaining frequency response characteristics of the considered type of mechanical systems. The method provides accurate solutions over a much broader frequency range than the harmonic balance method since it accommodates multiple harmonics. Although there are no theoretical limitations to the number of harmonics included into approximate solution, there is a computer time, which grows geometrically with the number of harmonics. The method is very good suited to parametric studies, because the new solution can be sought iteratively with the previous solution used as an approximation. The use of the incremental harmonic balance method is limited only to consideration of periodic vibrations.

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DYNAMIC BEHAVIOUR OF ARTICULATED TOWER IN WAVES

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Abstract - Articulated tower is a stiff, buoyant and simply supported marine structure. Its dynamic behaviour in regular and irregular waves, as a special case of flexible tower, is analyzed. A review of the developed theory and computer program is given. They are verified within correlation analysis using some available results from literature.

1. INTRODUCTION

In recent decades marine technology has been developing rapidly. A large number of offshore structures have been installed in deep water for the purpose of exploration and exploitation of sea goods [1, 2, 3]. Marine towers are specific structures which may be of framed gravity or self-buoyant type, flexible or stiff, simply supported or clamped at the sea bottom. In this article the articulated tower, which is a buoyant, stiff and simply supported structure, is considered as a special case of flexible marine tower, Fig.1. The numerical procedure for the dynamic analysis of marine towers exposed to regular and irregular sea waves may be found in [4, 5]. An outline of the procedure is given as follows.

2. WAVE LOAD AND MOTION EQUATION

The beam model of a tower buoy and its differential element with external loads and internal forces, is shown in Fig.2. Transverse load q is a result of the inertia and drag forces of fluid particles circulating in waves. It is defined by the modified Morison equation [3, 6]. Axial force N is resultant of the structure weight and its buoyancy, which depends on wave elevation. Referring to [4, 5], the complete nonlinear differential equation for large amplitude motion may be written in the pseudo-linear form

$$\frac{\partial^2}{\partial x^2} \left(E I \frac{\partial^2 w}{\partial x^2} \right) - N_s \frac{\partial^2 w}{\partial x^2} + C_s \frac{\partial w}{\partial t} + (m_s + m_a) \frac{\partial^2 w}{\partial t^2} = q \quad (1)$$

$$q = N_d \frac{\partial^2 w}{\partial t^2} + C_I \left[\frac{\partial^2 \xi}{\partial t^2} + \left(\frac{\partial \xi}{\partial t} - \frac{\partial w}{\partial t} \right) \frac{\partial^2 \xi}{\partial y \partial t} \right] + C_D \left| \frac{\partial \xi}{\partial t} - \frac{\partial w}{\partial t} \right| \left(\frac{\partial \xi}{\partial t} - \frac{\partial w}{\partial t} \right) \quad (2)$$

where

E - Young's modulus

I - moment of inertia of cross-section

N_s - static axial force

N_d - dynamic axial force

m_s - structure mass

m_a - added mass of surrounding water

C_s - coefficient of structural damping

C_I - inertia coefficient

C_D - drag coefficient

q - load

w - tower deflection

ξ - horizontal displacement of fluid particle

x - ordinate

t - time

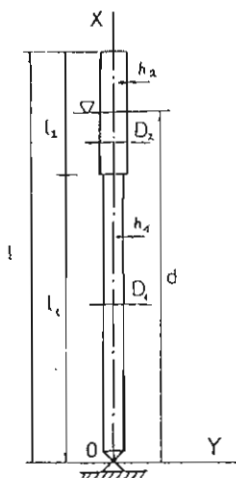


Fig.1. Marine tower

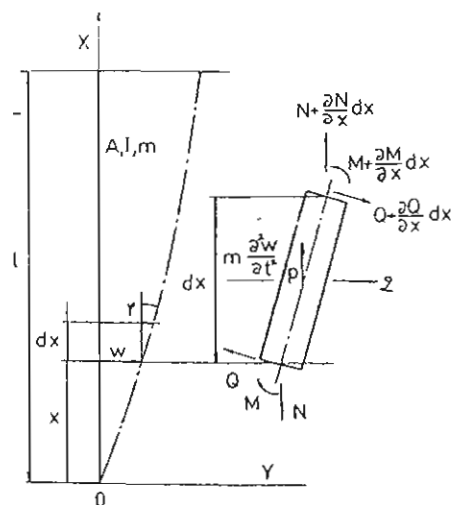


Fig.2. Bending of tower

The load q includes a component of dynamic axial force, local and convective fluid inertia and drag force. Furthermore,

$$C_l = (1 + c_a) \frac{\rho \pi D^2}{4}, \quad C_D = \frac{1}{2} c_d \rho D, \quad m_a = c_a \frac{\rho \pi D^2}{4} \quad (3)$$

where ρ is water density and c_a and c_d are nondimensional added mass and drag coefficient respectively [3, 6]. Dynamic axial force reads

$$N_d = \rho g A \eta \quad (4)$$

where g is gravity constant, A is area of water line and η is water surface elevation.

For a regular harmonic wave, the water surface elevation and the horizontal displacement of a fluid particle are given in the following form

$$\eta(y, t) = h \cos(ky - \omega_o t) \\ \xi(x, y, t) = -\xi_x \sin(ky - \omega_o t), \quad \xi_x = h \frac{\operatorname{ch} kx}{\operatorname{sh} kd} \quad (5)$$

where

$k = \omega_o^2 / g$	- wave number	H - wave height
ω_o	- wave frequency	d - water depth
$h = H/2$	- wave amplitude	$y = w$

In the case of an irregular wave presented by harmonic components one may write

$$\eta(y, t) = \sum_{n=1}^{\infty} a_n \cos(k_n y - \omega_n t + \varepsilon_n) \\ \xi(x, y, t) = -\sum_{n=1}^{\infty} \xi_{xn} \sin(k_n y - \omega_n t + \varepsilon_n) \\ \xi_{xn} = a_n \frac{\operatorname{ch} k_n x}{\operatorname{sh} k_n d}, \quad k_n = \frac{\omega_n^2}{g} \quad (6)$$

where n is index of component wave and ε_n random phase angle. Component wave amplitude depends on the sea energy spectrum $S(\omega)$ [3]

$$a_n = \sqrt{2 S_n(\omega_n) \varpi} \quad (7)$$

where $\varpi = \omega_n / n$ is frequency step.

The complete differential equation (1) is nonlinear and dedicated to large motion analysis. In the case of small amplitude one may take $y = 0$ in wave equations (5) and (6). Both tasks are solved in the time domain. However, if the small amplitude problem is linearized, differential equation (1) takes the form

$$\frac{\partial^2}{\partial x^2} \left(E I \frac{\partial^2 w}{\partial x^2} \right) - N_s \frac{\partial^2 w}{\partial x^2} + (C_s + \bar{C}_D) \frac{\partial w}{\partial t} + (m_s + m_a) \frac{\partial^2 w}{\partial t^2} = q \quad (8)$$

$$q = C_l \frac{\partial^2 \xi}{\partial t^2} + \bar{C}_D \frac{\partial \xi}{\partial t} \quad (9)$$

and it may be solved in the frequency domain. The linearized drag coefficient for regular and irregular waves according to /7/ reads

$$\bar{C}_D = \frac{8}{3\pi} \dot{r}_{\max}, \quad \bar{C}_D = C_D \sqrt{\frac{8}{\pi}} \sigma_r \quad (10, 11)$$

where $\dot{r}_{\max}$ and σ_r are amplitude and standard derivation of deterministic and stochastic relative velocity $\dot{\xi}$ - $\dot{w}$ respectively.

3. FINITE ELEMENT SOLUTION

Partial nonlinear differential equation (1) and its linearized transform (8) may be solved in space by the finite element method /8, 9, 10/. The ordinary beam element is used. Its properties, i.e. bending and geometric stiffness, damping and mass matrices, and load vector are specified in /4/.

The equilibrium of nodal forces and compatibility of nodal displacements result in the pseudo-linear matrix equation /11/

$$[K] \{\delta\} + [C] \{\dot{\delta}\} + [M] \{\ddot{\delta}\} = \{F(t)\} \quad (12)$$

where $[K]$, $[C]$ and $[M]$ are general stiffness, damping and mass matrix, respectively; $\{\delta\}$, $\{\dot{\delta}\}$ and $\{\ddot{\delta}\}$ are displacement, velocity and acceleration vectors, respectively; and $\{F(t)\}$ is nonlinear load vector which depends on $\{\delta\}$. Assuming response in the form

$$\{\delta\} = [\phi] \{X\} \quad (13)$$

where $[\phi]$ and $\{X\}$ are undamped mode matrix and generalized displacement vector respectively, the equation (12) may be reduced to a set of coupled modal equations

$$\omega_i^2 X_i + 2 \omega_i \xi_i \dot{X}_i + \ddot{X}_i = \varphi_i(t), \quad i = 1, 2, \dots, N \quad (14)$$

where

$$\begin{aligned} \varphi_i(t) &= \psi_i(t) - 2 \omega_i \sum_{j=1}^N (1 - \delta_{ij}) \xi_{ij} \dot{X}_j \\ \delta_{ij} &= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad \text{is Kronecker symbol} \end{aligned} \quad (15)$$

N is number of d.o.f., ω_i natural frequency of i -th mode, ξ_i coefficient of modal damping, and $\psi_i(t)$ modal excitation.

The coupled nonlinear modal equations (14) for regular and irregular wave excitation and large and small motion are solved by time integration. Applying the harmonic acceleration method, iteration integration algorithm reads /11/

$$\{Y\}_{i,j+1}^{(k+1)} = [T] \{Y\}_{i,j} + \{L\} \varphi_{i,j+1}^{(k)} \quad (16)$$

where $\{Y\} = \langle X, \dot{X}, \ddot{X} \rangle$ is response vector, $[T]$ integration transfer matrix, and $\{L\}$ load operator defined in /11/. The subscript $i (1, 2, \dots, N)$ is the mode index, while $j (0, 1, 2, \dots)$ and $k (0, 1, 2, \dots)$ denote the time and iteration step respectively.

In the case of small motion and linearized damping, the governing modal equations (14) may be solved in the frequency domain. Since an irregular wave is presented in the harmonic form (6), the modal load (15) yields in the same form, i.e.

$$\varphi_i(t) = \sum_{n=1}^{\infty} (\varphi_{sin} \sin \omega_n t + \varphi_{cin} \cos \omega_n t) \quad (17)$$

where $\omega_n = n \omega$, and ω is frequency step. The modal response reads

$$X_i(t) = \sum_{n=1}^{\infty} (X_{sin} \sin \omega_n t + X_{cin} \cos \omega_n t) \quad (18)$$

where substituting (17) and (18) into (14) one obtains

$$\begin{aligned} X_{sin} &= \frac{(1 - \beta_{in}^2) \varphi_{sin} + 2 \xi_i \beta_{in} \varphi_{cin}}{\left[(1 - \beta_{in}^2)^2 + (2 \xi_i \beta_{in})^2 \right] \omega_i^2} \\ X_{cin} &= \frac{(1 - \beta_{in}^2) \varphi_{cin} - 2 \xi_i \beta_{in} \varphi_{sin}}{\left[(1 - \beta_{in}^2)^2 + (2 \xi_i \beta_{in})^2 \right] \omega_i^2}, \quad \beta_{in} = \omega_n / \omega_i \end{aligned} \quad (19)$$

If the tower motion in a regular wave is analyzed, only one harmonic of the excitation and response is taken in account. The determination of relative velocity $\dot{r} = \dot{\xi} - \dot{w}$ for deterministic and random excitation is explained in /4/ and /5/ respectively.

4. COMPUTER PROGRAM

The summarized algorithms for the dynamic analysis of the marine tower were programmed in FORTRAN 77. The program includes the following seven moduli /12/:

1. PLUVIBR for the calculation of free undamped vibration by the subspace iteration method. Natural frequencies, mode shapes, bending moments and shear forces are determined.
2. PLUSTAT for the calculation of static response due to water stream load. The static part of equation (12) is used.
3. PRVIB for the calculation of linearized forced vibration in frequency domain, caused by regular wave. The modulus may also be used for a static analysis.
4. NELIN1 for the calculation of nonlinear forced vibration of small amplitude in time domain, caused by regular wave.
5. NELIN2 for the calculation of nonlinear forced vibration of large amplitude in time domain, caused by regular wave.
6. FRSTOH for the calculation of linearized forced vibration of small amplitude in frequency domain, caused by irregular wave.
7. TISTOH1 for the calculation of nonlinear forced vibration of small amplitude in time domain, caused by irregular wave.
8. TISTOH2 for the calculation of nonlinear forced vibration of large amplitude in time domain, caused by irregular wave.

The program was verified in the case of the tower buoy exposed to regular and irregular waves in /4, 5, 13/.

5. NUMERICAL EXAMPLES

5.1 Particulars of articulated tower

An articulated tower is usually a stiff vertical column, simply supported at the sea bottom. The upper part is buoyant, while the lower part is ballasted. The tower is dynamically balanced in such a way that its natural frequency is larger than the peak frequency of the wave energy spectrum.

A typical articulated tower analysed in /6, 14, 15, 16/ is taken into consideration, Fig.3. Its particulars are the following:

added mass coefficient: $c_a = 1$
cross-section area: $A = 1.456 \text{ m}^2$

moment of inertia: $I = 3.58 \text{ m}^4$
Young's modulus: $E = 200 \text{ GPa}$

The tower is divided into 50 beam elements. Since the column is very stiff only the first natural (rigid body) mode is taken into account in the mode superposition method.

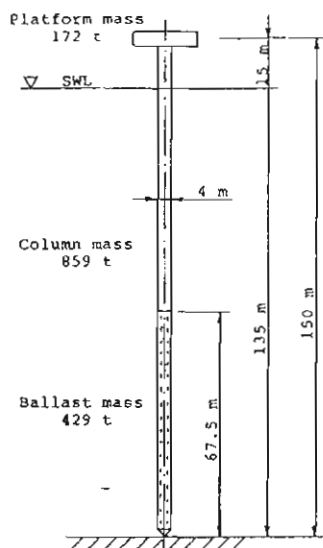


Fig.3. Articulated tower

5.2 Static analysis

The articulated tower is exposed to a steady and uniform current of a velocity $v = 0.9125 \text{ m/s}$ [6, 16]. The drag coefficient is $c_d = 0.5$. The heel angle calculated by the available moduli is compared to those known in literature, Table 1. The results show a very good agreement.

Table 1. Heel angle φ [rad]

PLUSTAT	PRVIB	NELIN2	Analytic Ref. /6/	Numerical Ref. /6/	Numerical Ref. /16/
0.0215	0.0217	0.0213	0.0216	0.0216	0.0205

The time history of the tower response due to suddenly imposed current load is shown in Fig.4. The static equilibrium is achieved after 200 s.

5.3. Free vibration

Using the analytical solution for the natural frequency of rigid tower oscillations derived in [14], one obtains $\Omega = 0.1282 \text{ rad/s}$, and the natural period reads $T = 49 \text{ s}$. The same values are obtained by modulus PLUVIBR. In order to get better insight into the tower dynamic behaviour, the initial conditions $\varphi_0 = -0.3 \text{ rad}$ and $\dot{\varphi}_0 = 0$ are imposed, taking $c_d = 1$. The free vibration calculated by NELIN2, Fig.5, is used to determine logarithm decrement and damping coefficient ξ in each half period. Its values are compared to those determined in [16], Table 2. After $3T/2$ there are no discrepancies of the results.

Table 2. Damping coefficient ξ

Half period	NELIN2	Ref. /16/
1	0.30	-
2	0.15	0.17
3	0.10	0.09
4	0.08	0.08
5	0.06	0.06

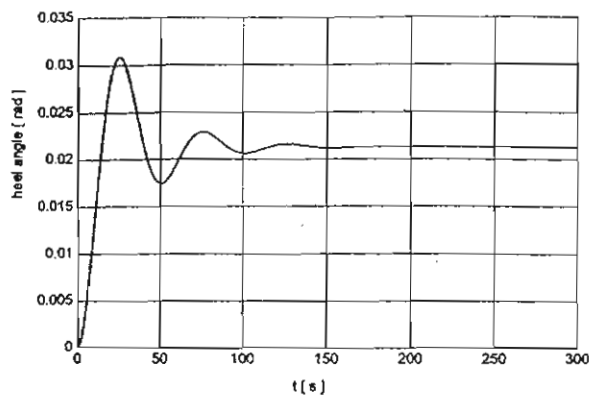


Fig.4. Heel angle due to suddenly imposed current load, NELIN2

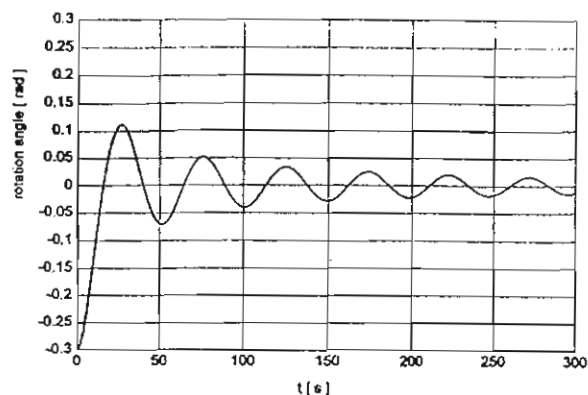


Fig.5. Free vibration of articulated tower, initial conditions $\phi_0 = -0.3$ rad, $\dot{\phi}_0 = 0$, NELIN2

5.4. Small motion in regular waves

The vibration of the articulated tower in regular harmonic waves is analyzed in the frequency and time domain assuming small motion. In Table 3 the amplitude of the rotation angle for different input data is compared with that quoted in literature /14, 15/. Also, the value of relative damping coefficient is given. The response determined in the frequency domain and that determined in the time domain are in a good agreement.

Table 3. Amplitude of rotation angle, regular wave, small motion

Wave height H [m]	Wave period T [s]	Drag coefficient c_d	Amplitude of rotation angle, ϕ [rad]			Damping coefficient ξ
			PRVIB	NELIN1	Ref./14,15/	
30	17	1.0	0.128	0.126	0.115	0.23
30	17	0.5	0.129	0.127	0.132	0.12
20	13.8	1.0	0.062	0.062	0.063	0.13
20	11.3	1.0	0.046	0.046	0.046	0.21
6.67	8	1.0	0.009	0.009	0.017	0.12

The time history of rotation angle obtained by modulus NELIN1 for the first case from Table 3 is shown in Fig.6. The response is stabilized after 100 s.

5.5. Large motion in regular waves

The response of the articulated tower in regular waves ($H = 30$ m, $T = 17$ s, and $c_d = 1$) is determined in the time domain by modulus NELIN2, Fig.7. A static inclination is noticed, which is not the case in the small amplitude solution, Fig.6. The results are similar to those given in /6, 16/.

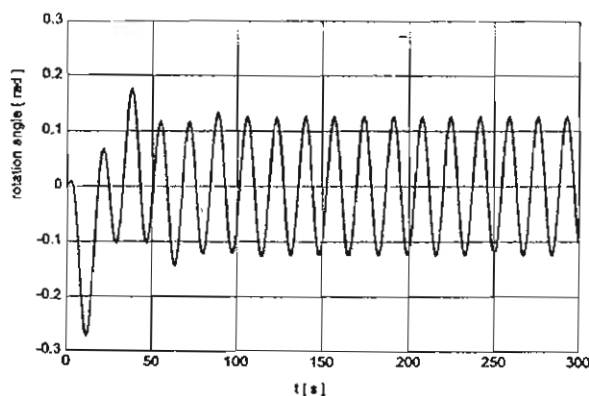


Fig.6. Rotation angle, regular wave $H = 30$ m, $T = 17$ s, small motion, NELIN1

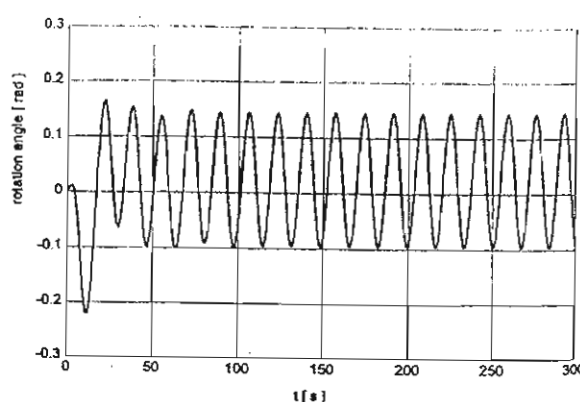


Fig.7. Rotation angle, regular wave $H = 30$ m, $T = 17$ s, large motion, NELIN2

5.6. Small motion in irregular waves

Rough sea is defined by the JONSWAP wave energy spectrum /3, 14, 15/

$$S_{\eta}(\omega) = \alpha g^2 \omega^{-5} \exp \left[-1.25 \left(\omega_p / \omega \right)^4 \right] \gamma \exp \left[-(\omega - \omega_p)^2 / 2 \tau^2 \omega_p^2 \right]$$

where $\alpha = 0.0081$, $\gamma = 3.3$, $\tau = 0.07$ for $\omega < \omega_p$, and $\tau = 0.09$ for $\omega > \omega_p$, $\omega_p = (2/H_{1/3})^{1/2}$ is peak frequency, $H_{1/3}$ significant wave height. The JONSWAP spectrum is a modification of the Pierson-Moskowitz spectrum, which is obtained taking $\gamma=1$ into account. The spectra are compared in Fig.8 in the case of $H_{1/3}=15$ m. The JONSWAP spectrum is narrow and peaked.

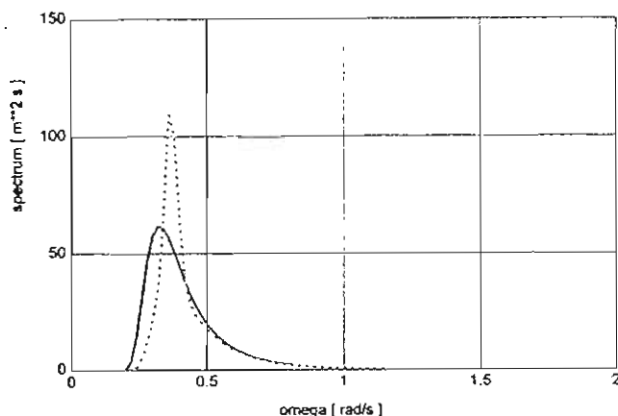


Fig.8. Wave energy spectrum,
 $H_{1/3} = 15$ m,
- - - JONSWAP,
— Pierson-Moskowitz

The response of the articulated tower for both JONSWAP and P-M spectra in the frequency and time domain is calculated by moduli TISTOH1 and FRSTOH respectively. Irregular wave is simulated by 100 harmonics up to the frequency $\omega = 2$ rad/s. Random phase angles are generated by a FORTRAN 77 routine. Time integration is performed up to 464 s, by the time step of 1 s. The first 150 s are excluded from the Fourier analysis in order to avoid the influence of the initial conditions. The energy spectra for the tower top displacement obtained for JONSWAP and P-M spectra are shown in Figs.9 and 10 respectively. The agreement of the results determined in the frequency and time domain is excellent. The peak of the response spectrum in the case of JONSWAP wave energy spectrum is somewhat higher than in the case of P-M spectrum.

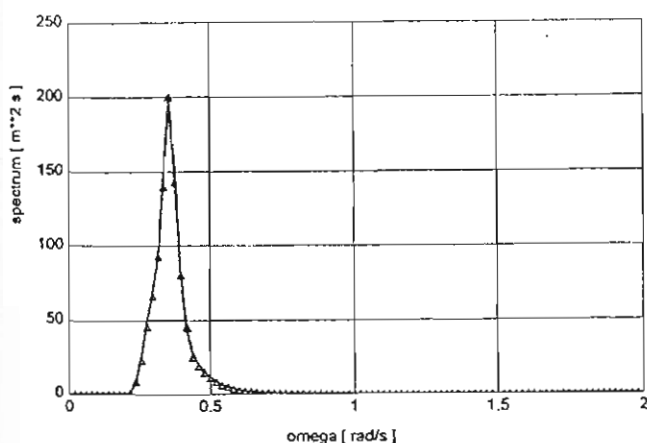


Fig.9. Energy spectrum of tower top displacement, small motion, JONSWAP spectrum, $H_{1/3} = 15$ m, Δ TISTOH1, — FRSTOH

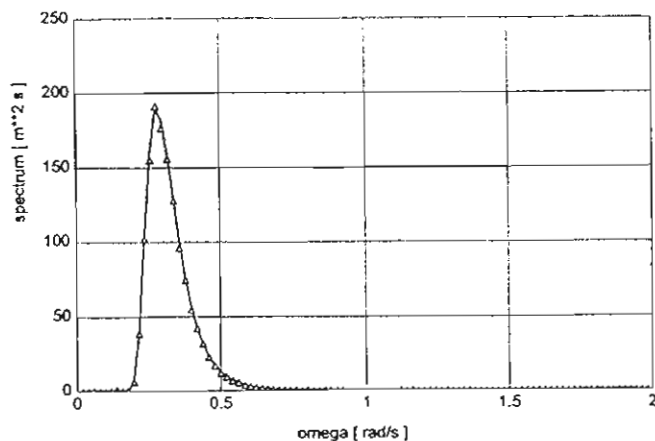


Fig.10. Energy spectrum of tower top displacement, small motion, Pierson-Moskowitz spectrum, $H_{1/3} = 15$ m, Δ TISTOH1, — FRSTOH

In order to get better insight into the correlation of the obtained results, the standard deviation of the energy spectra of the rotation angle, determined in the time and frequency domain in the case of JONSWAP and P-M spectra for different sea states given by significant

wave height, is listed in Tables 4 and 5 respectively. The variation of the results is also very low compared to those given in /15/. The influence of drag force is explored taking different values of drag coefficient c_d into account. A negligible change of the response may be noticed.

Table 4. Standard deviation of rotation angle of articulated tower, σ_φ [°], small motion, JONSWAP spectrum

$H_{1/3}$ [m]	c_d	TISTOH1	FRSTOH	Ref. /15/
15	1.0	1.67	1.69	1.65
15	0.5	1.68	1.70	1.73
9	1.0	0.68	0.69	0.68
5	1.0	0.25	0.24	0.23

Table 5. Standard deviation of rotation angle of articulated tower, σ_φ [°], small motion, Pierson-Moskowitz spectrum

$H_{1/3}$ [m]	c_d	TISTOH1	FRSTOH	Ref. /15/
15	1.0	1.98	2.01	1.97
15	0.5	2.00	2.02	2.12
10	1.0	0.97	0.98	0.98
10	1.6	0.96	0.97	0.95
5	1.	0.30	0.28	0.30

5.7. Large motion in irregular waves

The complete nonlinear response of the articulated tower in irregular waves defined by JONSWAP and Pierson-Moskowitz spectra in the previous example, Fig.8, is calculated by modulus TISTOH2. Two sets of random phase angles of harmonic component waves are taken into account, since their influence on nonlinear response is expected /17/. The energy spectra of the tower top displacement are shown in Figs.11 and 12. Two spectrum peaks are evident, the higher one at the wave spectrum peak frequency and the lower one at the natural frequency. The phase angles mostly influence the latter peak.

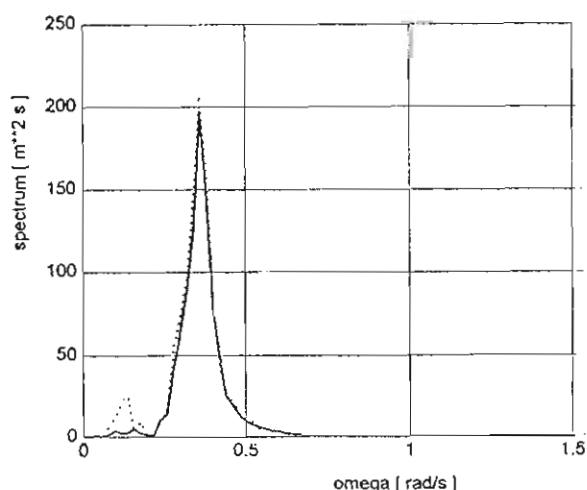


Fig.11. Energy spectrum of tower top displacement, large motion, JONSWAP spectrum, $H_{1/3} = 15$ m, - - - $\{\epsilon\}_1$, — $\{\epsilon\}_2$

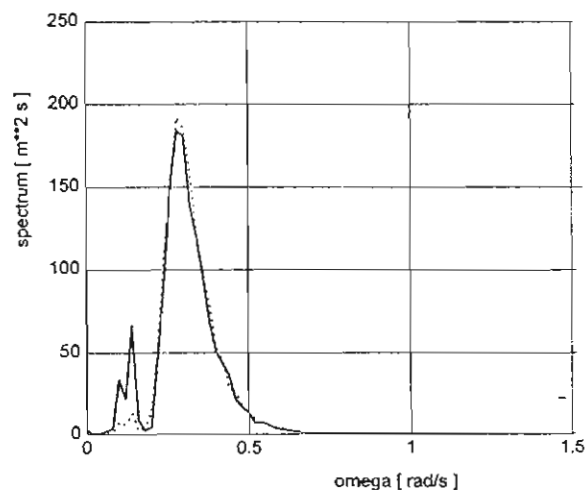


Fig.12. . Energy spectrum of tower top displacement, large motion, Pierson-Moskowitz spectrum, $H_{1/3} = 15$ m, - - - $\{\epsilon\}_1$, — $\{\epsilon\}_2$

6. CONCLUSION

An articulated tower is a stiff, buoyant simply supported structure. It is usually analyzed as a single degree of freedom system /6, 14, 15, 16/. However, it may be considered as a special case of flexible tower, and the theory developed in /4, 5/ may be applied. A review of the theory is presented in this paper together with the corresponding computer program consisting of eight moduli /12/. The application of the theory and the program is illus-

trated in the case of the articulated tower considered in references /6, 14, 15, 16/. Thus, static response due to current, free vibration and forced vibration of small and large amplitude in regular and irregular waves are analyzed. A correlation analysis with some published results is made. In such a way the developed theory and program are verified and prove useful for the analysis of more complex flexible marine towers.

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DYNAMIC THERMAL RESPONSE OF CONSTRUCTIONAL COMPLEXES

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ABSTRACT

User's requirements concerning temperature levels in living and working spaces are in most cases time dependent. During the period when the space is in use certain level of thermal comfort is needed. Many times different temperatures are required for optimal levels of thermal comfort. During the periods when the space is not occupied comfort level temperature is not needed. That means that from efficiency point of view energy delivery- supply could be reduced. The result is lower energy consumption and, depending on climate, either temperature drop or rise.

The influence of thermal mass on thermal response of different arrangements of selected layers in constructional complexes is treated in summer-time conditions, in August, in Ljubljana. The variables are colours: white, yellow, black; loadbearing construction: brick, concrete, lightweight concrete; position of thermal insulation; outside, inside, coat, core; and plaster: light, dense.

1 INTRODUCTION

Changes in the patterns of building use, new types of fuels and increased acceleration of distribution systems operation, together with new building materials and automatic control systems, demand dynamic approach in thermal analysis of the opaque constructional complexes.

The envelope of building is an influential element of the thermal response of building as a whole. It is well known that composition and arrangement of the structure's layers essentially influence on the thermal behavior of constructional complexes. From the quantitative evaluation of the energy efficiency of different compositions of constructional complexes the results of their dynamic thermal responses must be known. Application of the dynamic thermal response analyses of constructional complexes using two different regimes - outside daily cycle, inside specific users requirements - enables the research of new configurations of the building envelope. Constructional complexes can be adjusted to specific temperature conditions in the inner and in the outer environment as well.

This fact generates new design possibilities, taking into account dynamic thermal conditions on both sides (inner and outer) of the envelope.

2 TOOL

The tool for the analysis of the dynamic thermal response of the constructional complexes under the influence of two nonharmonic temperature or thermal flow regimes enables the research of the building envelope structures under real input data on both sides of the envelope. Besides real weather data, different patterns of space use by occupants can be taken into account.

The daily and seasonal cycles can cause considerable changes on the outer surface of the building envelope. These changes are caused by the outer air temperature oscillation, solar radiation, air movement and humidity. Consumers needs concerning the temperature levels in the living and working environment are more or less time dependent. Different rooms are usually used in some characteristic pattern. Bedrooms, kitchens, living rooms, offices, classrooms etc. are not occupied all the time. Maintaining constant and optimal temperature level in these cases is not rational. When a room is not occupied the temperature can be reduced and this diminishes the energy demand. For time dependent temperature course the regime of intermittent heating is suitable. The needed quantity of energy is in this case considerably influenced by the distribution and arrangement of different functional layers in the constructional complex, especially by the relation thermal insulation and thermal mass materials.

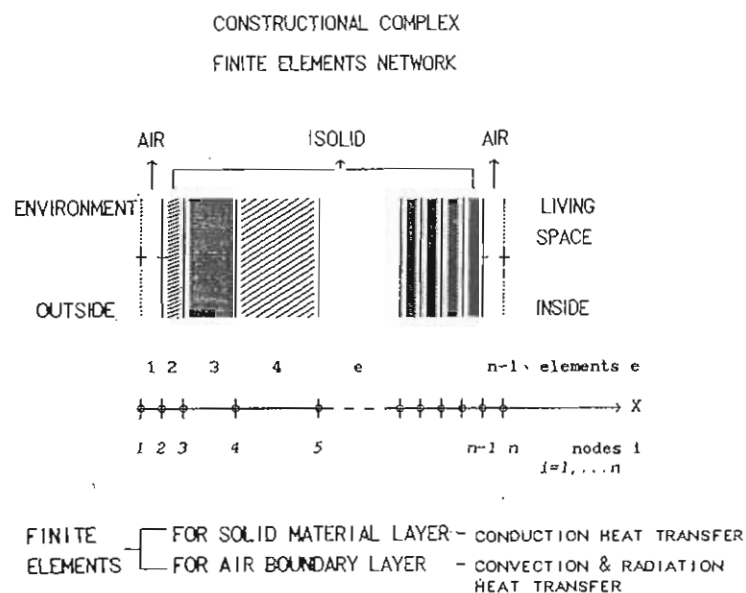


Fig. 1. Computer programme TEMPOL constructional complex finite elements network. Assumption: homogeneous isotropic material, constant thermodynamic characteristics.

A computer program TEMPOL was developed for one-dimensional dynamic thermal analysis of multilayered constructional complexes (Fig.1). The thermal phenomena in the building envelope can be studied in chosen intervals with optional duration. In the chosen time interval the course of the outer or the desired inner air temperature is time dependent. The temperature course over the layer (finite element of the constructional complex) is described with the thermal diffusion equation. It is assumed that single layer is homogeneous and isotropic, and that the thermodynamic properties of the material are independent of the temperature in the treated temperature

interval. The result is thermal response of multilayered constructional complex to optional temporary dependent conditions in outer - natural and inner - living environment in the form of temperature oscillations or thermal fluxes in 24 hour interval. On the basis of this data absolute quantities of thermal energy entering and/or living optional segments of constructional complex in 24 hour period can be calculated.

Simulations with program TEMPOL were tested in-situ. The results are very precise when constructional complexes without boundary air layers are treated. Parameters like intensive solar radiation, wind, outside air temperature changes and temporary dependent changeable temperature difference between outside air and, the envelope surface are difficult to simulate and thus they caused less precise results of simulations.

3 CASES

For simulations real meteorological data of outside temperatures and solar radiation for an August day in Ljubljana were used. On the basis of these data sol-air temperatures were calculated for white ($a=0.2$), yellow ($a=0.4$) and black colors ($a=0.9$) (Fig.2).

The influence of the position of thermal insulation regarding the thermal mass is represented for the group of cases with brick as loadbearing construction - thermal mass, polystyrene as thermal insulation with light and dense plaster on the inner and outer surface, respectively (Fig.3). The selected color is yellow.

In the first column T-d (temperature-real thickness of layers) diagrams are represented, and in the second column q-d (thermal flow-real thickness of layers) diagrams for treated cases with different positions of thermal insulation. Curves represent temperature and thermal flow distribution over constructional complexes in an one hour interval. The scale of T-d diagrams is the same for all cases, the scale for q-d diagrams covers the range from the minimal to the maximal values.

Graphical results provide a good impression of thermal behaviour of constructional complex with the influence of physical properties of particular layers in specific position and relation with the others. Temperature distribution and corresponding thermal flows can be compared layer by layer.

In the outside thermal insulation case temperature shift is within 1K through plaster layer on the outer and on the inner side. Through polystyrene thermal insulation layer the temperature swing shrinks to the smallest value and the rest of CC is under the influence of the inside air temperature. In the coat thermal insulation case the biggest temperature swing is also through the outer thermal insulation layer, but it diminishes moderately through the loadbearing construction where it is influenced by the inner insulation layer. The accumulation effects bend temperature curves in the inside thermal insulation and core thermal insulation cases. While temperature swing is diminishing through the whole loadbearing construction layer in the inside thermal insulation case it is interrupted and significantly diminished by the thermal insulation layer in the core thermal insulation case. Through the outer half of the inner loadbearing construction there is practically no

temperature change, and in the inner half there is the influence of the inside air temperature oscillations.

On q-d diagrams (thermal flow through constructional complex) thermally active parts of CC and their orientation towards the inner or outer side can be seen. In all treated cases the temperature is rising and thermal flows coming into (+) constructional complex are bigger than thermal flows coming out (-) of it. Thermal balances for "inner" and "outer" part can be calculated.

The inside air temperature oscillation is between 16.8°C and 26.3°C. Temperature difference on the inner surface is 4K for TI outside and TI core, 7.5K for TI coat and 8K for TI inside. During 24 hour period thermal inflow - 17.65 Wh/m²day, outflow + 7.30 Wh/m²day and resulting thermal accumulation - 10.35 Wh/m²day for TI outside, - 15.94 Wh/m²day, + 7.38 Wh/m²day and - 8.56 Wh/m²day for TI core, - 7.73 Wh/m²day, + 0.94 Wh/m²day and - 6.78 Wh/m²day for TI coat, and - 5.55 Wh/m²day, + 0.56 Wh/m²day and resulting thermal accumulation - 4.99 Wh/m²day for TI inside. The biggest thermal flow oscillation causes the smallest temperature oscillation and vice versa.

material	thickness (m)	λ (W/mK)	ρ (kg/m ³)	c J/kgK
brick	0.25	0.760	1800.	920.
concrete	0.25	2.040	2400.	960.
leightweight concrete	0.25	0.350	800.	1050.
polystyrene	0.05	0.041	25.	1260.
plaster light	0.02	0.850	1700.	1050.
plaster dense	0.02	0.700	1900.	1050.

Table 1. Physical properties of the materials used in treated constructional complexes.

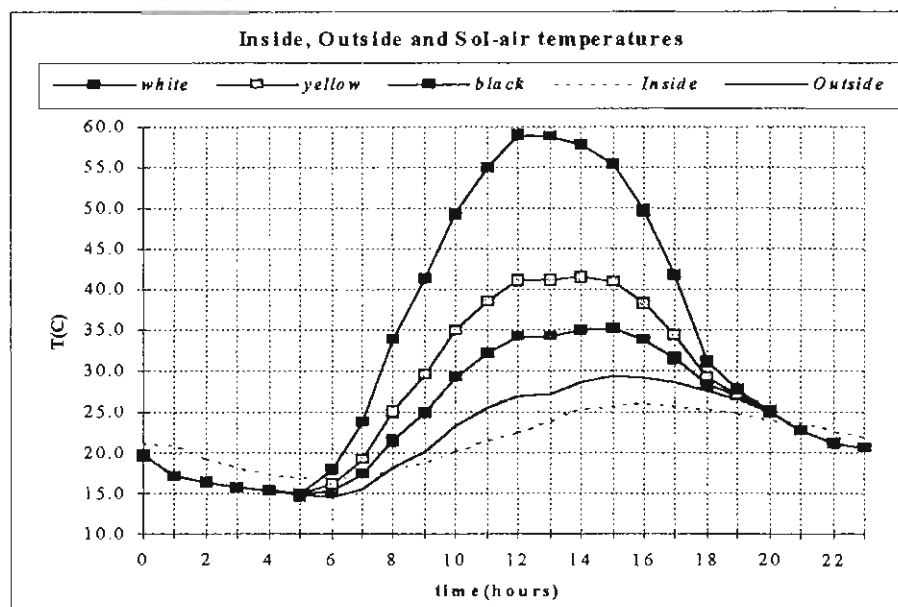


Fig. 2. Inside and outside temperatures in August day in Ljubljana used for simulations.

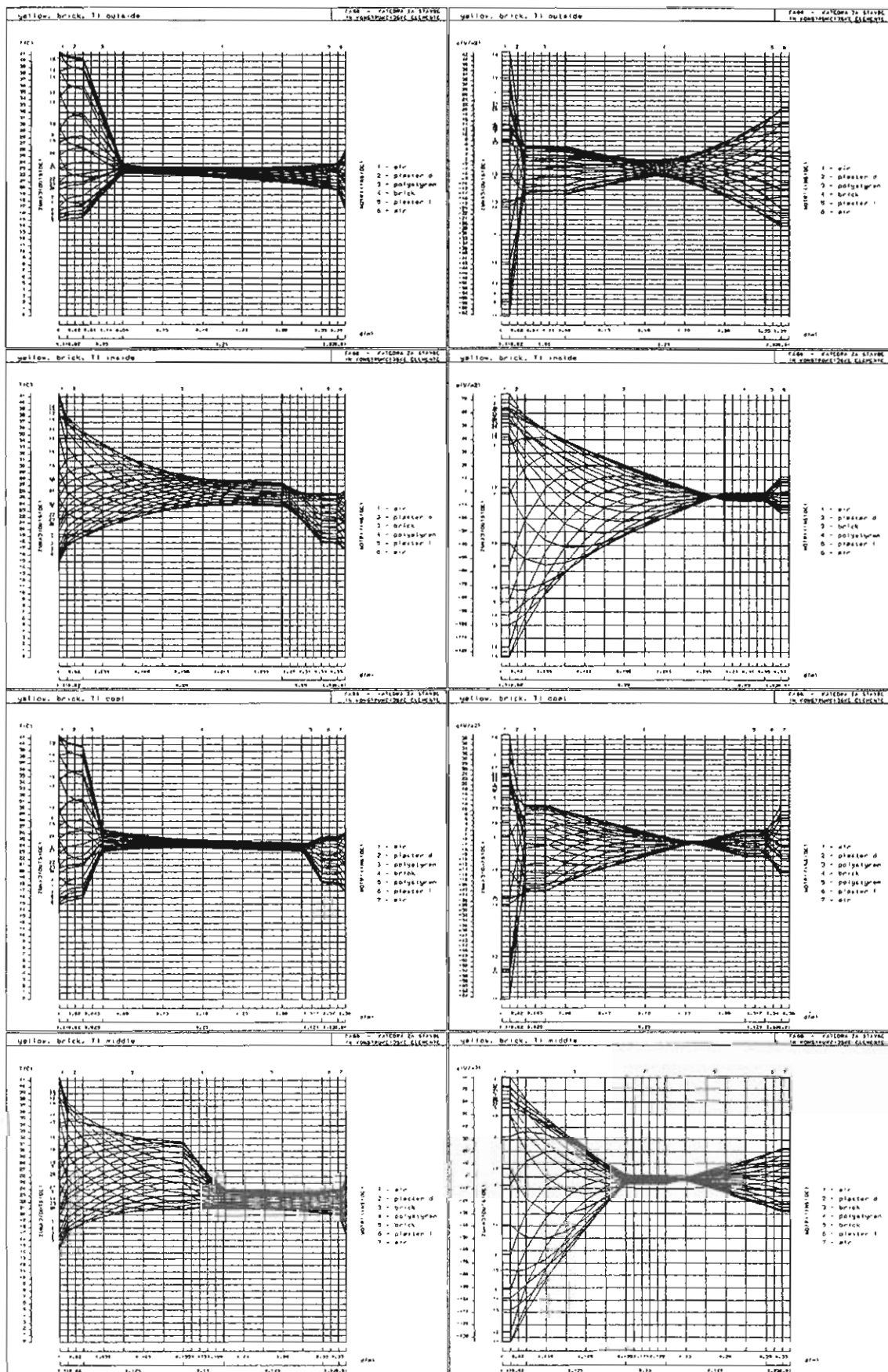


Fig. 3. T-d and q-d diagrams for treated dispositions of thermal insulation - thermal mass layers in constructional complexes.

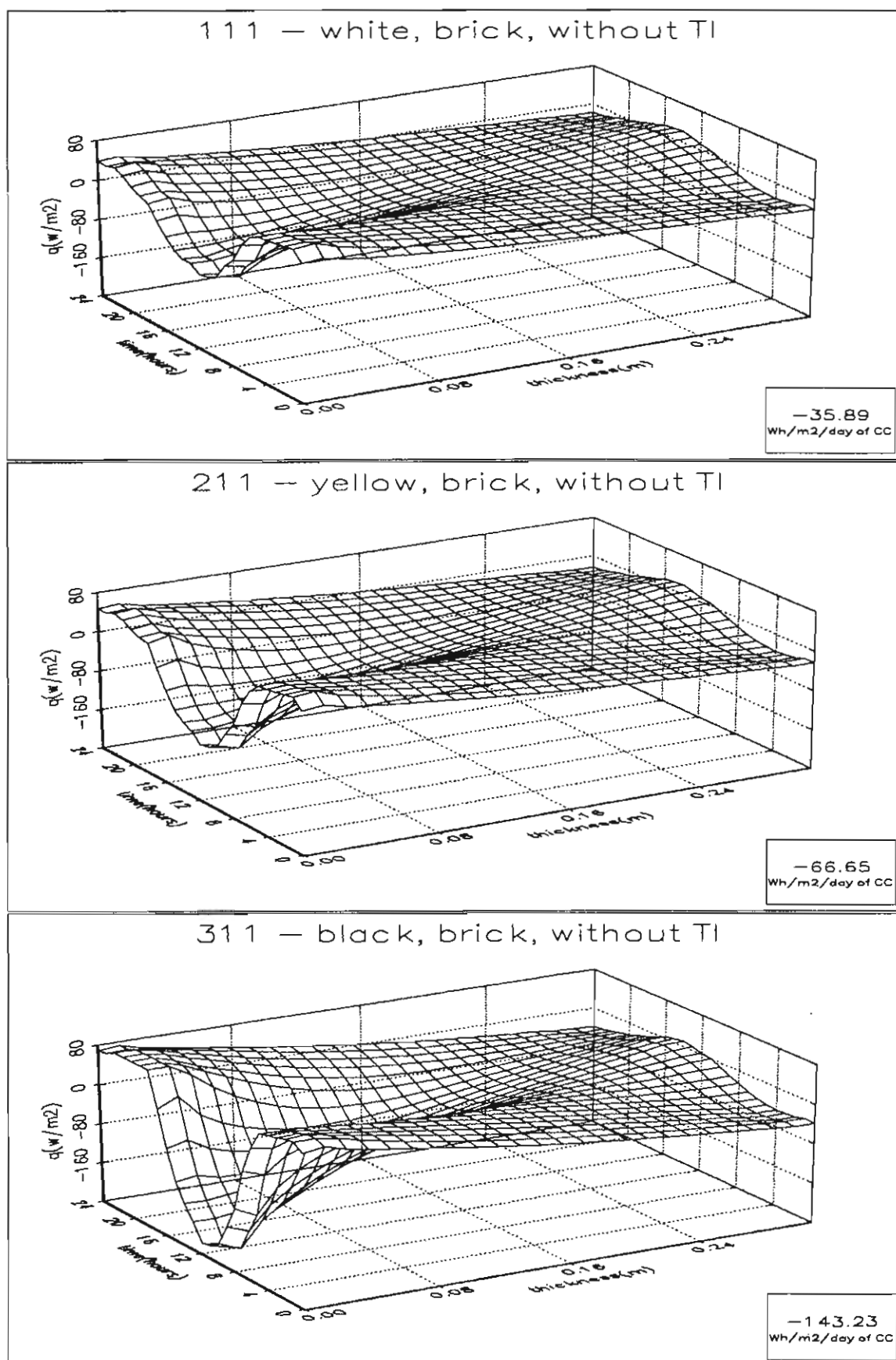


Fig. 4. Thermal flow distribution in brick wall for different colours.

Under given circumstances the inner surface temperature oscillates between 20 and 24°C for TI outside and TI core cases, but energy accumulation is 20% bigger for TI outside case. The smallest energy accumulation is for TI inside case (- 4.99 Wh/m²day) but the result is the highest inner surface temperature oscillation between 18 and 26°C.

Thermal flow distribution in brick wall without thermal insulation is represented for white, yellow and black colors, within 24 hours period on Fig.4. There are big differences between thermal flows on the outer surface with some influence also on the inner surfaces. Energy accumulation, i.e. difference between entering and leaving thermal flows, is 1.85 times bigger when yellow color and 3.99 times bigger when black color is used on the outside surface, as compared to white color. These values are useful as average comparative values and as a starting-point for the next 24 hours period. Direct influence on the inner space have values of thermal flows on the inner surface of the constructional complex, because outer part of the envelope is exchanging energy with the outer environment. Energy response for 24 hour period on the inner surface is: inflow - 16.09 Wh/m²day, outflow + 4.33 Wh/m²day, resulting thermal accumulation - 11.76 Wh/m²day for white color, - 18.25 Wh/m²day, + 2.16 Wh/m²day, resulting thermal accumulation - 16.09 Wh/m²day for yellow color and - 34.88 Wh/m²day, + 0.30 Wh/m²day, resulting thermal accumulation -34.58 Wh/m²day for black color.

In cases with thermal insulation in any position, energy accumulation is 2.75 times smaller, as compared to cases without thermal insulation.

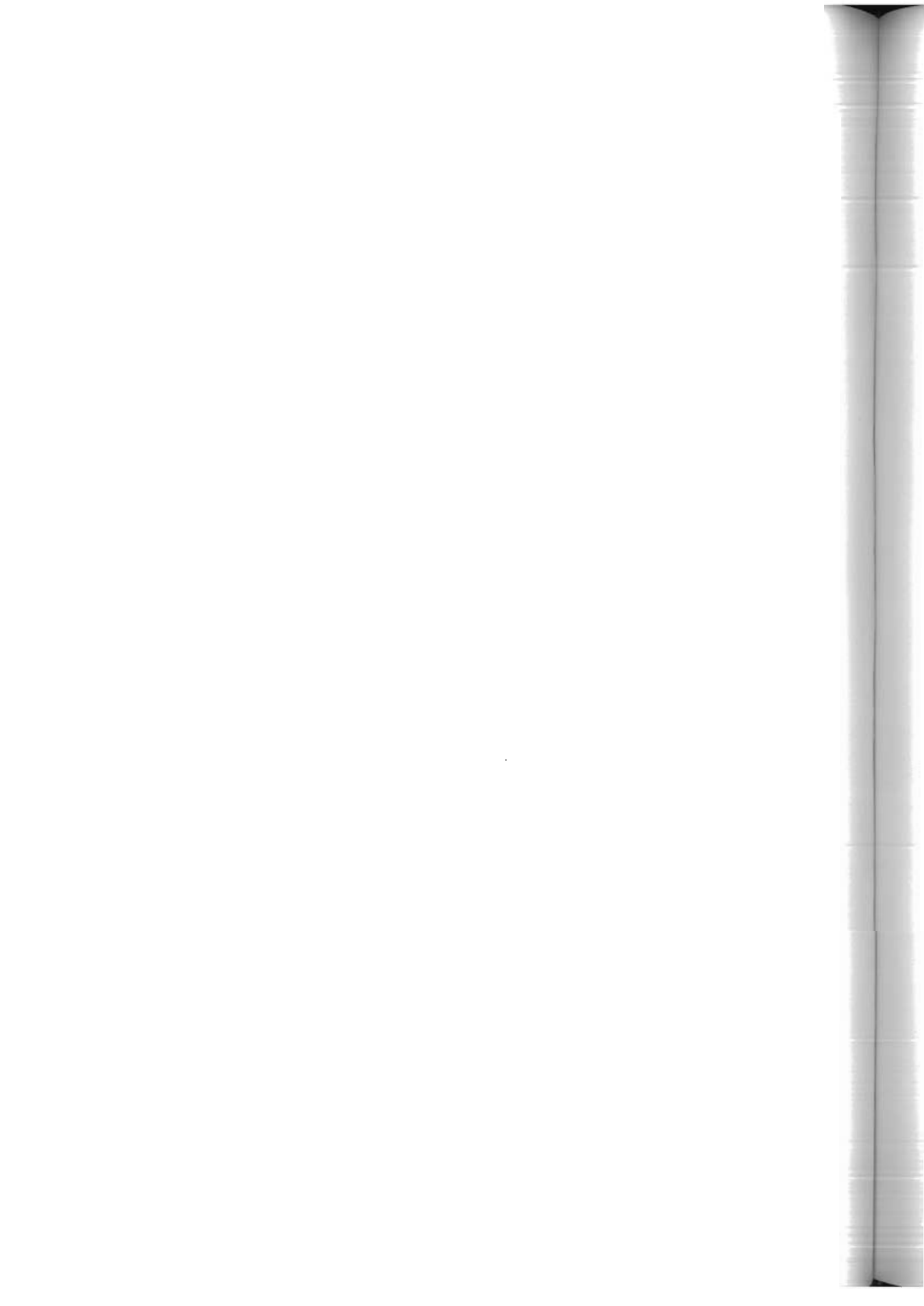
CONCLUSIONS

The computer programme for the calculation of dynamic response of multi-layered constructional complexes being dependent on the optional oscillation of temperature of the outer and inner air was designed using the finite elements method. The thermal dynamic in the building envelope can be studied in the chosen intervals with optional duration.

The dynamics of thermal response of constructional complexes are significantly different for various distributions of functional layers. The results of analyses confirm considerable dependence of thermal flow levels on the layers sequences. The biggest problem is the definition of the influencing parameters in the framework of the boundary air layers. The criterion for reasonable constructional complexes use in dynamic and nonharmonic conditions of the inner and outer environment is not only used energy but also necessary thermal flows when either cooling or heating is needed. It is practically impossible to assess optimal thermal response of constructional complexes both in summer and in winter conditions.

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Damage Detection in Beams and Framed Structures Using Natural Frequencies

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Abstract

The method of the identification of localised reductions in the stiffness of beams and framed structures using measured natural frequencies are presented. The sensitivity of the eigenvalue to localised changes in the stiffness has been developed as a set of underdetermined equations. These equations and set of additional equations, which ensure that the change in the stiffness is nonpositive, are used as constraints in an optimisation problem, which minimise the residuals of the eigenvalue problem of the damaged and undamaged structure. The solution of putted problem of the non-linear mathematical programming represents the locations and severity of the fractional changes in the stiffness of the structure. Only the small number of measured frequencies is required for detection of localised damages. The finite element code "DT1" is developed for solving this problem.

1. INTRODUCTION

During the past two decade, modest attention has been focused on the possibility of using changes in the vibrational characteristics of structures as an indication of structural modifications. The vibration approach is based on the fact that, for any structure, the modal parameters (natural frequencies, modal vectors and modal damping) depend only on the mechanical characteristics of the structure and not on the excitation applied to the structure. The vibrational approach is attractive because vibrational characteristics can be obtained by taking measurements at only a single location on the structure. Structural damage may be defined, as deviations of geometric or material properties of structure that may cause unwanted displacements, distortions or vibrations in the structure. In case of concrete structure, the severity of damage can be identification over stiffness less at one or more locations in structure. This important fact is followed that the changes of dynamic characteristics are very sensitive to changes in the mass and stiffness of the structure. Therefore, data of the changes in the dynamic parameters can be used to calculate local decrees in the stiffness, which indicate structural damage.

In this paper the first order perturbation of the eigenvalue problem is used to yield the relationship between the variation in global stiffness matrix and the eigenvalues.

2. THE PERTURBATION METHOD IN EIGENVALUES SENSITIVITY ANALYSIS

The sensitivity of the vibration eigenvalues to local stiffness variation leads to a relationship between the observed changes in the natural frequencies and the damage of the structure. Eigenvalue problem of undamped dynamic system with n -degree of freedom

$$(\mathbf{K} - \lambda_i \mathbf{M})\phi_i = 0 \quad (i = 1, 2, \dots, n) , \quad (1)$$

where $\mathbf{K}$ is on $(n \times n)$ stiffness matrix, $\mathbf{M}$ is an $(n \times n)$ mass matrix, λ_i is the square of the circular natural frequency and ϕ_i is corresponding natural mode shape of the system. The made shapes can be normalised so that

$$\phi_i^T \mathbf{M} \phi_j = \delta_{ij} , \quad (2)$$

where δ_{ij} is the Kronecker delta symbol. Equation (1) describes a relationship between the dynamic parameters λ_i and ϕ_i against the structural parameters $\mathbf{K}$ and $\mathbf{M}$. Damage of the original structure is defined here to be the change of its stiffness matrix by an amount $\delta\mathbf{K}$. In dynamics of structures, in most cases, a change in mass will be neglected.

However, a change in stiffness produces changes in the eigenvalue, symbolised here as $\delta\lambda_i$ and changes in the eigenvectors, $\delta\phi_i$. Using perturbation method an equation (1), the eigenvalue problem of the damaged structure with unchangeable of mass structure is given by

$$(\mathbf{K} + \delta\mathbf{K} - (\lambda_i + \delta\lambda_i)\mathbf{M})(\phi_i + \delta\phi_i) = 0 . \quad (3)$$

With transformation of equation (3) and neglecting the high order terms we have

$$\mathbf{K}\delta\phi_i - \lambda_i \mathbf{M}\delta\phi_i = -\delta\mathbf{K}\phi_i + \delta\lambda_i \mathbf{M}\phi_i . \quad (4)$$

Equation (4) can be premultiplied by ϕ_i^T to obtain

$$\phi_i^T (\mathbf{K} - \lambda_i \mathbf{M})\delta\phi_i = -\phi_i^T \delta\mathbf{K}\phi_i + \delta\lambda_i . \quad (5)$$

Nothing that $\phi_i^T (\mathbf{K} - \lambda_i \mathbf{M}) = 0^T$ the sensitivity of an eigenvalue to changes in the stiffness matrix is given by

$$\delta\lambda_i = \phi_i^T \delta\mathbf{K}\phi_i , \quad (6)$$

where we have linear relationship between changes in eigenvalue and global stiffness matrix. This is the a elementary equation in damage detection theory.

To relate the change in the eigenvalue to the change in the stiffness of a local elements, the change in the stiffness matrix are expressed in terms of the changes of the independent structural elements. If $\mathbf{K}_j^e$ is the contribution of the j-th element to the total stiffness matrix of the structure, the stiffness of the damaged structure can be written as

$$(\mathbf{K} + \delta\mathbf{K}) = \sum_{j=1}^{N_e} \mathbf{K}_j^e (1 + \delta k_j) \quad (7)$$

where N_e is the number of elements of n-degree of freedom structure, and δk_j is the proportional change in the stiffness of element j. This $\delta\mathbf{K}$ can be written as

$$\delta\mathbf{K} = \sum_{j=1}^{N_e} \mathbf{K}_j^e \delta k_j \quad (8)$$

It is very important in finite element formulation that the change in global stiffness matrix depends on the changes of the element stiffness matrices.

This formulation guarantees the symmetry of the stiffness matrix and structural connectivity.

The substitution of equation (8) into (6) gives the set of simultaneous equations.

$$\delta\lambda_i = \sum_{j=1}^{N_e} \phi_i^T \mathbf{K}_j^e \phi_i \delta k_j \quad (i = 1, 2, \dots, M_e) \quad (9)$$

where M_e is the number of used modes, or in matrix form

$$\mathbf{S} \delta\mathbf{k} = \delta\lambda \quad (10)$$

where $\mathbf{S}$ is an $(M_e \times N_e)$ matrix of elements $s_{ij} = \phi_i^T \mathbf{K}_j^e \phi_i$, $\delta\mathbf{k}$ is an N_e - vector of the unknown changes in stiffness δk_j and $\delta\lambda$ is an M_e - vector of the changes in the eigenvalues $\delta\lambda_i$.

If the elements of vector $\delta\lambda$ are measured and elements of matrix $\mathbf{S}$, s_{ij} are determined analytically or numerically then the solution of the system (10) gives the changes in the elements stiffness. Theoretically if the number of available changes in frequency M_e is equal to the number of elements N_e , a solution may be determined uniquely. However, only a small number of natural frequencies can usually be measured. So the number of measured changes in the eigenvalues is less than the number of elements ($M_e < N_e$) and we have equations undetermined. They can be solved only after the introduction of an suitable optimality criterion.

Between more optimal criterion, the most suitable will be the one which minimises the difference between the eigenvalue problem of the undamaged and damaged structure.

3. INTRODUCTION OF THE OPTIMAL CRITERION

It will assumed that a good approximation for on eigenvector of the damaged structure is the corresponding eigenvector of the undamaged structure. Then the eigenvalue problem of the damage structure can be stated as

$$(\mathbf{K} + \delta\mathbf{K})\phi_i - (\lambda_i + \delta\lambda_i)\mathbf{M}\phi_i = \mathbf{D}_i \quad (11)$$

where $\mathbf{D}_i$ is a vector of residuals. Simplification of equation (11) leads to

$$\mathbf{D}_i = \delta\mathbf{K}\phi_i - \delta\lambda_i\mathbf{M}\phi_i \quad (12)$$

If the square of the magnitude of the residual is minimised, than we have

$$\|\mathbf{D}_i\|^2 = \mathbf{D}_i^T \mathbf{D}_i = \phi_i^T (\delta\mathbf{K})^2 \phi_i - 2\delta\lambda_i \phi_i^T \delta\mathbf{K} \mathbf{M} \phi_i + \delta\lambda_i^2 \phi_i^T \mathbf{M}^2 \phi_i \quad (13)$$

If replace δK from equation (8), the throw constant part and summation over the number of measured changes in the eigenvalues M_e and expressed in matrix form we have the objective function as

$$f = \sum_{i=1}^{M_e} \|D_i\|^2 = \delta \mathbf{k}^T \mathbf{Q} \delta \mathbf{k} + 2\delta \mathbf{k}^T \mathbf{c} , \quad (14)$$

where $\mathbf{Q}$ is an $(N_e \times N_e)$ matrix with elements

$$q_{ij} = \sum_{k=1}^{M_e} \phi_k \mathbf{K}_i^e \mathbf{K}_j^e \phi_k \quad (i, j = 1, 2, \dots, N_e) \quad (15)$$

and $\mathbf{c}$ is an N_e - vector with elements

$$c_i = -\sum_{k=1}^{M_e} \delta \lambda_k \phi_k^T \mathbf{K}_i^e \mathbf{M} \phi_k . \quad (16)$$

The optimisation problem can now be stated as

$$\begin{aligned} \min f &= \frac{1}{2} \delta \mathbf{k}^T \mathbf{Q} \delta \mathbf{k} + \delta \mathbf{k}^T \mathbf{c} \quad \text{-objective} \\ S \delta \mathbf{k} &= \delta \lambda \\ \delta \mathbf{k} &\leq 0 \quad \text{-constraints} \end{aligned} \quad (17)$$

The second constraint ensures that the change in the stiffness is always negative, since a positive change in the stiffness can never be produced by damage.

So the problem is a quadratic programming problem with equality and inequality constraints. Matrix $\mathbf{Q}$ is positive definite and problem is strictly convex, so the unique solution for $\delta \mathbf{k}$ can be found using efficient algorithms developed in the field of linear and non-linear mathematical programming.

4. THE FINITE ELEMENT CODES "DT1" AND "DT2" FOR DAMAGE DETECTION ANALYSIS

Program "DT1", a finite element code for damage detection analysis of beams and framed structures is developed by Rak [8]. Program is assembled of three segments. In the first segment of program "DT1", the experimental and structural data are inputted. After the standard eigenvalue problem of beams and framed structures is solved. In the second segment the required matrices and vectors $(s_{ij}, q_{ij}, \mathbf{K}_i^e, \mathbf{K}_j^e, \mathbf{c}_i)$ from chapters 2. and 3 are solved. In the third segment, the optimisation problem is solved and the optimal solution $\delta \mathbf{k}$ is written. Flow diagram of FORTRAN program "DT1" is shown in fig.1.

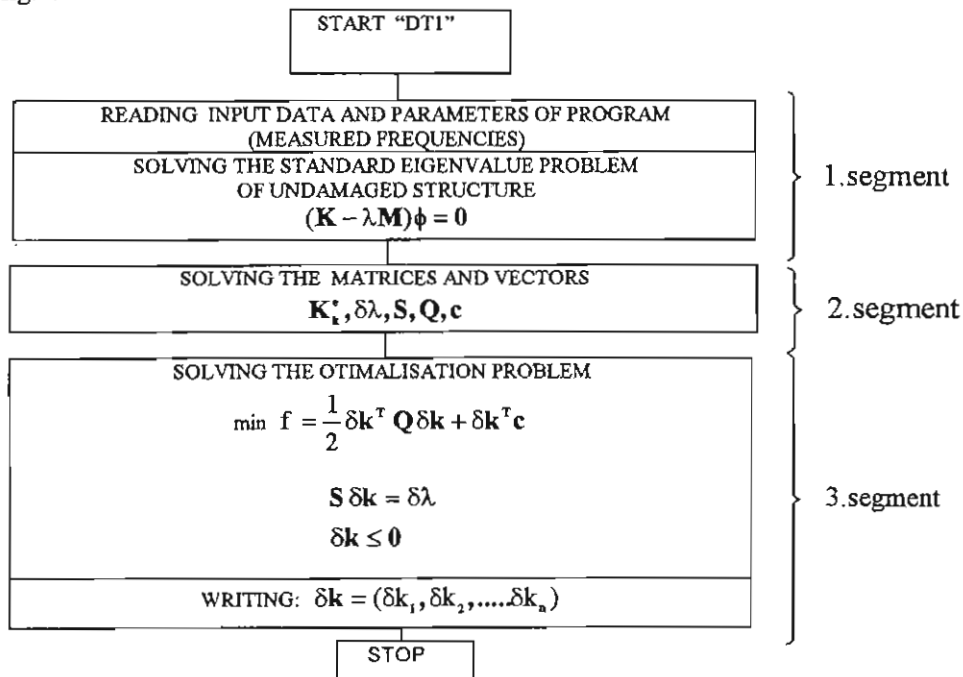


Fig 1. Flow Diagram of FORTRAN Program "DT1"

Program "DT2" is also developed in the work [8] and used in theoretical analysis of the damaged structure. The damages can be initiated in one or more locations with the same or the different magnitudes. The output data of this program are eigenvalues and eigenvectors of the damage structure which use as the input data in the computer program "DT1".

5. NUMERICAL EXAMPLES

5.1. Identification of Damage in Beam Using Test Data

5.1.1. Example 1. (Initiated Damages)

Measurements of the set of natural frequencies before and after the damage of the simply supported beam are presented here. The experimental values of the change in the frequencies and the known location and magnitudes of the damages will be used here to test the theory presented in the chapters 2. and 3.

The subject of the experiment was simply supported beam with 125 cm span, 8 cm wide and 10 cm height, as shown in Fig.2. The beam was made of reinforced concrete with Young's modulus of $3.4 \cdot 10^4$ MPa and mass density of $2.4 \cdot 10^{-5}$ N/mm³. The response of the beam to an sinusoidal excitation was measured using accelerometers. Spectral analysis of the data was used to obtain the first three natural frequencies. The beam was subsequently damaged using a grinding machine. The measured frequencies of the undamaged and damaged structure are shown in Table 1. The beam was divided into ten parts, as finite element in theoretical analysis. Severity of damage on the elements of beam was about 27%.

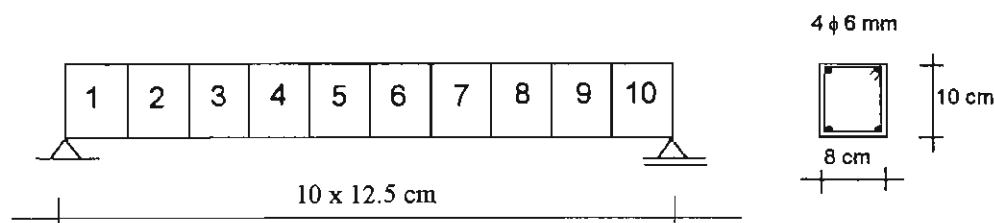


Fig 2. Reinforced Concrete Beam

Table 1. Measured Natural Frequencies of Beam (Hz)

Mode	Frequencies of Undamaged Beam	Damage $\cong$ 27%		
		Elements (5 -6)	Elements (4 -7)	Elements (3 -8)
1	105,47	98,88	95,21	92,77
2	437,50	429,69	416,02	393,07
3	1066,41	1025,40	1021,50	987,55

Using finite element code "DT2", the natural frequencies were calculated for the same scenarios as in experiments. The beam was divided into ten finite elements. The results of this calculations are shown in table 2.

Table 2. Calculated Natural Frequencies of Beam (Hz)

Mode	Frequencies of Undamaged Beam	Damage $\cong$ 27%		
		Elements (5 -6)	Elements (4 -7)	Elements (3 -8)
1	112,32	105,04	100,06	97,25
2	449,33	445,32	425,62	400,70
3	1011,41	965,10	960,60	924,50

Using data from Table 1. and Table 2., the locations and severity's of damages were estimated by finite element code "DT1" and "DT2". The results of this analysis are presented by diagrams in Fig 3.

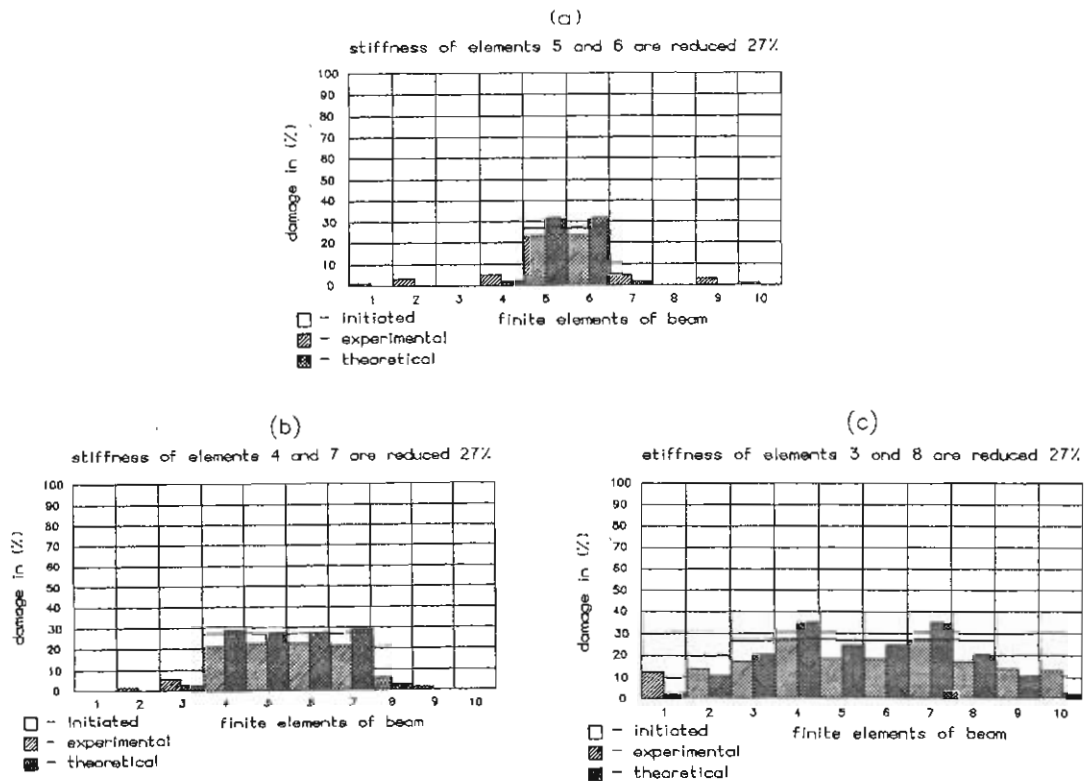


Fig 3. Theoretical and Experimental Values of Damages Compared with Initiated:
a) Elements (5-6), b) Elements (4-7), c) Elements (3-8)

5.1.2. Example 2. (Damages Caused by Impact Load)

In this example the reinforced concrete beam was subjected to an impact load. The impact load level was determined by dropping the steel hammer ($G = 10\text{ kg}$) from different highs (5 cm, 10 cm, 20 cm, 30 cm) Fig.4.

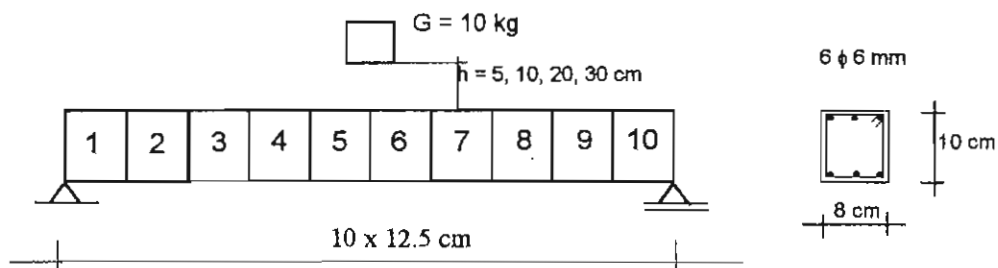


Fig. 4. Impact Load of Reinforced Concrete Beam

The four beams was treated, one for every high. Response of the beams to an impact load was measured using accelerometers. Spectral analysis of data was used to obtain the first three natural frequencies in every experiment. The measured frequencies of the undamaged and damaged structure are shown in Table 3.

Table 3. Measured Natural Frequencies of Beam

Mode	Frequencies of Undamaged Beams	h (cm)			
		5	10	20	30
1	109,18	105,47	101,56	95,70	91,80
2	449,28	443,40	437,50	425,78	408,20
3	1038,44	1021,48	1005,86	986,33	945,31

Using computer program "DT1", the results of the local decrease of stiffness are shown in Fig.5.

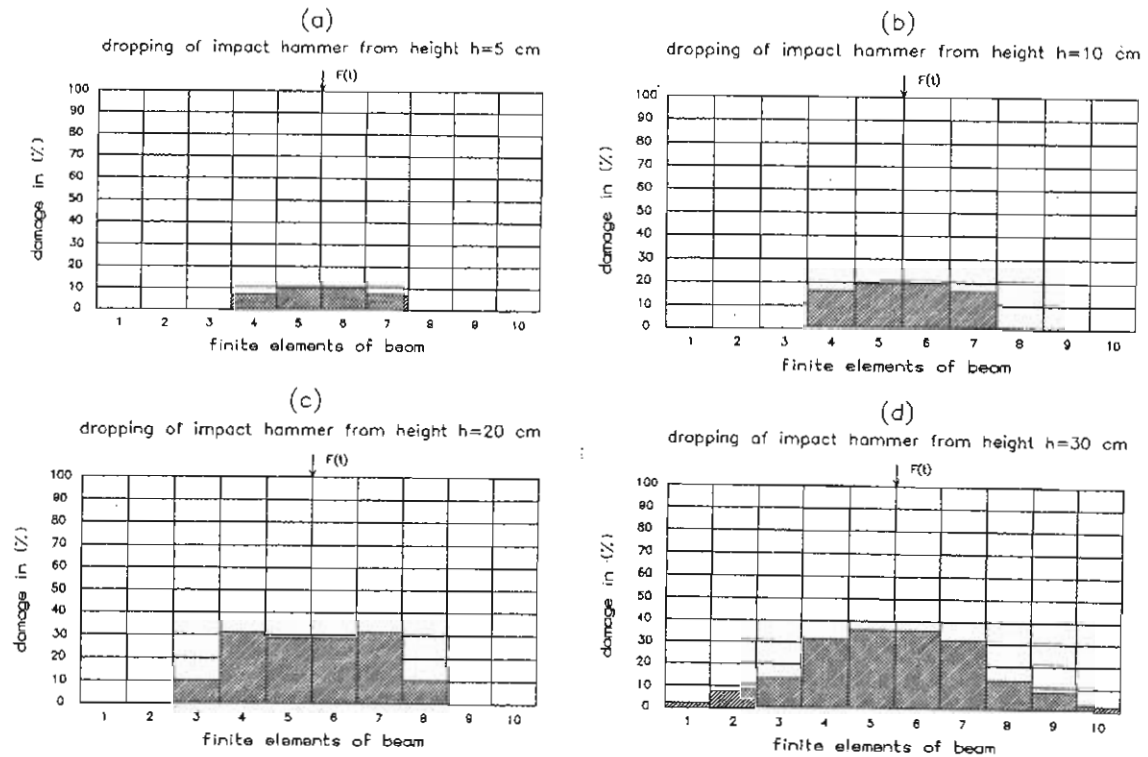


Fig 4. Damage Diagrams of Beams Caused by Impact Hammer from Different Highs:
a) $h = 5$ cm, b) $h = 10$ cm, c) $h = 20$ cm, d) $h = 30$ cm

CONCLUSIONS

The structural sensitivities of the eigenvalues to localised changes in stiffness were presented as set of underdetermined equations, which were used as the constraints in the optimisation problem. The second set of the constraint equations ensures that the changes in stiffness are always negative. The objective function is derived from the residuals of the eigenvalue problem. The solution of the non-linear problem of mathematical programming represents the locations and severity of the stiffness changes in the independent structural elements. In this work, only three measured natural frequencies are required in identification of stiffness change. The method, explained here, is very successfully used to obtain the location and severity of damage in simple structures. But, for large structures there are problems in measuring a big number of modal frequencies, which must be used for identification changes in stiffness.

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MULTISTOREY RC BUILDING STRUCTURES: EFFECT OF PLAN IRREGULARITY

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ABSTRACT

The paper investigates the nonlinear behavior of asymmetric multistorey frame-wall buildings subjected to horizontal loading. Thirteen building layouts with different positions of load resisting elements were subjected to monotonically increasing lateral loading applied at the center of masses. Within a parametric study the strength, the displacements and the ductility demand of elements for different asymmetric buildings are compared with results obtained for the symmetric building. The analyses were performed with the computer program for simplified push-over analysis of buildings that was recently developed at the University of Ljubljana. The results obtained with a simplified method are compared with the results of a general 3D nonlinear computer program (program CANNY-E).

1. INTRODUCTION

The simplified method for push-over analysis is based on the extension of a pseudo three-dimensional mathematical model of a building structure into the nonlinear range. The method is intended to achieve a satisfactory balance between required reliability and applicability for everyday design use. It was implemented into an interactive and user friendly computer program NEAVEK that is suitable for practical analysis of complex multistorey structures. The developed program can be used as a part of the new methodologies for the seismic design (e.g. Moghdam and Tso, 1995; Fajfar and Gašperšič, 1996) or it can be used as a separate tool to determine the nonlinear behavior of a building, as it will be presented in this paper.

In the paper the used mathematical model and the simplified method for push-over analysis are only briefly summarized. The paper focuses on the parametric study of a 4-storey frame-wall building with different degree of irregularity in plan. The stiffness eccentricity of the building is changed by shifting the wall group (i.e. stair case) toward the edge of a building. To ease the comparison of different building layouts and to minimize the number of parameters involved, the reinforcement for all asymmetric cases is equal to the reinforcement that is required for the symmetric case. The investigated parameters are the strength, the displacements on the flexible and on the stiff side, and the ductility demand on frames and walls. In all cases the push-over analysis was performed up to the point when the top displacement at the center of masses is equal to 1% of building height.

2. MATHEMATICAL MODEL

The program NEAVEK is based on the extension of a pseudo three-dimensional mathematical model of a building structure into the inelastic range. A pseudo three-dimensional mathematical model consists of assemblages of two-dimensional macroelements (substructures) such as frames, walls, coupled walls and walls on columns that may be oriented arbitrarily in plane (Figure 2). Each macro-element is assumed to resist load only in its own plane, but the building as a whole can resist load in any direction. The macroelements are connected at each floor level by floor slabs that are assumed to be rigid in their own planes and have no out-of-plane flexural stiffness. The model has three degrees of freedom for each floor level (two horizontal translations and one rotation about the vertical axis). It is considered that for most buildings this is an acceptable approximation, with the possible exception of some tall slender buildings or tube-type structures. An example of pseudo three dimensional model is shown in the Figure 1. In spite of some limitations of the pseudo three-dimensional model its advantages over a fully three-dimensional model remain easier data preparation, easier interpretation of the results, and higher computational efficiency. In Slovenia, the pseudo three-dimensional model was implemented in the EAVEK program (Fajfar 1978). Recently the program EAVEK was equipped with an interactive graphic preprocessor (Kilar, 1991), that substantially simplifies the data preparation (structure can be drawn with the mouse). The latest preprocessor version includes also the program for push-over analysis (Kilar, 1995) that it is able to call automatically the program EAVEK and to browse its results. In this way the preprocessor was changed in an interactive graphic environment that can be used to perform the elastic and nonlinear push-over analysis of the prepared mathematical model of the building structure.

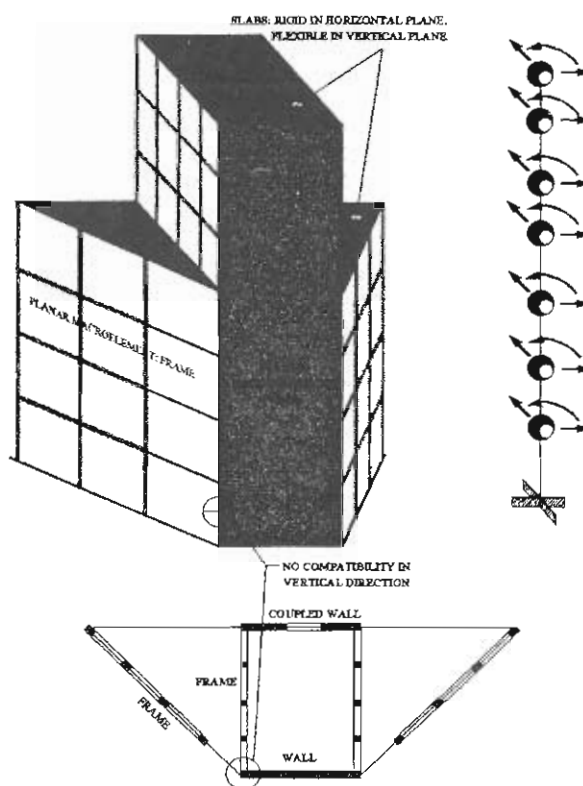


Figure 1: Pseudo three-dimensional model.

3. PUSH-OVER ANALYSIS METHOD

The analysis is performed in steps. In each step the structure is loaded with a selected increment of external loading (usually in the form of inverted triangle). All static response quantities are computed with program EAVEK. For each macroelement the most typical plastic mechanisms are predicted. It is assumed that plastic hinges can develop only cross sections (for example at the wall base, at the end of beams) that are usually the most affected due to strong horizontal loading (Figure 2). In each step a scale factor is calculated for all possible plastic mechanisms. The smallest factor

determines the plastic mechanism that occurs next. The increment of external loading and other response quantities are multiplied with the scale factor and added to the results of the previous step. At the end of each step the top displacements of all macro-elements are checked. If the new top displacement exceeds the prescribed maximum allowable macro-element top displacement (1% of height for walls and 2% for frames), the user can decide to terminate the analysis. After the plastic mechanism of the specific macroelement occurs, the stiffness of the macroelement is changed and the procedure is repeated. In this way the analysis progresses until the plastic mechanism of the whole structure is reached. The proposed procedure is in principle applicable for building structures of any material, however the ductile behavior of critical cross-sections is assumed. For everyday practice work it is important that the elastic and the push-over analysis are performed on the same mathematical model. Additional data for push-over analysis are only the data about the yield moments of critical cross-sections. The method for push-over analysis is described in detail in (Kilar and Fajfar, 1997).

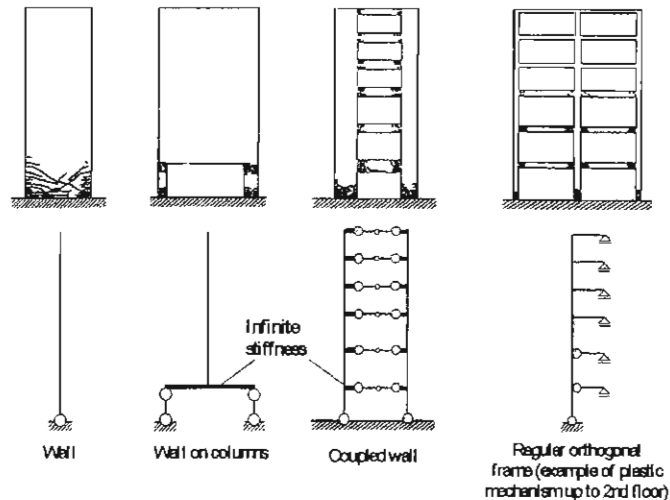


Figure 2: Considered locations of damage due to a strong horizontal loading and assumed mathematical models with plastic mechanisms.

4. PARAMETRIC STUDY

As a numerical example a nonlinear behavior of a four-storey RC frame-wall building is investigated (Figure 3). Four different layouts of a building with different positions of the wall group (walls W1, W2, W3 and W4) are examined. The X coordinate of the wall group varies: $X_g=0$ (Figure 3), $X_g=3\text{m}$, $X_g=6\text{m}$ and $X_g=9\text{m}$. The thickness of the walls is considered to be: $t=0.1\text{m}$, $t=0.2\text{m}$ and $t=0.3\text{m}$ (Figure 4). The case $t=0$ represents the building without shear walls. The buildings are loaded with monotonically increasing horizontal loading in Y direction. The loading is distributed in the form of inverted triangle and it is always applied in the mass center of the building ($X=0$ and $Y=0$). The elasticity modulus is $E=3 \cdot 10^7 \text{ kN/m}^2$, shear modulus is $G=0.4E$ and considered $M_B=30 \text{ MPa}$.

Considering different positions of the wall group and different thickness of the walls, thirteen different building layouts were analyzed. In all cases the push-over analysis with program NEAVEK was performed up to the top displacement 12 cm (1% of building height). The last recorded event in all cases was the plastification of the frame F3. The symmetric building structures were designed according to the formal Yugoslavian code for the MSK intensity VII and moderate quality of soil ($k=0.025$). The mass in all stories amounted to 432 tons. The accidental eccentricity was not taken into account. The basic properties of columns, beams and walls, needed for linear and nonlinear push-over analysis are as follows:

	Dimensions (m)	Reinforcement (R400/500)	Axial force (kN)	Yield moment (kNm)
Columns	0.4 / 0.4	1. story 8R Φ 28	1440	391.1
		2. story 8R Φ 28	1080	371.7
		3. story 8R Φ 28	720	335.6
		4. story 8R Φ 28	360	296.1
Beams	0.3 / 0.4	bottom / top 6R Φ 22	0	300.6
Walls	t / 2.0	t=0.1, edges 4R Φ 12, distributed 18R Φ 6	720	900.0
		t=0.2, edges 4R Φ 14, distributed 36R Φ 6	720	1162.9
		t=0.3, edges 6R Φ 14, distributed 53R Φ 6	720	1501.2

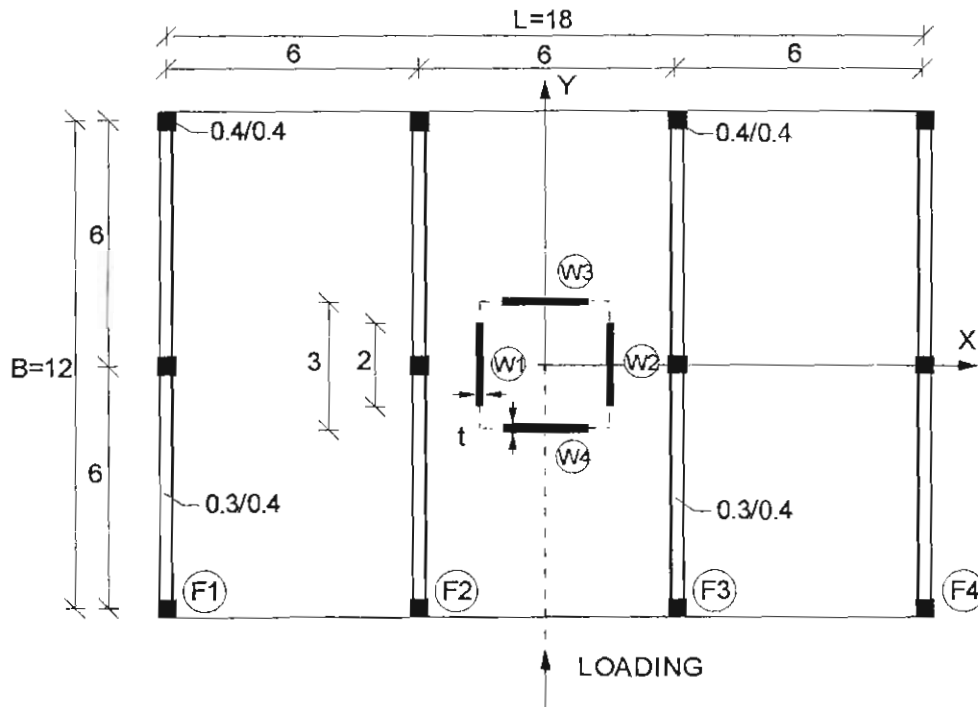


Figure 3: Schematic plan of examined building (units in m).

In the symmetric cases ($X_g/L=0$) the plastification at the bottom of the walls W1 and W2 occurs at the same time. After, the plastic mechanism of all frames develops at the same time. In asymmetric cases ($0 < X_g/L \leq 0.5$) the sequence of the first four plastifications (events) is always the same: wall W1, wall W2, frame F1 and frame F2. With further increase of the load two possibilities were observed: (a) frame F3 or (b) wall W3, wall W4 and finally the frame F3. The plastification of the walls W3 and W4 (perpendicular to the direction of the load) was noted only in the cases $t=0.2$ and $t=0.3$ for $X_g/L \geq 0.33$.

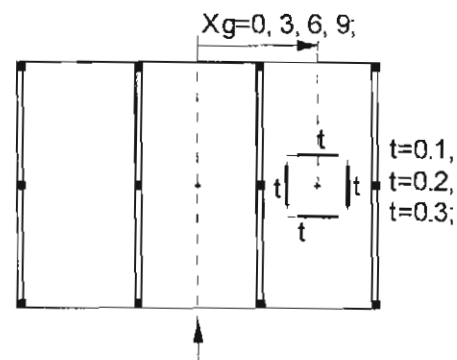


Figure 4: Considered variants of the prototype building

In order to justify the results of program NEAVEK, all variants of the structure were analyzed also with the program CANNY-E (Li, 1997). In CANNY program the structure is modeled with a fully three-dimensional model. The walls are modeled as shear panels with in- and out-of-plane flexural stiffness, shear stiffness and axial stiffness.

The columns are modeled as multi-spring elements that takes into account bending and shear deformations in two directions as well as axial deformations.

Figure 5 and Figure 6 show the results for the building with $t=0.3$ and $X_g/L=0$ (the stiffest building variant) and for the building with $t=0.1$ and $X_g/L=0.5$ (the weakest building variant). The results for all other considered building structures lie between the values obtained for this boundary examples. The obtained correlation for both building structures is very good. It can be also noted the increased asymmetry of the structure does not reduce significantly the maximum building strength, however it substantially increases the torsional rotations of the building. In the case $t=0.1$ and $X_g/L=0.5$ the contribution of torsion is as high as 48%.

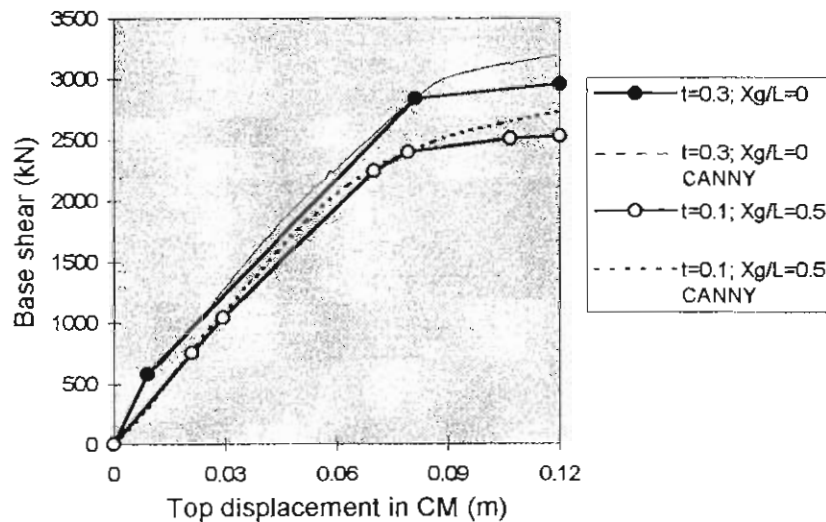


Figure 5: A comparison of base shear-top displacement relationships obtained with program NEAVEK and program CANNY for to boundary building variants.

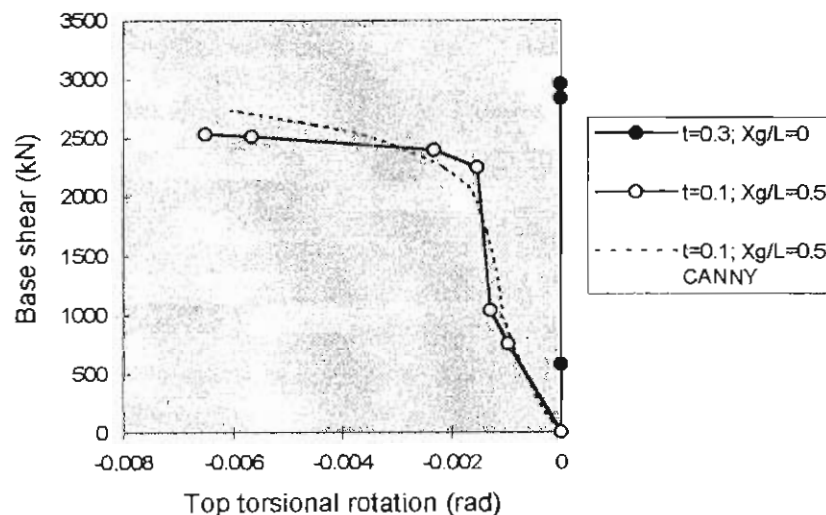


Figure 6: A comparison of base shear-top torsional rotation relationships obtained with program NEAVEK and program CANNY for building with $t=0.1$ and $X_g/L=0.5$.

Figures 7, 8 and 9 show the maximum strength ratio and maximum displacement ratio (for flexible and for stiff edge) with the respect to the position of the wall group (X_g/L). The maximum strength ratio is defined as:

$$\text{MAXIMUM STRENGTH RATIO} = \frac{\text{MAXIMUM BASE SHEAR (Y Direction)}}{\text{MAXIMUM BASE SHEAR for } X_g = 0 \text{ and } t = 0.2}$$

and the maximum displacement ratio is defined as:

$$\text{MAXIMUM DISPLACEMENT RATIO} = \frac{\text{MAXIMUM TOP DISPLACEMENT}}{\text{MAXIMUM TOP DISPLACEMENT for } X_g = 0 \text{ and } t = 0.2}$$

where maximum top displacement is obtained for respective edge of the building (frame F1 or frame F4). For the comparison also the results for building without walls ($t=0$) are presented (only for $X_g=0$). For the building variants with $t=0.2$ m, also the results of program CANNY are presented.

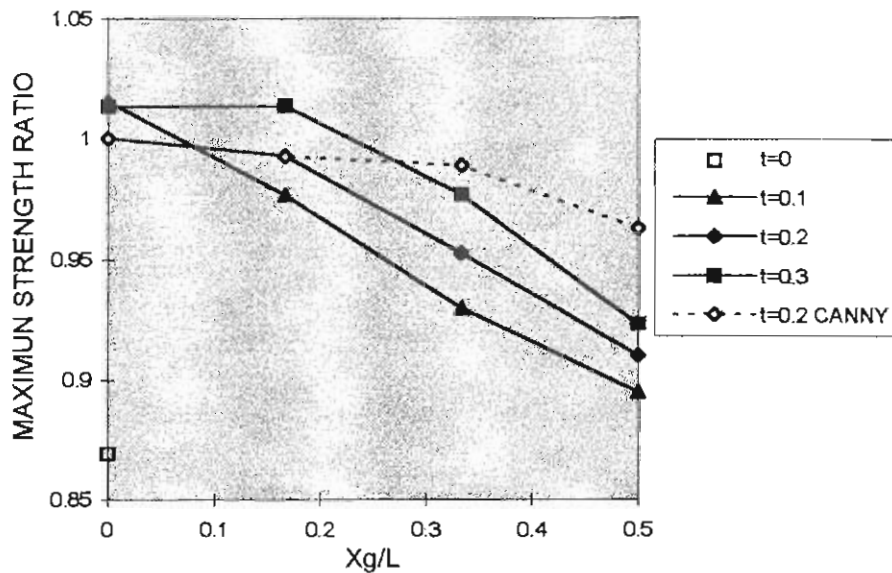


Figure 7: Maximum strength ratio.

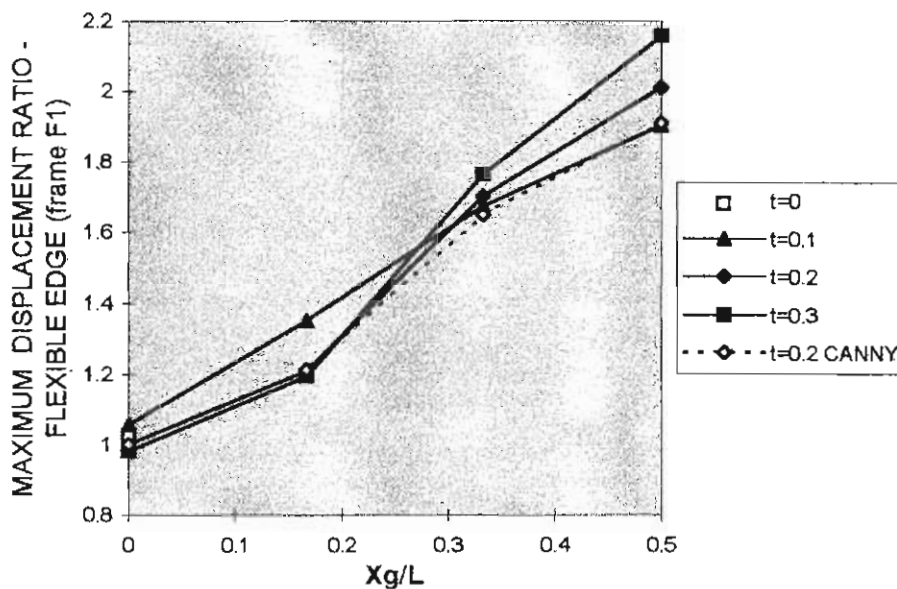


Figure 8: Maximum displacement ratio for the flexible side of the building structure.

From Figure 7 it can be observed that with the increasing asymmetry the maximum strength of the building decreases. However, the strength decrease is small and it does not exceed 90% of the strength of symmetric building with $t=0.2$ (except for the case $t=0$). It is necessary to point out, that the plastic mechanisms for frames differs from case to case. That can be clearly seen for symmetric case, where the strength for the case $t=0.1$ is bigger than for the case $t=0.2$ and the case $t=0.3$. The reason is that in the case $t=0.1$ the global mechanism of all frames occur. In other two cases the mechanisms through columns in the first story occurs in all frames (because different stiffness of the walls causes different distribution of forces on frames) and consequently the strength of the whole building is smaller.

From Figure 8 we can see that the maximum top displacement at the flexible edge can be up to 2.15 times bigger than for the symmetric structure with $t=0.2$. Figure 9 shows that the maximum top displacement at the stiff edge decreases with the increasing asymmetry. Similar results were obtained also with program CANNY (shown only for $t=0.2$ m), however the results coincide better for smaller values X_g/L . It is interesting to note, that for $t=0.2$; $X_g/L=0.33$ and $t=0.3$; $X_g/L=0.5$ also the plastification of walls W3 and W4 occurs. Therefore in this cases the torsional mechanism of the structure is formed. In the case of the analysis with the program CANNY the plastification of walls W3 and W4 occurs for all asymmetric cases.

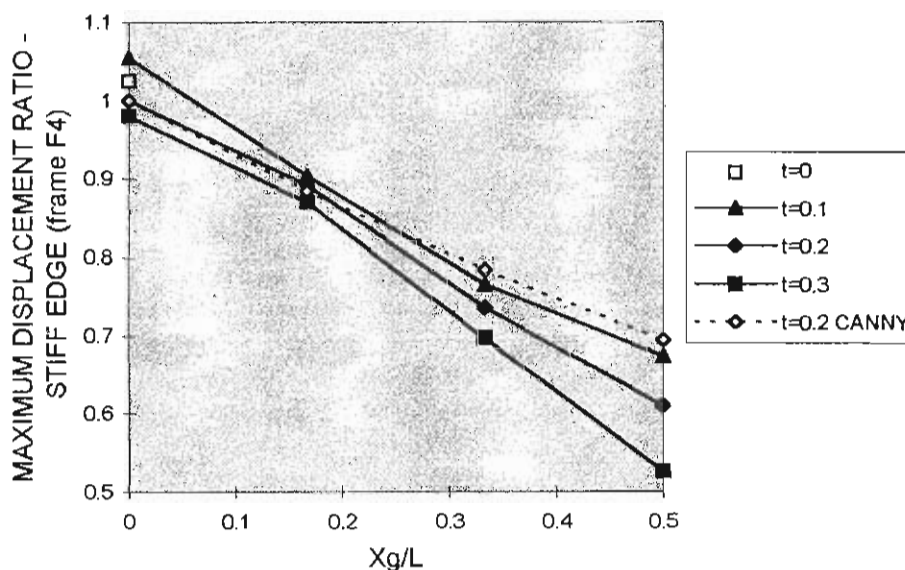


Figure 9: Maximum displacement ratio for the stiff side of the building structure.

In the program NEAVEK the ductility factor can be defined only approximately on the macroelement level and it is calculated as a ratio between the ultimate top displacement and the top displacement at the plastification of the macroelement. Figure 10 shows the ductility factors for different macroelements for the building with wall thickness $t=0.2$. It can be noted that the required ductility factors for wall W1, at the time when the top displacement at center of masses becomes 1% of building height, are rather high, especially in cases $t=0.2$ and $t=0.3$. In this case it is therefore possible that the collapse of wall W1 occurs before the prescribed top displacement is reached. For the real design the ductility demand (that was in our example set to the displacement equal to 1% of building height) should be determined by analysis, for example with the nonlinear dynamic analysis of equivalent one-degree-of-freedom system.

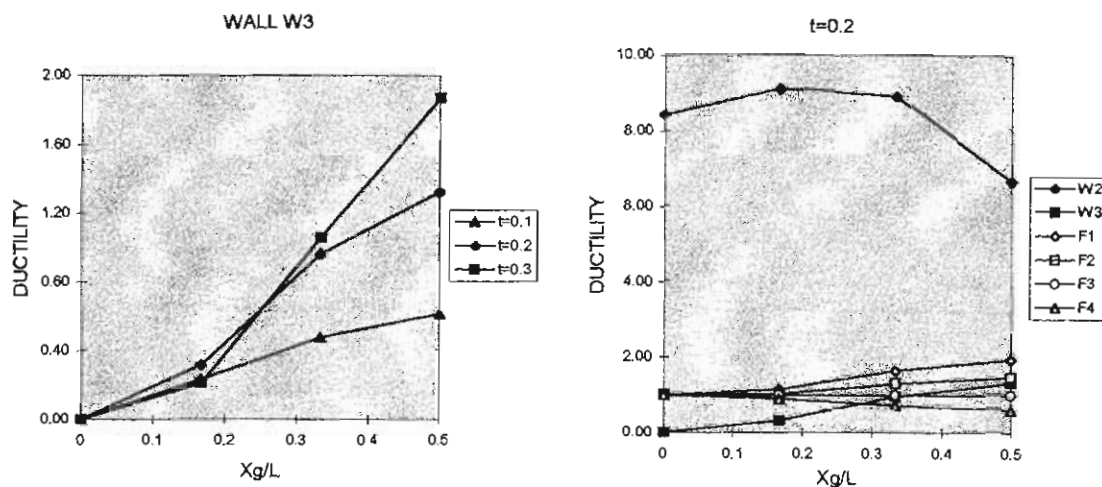


Figure 10: Ductility factors for wall W3 and for building variants with wall thickness $t=0.2$.

5. CONCLUSIONS

The parametric study confirms that the structural irregularity in plan is unfavorable. In the presented parametric study the asymmetric distribution of walls caused up to 10% reduction of maximum strength and up to 115% increase of the maximum top displacement, with the respect to symmetric distribution of walls. It was also shown that the increase of stiffness of asymmetrically positioned walls does not significantly improve the nonlinear behavior of the structure. In some cases such strengthening can cause negative effects, such as the increase of maximum displacement at the flexible side of the building and the change of the plastic mechanism of frames from global type to less desirable partial type mechanisms (e.g. thru the columns in the first story).

The comparison of results of program NEAVEK and program CANNY-E showed that the correlation of results in all cases is satisfactory. It should be noted however, that the efforts needed for data preparation, computation and interpretation of the results are much smaller when the program NEAVEK is used.

ACKNOWLEDGEMENT

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Review of Studies on Analysis of Structural Pounding

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1. Abstract:

The paper is related to the effect of earthquake induced pounding, i.e. the damage caused by two buildings or different parts of buildings hitting one another, due to an earthquake ground motion. Structural pounding refers to the lateral collision of adjacent buildings and occurs when building separations are insufficient to accommodate the relative motion of adjacent buildings. Investigations after the few strong past earthquakes, (Mexico City 1985, Loma Prieta 1989), showed that damage caused by pounding can be very significant and many researchers were trying to solve the problem of minimum separations distance to avoid pounding. This area does not appear to be well investigated, and need further research.

This is a short review of the articles to structural pounding which have been published in many relevant journals during the last fifteen years.

- reviews work

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2. Introduction:

The problem of mutual pounding between adjacent buildings during earthquakes has received increased attention in recent years. This is due to incidences of building damage, structural and non-structural, caused by pounding in strong earthquakes that have struck major cities around the world during recent years. The earthquake that struck Mexico City in 1985 provides a stark example of the seismic hazard that pounding poses. Pounding was identified in over 40% of the 330 collapsed or severely damaged buildings surveyed and in 15% of all cases it led to collapse, (Rosenblueth and Meli 1986, [17]).

During the earthquakes buildings sway dynamically, and lateral collision occurs between structures that are close. This is commonly called structural pounding. Structural pounding occurs when due to buildings' different dynamic characteristics, adjacent buildings vibrate out of phase and there is insufficient separation between them.

There are several factors contributing to this: most building codes contain only vague provisions for the necessary building separations, required to preclude pounding and do not take into account large inelastic deformations which can occur during major earthquakes, and many buildings are designed under the false impression that pounding will not happen. Buildings codes generally permit the allowed distance to be around 5 % of the height of the building, but recent findings suggest that the real distance due to an earthquakes, with ground acceleration of around 40% of the acceleration of gravity, can be as much as ten times higher. The cost of land in high densely populated areas is very high, and the tendency to increase the distance between buildings is contrasted with the market value of the usable space.

Clearly, pounding can be a serious seismic hazard and it is important to understand the characteristics of pounding in order to provide guidance in future building designs. Because as little is known, further research into the effect to pounding during strong earthquakes is essential.

3. Historical review:

One of the first analytical and numerical analyses due to pounding was the work of S.A. Anagnostopoulos [1], who analyzed a simplified model of several buildings in a block. Each structure was modelled as a single degree of freedom, (SDOF) system, with mass m_i , viscous damping constant c_i , initial stiffness K_i , yield level R_{yi} and post-yield stiffness pK_i . Pounding was simulated by means of linear viscoelastic impact elements, (spring-dashpot), that are introduced between the masses and act only when the masses are in contact. These element are idealized by linear spring constants s_{ij} and the dash constants c_{ij} , see Fig 1.

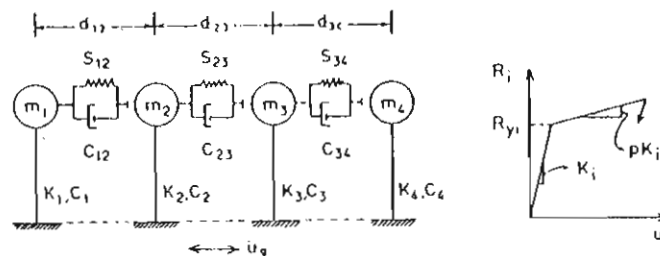


Figure 1. Idealisation of several adjacent structures

Equation of motion, due to displacement u_i of mass m_i relative to the ground displacement u_g is:

$$m_i \ddot{u}_i + c_i \dot{u}_i + R_i - F_{i-1,i} + F_{i,i+1} = -m_i \ddot{u}_g \quad \text{eqn. 1.}$$

R_i = non-linear resistance of the system, a function of relative displacement u_i

$F_{i-1,i}$ $F_{i,i+1}$ = impact forces due to pounding of masses i with masses $i-1$, and $i+1$ respectively;
they are present only if some masses are in contact
 $\delta_i = u_i - u_{i+1} - d_{i,i+1}$ eqn. 2.

Condition for contact between masses i and $i+1$ is

if $\delta_i > 0$:

$$\begin{aligned} F_{i-1,i} &= 0 & \delta_{i-1} &\leq 0 \\ F_{i-1,i} &= s_{i-1,i} \delta_{i-1} + c_{i-1,i} \delta_{i-1} & \delta_{i+1} &> 0 \\ F_{i,i+1} &= 0 & \delta_i &\leq 0 \\ F_{i,i+1} &= s_{i,i+1} \delta_i + c_{i,i+1} \delta_i & \delta_i &> 0 \end{aligned} \quad \text{eqn. 3.}$$

In matrix form, the system of equations is written as:

$$\begin{aligned} [M]\{\ddot{U}\} + [C]\{\dot{U}\} + \{R\} + [S]\{U\} + \{D\} &= -\ddot{u}_g \{m\} \\ [C] &= [C_s] + [C_p] \end{aligned} \quad \text{eqn. 4.}$$

$\ddot{u}_g$ - ground acceleration,

$\{R\}$ - vector of structural resistance R_i

$[S]$ - stiffness matrix corresponding to the impact element

$[C_s]$ - diagonal damping matrix with element damping constants C_i

$[C_p]$ - damping matrix corresponding to the impact elements

The equations of motion (eqn. 4.) were solved numerically using central differences, (constant velocity method), and the linear acceleration method to start the solution. Two different time-steps were used: a large time-step 0.01 sec, and finer time-step 0.0005 sec which was applied whenever one or more masses were in contact.

Parametric investigation is obtained for several parameters to obtain results of a more general nature:

1. system configuration, number of systems in a row and their periods
2. gap size
3. strength (yield level of the system)
4. relative size of system masses
5. impact element damping
6. impact element stiffness

On the basis of the results obtained the following conclusions are summarized:

1. Effect of earthquake induced pounding on the global response of a structure in a row of several adjacent structures depends primarily on:
 - a) the properties of the structure itself and in relation to the properties of the adjacent structures
 - b) whether the structure is subjected to one- or two-sided impact, (exterior and/or interior structure)
 - c) gap size
2. Exterior buildings are subjected to one-sided impact and as a rule experience a quite substantial response amplification factor. Interior structures subjected to the two-sided impact can produce either increasing or reduction of response, depending on the ratio of their periods to the period of adjacent structures. For a ratio smaller than one, pounding is amplified in most cases. The amplification is lower than that of exterior structures. When the ratio is greater than one the response of interior structures is typically reduced to pounding.
3. Increasing the gap size decreases the effect of pounding. Gap size equal to the square-root-of-sum-of-squares, (SRSS) could be sufficient to avoid pounding
4. Pounding causes similar effects on elastic and inelastic structures
5. Larger differences in the masses of two adjacent structures make the effect of pounding more pronounced for the smaller mass structures
6. Computed displacement amplifications due to pounding are not very sensitive to changes in the parameters of the impact elements.
7. Use of soft viscoelastic material between structures can significantly reduce the effect of pounding.

In their later works on pounding S.A. Anagnostopoulos and K.V. Spiliopoulos [2], modelled each of a number of adjacent buildings as a MDOF close-coupled, (shear beam type), system with bilinear interstorey resistance and masses lumped at the floor level. Rayleigh-type damping is further specified and foundation compliance is accounted for by means of appropriate rocking and translations spring-dashpot elements. The floor elevations are the same for all buildings and pounding, simulated with viscoelastic impact element, s_j and c_j . One translations degree of freedom is allowed for every lumped mass in each system, except for the mass of the foundation, which in addition has a rotational DOF to permit rocking motion, see Fig.2.

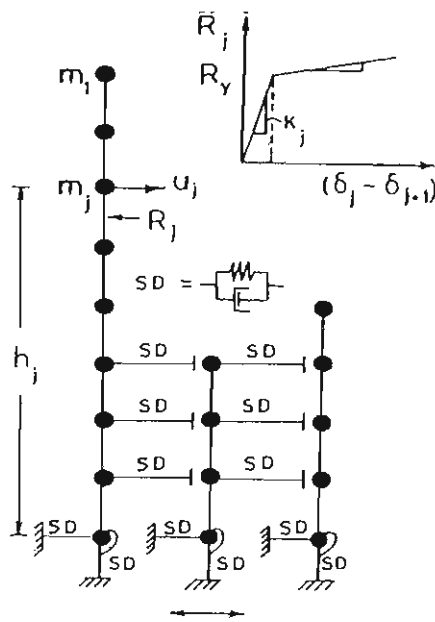


Figure 2. Idealization of adjacent building

Equation of motion in matrix form:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [S]\{U\} + \{R\} = -\ddot{u}_g \{m\} \quad \text{eqn. 5.}$$

$[C] = [C]_R + [C]_I$ damping matrix

$[C]_R = \alpha[M] + \beta[K]$ matrix for Rayleigh type of structural damping

$[C]_I$ = damping matrix associated with impact dashpot

$[K]$ = elastic stiffness matrix used only for an initial eigenvalue solutions

$\{R\}$ = vector of structural resistance

$\{U\}$ = vector of unknown DOF

The system of equations of motion eqn 5. is solved numerically using central differences, with two different time-steps. Large time-step applied when no impact, and small one when buildings are in contact

In example applications two 5-storey buildings at 5 mm distance are observed, and results are expressed as ratios μ/μ_0 , storey ductility with or without pounding respectively, for the five earthquakes. Stiffer buildings exhibit large increases in ductility demands, as a result of pounding. In the second example, with four buildings in the row, exterior buildings are penalized more than the interior building and this is far greater for the stiffer building. In the example with two adjacent buildings with unequal heights, 10 and 5 stories with $T=1.03$ and $T=0.52$, the most noticeable effect is seen in the top storey of the small building. In the case of extremely stiff structures, in analyses of 10 and 5 stories buildings the collision can be catastrophic.

In the third instance S.A. Anagnostopoulos and K.V. Spiliopoulos, [3], continued their work on pounding; using five real earthquake motions, the effect of the several factors were investigated: building configuration and relative size, seismic separation distance and impact element properties.

Equations of motion: If u_i is the horizontal displacement of mass m_i of the systems relative to the ground, u_b and ϕ_b the translation and rotation of the foundation, respectively, and δ_j the part of u_j due to structural deformations then:

$$u_j = \delta_j + u_b + h_j \phi_b \quad \text{eqn. 6.}$$

equations of motions of the mass m_j :

$$m(\ddot{\delta}_j + \ddot{u}_b + h_j \ddot{\phi}_b) + \sum_{i=1}^n c_{ji} \dot{\delta}_i + F_j + R_j = -m_j \ddot{u}_g \quad \text{eqn. 7.}$$

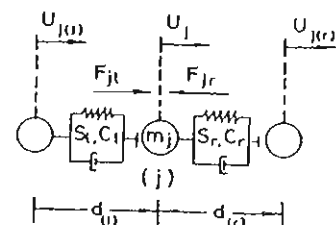
c_{ji} - damping coefficient for Rayleigh type of damping

F_j - impact forces acting on the one or two side

R_j - restoring forces due to structural resistance

$\ddot{u}_g$ - ground acceleration

Impact forces in that case are:



$$F_j = F_{j,r} - F_{j,l} = (s_r v_r + c_r \dot{v}_r) - (s_l v_l + c_l \dot{v}_l) \quad \text{eqn 8.}$$

Non-linear restoring forces R_j can be computed numerically from bilinear force-deformation relationship of each storey:

$$R_j = k_j'(\delta_j - \delta_{j+1}) - k_{j-1}'(\delta_{j-1} - \delta_j) \quad \text{eqn. 9.} \quad k_j' - \text{tangent stiffness of story } j$$

In matrix form equations of motion look like:

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [S]\{U\} + \{R\} = -\ddot{u}_g \{m\} \quad \text{eqn. 10.}$$

$$[C] = \alpha[M] + \beta[K] + [C]_I$$

$[C]$ - damping matrix

$[M]$, $[K]$, - mass matrix and elastic stiffness matrix respectively

$[C]_I$ - damping matrix associated with impact dashpot

It was found that pounding can cause high overstress, mainly when the colliding buildings have significantly different heights, periods and masses.

B. F. Maison, K. Kasai [4], in their work, analyzed MDOF equations of motion for a type of structural pounding, at a single floor level with a rigid adjacent building. The floor is idealized as rigid in plane with three degrees of freedom to each floor level at the floor center of mass, two lateral translation and one rotation about vertical axis. (The structural stiffness can be reduced to these degree of freedom by standard matrix condensation techniques).

main assumptions:

1. Building of interest vibrates dynamically and collides laterally with an adjacent structure that is rigid, building of interest is very flexible and with low mass relative to the adjacent building.
2. The pounding occurs at a single floor level in the building of interest at the floor diaphragms are rigid in plane. The collision forces are distributed to all structural elements at the floor level
3. A single linear spring represents the local flexibility
4. The pounding problem is idealized to two linear states. The first is that the building of interest vibrates by itself, and second the buildings are in contact.

1. If buildings are not in contact, (state 1.), the MDOF equations of motion are

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -mr\ddot{u}_g(t) \quad \text{eqn. 11.}$$

where : $\ddot{u}(t)$, $\dot{u}(t)$, $u(t)$ are vector of acceleration, velocity and displacement respectively,

m, c, k , are matrices of mass, damping and stiffness, $\ddot{u}_g(t)$ is translations earthquakes acceleration at the base, and r is the earthquake influence coefficient column vector containing the dynamic degrees of freedom displacement associated with a unit horizontal earthquake translation at the building base. The equations are linear and can be uncoupled by transformation to normal coordinates, using undamped modal properties.

The uncoupled equations of motions can be solved explicitly using a time step-by-step solution scheme

2. (state 2.) the buildings are in contact, and MDOF equations of motion are:

$$m\ddot{u}(t) + \bar{c}\dot{u}(t) + \bar{k}u(t) = -mr\ddot{u}_g(t) + b \quad \text{eqn. 12.}$$

where b is a column vector of static preload forces to account for the building separation and

$\bar{c}, \bar{k}$ are, respectively, damping matrix assumed as a linear combination of the mass and stiffness matrix, and stiffness matrix of building of interest.

The uncoupled equations of motion can be solved explicitly in a manner similar to previous, except here the generalised displacement must be separated into static and dynamic components.

The solution methodology is implemented into the microcomputers program SLAM, (Maison and Kasai 1988) which uses the building dynamic properties. In several SDOF and MDOF systems analyzed the following conclusions are obtained: Neglecting the effect of possible pounding leads to unconservative building design. The sample analyses response behavior trends are robust, thereby

providing insight into the response behavior of other buildings, and pounding can be studied on a case-by-case basis using the analytical tools presented in the paper.

At the Fourth U.S. National Conference on Earthquake Engineering 1990, Kasai, Maison and Patel [5], presented a technique to predict building peak response for a type of pounding, (work based on previous [4]). It involved the scaling of linear response spectrum analyses results based on energy determined from seismic pseudo-velocity spectra. The method is based on extensive numerical experiments involving the pounding response to both SDOF and MDOF systems subjected to a series of artificial earthquakes generated from SEAOC spectra, (Structural Engineers Association of California). Estimates of peak pounding responses are obtained by scaling the no-pounding and fixed-spring response spectrum results, (displacements, drifts, shears and overturning momentum).

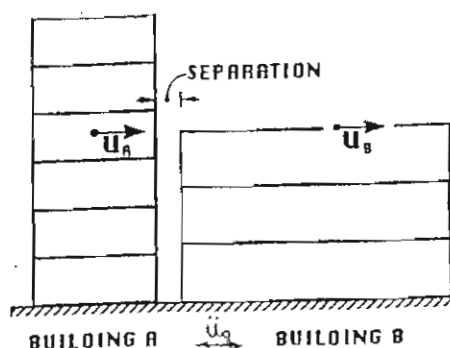
Many authors observed the structural pounding damage after strong earthquakes. In their work, Kasai and Maison (1990), observed pounding damage from 1989 Loma Prieta earthquake [6]. The observed area was very large, and classification of pounding damage were made. Four main types of damage were observed: type1.-major structural damage, type2.-failure and falling of building appurtenances creating a life-safety hazard, and type3.- loss of function due to failure of key mechanical, electrical and similar systems. Type 4.- architectural damage. Initial work based on the survey of building pounding caused by Loma Prieta earthquake were presented in 1996 by Kasai and Maison [7].

Kasai, Maison, Jeng, D.J. Patel and P.C. Patel, [8] studied a spectrum method to determine required building separations to preclude pounding, and peak pounding response. The analysis is obtained by two computer programs SLAM and SLAM2, which idealized structure as a three-dimensional MDOF system. The SLAM assumes that a building laterally collides with a rigid adjacent building, and SLAM2 that both buildings are flexible. The pounding is idealized as in the previous work, [4],[5] Spectrum Difference Method based on random vibration theory and considering a first mode approximation for displacement of elastic multi-storey buildings, is presented by authors. A simplified method to obtain an accurate estimate of the required building separations s , to preclude pounding, considers the difference of vibration phase between adjacent building A and B:

$$s = \sqrt{u_A^2 + u_B^2 - 2\rho_{AB}|u_A||u_B|} \quad \text{eqn. 13.}$$

u_A, u_B are the peak lateral displacements at the possible pounding location under the no-pounding condition, in building A and B respectively, the magnitude obtained from commonly used spectrum approach. ρ_{AB} is a cross correlation coefficient.

Van Jeng, Eeri, Kasai and Eeri, Maiso, Eeri, 1992, [9] presented a spectral difference method to estimate building separations to avoid pounding. Their method is based on random vibration theory. The required separation distance to preclude pounding can be obtained through dynamic time history analysis, of the adjacent buildings, involving the calculation of relative displacement between the potential pounding locations of the buildings. The accuracy is demonstrated by numerical experiments using 9 artificial and 6 actual earthquakes. Relative displacement of adjacent buildings according to Fig.3. is:



$$u_{REL}(t) = u_A(t) - u_B(t) \quad \text{eqn. 14.}$$

Pounding occurs when for separation distance s
 $u_A(t) - u_B(t) > s$

Using the fundamental mode as an approximation for MDOF response of buildings and recently developed Double Difference Combination (DDC), [8]

Figure 3. Relative Displacement

$$u_{REL}(DDC) = \sqrt{u_A^2 + u_B^2 - 2\rho_{AB}|u_A||u_B|} \quad \text{eqn. 15.}$$

Two other possibilities for estimation the relative displacement are:

The absolute sum (ABS)

$$u_{REL}(ABS) = u_A + u_B \quad \text{eqn.16.}$$

The square-root-of-sum-of-squares, (SRSS)

$$u_{REL}(SRSS) = \sqrt{u_A^2 + u_B^2} \quad \text{eqn. 17.}$$

The proposed method is verified through several parametric investigations.

Several other authors investigated separation distance between buildings to avoid pounding. A.Filitrault and M. Cervantes 1993, [10], used nonlinear time-step dynamic analyses. The required separations obtained by subtracting the time-histories of lateral displacements between adjacent floor level were compared with the static method of the National Building Code of Canada. Five different wall buildings were analysed.

Studies on structural pounding were presented by A.J.Carr and P.J. Moss, [11], B.Westermo, 1988,[12], R.O.Davis, [13], J.P.Wolf nad P.E. Skrikerud, [14], and lot of others.

4. Conclusion:

The problem of pounding during strong earthquake is bigger than was realized until recent years. The surveys from two recent earthquakes, Loma Prieta 1989, Mexico City 1985, showed that damage from pounding is significant and can not be neglected. Several methods to preclude pounding, are presented by authors, but further research is more than necessary, especially on minimal separation distance in high densely populated areas, to avoid pounding, either in elastic and inelastic areas and their results should be incorporated into building design codes.

5. Acknowledgment

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Numerical model for the dynamic analysis of pillar jib crane with rotating flexible jib

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Abstract

This paper deals with dynamic analysis of a pillar jib crane with rotating flexible jib. A crane with electric hoist suspended from the jib end is rotated by hand is presented. An analysis has been carried out in order to find out the dynamic behaviour of the crane due to loads that may arise during starting operation of the hoisting mechanism. The results have been obtained for the structure exposed to static and dynamic load. The actual conditions were simulated using FEM package IDEAS.

1. Introduction

Hoisting and conveying machinery is of great importance in modern production systems. Displacing parts and products from one work station to another, this machinery enables the process to run without interruptions and at a steady pace. Jib cranes are widely used in various industrial applications, especially in smaller workshops. The jib cranes (in further text crane = jib pillar crane) are cranes with a load-handling device suspended from a cantilever (a jib) capable of rotating on a pillar fixed at its base to a foundation, or secured to a pillar which can rotate in a support socket in its foundation. For load lifting different types of winches (hoists) are used, which can be hand-, electric-, or air- powered. A typical pillar jib crane with a flexible jib is shown in Fig. 1.

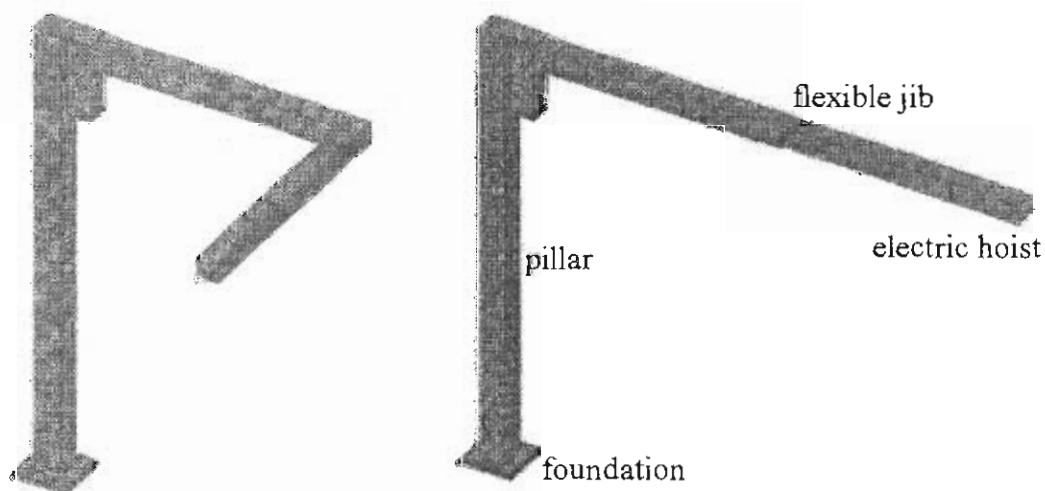


Fig.1: Pillar jib crane with flexible jib in two different working positions

2. Basic characteristic of the pillar jib crane

The pillar jib crane serves the purpose of moving loads vertically as well as transferring them from one point of the area covered by the device into another. The presented crane is constructed from standard rolled steel shapes. The basic performance and geometric parameters of analysed crane are given in Table 1.

Table 1: Basic performance parameters of crane

Lifting capacity	2500 [N]
Span-reach (max,min)	4 m (1.15m)
Lift	2.5 m
Height of pillar	3.4 m
Length of inner jib	1.8 m
Length of outer jib	2.0 m
Hoisting speed	10 m/min
Weight of hoist	38 kg

3. Dynamic analysis process

The dynamic analysis process can be performed by the steps as are represented in Figure 2. The analyst must evaluate the finite element model in terms of the type of dynamic loading to be applied to the structure. An overall system design is set up considering the dynamic environment. As part of the evaluation process, a finite element model is created. The first step in dynamic analyses is usually a modal analysis to determine the structure's natural frequencies and mode shapes.

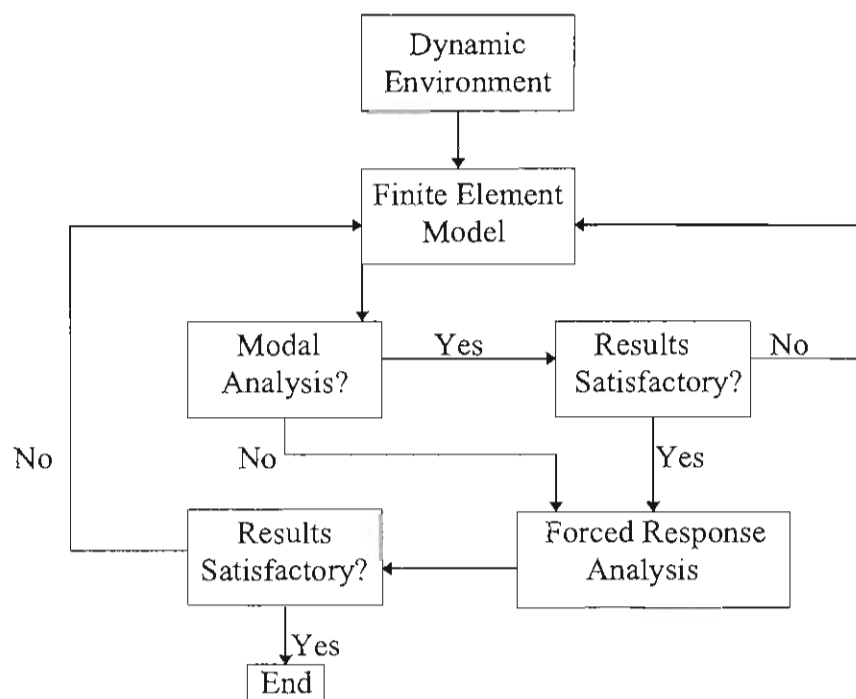


Fig.2: Overview of Dynamic Analysis Process

Forced response is the next step in the dynamic evaluation process. The results of a forced response analysis are evaluated in terms of the system design. Necessary modifications are made to the system design. These changes can then be applied to the model and analysis parameters to perform another iteration on the design. The process is repeated until an acceptable design is determined, which completes the design process.

3. FEM Model

Finite element model of the crane is made by using 3-D beam elements (35 elements). The properties of the beams are modelled as realistic as possible. The numerical model with corresponding cross-sections of beam elements and some boundary conditions is shown in Figure 3.

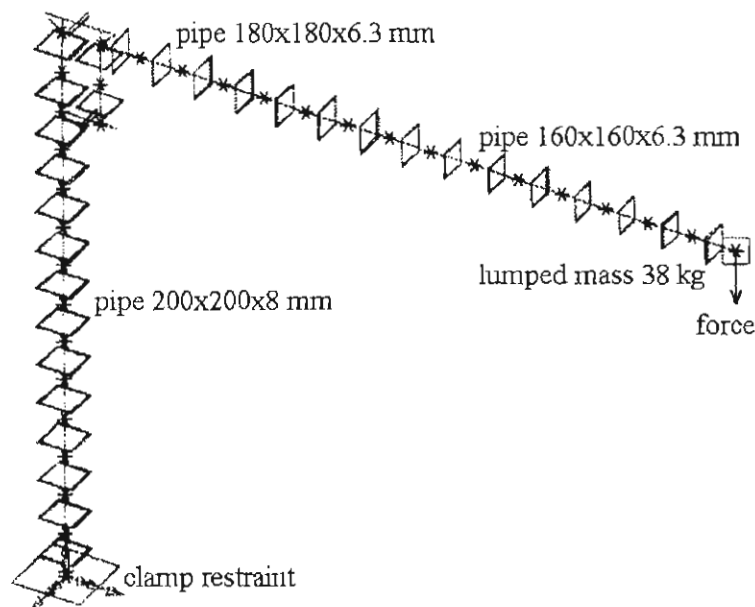


Fig.3: Numerical model with cross-sections of beam elements

In the calculations we took into account material properties of steel with: Modulus of elasticity $E=210000$ MPa, Poisson's ratio $\nu=0.3$ and density $\rho=7850$ kg/m³.

4. Static and Normal modes analysis

In the first stage static analysis have been made in order to obtain displacements, because this type of cranes are very sensitive to considerable displacements. Two different cases with the same load combination have been analysed:

case a): outer jib in extended position (maximum reach)

case b): outer jib perpendicular to the inner jib (maximum torsion)

load: force $F=2875$ N (included 15% overload according to standards) at the end of outer jib, dead weight of crane structure and lumped mass 38 kg at the end of outer jib

Results of static analysis are given in Figure 4 and Figure 5. We can see that the maximum displacement at the end of the outer jib is high and stresses within allowable range (under 70 MPa).

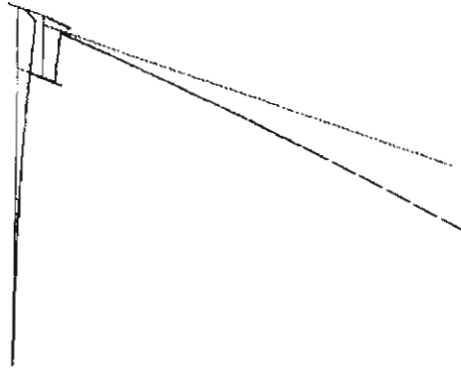


Fig 4: Displacement (case a)-dotted line
(max. magnitude 28.9 mm)

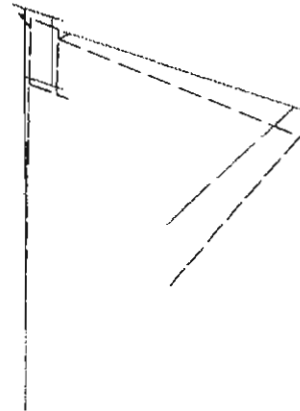


Fig.5: Displacement (case b)
(max. magnitude 18.9 mm)

For safe operation of crane is important that all the loads that a structure has to carry are estimated and included in structural analysis. Therefore dynamic effects of hoisting the load were taken into account by performing dynamic analysis. The first step in performing a dynamic analysis was determining the natural frequencies and mode shapes of the structure with damping neglected in order to find out the basic dynamic behaviour of the structure. Computation of the natural frequencies and mode shapes is performed by solving an eigenvalue problem with governing equation (1) written in matrix form.

$$[M] \cdot \{\ddot{u}\} + [K] \cdot \{u\} = 0 \quad (1)$$

where $[M]$ is mass matrix, $[K]$ is stiffness matrix, $\{\ddot{u}\}$ is acceleration vector of mass and $\{u\}$ is displacement vector of mass. Different methods of real eigenvalue extraction are provided in numerical approaches [1],[2] (seven methods in MSC/NASTRAN and three algorithms in I-DEAS). The results for some first natural frequencies for given structural properties and boundary conditions are given in Table 2 and corresponding mode shapes in Fig.6 and Fig. 7.

Table 2: Frequency results in [Hz]

Normal mode	Model solution case a	Model solution case b
1	4.13	5.0
2	4.9	5.7
3	12.1	11.8
4	12.1	12.7
5	49.5	28.3
6	49.8	32.9
7	91.4	83
8	97.4	96

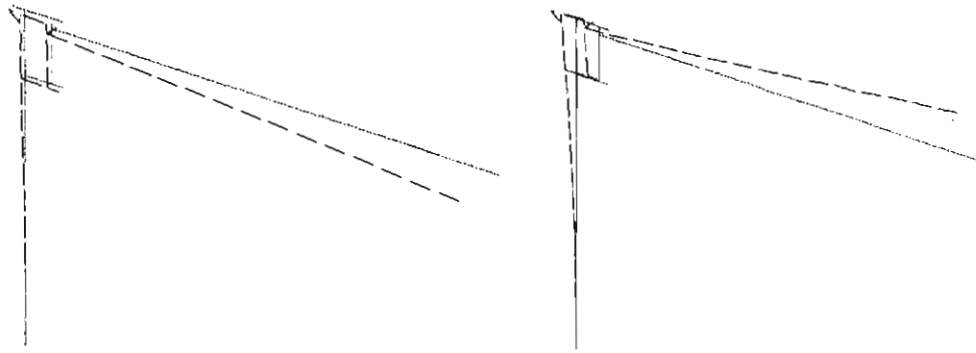


Fig 6: First two mode shapes (case a): ($f_1=4.1$ Hz and $f_2=4.9$ Hz)

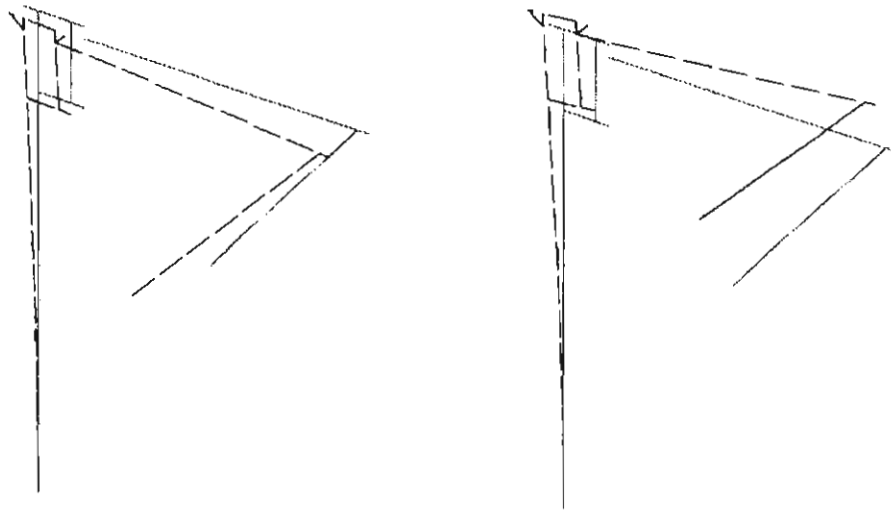


Fig 7: First two mode shapes (case b): ($f_1=5.0$ Hz and $f_2=5.7$ Hz)

5. Transient response analysis

The purpose of a transient response analysis was to compute the behaviour of a structure subjected to time-varying excitation. Dynamics (or vibrations) in a system is the result of energy being transferred back and forth between kinetic energy (energy stored in the motion of masses) and potential energy (energy stored in spring stiffness). The equations of motion for a dynamic system can be written in matrix form:

$$[M] \cdot \{\ddot{u}\} + [B] \cdot \{\dot{u}\} + [K] \cdot \{u\} = \{P\} \quad (2)$$

where $\{P\}$ is a vector of forces, $[B]$ is viscous damping matrix and other values have the same meaning as in equation (1).

When a rated load is being hoisted at an acceleration, the crane is subject to the combined effect of static and dynamic forces. The transient excitation is explicitly defined in the time domain. Applied loads are in accordance with Fig.8, with variation of time in which maximum load can occur. We wanted to simulate the influences on

dynamic behaviour of the structure during the time within the electric drive is brought up to the appropriate hoisting speed. 3% of the critical modal damping has been assumed in the analysis.

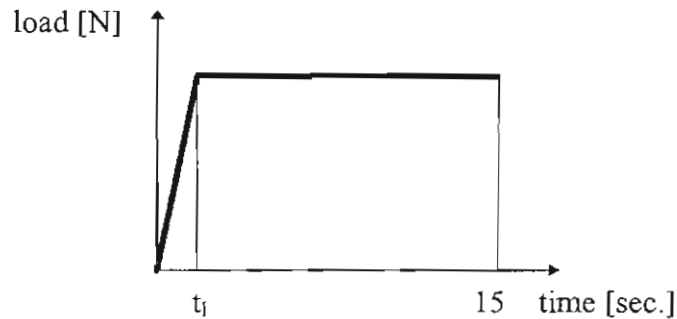


Fig.8: Applied dynamic load

Load patterns applied in the analysis at the end of jib in vertical direction are given in Table 3 in accordance with Fig.8. It was supposed that full load occurs in time t_1 , but in cases A4 and B4, where we supposed that in time 0 - 0.1 sec. 20% of load and in time 0.1 - 1 sec. full load occurs.

Table3: Dynamic Load

Case	Time t_1 [sec]	Load [N]
A1 and B1	0.01	2500
A2 and B2	0.1	2500
A3 and B3	1.0	2500
A4 and B4	0.1 and 1.0	500, 2500

Model A): outer jib in extended position (maximum reach)

Model B): outer jib perpendicular to the inner jib (maximum torsion)

The results for responses of crane structure obtained at the end of jib in the vertical direction are shown in Fig.9, 10, 11 and 12.

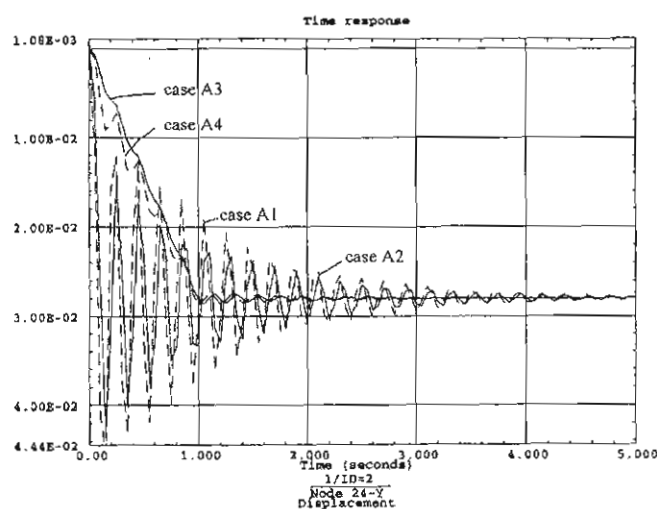


Fig.9: Responses of model A (displacements)

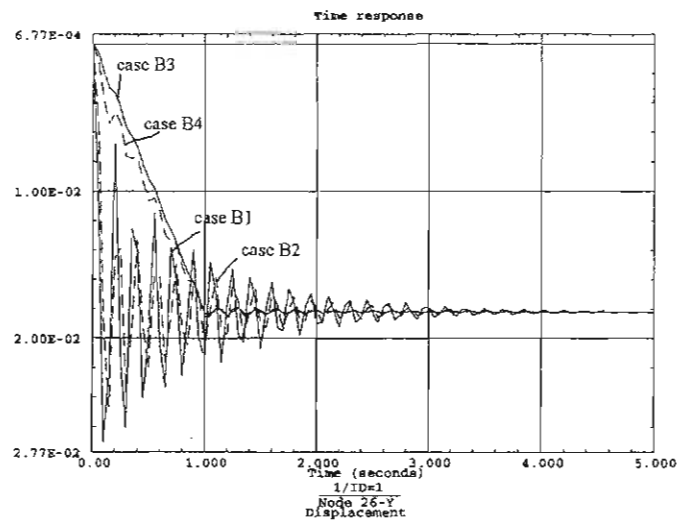


Fig.10: Responses of model B (displacement)

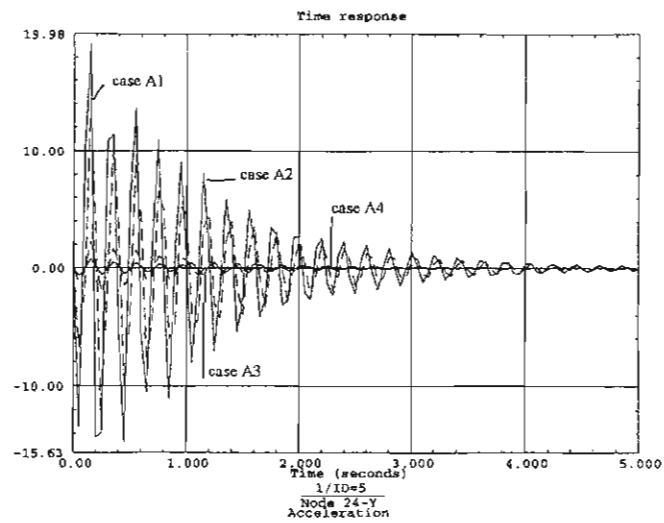


Fig.11: Responses of model A (acceleration)

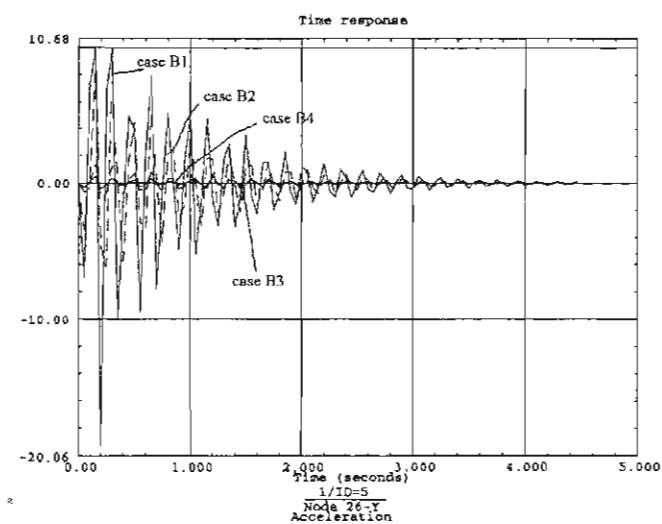


Fig.12: Responses of model B (acceleration)

6. Conclusion

With the used numerical analysis of the pillar jib crane we have tried to simulate as precisely as possible the actual conditions during starting of the hoisting mechanism. In spite of certain assumptions and simplifications (pliability of rope is neglected etc.) we can see that the motor control during the start of hoisting is very important. When a rated load is being hoisted, the crane is subjected to static and dynamic forces. After a period the fluctuating loading with corresponding displacements gradually subside to the values equal to static force. It is seen from the results that the recommended values of acceleration in starting the hoisting mechanisms can quickly be overcome. We conclude that for lifting loads close to the lifting capacity it is recommended to use an electric motor capable of two-speed operation. The starting speed of the motor is the maximum hoisting speed, but when the load arises, for instance, over 20 percent of rated load the automatic control device changes to slower hoisting speed. To prevent the crane overloading to a point causing rope or chain breakage, hook failure or irrecoverable deformation of structural parts, an overload protection device must be provided. This device automatically stops the hoisting motion when the load lifted exceeds 10 percent of the rated capacity.

For further work it will be recommended to evaluate the model with a dynamic experiment before performing more complex analysis (fatigue analysis, optimisation etc.)

7. References

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OPTIMIZATION OF DAMPING OF LAVAL ROTOR BEARINGS

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Abstract

Optimal damping in Laval rotor bearings, based on a model of a rotor in isotropic bearings with one-and-half-degree-of-freedom is found. Eigenmotions, in general, are damped vibrations and an exponentially decreasing motion. Only in a limited range of parameters there are not a damped vibrations, but all three eigenmotions are exponential motions with various decreasing rates. The most influence on resonant maximum has the damping. Minimal maximum depends of damping in bearings. The obtained damping is optimal, but it is difficult to realize in praxis.

1. Introduction

The rotor vibration at resonance directly depends on damping in its bearings. Adding of damping in bearings is the most easily way for decreasing of the resonant amplitudes. The additional damping has to minimize amplitudes of forced vibrations or some other vibration characteristics. This optimization problem can be solved by mathematical methods, for that, limited number of possibilities are existed in praxis. Useful direction for selection of damping gives a simplified model of the Laval rotor (see Figure 1).

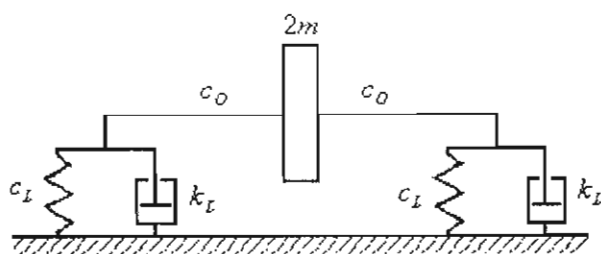


Figure 1 Isotropic Laval rotor

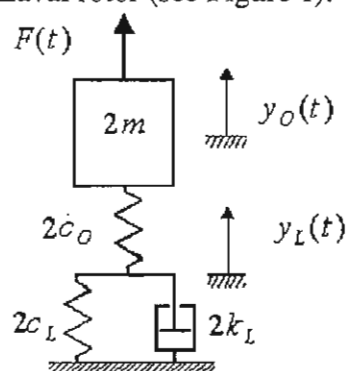


Figure 2 Vibration model of isotropic Laval rotor with one-and-half-degree-of-freedom

In this paper are analysed eigen- and forced vibrations of Laval rotor. The obtained optimal damping in bearings gives a good estimate for praxis.

2. Equation of motion of Laval rotor in isotropic bearings

The Laval rotor in isotropic bearings is a plane model with one-and-half-degree-of-freedom. The equations of motion for displacements of disk y_O and bearings y_L (see Figure 2) are

$$2m \ddot{y}_O + 2c_O y_O - 2c_O y_L = F(t),$$

$$2k_L \dot{y}_L - 2c_O y_O + 2(c_O + c_L) y_L = 0. \quad (1)$$

2.1 Eigenvibrations

For $F(t) = 0$ Eqs. (1) gives eigenvibrations of Laval rotor. Assuming the solutions in the exponential form

$$y_O(t) = v_O e^{\lambda t}, \quad y_L(t) = v_L e^{\lambda t}, \quad (2)$$

with normalized parameters

$$\zeta = k_L / \sqrt{c_O m}, \quad \lambda^* = \lambda / \omega_O, \quad \kappa = c_L / c_O, \quad (3)$$

where $\omega_O = \sqrt{c_O / m}$, Eqs. (1) becomes,

$$\begin{aligned} (1 + \lambda^{*2}) v_O - v_L &= 0, \\ -v_O + (1 + \kappa + \zeta \lambda^*) v_L &= 0. \end{aligned} \quad (4)$$

The corresponding characteristic equation is

$$\lambda^{*3} + \frac{1 + \kappa}{\zeta} \lambda^{*2} + \lambda^* + \frac{\kappa}{\zeta} = 0. \quad (5)$$

The solutions of Eq. (5) can be:

a) a pair of conjugated complex values and a single negative real value

$$\lambda_1^* = \alpha^* + j\omega^*, \quad \lambda_2^* = \alpha^* - j\omega^*, \quad \lambda_3^* = \alpha_3^* \quad (6a)$$

or,

b) all three real negative values

$$\lambda_k^*, \quad v_{Ok}, \quad v_{Lk}, \quad k = 1(1)3. \quad (6b)$$

From Eqs. (4) with eigenvalues λ_k^* , $k = 1(1)3$, is obtained

$$v_{Lk} = (1 + \lambda_k^{*2}) v_{Ok}. \quad (7)$$

The solutions, Eqs. (2), introducing normalized time $\tau = \omega_O t$, become

$$y_O(\tau) = \sum_{k=1}^3 v_{Ok} e^{\lambda_k^* \tau}, \quad y_L(\tau) = \sum_{k=1}^3 (1 + \lambda_k^{*2}) v_{Ok} e^{\lambda_k^* \tau}. \quad (8)$$

Using eigenvalues given in Eqs. (6a), resulting v_{O1} and v_{O2} are conjugated complex values and v_{O3} is a real value. Introducing

$$v_{O1} = \frac{1}{2}(C - jS), \quad v_{O2} = \frac{1}{2}(C + jS), \quad v_{O3} = K,$$

the solutions, Eqs. (8), have the form

$$\begin{aligned} y_O(t) &= e^{\alpha t} (C \cos \omega t + S \sin \omega t) + K e^{\alpha_3 t}, \\ y_L(t) &= e^{\alpha t} [(aC + bS) \cos \omega t + (aS - bC) \sin \omega t] + (1 + \alpha_3^{*2}) K e^{\alpha_3 t}, \end{aligned} \quad (9)$$

where

$$\alpha = \alpha^* \omega_0, \quad \omega = \omega^* \omega_0, \quad \alpha_3 = \alpha_3^* \omega_0, \quad a = 1 + \alpha^{*2} - \omega^{*2}, \quad b = 2\alpha^* \omega^*. \quad (10)$$

The real parts of α^* and α_3^* are always negative. This means that eigenmotions of the disk and bearings are damped vibrations with natural frequency ω and damping constant $-\alpha$, and exponentially decreasing motion with damping constant $-\alpha_3$. The constants C , S , K depend on the initial conditions.

In the case of all three real eigenvalues, the eigenmotions are the exponentially decreasing motions

$$y_0(t) = \sum_{k=1}^3 K_k e^{\alpha_k^* t}, \quad y_L(t) = \sum_{k=1}^3 (1 + \alpha_k^{*2}) K_k e^{\alpha_k^* t}. \quad (11)$$

This case occurs only for a limited range of parameters κ and ζ of the model (see Figure 3).

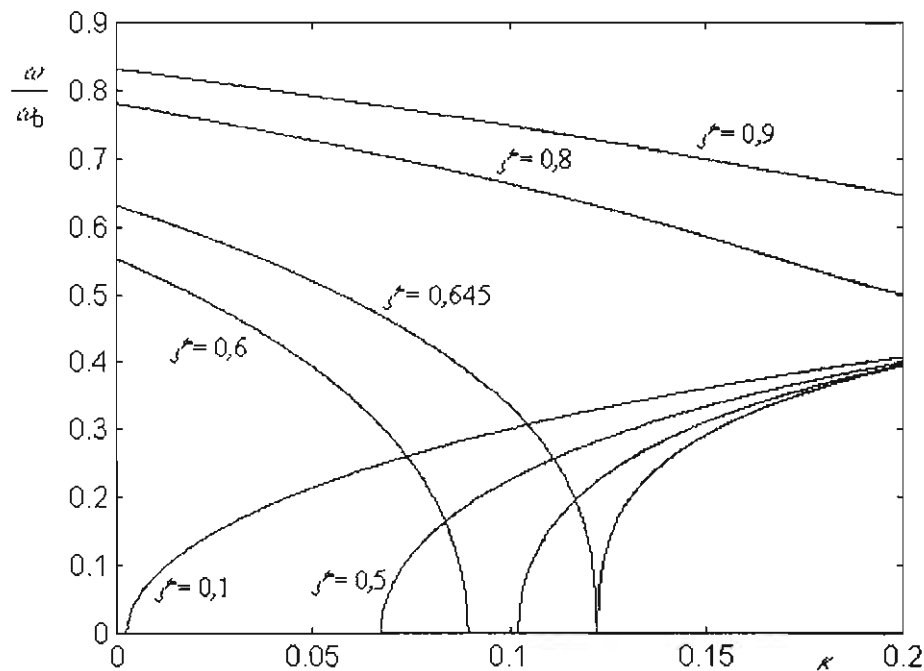


Figure 3 The natural frequency as function of stiffness constants ratio and normalized damping in the bearings

For eigenvalues Eqs. (6a), the functional dependence of stiffness constants ratio is given in Figures 3, 4 for various values of normalized damping. From Figure 4 it can be seen that for $\zeta > 1$ the graphs have a minimum. Figure 3 corresponds to a range of small bearings stiffness. Apparently for $\zeta < 0,645$ there is a limited range of stiffness constants corresponding to eigenfrequency equal to zero. For this range of parameters the characteristic equation has three real eigenvalues given by Eqs. (6b) and model has the motions according to Eqs. (11).

The graphs of normalized damping constants, Figure 5, are given as function of the stiffness ratio for normalized damping in the bearings $\zeta > 1$.

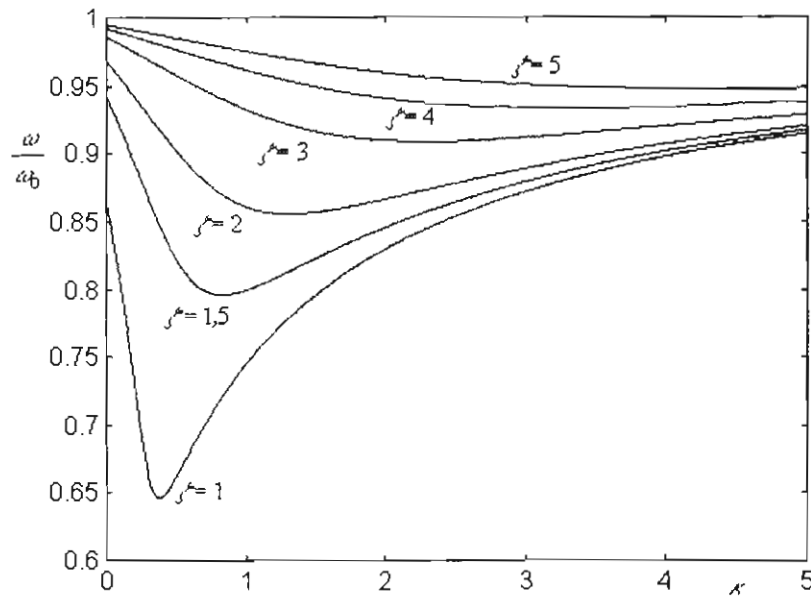


Figure 4 The natural frequency as function of stiffness constants ratio and normalized damping in the bearings

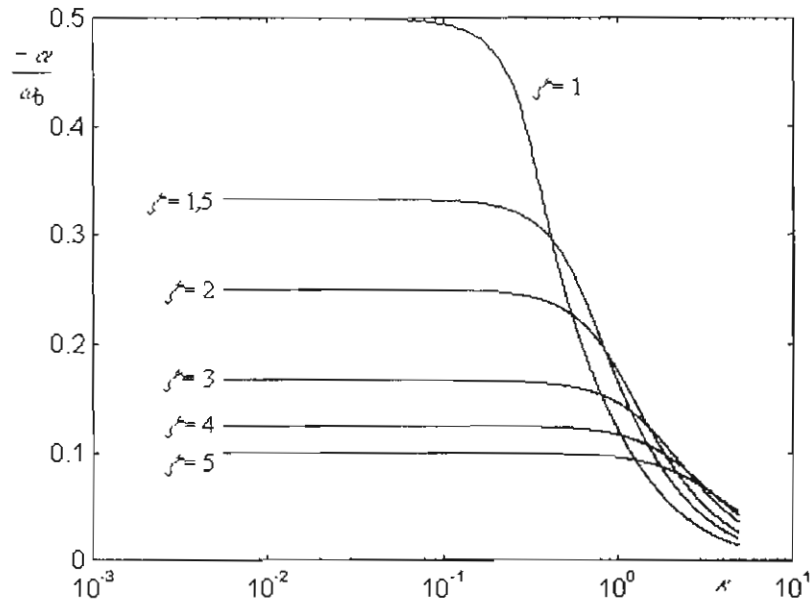


Figure 5 The damping constants as function of stiffness constants ratio and normalized damping in the bearings

2.2 Forced vibrations due to mass unbalance

The rotor excitation due to the disk mass unbalance is

$$F(t) = 2me\Omega^2 \cos \Omega t, \quad (12)$$

where e and Ω are eccentricity and forcing frequency, respectively. The rotor excitation written in complex form is

$$\underline{F}(t) = 2me\Omega^2 e^{j\Omega t}. \quad (13)$$

Steady-state solutions of Eqs. (1) are now

$$\underline{y}_o(t) = \underline{\hat{y}}_o e^{j\Omega t}, \quad \underline{y}_L(t) = \underline{\hat{y}}_L e^{j\Omega t}. \quad (14)$$

The amplitudes of steady-state vibrations of the disk and bearings are

$$\frac{\hat{y}_o}{e} = \frac{\eta^2 \sqrt{(1+\kappa)^2 + (\zeta \eta)^2}}{N}, \quad \frac{\hat{y}_L}{e} = \frac{\eta^2}{N} \quad (15)$$

and amplitude of force transmitted to the ground is

$$\frac{\hat{F}_{pod}}{2ec_o} = \frac{\eta^2 \sqrt{\kappa^2 + (\zeta \eta)^2}}{N}, \quad (16)$$

where $N = \sqrt{[\kappa - (1+\kappa)\eta^2]^2 + [\zeta \eta(1-\eta^2)]^2}$, $\eta = \Omega/\omega_o$.

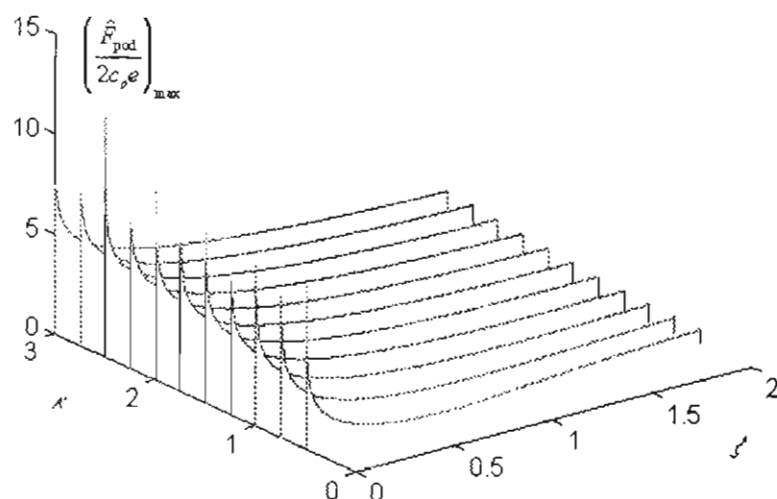


Figure 6 Maximal force transmitted to the ground as function of normalized damping in the bearings and the stiffness constants ratio

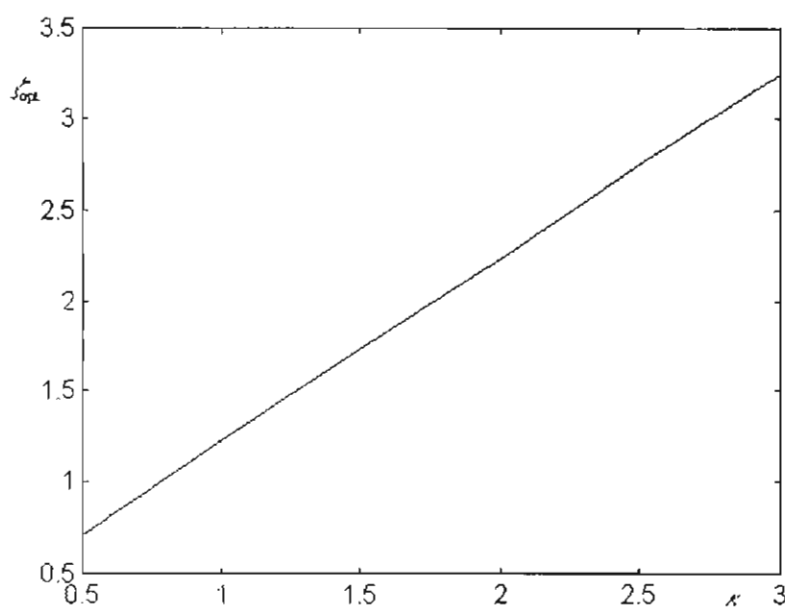


Figure 7 Dependence of the optimal normalized damping in the bearings on the stiffness constants ratio based on the criteria of minimum amplitude of the force transmitted to the ground

The maximum amplitudes of the force transmitted to the ground are given as function of normalized damping in bearings and the stiffness constants ratio of the shaft and bearings, Figure 6. The results of optimization based on the criterion of minimal maximum of amplitude of the force transmitted to ground are shown in Figure 7.

3. Bond-graph model of Laval rotor

The obtained results are further studied by computer simulation based on bond-graph modelling approach. Bond-graph model of isotropic Laval rotor is given in Figure 8. The disk inertia is represented by inertial process **I**, two springs corresponding to shaft and bearings stiffness by capacitive processes **C** and damping in the bearings by resistive process **R**. The force excitation is modelled using source **SE**. The physical interaction between processes are described by branches **e** (forces) and **f** (velocities). Joining bond lines assume power which transfers between processes. The disk displacement, bearings displacement and force transmitted to the ground are chosen for monitor display. Generation of bond-graph model as well as execution of simulation runs is conducted using modelling and simulation package BondSim developed by one of authors.

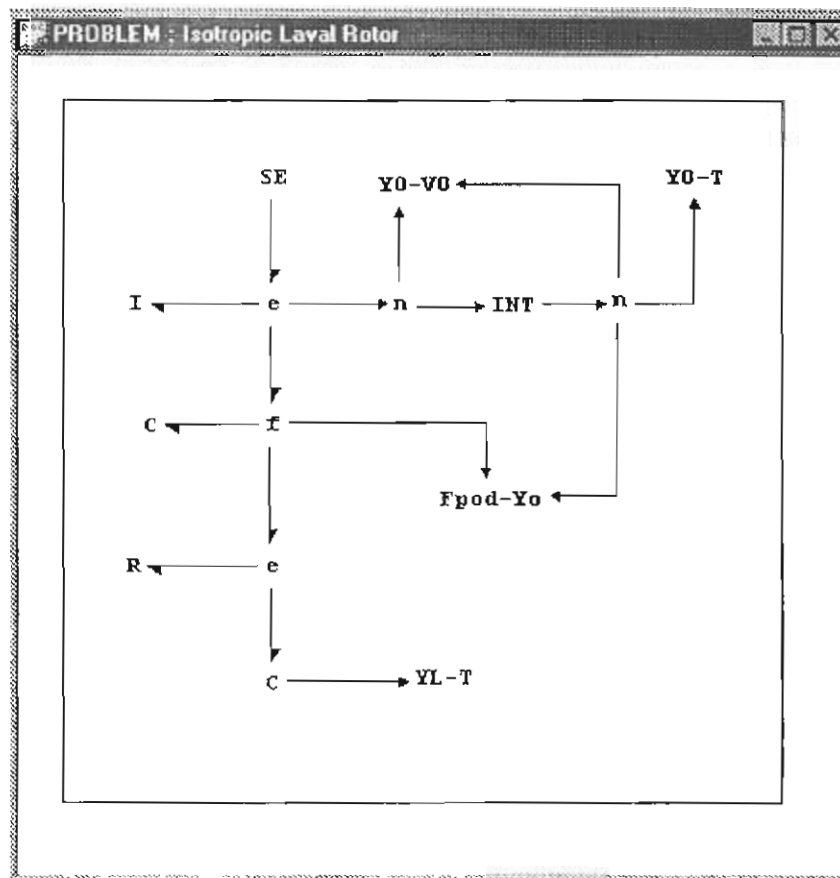


Figure 8 Bond-graph model of Laval rotor

Behaviour of system is studied for various combination of model parameters. In Figures 9, 10, 11 are shown some results corresponding to values of model parameters $k = 1$ and $\omega_0 = 1$. The damping ratio $\zeta = 1,3$ corresponds to optimized value (see Figure 7) and normalized excitation frequency $\eta = 0,78$ to resonant frequency (see Figure 4).

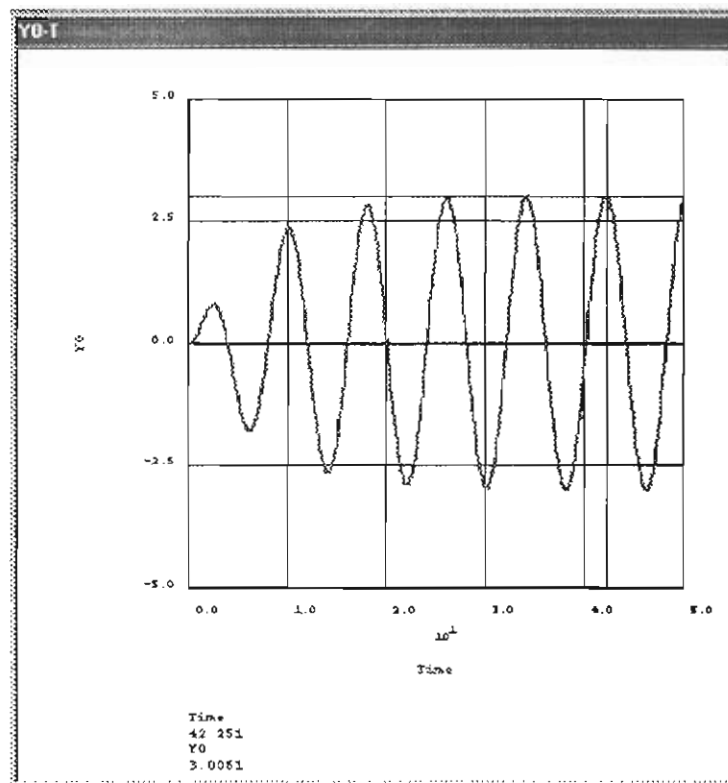


Figure 9 The time response of disk displacement to excitation due to disk mass unbalance

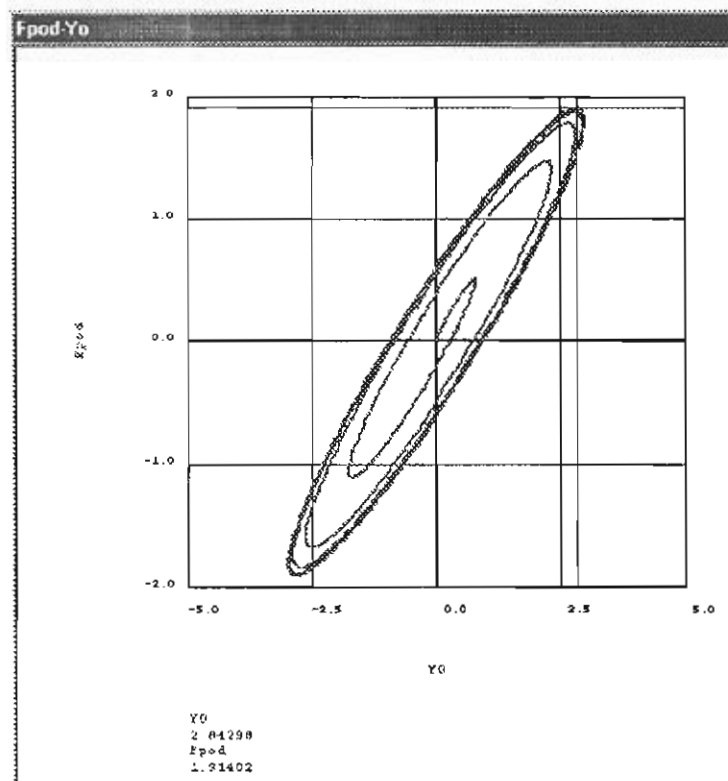


Figure 10 Trajectory of transmitted force to disk displacement

4. Conclusion

Approximate determination of optimal damping in the bearings of Laval rotor is based on the model with one-and-half-degree-of-freedom, Figure 2. The steady-state resonant amplitudes of forced vibrations due to centrifugal excitation depends on damping in the bearings and reach its minimal values for optimal damping. The optimal normalized damping depends on the stiffness constants ratio of shaft and bearings. Exact values of damping is difficult to realize in praxis. Dependence of optimal damping to the stiffness constants ratio is approximately linear (see Figure 7). Overall behaviour of complete assembly can be effectively studied using bond-graphs modelling approach and computer simulation.

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INVESTIGATIONS IN TORSIONAL-BENDING DYNAMIC ANALYSIS OF THE CONTAINERSHIPS

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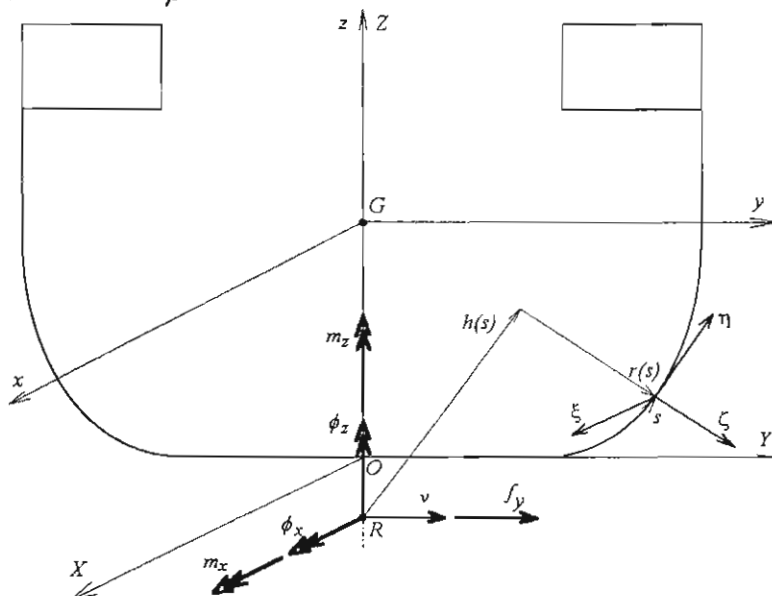
Abstract

The large decks openings of the containerships are reducing the lateral and torsional stiffness of the hull. So, wave-excited distortions occur because the resonance frequencies of antisymmetric responses are reduced from values that would be found in the general cargo ships. In this paper, numerical and experimental investigations into the coupled lateral and torsional dynamic analysis of an wet containership model are described. The experimental results are compared with the results obtained with the methodology used in [1], where a thin-walled beam finite element is presented. The added masses of the water are computed with FEM using the strip theory.

The dynamic loads develop dynamic torsional moments, acting on the ship hull. For general dynamic response the ship is treated to be a thin-walled prismatic body. A mathematical model of a flexible ship hull must embody simplifications and assumptions about the structure-fluid interaction. The ship hull has symmetric cross section. For the containerships it is important to shape the warping boundary conditions that are chosen such as the numerically calculated for the warping function at the ends correspond to the experimentally observed values.

1. The finite element model

The current cross section of the element has 4 degrees of freedom (generalised displacements of the cross section): the displacement of the section, projected to the y axis, noted with v , the rotation of the section around the x axis, noted with ϕ_x , the rotation around the z axis, noted with ϕ_z , defined in the torsion centre of the section, R , and the warping function of the section, noted with β .



The projections to the coordinate axis of the absolute displacement of a current point P , has the components $\xi(x, s)$ on the normal to the section, $\eta(x, s)$ to the tangent on the section outline and $\zeta(x, s)$ to the normal of the section outline, as it is shown in fig.1.

These displacements may be written depending on the generalised displacements of the section:

$$\xi(x, s) = -\phi_z(x) y(s) - \beta(x) \omega(s) \quad (1)$$

$$\eta(x, s) = v(x) \frac{dy}{ds} + \phi_x(x) r(s) \quad (2)$$

$$\zeta(x, s) = v(x) \frac{dz}{ds} - \phi_x(x) h(s) \quad (3)$$

That is

$$\{v\} = [S(s)] \{u\} \quad (4)$$

where $\{v\}$ is the displacements vector of the point P

$$\{v\} = [\xi \quad \eta \quad \zeta]^T \quad (5)$$

$\{u\}$ is the cross section generalised displacements vector

$$\{u\} = [v \quad \phi_x \quad \phi_z \quad \beta]^T \quad (6)$$

and $[S(s)]$ is the matrix of the variation functions of the displacements of the point P , on the section.

The variation of the cross section generalised displacements, corresponding to the bending, taking into consideration the shearing in the xy plane, is determined from the equilibrium equations of these loading types.

It is noted with $\{D\}$ the vector of the generalised displacements of the thin-walled beam element, having the following form

$$\{D\} = [v_1 \quad \phi_{x1} \quad \phi_{z1} \quad \beta_1 \quad v_2 \quad \phi_{x2} \quad \phi_{z2} \quad \beta_2]^T \quad (7)$$

The solutions of the differential equations may be written in the matrix form as

$$\{u\} = [X(x)]\{D\} \quad (8)$$

where $[X(x)]$ is the matrix of the approach functions, along the x axis, of the generalised displacements of the current section.

The expression of the displacements of the points from the section is obtained by substituting the equation (8) into (4). Thus

$$\{v\} = [N(x,s)]\{D\} \quad (9)$$

where

$$[N(x,s)] = [S(s)][X(x)] \quad (10)$$

is the matrix of the approach functions of the displacements.

Based on the FEM theory the stiffness matrix is determined by the equation

$$[k] = \int_V [B]^T [C] [B] dV$$

where $[B]$ is the matrix of the approach functions of the strains and $[C]$ is the matrix of the elastic characteristics of the material.

2. The supplementary inertia of the fluid

The effect of the supplementary inertia of the fluid appears from the hydrodynamic forces. These forces act on the immersed hull as hydrodynamic pressures. Because the supplementary inertia is calculated by the finite element method, these pressures may be expressed in the form

$$p = [N]\{P\} \quad (11)$$

where $[N]$ is the matrix of the approach functions, and $\{P\}$ is the vector of the nodal pressions.

2.1 The supplementary inertia of the fluid at the lateral displacements

During the lateral horizontal vibration of the section, the point P has the acceleration

$$a_n = v \cos \alpha_n \quad (12)$$

In this case the "load" vector has the expression

$$\{Q\} = \rho v \int_C [N]^T \cos \alpha_n ds = \rho v \{Q_v\} \quad (13)$$

where

$$\{Q_v\} = \int_C [N]^T \cos \alpha_n ds \quad (14)$$

The projection of the hydrodynamic force on the direction of the moving is

$$q_p = p \cos \alpha_n \quad (15)$$

The hydrodynamic resulted force, acting on the all immersed outline of the cross section is

$$Q_p = 2 \int_C q_p ds = 2 \int_C p \cos \alpha_n ds \quad (16)$$

By substituting the pression in the above equation it may write

$$Q_p = 2 \int_C [N] \{P\}_V \cos \alpha_n ds = 2 \int_C [N] \cos \alpha_n ds \{P\}_V \quad (17)$$

The subscript **V** indicates the moving in the lateral displacement.

By substituting (14) in above expression, the hydrodynamic force expression is obtained

$$Q_p = 2 \{Q_V\}^T \{P\}_V \quad (18)$$

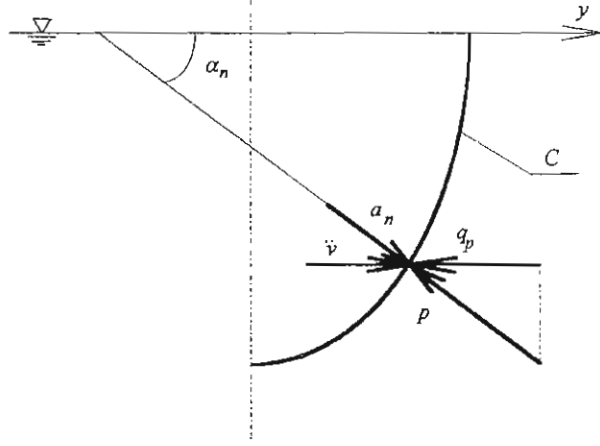


Fig. 2

After the solving of the equation system, so formed, the pression values in the nodal points are obtained as

$$\{P\}_V = \rho v [K]^{-1} \{Q_V\} \quad (19)$$

By substituting the equation (17) in (16), the hydrodynamic force expression is obtained

$$Q_p = 2 \rho v \{Q_V\}^T [K]^{-1} \{Q_V\} \quad (20)$$

This force, acting on the section at coordinate x , is introduced in the equation of dynamically equilibrium of the horizontal vibrations. In this

case, the distributed additional mass on the unit hull length is obtained

$$m_v = 2 \rho \{Q_V\}^T [K]^{-1} \{Q_V\} \quad (21)$$

2.2 The supplementary inertia of the fluid at the torsional rotation

During the lateral horizontal vibration of the section, the point P has the acceleration

$$a_n = -r_R \phi_x \sin \alpha_n \quad (22)$$

where, as it may be seen in fig. 1

$$r_R = \sqrt{r^2 + h^2} \quad (23)$$

The "load" vector has the expression

$$\{Q\} = -\rho \phi_x \int_C r_R [N]^T \sin \alpha_n ds \quad (24)$$

Because, in accordance with fig. 3

$$h = r_R \sin \alpha_n \quad (25)$$

noting

$$\{Q_X\} = -\int_C h(s) [N]^T ds \quad (26)$$

it may write

$$\{Q\} = \rho \phi_x \{Q_X\} \quad (27)$$

The subscript **X** indicates the moving around the x axis.

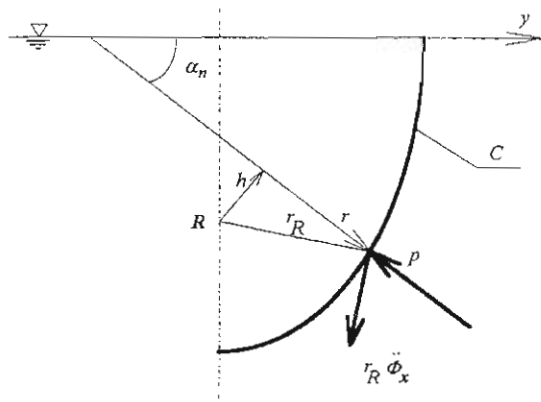


Fig. 3

The elementary moment of the force p reported to the torsion centre R is

$$m_R = -ph \quad (28)$$

In the zones where the ship hull is prismatic, and the normal to the hull side there is not in the plane of the cross section, the longitudinal projection of the hydrodynamic forces perturb the warping vibration of the section.

2.3 The supplementary inertia of the fluid at the warping of the sections

The warping of the section is

$$\xi = -y\phi_z - \omega\beta \quad (29)$$

where ϕ_z is the rotation angle of the section due to the bending of the hull between the vertical axis z ; β is the warping function of the section and ω is the area coordinate of the point P .

During the warping vibration of the section, the point P has the acceleration

$$\ddot{\xi} = -y\ddot{\phi}_z - \omega\ddot{\beta} \quad (30)$$

The normal to the hull surface as being it is considered in the plane yOx (fig.4). In this case, the acceleration along this direction has the expression

$$a_n = \ddot{\xi} \cos \alpha \quad (31)$$

where α is shown in fig.4, or

$$a_n = -y(s)\ddot{\phi}_z \cos \alpha - \omega(s)\ddot{\beta} \cos \alpha \quad (32)$$

The determination of the angle α may be done by adopting a mean value along the thin walled beam element, based on the points which defined its end cross sections.

$$\operatorname{tg} \alpha' = \frac{y_{i+1}(s) - y_i(s)}{l_i} \quad (33)$$

$$\alpha(s) = \frac{\pi}{2} - \arctg \alpha' \quad (34)$$

The load vector has the following expression

$$\{Q\} = -\rho\ddot{\phi}_z \int_C y(s)[N]^T \cos \alpha(s) ds - \rho\ddot{\beta} \int_C \omega(s)[N]^T \cos \alpha ds \quad (35)$$

where C is the outline of the section at the coordinate x .

Noting

$$\{Q_\phi\} = -\int_C y(s)[N]^T \cos \alpha(s) ds \quad (36)$$

$$\{Q_\beta\} = -\int_C \omega(s)[N]^T \cos \alpha(s) ds \quad (37)$$

the equation (35) may be written as

$$\{Q\} = \rho \ddot{\phi}_z \{Q_\phi\} + \rho \ddot{\beta} \{Q_\beta\} \quad (38)$$

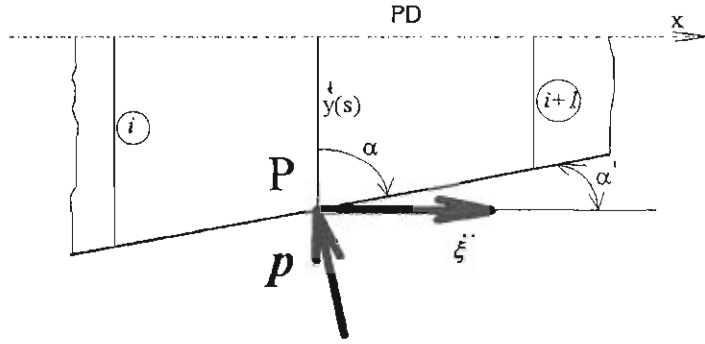


Fig.4

The bimoment produced by the projection to the longitudinal direction of the elementary hydrodynamic force p_i is

$$b_\omega = -\omega(s) p(s) \cos \alpha \quad (39)$$

The resulted bimoment of the all hydrodynamic forces in the cross section has the

expression

$$B_\omega(x) = - \int_C \omega(s) p(s) \cos \alpha(s) ds \quad (40)$$

By substituting the equation (4) in (40) it is obtained

$$B_\omega(x) = - \int_C \omega(s) [N] \cos \alpha(s) ds \{P\} \quad (41)$$

Based on the equation (17) it may write

$$B_\omega(x) = - \{Q_\beta\}^T \{P\} \quad (42)$$

After the substitution, the equation (38) in (6) and solving the equation so formed, it is obtained

$$\{P\} = \rho \ddot{\phi}_z [K_f]^{-1} \{Q_\phi\} + \rho \ddot{\beta} [K_f]^{-1} \{Q_\beta\} \quad (43)$$

or

$$\{P\} = \rho \ddot{\phi}_z \{P_\phi\} + \rho \ddot{\beta} \{P_\beta\} \quad (44)$$

By substituting (44) in (42) the bimoment equation produced by the hydrodynamic forces in the cross section is obtained

$$B_\omega = \rho (\ddot{\phi}_z \{Q_\beta\}^T \{P_\phi\} + \ddot{\beta} \{Q_\beta\}^T \{P_\beta\}) \quad (45)$$

During the warping of the section, the projection in the longitudinal direction of the elementary hydrodynamic force produces an elementary bending moment along the axis z

$$m_z = -y(s) p(s) \cos \alpha \quad (46)$$

The resulted moment is

$$M_z = \int_C m_z ds = - \int_C y(s) p(s) \cos \alpha ds \quad (47)$$

By substituting $p(s)$ from (11) in (47), it is obtained

$$M_z = - \int_C y(s) \cos \alpha [N] ds \{P\} \quad (48)$$

Based on equation (36), the moment M_z may be written as

$$M_z = \{Q_\phi\}^T \{P\} \quad (49)$$

By substituting (44) into (49) it is obtained

$$M_z = \rho (\ddot{\phi}_z \{Q_\phi\}^T \{P_\phi\} + \ddot{\beta} \{Q_\phi\}^T \{P_\beta\}) \quad (50)$$

The bimoment from the equation (45) and the moment from the equation (50) must be introduced in the equation of dynamic equilibrium of the hull element. An equation system, inertial coupled, due to the added mass matrix is obtained. The added mass matrix has the following form

$$[m_a] = \begin{bmatrix} m_{\phi\phi} & m_{\phi\beta} \\ m_{\beta\phi} & m_{\beta\beta} \end{bmatrix} \quad (51)$$

where

$$\begin{aligned} m_{\phi\phi} &= \rho \{Q_\phi\}^T \{P_\phi\}; & m_{\phi\beta} &= \rho \{Q_\phi\}^T \{P_\beta\}; \\ m_{\beta\phi} &= \rho \{Q_\beta\}^T \{P_\phi\}; & m_{\beta\beta} &= \rho \{Q_\beta\}^T \{P_\beta\}. \end{aligned} \quad (52)$$

3. Numerical and experimental results

By coupling the fluid effect on the warping, torsion and lateral displacement of the section during the hull vibration, the matrix of inertia of the element is made.

The equation of the vibration of the element is

$$([M_h] + [M_w])\{\ddot{\Delta}\} + [K]\{\Delta\} = \{0\} \quad (53)$$

The influence of the hydrodynamic forces is taken into consideration by the "added mass" matrix of the water, $[M_w]$. For the usual ships, the natural frequencies, associated to the elastic deformations of the hull, are so high so that hydrodynamic damping corresponding to the sway and roll of the ship cross sections may be neglected in comparison to structural damping.

For the structural vibration, the terms of the hydrostatic stiffness, associate to torsion and bending movies may be neglected in comparison to structural damping. So, only the hydro-dynamic inertia terms are taken into account.

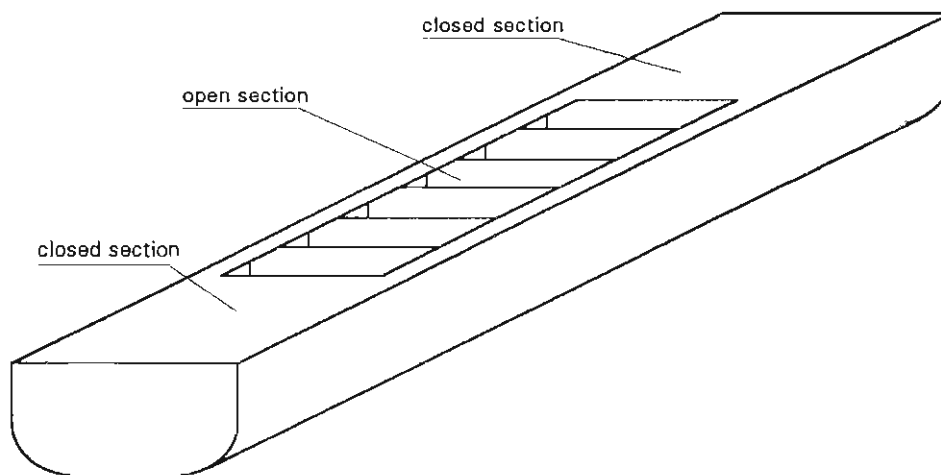


Fig. 5

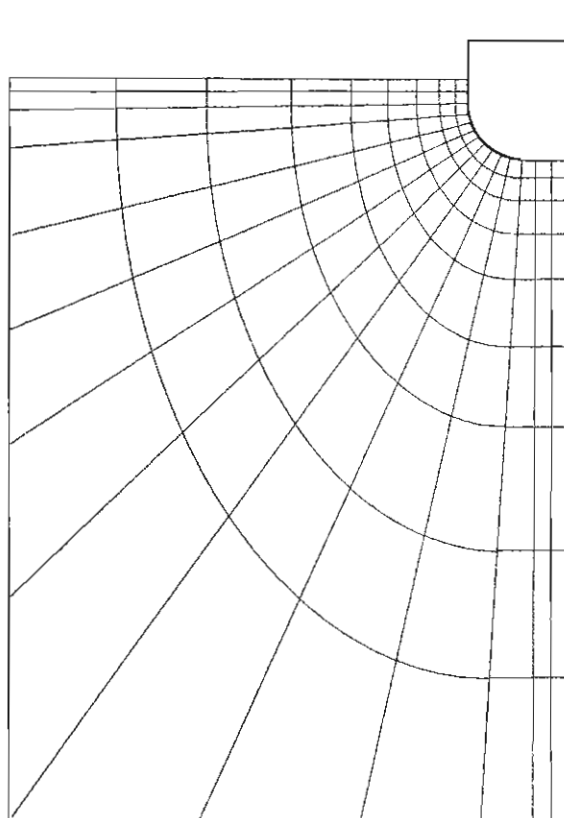


Fig. 6

The movies of the fluid particles, produced by the hull vibration are small and fast. The fluid is considered as incompressible. In the dynamic equations of the fluid the convective terms are neglected. In these hypothesis the dynamic pressure, p , is determined by the FEM.

Also, the influence of the hydrodynamic forces is considered. The abrupt changes in cross-sectional properties which occur at the transitions between open and closed parts of the ship and where deck beams are situated, are introduced as discontinuity conditions. These discontinuity conditions introduce strong coupling between torsion and bending. The simple description of the container ship structure was made.

The model is a simple box girder, round form, having thin-walled closed sections at the ends and open section on the middle. The deck structure was made with an opening which is limited at the sides by narrow longitudinal deck strips.

The model is steel made, having the hull thickness of 2 mm, and the deck shell of 3 mm.

The fluid domain around a cross section is discretised by finite elements (fig.6).

The natural frequencies of the model were experimentally determined by impact method. To avoid the distortions produced by the occurred local vibrations, the impact points were chosen at the intersection between the side and the deck.

The results obtained by the desk top computer program TORS for the coupling conditions and with the concentrated inertia of the transverse bulkheads inertia were nearest the experimental results.

For these cases the first three natural frequencies are the following:

$$f_1 = 138.9 \text{ Hz}, f_2 = 404.3 \text{ Hz}, f_3 = 596.2 \text{ Hz}$$

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DYNAMIC INTERACTION OF LIQUID-SOLID COUPLED SYSTEM INCLUDING WATER PRESSURE IN CRACKS

Summary:

Numerical model for simulation the dynamic interaction of liquid-solid coupled system, which includes the liquid pressure in cracks, is briefly presented. The applicability of exposed model and developed software is illustrated on analysis of Koyna dam.

1. Introduction

The numerical simulation of the dynamic interaction of the liquid-solid system represents one of the most complex tasks of numerical analyses. In other words, in such coupled problems the behavior of one field is under a continuous influence of the other fields; hence, in order to ensure precise analyses these fields cannot be considered independently and their interaction should be taken into account. Modeling the interaction of the liquid-solid system represents a complex nonlinear task even when the linear material models and the geometry model are applied to a single medium. The numerical analysis becomes more complex with the simulation of material and geometry nonlinearity of each medium.

Papers [1-5] present the development of a nonlinear model for the numerical simulation of the dynamic liquid-solid interaction for planar structures. The solution of the interaction problem is based on the so-called method of single solutions. Therein, within the nonlinear iterative procedure, the fields are analysed separately, so that each iterative step includes the influence of the interaction forces upon the joining surfaces. This approach enables the use of ordinary models and the respective computational programs for the analysis of separate media, with a minor expansion for the introduction of the influence of field interaction. It is also possible to simulate the most important non-linear effects of each medium. Some possibilities and basic data of the developed numerical model and the respective computational program are stated, as follows:

- Displacement formulation for the solid and the pressures or potential formulation for the liquid.
- Linear and non-linear (large displacements) models for the solid geometry, and the linear model for the liquid geometry.
- Classical elastic, elasto-plastic and elasto-viscoplastic material solid model.
- Special material model for reinforced concrete which includes:
 - simulation of the cracks opening in tension and crack closing in compression (model of distributed cracks),
 - concrete yielding in compression,
 - tensile and shear stiffness of cracked concrete,
 - non-linear behaviour of reinforcement in tension and compression,
 - influence of the strain rate upon the behaviour of steel and concrete.
- Linear liquid model and non-linear model which simulates the occurrence of cavitation in liquids.
- Simulation of the structural and radiation damping of the solid and radiation damping of the liquid.
- Explicit, implicit and explicit-implicit time integration.
- Solution of the problem of separated and coupled fields.
- Arbitrary external forces.
- Diagonal and consistent mass matrix.

- Eight-node isoparametric elements for the solid and liquid.
- Simple input of a relatively small number of input data.
- Numerical and graphical presentation of input and output data (geometry, displacement, pressures, cracks, rates, accelerations etc.).

This paper briefly presents further improvement of the described model with the possibility of simulating the effects of the water pressure upon the cracks in the solid, which has been dealt with in detail in the previous paper /6/. A short presentation of the considered physical phenomenon is given in Section 2, and the model of its simulation in Section 3. The application of the improved model and the computation program together with the analysis of the effects of water pressure in the solid cracks are illustrated by the example of the numerical analysis of the Koyna dam in Section 4.

2. Brief description of the problem

The appearance of cracks along the surface of solid which is in contact with the liquid brings about the penetration of the liquid into the open cracks (Figure 1). The depth of the penetration is influenced by the limited width of the crack, surface tension within the crack plane, viscosity of the liquid and other influences. The hydrostatic and hydrodynamic pressure of the liquid, under dynamic excitation, act upon the crack sides and push them apart, i.e. they result in further opening of cracks. On the other hand, the pressure in the crack can counteract the tendency of its closing, which occurs under certain conditions of external dynamic action and relative solid displacement.

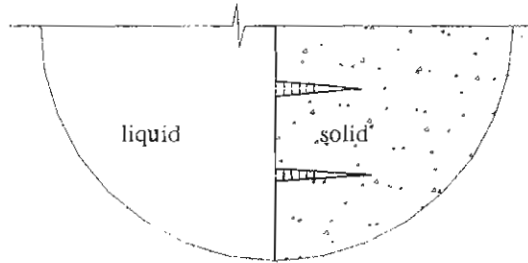


Figure 1 - Scheme representing the penetration of the liquid into the open cracks of the solid

When the degree of solid cracking is high, the crack can run along the entire structure. The distribution of total liquid pressures along the crack depth in this case represents another unknown.

Evidently, the penetration of the liquid into the open cracks in the solid influences the behaviour of the liquid-solid system under dynamic conditions.

3. Brief description of the developed model

The behavior of the fluid solid system (where the term fluid includes liquids and the term solid implies the considered section of the foundation soil), under dynamic loading, can be described by a system of two differential second-order equations. If the formulations for the structure displacement ($\mathbf{u}$) and for the fluid displacement potential (Ψ) are used, then the dynamic equilibrium equations have the following form

$$\begin{aligned} \mathbf{M}_s \ddot{\mathbf{u}} + \mathbf{C}_s \dot{\mathbf{u}} + \mathbf{K}_s \mathbf{u} &= -\mathbf{M}_s \ddot{\mathbf{d}} + \mathbf{f}_s + \mathbf{f}_{cs} \\ \mathbf{M}_f \ddot{\Psi} + \mathbf{C}_f \dot{\Psi} + \mathbf{K}_f \Psi &= \mathbf{f}_f + \mathbf{f}_{cf} \end{aligned} \quad (1)$$

where

$$\begin{aligned} \mathbf{f}_{cs} &= -\mathbf{Q} \ddot{\Psi} \\ \mathbf{f}_{cf} &= -\rho_f \mathbf{Q}^T (\mathbf{u} + \mathbf{d}) \end{aligned} \quad (2)$$

In the above equations $\mathbf{M}_s, \mathbf{C}_s, \mathbf{K}_s$ represent mass, damping and stiffness matrices for the solid, while $\mathbf{M}_f, \mathbf{C}_f, \mathbf{K}_f$ represent the matrices for the fluid; $\ddot{\mathbf{u}}, \dot{\mathbf{u}}, \ddot{\Psi}, \dot{\Psi}$ represent time derivations for the solid displacement, i.e. the fluid displacement potential; $\mathbf{d}, \ddot{\mathbf{d}}$ are the displacement vector and base acceleration, while $\mathbf{f}_f$ is the vector of nodal forces upon the fluid; $\mathbf{f}_{cs}$ represents the interaction forces vector upon the solid, and $\mathbf{f}_{cf}$ upon the fluid, while $\mathbf{Q}$ is the matrix of the fluid-solid interaction. The formation of these matrices and vectors can be found in /1/. Subsequently, the introduction of the fluid pressure in the solid cracks will be described briefly, while further details of this model can be found in /6/.

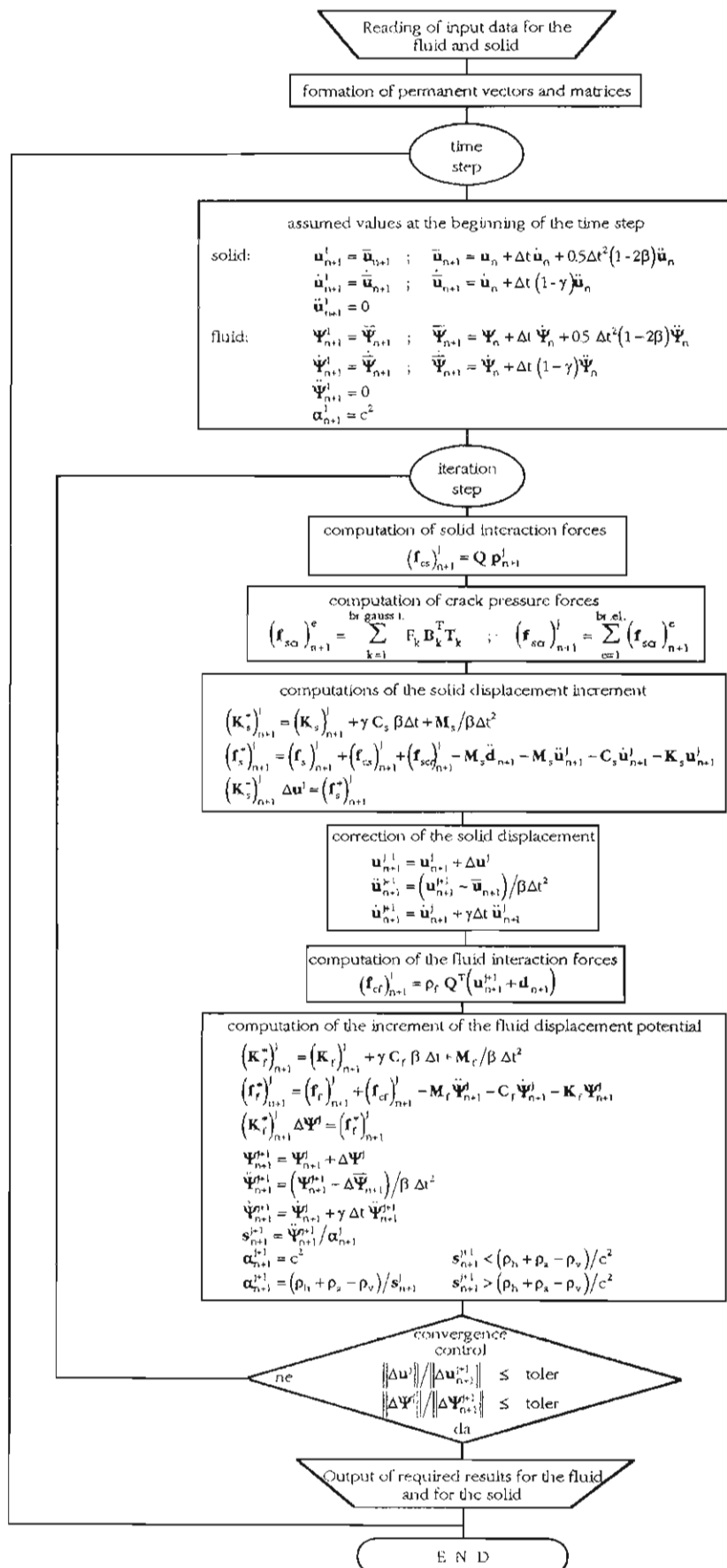


Figure 3 - Flow chart for solving the problem of the fluid-solid interaction including the pressures in the solid cracks.

The used model of distributed cracks implies the "distribution" of cracks across the considered representative area and does not model the discontinuity of the displacement. The section of the finite element represented by the Gaussian integration point, if the criterion of the cracks occurrence is satisfied, is assumed to have a series of parallel cracks, placed at short intervals (Figure 2a). It is assumed that forces between adjacent cracks counteract each other, so that it can be assumed that there exists only one crack passing through that point (Figure 2b).

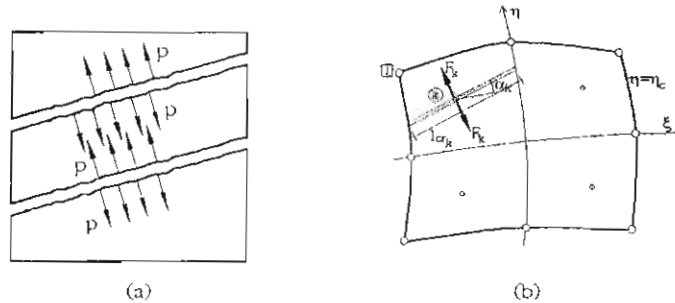


Figure 2 - Equivalent forces caused by water pressure in the cracks

The magnitude of the resultant force F_k , resulting from the action of the fluid pressure in the considered integration point, can be determined by means of

$$F_k = b \int_{l_{cr}} p_k dl_{cr} \quad (3)$$

where b is the thickness of the element, l_{cr} the crack length and p_k total pressure (hydrostatic + hydrodynamic) in the fluid at the level of the considered integration point k .

According to forces F_k in each integration point, it is possible to compute the vector of the nodal element forces by

$$(\mathbf{f}_{scr})_l^e = \sum_{k=1}^m F_k \mathbf{B}_k^T \mathbf{T}_k \quad ; \quad \mathbf{T}_k^T = [\cos^2 \alpha_k, \sin^2 \alpha_k, -\sin \alpha_k \cos \alpha_k] \quad (4)$$

where $\mathbf{B}_k^T$ is the matrix of the shape functions derivations, $\mathbf{T}$ is the rotation matrix and α_k the slope of the crack plane in the considered integration point, while m is the number of integration points on the element.

According to the nodal element forces $(\mathbf{f}_{scr})_l^e$, it is possible to compute the vector of nodal forces of the entire structure resulting from the action of the fluid in the cracks $\mathbf{f}_{scr}$. Therein, it is necessary to take into account the real possibility of the fluid penetration into each integration point of each element.

After computing the vector of additional nodal forces upon the structure resulting from the fluid pressure in cracks $\mathbf{f}_{scr}$, the procedure is analogous to the previously developed model /1-5/, so that in each iterative step j of each time increment n , the vector $(\mathbf{f}_{scr})_{n+1}^j$ is added to the vector of effective nodal forces of the structure $(\mathbf{f}_s^*)_{n+1}^j$ - see flow chart for solving the problem in Figure 3.

4. The analysis of the Koyna dam

The application of the developed numerical model and the respective computational program are illustrated by the Koyna dam. The analysis includes the contribution of the water pressure effects in the dam cracks to the dynamic response of the fluid-dam-soil system.

The problems of the Koyna dam have been dealt with in literature (c.g. /1/) and will not be considered in this paper. The cross-section of the considered non-spillway section of this gravity dam is presented in Figure 4, and its spatial discretization in Figure 5.

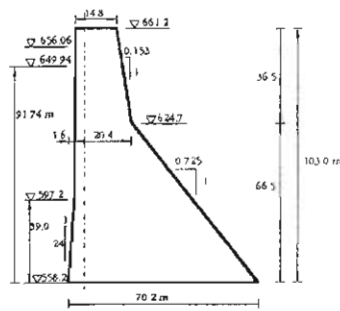


Figure 4 -
Characteristic non-spillway
cross-section of
the Koyna dam

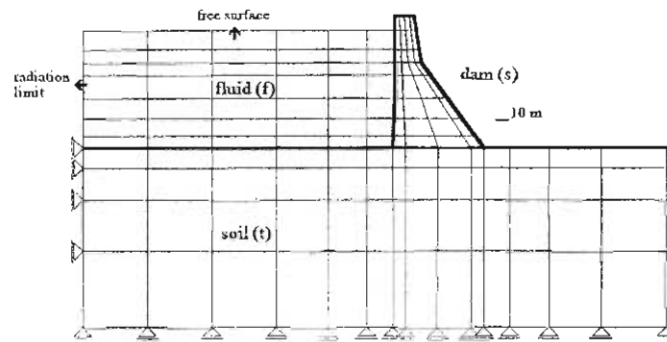


Figure 5 - Spatial discretization of the considered dam
cross-section

The dam is exposed to the acceleration of the foundation according to the recorded accelerogram since its damage by earthquake [1]. The linear geometry model and the non-linear material models for the fluid (including cavitation) and dam/soil (special model for reinforced concrete were used). The computation implied 5% of structure damping, implicit time integration with a time increment of $t=0.005$ s, diagonal mass matrix and 2x2 numerical integration. The fluid-dam-soil system was analysed separately first by taking and then by not taking into account the effects of the water pressure in the dam cracks.

Graphic interpretation of some of the obtained results is presented in Figures 6-9. Figure 6 presents the horizontal displacement of the dam top in time.

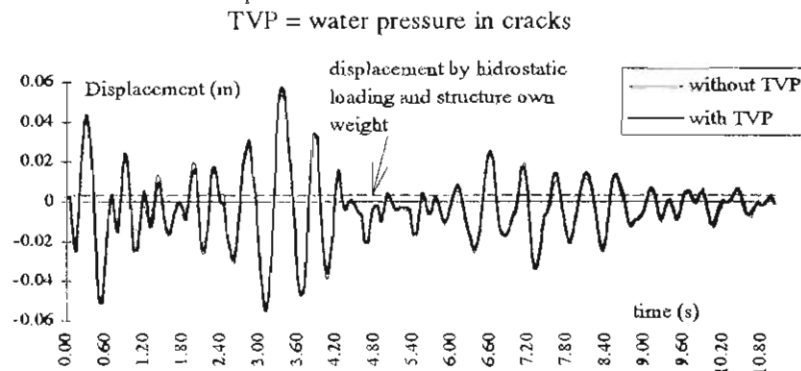


Figure 6 - Horizontal displacement of the dam top in time

The resulting hydrodynamic force acting upon the dam is presented in Figure 7 and the hydrodynamic pressure near the dam top in Figure 8. Figure 9 presents the final state of cracks.

According to the obtained results it can be concluded that the influence of the water pressure in the dam cracks has not been significant in this case, as could be expected.

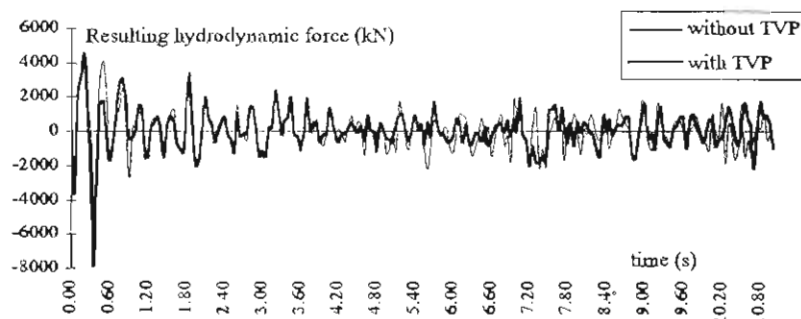


Figure 7 - Resulting hydrodynamic force acting upon the dam

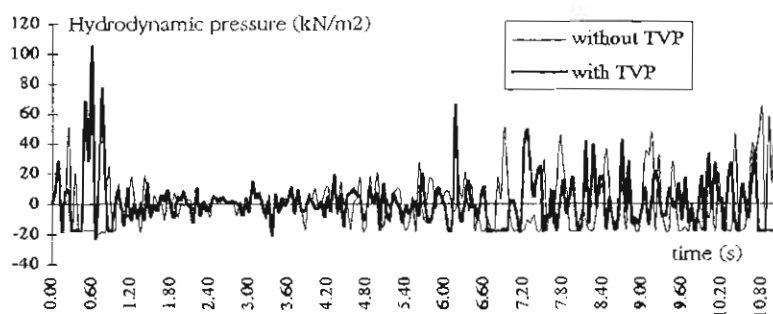
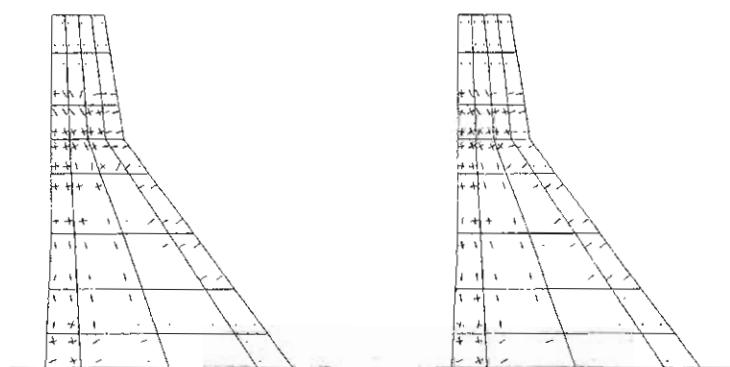


Figure 8 - Hydrodynamic pressure near the dam top



a) Without water influence in cracks

b) With water pressure in cracks

Figure 9 - Final state of cracks

5. Conclusion

The presented numerical modes and the respective computation program make possible a complex nonlinear analysis of planar fluid-solid system under dynamic conditions. The model is based upon basic material parameters for each medium and is primarily intended for application in engineering practice. It requires a small number of input parameters while the great number of output data, obtained by a reliable nonlinear analysis, can be presented in graphical and numerical forms. The model application is particularly useful in the design and computation of gravity dams, off shore structures, reservoirs, tunnels, supporting walls and similar 2D structures.

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BASE BLEED MODEL

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The approach to the modeling of the base bleed is experimental as it is in the references [4], [5], [6] and others, with exception in reference [3]. We need more accurate theoretical model to control the base bleed on the ground in the static conditions then in reference [3]. For the model of the base bleed in this paper, the initial velocity of the gas in the chamber of the base bleed and the jet contraction parameter are introduced. The jet contraction parameter is fitted to have the same total impulse calculated and measure on the ground.

Notations

A_c	surface of the burning	S_{ref}	surface of the reference
$C_{pb} = \frac{p_b - p_\infty}{q_\infty}$	pressure coefficient on the base	T	temperature
e	web	U	velocity of the burning
g	gravity	V	velocity of the of the gas flow in the base bleed
I	injection parameter	V_∞	aerodynamic velocity
m	mass of the solid propellant	ω	angular velocity of the projectile
Ma	Mach number	ρ	density
p	pressure	$\gamma = \frac{C_p}{C_v}$	
p_0	pressure when the velocity of the burning is U_0	“C”	state of the combustion
$q_\infty = \frac{\rho_\infty V_\infty^2}{2}$		“j”	state in the exit
R	gas constant per unit mass	“b”	base of the projectile
		“ ∞ ”	freestream
		“S”	solid

1. Introduction

There are many paper about the base bleed. Most of them are related to the decreasing of the base drag by the base bleed effect. Only in the reference [3] the ratio of the mass flow is given

$$\dot{m} = K \sqrt{\frac{\gamma}{\gamma-1} \left[\left(\frac{p_c}{p_b} \right)^{(\gamma-1)/\gamma} - 1 \right]}$$

based on the equation of the expanded gas from the base bleed chamber. In this equation it is assumed that the gas has not the velocity in the base bleed chamber. We shall not assume the velocity in the base bleed chamber to be equal zero. Also we shall introduce the reactive force of the base bleed.

2. The equations of base bleed

To find the injection parameter “I” during the flight, we need mass flow creating by burning

$$-\dot{m} = U_c \rho_s A_c \quad (1)$$

The burning mass on the surface of the burning $A_c U_c \rho_s$ and the mass of the developed gas on the same surface $A_c V_c \rho_c$ in the base bleed chamber are equal. From this equality we get the initial velocity of the gas

$$V_c = U_c \frac{\rho_s}{\rho_c} \quad (2)$$

So we know three variables from this state “C” in the base bleed chamber:

- the surface as a geometrical function of the web $A_c(e)$
- the velocity of the gas V_c from this surface,
- the temperature of the gas T_c on this surface as the temperature of the combustion.

We need two more variables: p_c and ρ_c . The equation of the state,

$$\rho_c = \frac{p_c}{RT_c} \quad (3)$$

in which the gas constant R of the burning product is known, and gives us one relation between these two variables.

To determine the exit velocity, we need variables of the state “j” in the exit orifice. Between state “C” and “j” we have: Bernoulli’s equation, adiabatic equation and the continuity equation

$$\frac{V_c^2}{2} + \frac{\gamma}{\gamma-1} \cdot \frac{p_c}{\rho_c} = \frac{V_j^2}{2} + \frac{\gamma}{\gamma-1} \cdot \frac{p_j}{\rho_j} \quad (4)$$

$$\frac{p_c}{p_j} = \left(\frac{\rho_c}{\rho_j} \right)^\gamma \quad (5)$$

$$V_c \rho_c A_c = V_j \rho_j A_j \quad (6)$$

We have also the equation of the state at the exit orifice

$$T_j = \frac{p_j}{R \rho_j} \quad (7)$$

These five equations from (3) to (7) relate five variables of the state "C" : $A_c, V_c, T_c, p_c, \rho_c$ and five variables of the state "j" : $A_j, V_j, T_j, p_j, \rho_j$. Three variables of the state "C" are known A_c, V_c, T_c , and also two of the state "j": A_j, p_j . The variable A_j is a constructive characteristics, and we accept the pressure at the exit orifice to be equal to the pressure on the base of the projectile ($p_j = p_b$).

$$p_b = p_\infty + q_\infty C_{pb}(Ma, T_j, I) \quad (8)$$

The base pressure coefficient C_{pb} is given in references [2] and [3] as a function of: Mach number $Ma = V_\infty / a_\infty$, temperature of the injections gas T_j and the parameter of injection

$$I = \frac{\dot{m}}{V_\infty \rho_\infty S_{ref}} \quad (9)$$

Finely, we have five equations to find five variables p_c, ρ_c, T_j, V_j and ρ_j

In this base bleed model we take the velocity of the burning as a function of the pressure and the rotation as in reference [3]

$$U_c = U_0 \left(\frac{p_c}{p_0} \right)^n \left[1.0625 + 2.687 \left(\frac{\omega}{6283} \right) + 3.58 \left(\frac{\omega}{6283} \right)^2 \right] \quad (10)$$

The velocity of the rotation is known along the trajectory, and in static condition $\omega = 0$. The pressure in the base bleed chamber p_c is given by this model. This model gives us also the temperature T_j of the injected gas, that we need in the equation (3).

3. Static experiment

In the static condition on the ground the base pressure p_b is equal to the environment pressure p_∞ and we have $V_\infty = 0$; $\omega = 0$. Therefor the base bleed model in static condition is

$$-\frac{dm}{dt} = U_c \rho_s A_c = V_j \rho_j A_j$$

$$V_c = U_c \frac{\rho_s}{\rho_c}$$

$$\frac{V_c^2}{2} + \frac{\gamma}{\gamma-1} \cdot \frac{p_c}{\rho_c} = \frac{V_j^2}{2} + \frac{\gamma}{\gamma-1} \cdot \frac{p_j}{\rho_j}$$

$$\frac{p_c}{p_j} = \left(\frac{\rho_0}{\rho_j} \right)^\gamma$$

$$V_c \rho_c A_s = V_j \rho_j A_j$$

$$\frac{de}{dt} = U_c$$

$$U_c = 1.0625 \cdot U_0 \left(\frac{p_c}{p_0} \right)^n$$

$$m = m(e)$$

We have to solve these nine equations to find nine values

$$U_c, V_c, p_c, \rho_c, V_j, \rho_j, T_j, e(t), m(t)$$

To solve this model we need the constants $\rho_s, p_0, U_0, n, R, T_c, A_j, p_j$ and the functions $A_c(e)$ and $m(e)$ which are the geometrical characteristics of the shape of the base bleed grain.

In our experiment on the ground, we measure the weight as a function of time $m(t)$ by the weight balance in two experiments. In the first, we have put the base bleed axis horizontal, and in the second experiment the base bleed axis was vertical. In the first experiment, the measurement of the weight was perpendicular to the base bleed axis, and we have record only the decreases due to the diminution of the mass (fig.1). So the measurement with the horizontal base bleed axis (fig.1) give us only mass $m(t)$ as a function of the time.

In the second experiment we have record the sum of the weight and the additional reactive force $- \dot{m} V_j$ as it follows from the theory "a control surface" in reference [1]. The measurement in the case with the vertical base bleed axis (fig.2) gives us the sum $\dot{m} V_j / g + m$ as a function of time.

On these two diagrams we have plotted also the same theoretical curves given by our model in the static conditions. The experimental and theoretical curve in the first experiment on the fig.1 are in very good agreement. It proves that the calculation of the

masse as a function of the time is correct in the model, and therefor the mass flow $\dot{m}(t)$ calculated is good. But the second experiment on the fig.2 (curve $c_v = 1$) shows clearly that the calculated reactive force of the base bleed is not good. As we can see at the beginning we have non steady flow and our model is not valid. After some second the

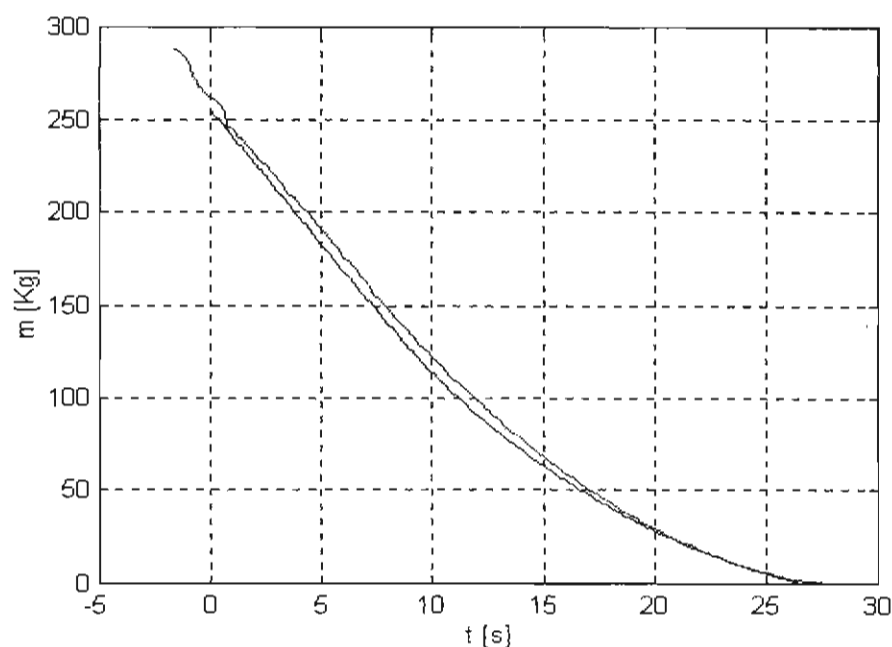


Fig.1 Weight measure as a function of time perpendicular on the base bleed axis

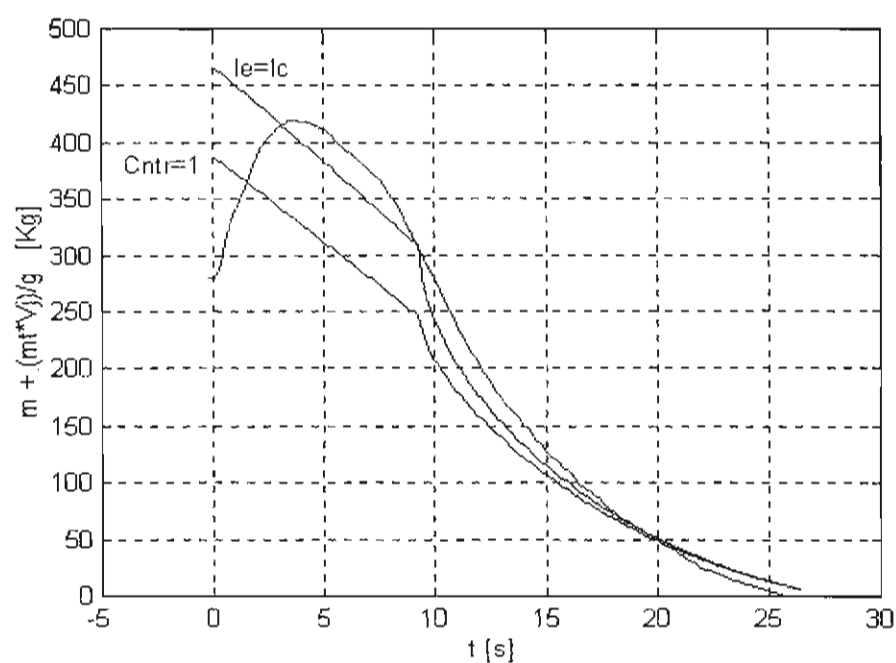


Fig.2 Weight measure as a function of time in the direction of the base bleed axis

steady state flow is realized and the model is correct. The expression of the reactive force $F^r = -\dot{m}V_j$ is exact, and we saw that the mass flow is correct, it follows that the exit velocity V_j in the model is not enough accurate. For the prediction of the range and for the fitting of the range, the influence of the total impulse $I_t = \int F^r dt$ is more important than the shape of the function $F^r(t)$. If we introduce the coefficient of the contraction of the exit jet $c_v \neq 1$

$$A_j = \left[1 - (1 - c_v) \frac{V_j}{V_{jmax}} \right] \frac{d_j^2 \pi}{4}$$

we can ameliorate the model. The total impulse from experiment 1 and 2 is

$$I_t = \int_0^{t_k} \dot{m} V_j dt = g \int_0^{t_k} m + \left(\frac{\dot{m} V_j}{g} \right) dt - g \int_0^{t_k} m dt = g(S_2 - S_1)$$

S_1 and S_2 are the area beneath the experimental curves in the first and second experiment on the ground. In our experiment, for $c_v = 0.63$ the calculated total impulse I_t is equal to the measure $g(S_2 - S_1)$. This value of the parameter will be used to fit the calculated range with the experimental range.

4. Conclusion

This is the proof that we have to take in account the reactive force $\dot{m}V_j$ created by the base bleed. The experiment on the ground in the static conditions gives possibilities to fit two parameters of the base bleed. From the first experiment we get the velocity of the burning U_0 , and from the second and the first we get the parameter c_v . This model of the base bleed, fitted on the ground, and the reduction of the base drag as in reference [3], have to be used with the modified point mass model for the prediction of the range and for fitting calculated range with the experimental.

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ANGULAR RATE GYROS AND MASS FLOW RATE SENSORS: A UNIFIED APPROACH

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ABSTRACT

In this paper a unified approach of using the Coriolis force on moving fluid for the purpose of the angular rate and the mass flow rate measurement is presented. The operation of the laminar jet angular rate sensor and the Coriolis mass flow rate meter has been analyzed. In both sensors the Coriolis force was used to generate the metering signal. As a result a new possibility for the angular rate measurements is foreseen. The new method employs an incompressible fluid circulation or oscillation in a closed loop. Due to the fluid motion in the presence of sensor rotation the Coriolis force is generated. The tube deflects under the action of the Coriolis force and its motion is used to produce the sensor output signal. An approximate theory was used to analyze the operation of the sensors.

1. INTRODUCTION

The spring restrain gyros with a constant spinning rotor and the fluid flow meters with turbine rotor for decades have been recognized as the most accurate and reliable measuring instruments for angular velocity and fluid flow rate measurements, respectively. However the presence of bearings makes these instruments unreliable on the long term or at least requires frequent calibration. In the mid of the 1970-s the new principles have been employed for angular rate measurements based on deflection of the stream lines. Two fluidic types of devices appeared known as the laminar angular rate sensor and the vortex angular rate sensor. The laminar angular rate sensors [1,2,3] use a laminar jet deflection under action of the Coriolis force. The vortex angular rate sensors [4,5] use a thin cylindrical chamber with the radial flow from the peripheral wall to the centrally located output. Through the uniformly spaced vanes at the peripheral wall the sensor housing imparts the input rotational motion of the fluid inside the chamber. The output from these devices was in the form of pressure difference or velocity difference. It was feasible to design a complete fluidic control system. Since the beginning of the 1980-s these fluidic sensors have been used in some military applications: combat vehicle and aircraft stabilization systems, ship rolling stabilization, anti-lock braking systems, missile stabilization system and others.

In a similar way a jet oscillation instead of a turbine rotation was introduced [6]. The frequency of jet oscillation was found to be proportional to the volume flow rate. The major break through the mass flow measurement occurred when the Coriolis force was applied to produce the output signal proportional to the mass flow rate [7,8,9,10,11,12]. Even though the principles have been known for decades [7] the application was rare because it required the rotating tube. The essential improvements occurred at the beginning of the 1980-s when a vibrating tube was introduced instead of the rotating tube [8].

In this paper a unified approach to the analysis of the fluidic jet angular rate sensor and the Coriolis mass flow rate sensors has been established. As a result a new possibility for the angular rate measurements is foreseen. The new principles use an incompressible fluid circulation or oscillation in a closed loop. Due to the fluid motion in the presence of sensor rotation the Coriolis force is generated. The tube deflects under the action of the Coriolis force and its motion is used to produce the sensor output signal. An approximate theory was employed to analyze the operation of the sensors.

2. FLUIDIC ANGULAR RATE SENSOR

The fluidic angular rate sensor produces the output signal proportional to the angular velocity of the sensor. In the laminar angular rate sensor either the two dimensional planar jet or the axially symmetric jet is used. The laminar flow is formed from the supply nozzle if the jet Reynolds number is sufficiently low (much lower than the pipe critical Reynolds number). If the nozzle aspect ratio is sufficiently large and the side boundaries are far away from the jet axis, the flow downstream of the nozzle exit plane is approximately two-dimensional as shown in Figure 1. The jet spreads outwards in the downstream direction while the centerline velocity decreases due to the action of viscosity on its periphery. At the distance L from the nozzle exit plane two Pitot pressure tubes are installed to measure the local stagnation pressures. Instead of the Pitot stagnation tubes the hot wire sensor could be used.

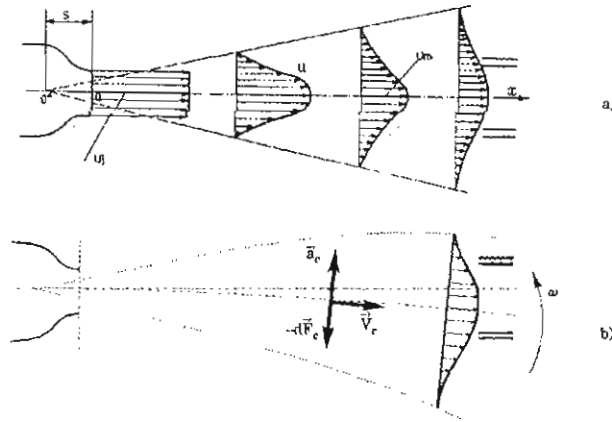


Figure 1. Velocity distribution in the laminar two-dimensional jet
a) Angular velocity $\omega = 0$, b) Angular velocity $\omega \neq 0$

In the case of no angular rotation the pressure difference between two Pitot tubes is equal to zero. If the jet rotates with an angular rotation $\vec{\omega}$ as shown in Figure 1.b, a fluid particles of mass dm that moves with the relative velocity $\vec{V}_r$ is exposed to the action of the Coriolis force $d\vec{F}_c = 2 \cdot dm \cdot \vec{\omega} \times \vec{V}_r$. The jet deflects and causes the pressure difference at two output Pitot tubes. The pressure difference is proportional to the angular rate ω in the limited range, depending on the geometric and fluid flow parameters. The basic design characteristics of the laminar angular rate sensor are shown in Figure 2. [2]. The laminar jet is coming out from the two dimensional nozzle into "V" shaped chamber. At the distance L the two receivers and the pressure output ports Po1 and Po2 are located. Three vent ports are provided to maintain a constant pressure in the chamber.

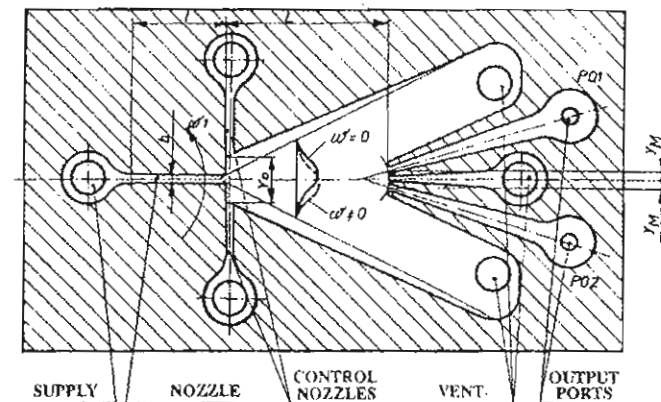


Figure 2. Laminar angular rate sensor (LARS)

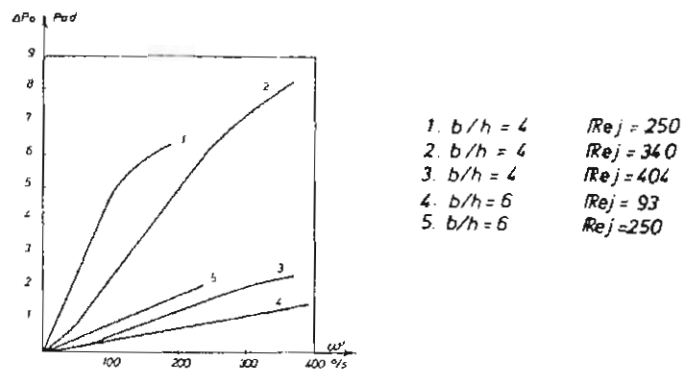


Figure 3. The experimental results: Pressure difference as a function of angular rate (the curves are drawn through the experimental results)

The experimental results presented in Figure 3. [2] show the relationships between the input angular rate and the output pressure difference for five jet Reynolds numbers and two nozzle aspect ratios. It is evident that in the range of interest (0 to 500 °) the input/output characteristics can be made linear.

3. CORIOLIS MASS FLOW RATE SENSORS

Rather an old application [8] used a rotating "T" tube to impart the inertial Coriolis force on the two legs of the tube.

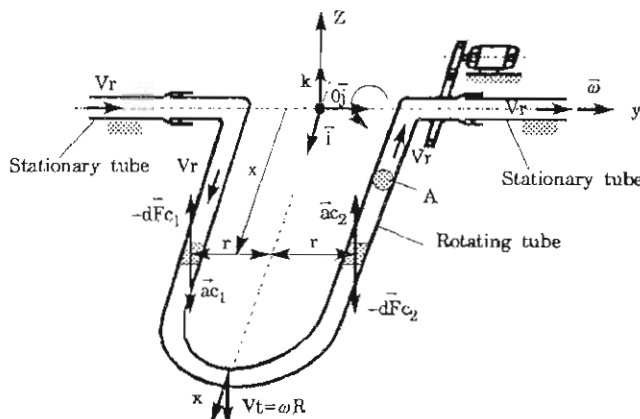


Figure 4. Rotating tube mass flow rate sensor

Similarly, the same principles can be applied to a rotating "U" tube (instead of rotating "T" tube) as shown in Figure 4. The xyz coordinate system is fixed to the rotating tube and rotates with a constant angular rate ω . If there is no flow through the tube ($V_r = 0$), neglecting the drag resistance due to a motion of the tube through the surrounding fluid, the "U" tube remains in the xy plane (see Figure 5.a). When there is a flow in the tube having a relative velocity V_r , the inertial Coriolis force is generated. Its direction is perpendicular to the xy plane. On the outward going fluid mass element dm the magnitude of the Coriolis force is as above, $dF_{c1} = 2 \cdot (dx) \cdot \omega \cdot (V_r \cdot A \cdot \rho)$. On the opposite side of the tube there is an inward going fluid mass element that imparts on the tube the force of the same magnitude $dF_{c2} = dF_{c1}$ but in opposite direction. At the same time the inertial Coriolis forces $d\bar{F}_{c1} = -d\bar{F}_{c2}$ are acting on the tube. The inertial forces $-d\bar{F}_{c1}$ and $-d\bar{F}_{c2}$ generate an elementary moment about x axis $d\bar{M}_x = -2 \cdot (dx) \cdot \omega \cdot (V_r \cdot A \cdot \rho) \cdot (2 \cdot r) \cdot \bar{i}$. The total moment $\bar{M}_x$ is obtained by integration, leading to the expression $\bar{M}_x = -4 \cdot M \cdot R' \cdot \omega \cdot r \cdot \bar{i}$. The reduced "U" tube radius R' takes into account a circular end of the tube and M represents the mass flow rate ($V_r \cdot A \cdot \rho$).

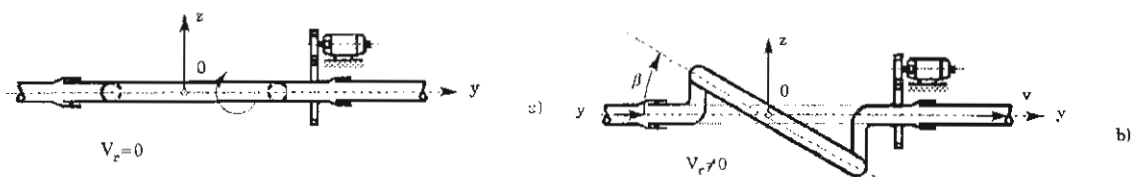


Figure 5. Twisting of the rotating tube sensor

The moment $\bar{M}_x$ causes an angular twisting of the "U" tube as shown in Figure 6.b. The deflection angle β depends on the moment magnitude $\bar{M}_x$, geometric and elastic characteristics of the tube. For small deflection angles, defined at some distance x , a resisting moment can be presented with $\bar{M}_e = C_e \cdot \beta$, where the stiffness coefficient C_e is a function of tube elastic and geometric properties. Because in the steady state equilibrium $\bar{M}_x = \bar{M}_e$, one concludes that the mass flow rate is proportional to the deflection angle.

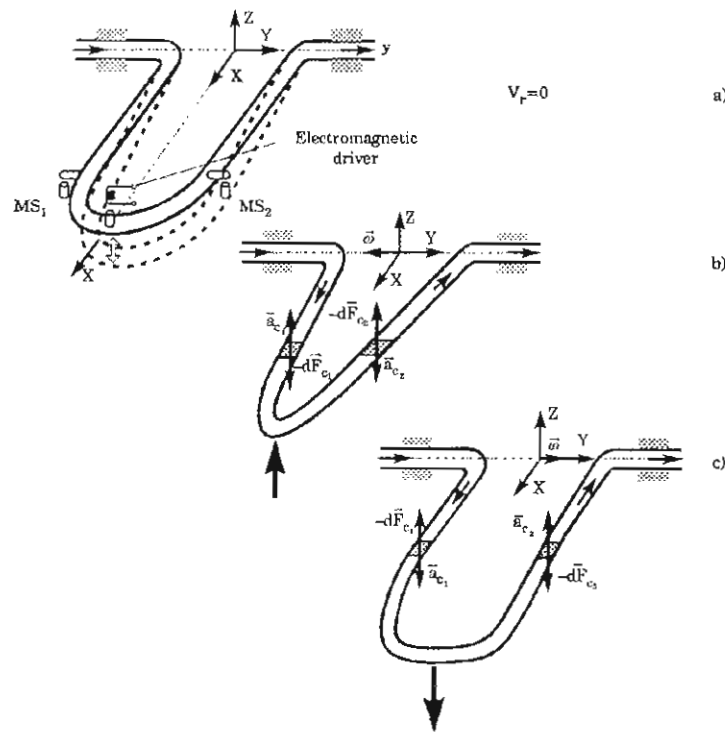


Figure 6. Vibrating tube circulatory mass flow sensor

a) no flow $V_r = 0$; b) $V_r \neq 0$, tube moves upwards; c) $V_r \neq 0$, tube moves downwards

In spite of the conceptual simplicity neither rotating "T" tube nor rotating "U" tube have found significant applications because of the difficulties in providing a constant angular rate for a large rotating mass (large flow rates) and relative displacement measurements.

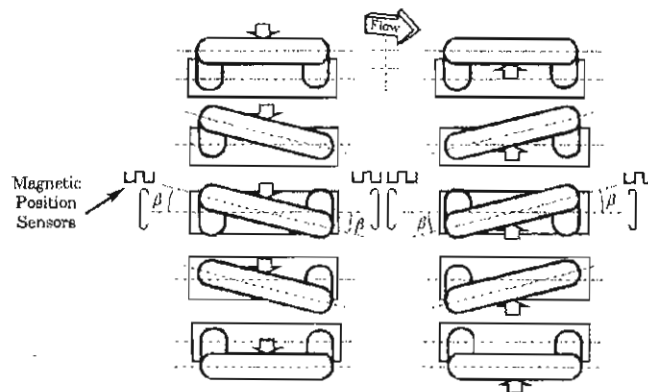


Figure 7. Complete cycle of sensor tube; measurement of twist angle β

The major progress in the application of the Coriolis force for a mass flow measurement occurred when the rotating tube was replaced with a vibrating tube [8,9]. Sultan and Hemp [9] modelled the meter by using the theory of vibrating beams. Here a lumped parameter approach is adopted. The principles are explained with the help of Figure 6. The tube is forced to vibrate using an electromagnetic driver located at the largest distance from the Y axis. Viewing from an inertial frame of reference XYZ it appears that the "U" tube rotates with a variable angular rate $\omega = \omega^0 \cdot \sin(\Omega \cdot t)$ about Y axis. With no flow each half of the "U" tube moves upward and downward simultaneously (see Figure 6.a.). With flow passing through the tube the variable Coriolis acceleration $a_c = 2 \cdot \omega \cdot V_r$ due to ω variation causes a variable Coriolis force. Figures 6.b. and 6.c. show that when the motion direction changes from upward to downward the direction of the Coriolis force changes too. The opposing forces on both tube legs create the bending moment about X axis that imparts a twisting oscillation superimposed on the vibration induced by the electromagnetic driver. The complete cycle of twisting during one period of forced oscillation is shown in Figure 7 [8]. The magnitude of the maximum twist angle is proportional to the

mass flow rate. Various manufacturers developed different techniques to detect the twist angle. The most frequent method uses measurement of the velocity time leg between two halves of the tubes. The time difference is proportional to the mass flow rate.

4. FLUID ANGULAR RATE SENSORS

In the Section 2. was shown how the effect of Coriolis force in a free jet flow field is used for measurement of the angular flow rate. In the Section 3. the action of the same force in a tube was used to generate the output signal proportional to the mass flow rate. By introducing tube oscillation instead of rotation the mass flowmeter has been designed. The Coriolis force is proportional to the (vector) product of the relative velocity V_r and the angular rate ω . By providing a constant velocity V_r the Coriolis force proportional to the angular rate ω can be imparted. Therefore, it should be feasible to design the fluid angular rate sensor (fluid rate gyro) based on fluid motion in a closed conduit. To generate the Coriolis force when the angular rate is sensed a mass motion must be provided. The two possibilities have been analyzed. The first one uses a constant mass flow rate circulation in a closed tube and the second one employs the oscillatory fluid flow in a closed tube.

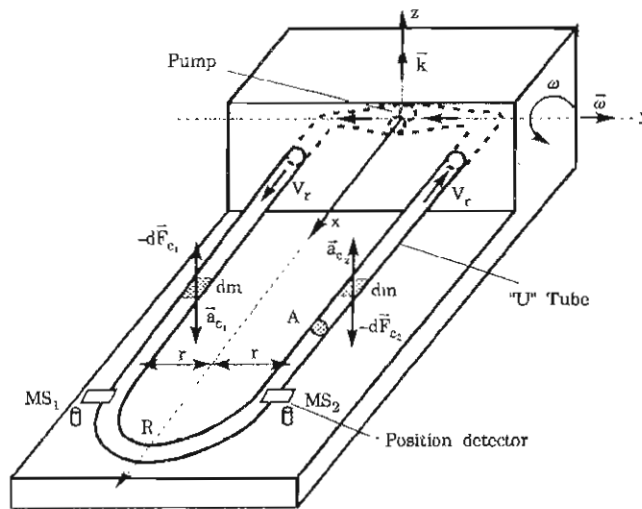


Figure 8. Circulating fluid angular rate sensor

The operational principles of the circulatory fluid angular rate sensor in a closed "U" tube are illustrated in Figure 8. The elastic "U" tube is connected to the circulatory pump driven by a constant *RPM* electric or mechanical motor. At the selected positions two motion-displacement detectors MS1 and MS2 are installed. The output signal from sensor is based on the signal difference from two motion detectors. The Coriolis acceleration of a fluid particle dm moving with a relative velocity $\vec{V}_r$, imparted by the pump, in a body (sensor) that rotates with the angular velocity $\vec{\omega}$ is defined by

$$\vec{a}_c = 2 \cdot \vec{\omega} \times \vec{V}_r \quad (01)$$

The definition (01) applied to the outward and inward moving fluid particles in a closed "U" tube gives

$$\vec{a}_{c1} = 2 \cdot \omega \cdot V_r \cdot (-\vec{k}) \quad (02)$$

$$\vec{a}_{c2} = 2 \cdot \omega \cdot V_r \cdot (+\vec{k}) \quad (03)$$

The corresponding inertial Coriolis forces on the fluid particles having mass $dm = A \cdot dx \cdot \rho$, where ρ stands for fluid density and A for a tube cross sectional area, are

$$d\vec{F}_{c1} = -d\vec{F}_{c1} = -dm \cdot \vec{a}_c = 2 \cdot (dx) \cdot \omega \cdot (V_r \cdot A \cdot \rho) \cdot (+\vec{k}) \quad (04)$$

$$d\vec{F}_{c2} = -d\vec{F}_{c2} = -dm \cdot \vec{a}_c = 2 \cdot (dx) \cdot \omega \cdot (V_r \cdot A \cdot \rho) \cdot (-\vec{k}) \quad (05)$$

Two forces given by the equations (04) and (05) develop a moment $dM_x = -4 \cdot \rho \cdot r \cdot V_r \cdot dx \cdot \omega \cdot \vec{i}$ about x axis. The total moment for the entire "U" tube is obtained by integration, leading to

$$\vec{M}_x = -4 \cdot R' \cdot V_r \cdot A \cdot r \cdot \vec{i} \quad (06)$$

where R' denotes the reduced length of the "U" tube that takes into account the circular end of the tube. The moment M_x is resisted by the elastic force moment M_e which for a small deformation can be presented as

$$\vec{M}_e = C_e \cdot \varphi \cdot \vec{i} \quad (07)$$

where the coefficient C_e is a function of the tube geometry and elastic properties of tube material. In the steady state rotation $M_e = M_x$, giving

$$4 \cdot \rho \cdot V_r \cdot A \cdot r \cdot R' \cdot \omega = C_e \cdot \varphi \quad (08)$$

or

$$\omega = \left(\frac{C_e}{4 \cdot r \cdot R' \cdot M} \right) \cdot \varphi \quad (09)$$

Hence, for the defined "U" tube geometry and its elastic properties, providing a constant mass flow rate the twist angle φ is proportional to the angular rate ω . The equation (09) forms the basis for the angular rate measurements. Figures 9.a. and 9.b. show the "U" tube deformation for the case of no angular rotation ($\omega = 0$) and with angular rotation ($\omega \neq 0$). Instead of the twist angle some other quantity like displacement or force could be used. Also, many other forms of elastic tubes like "S" shaped or winded tubes could be used. The difficulty in application of the equation (09) comes from the need of providing a constant mass flow rate on a long run.

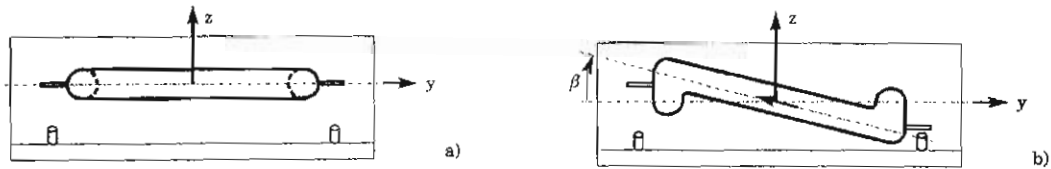


Figure 9. Tube twisting
a) angular rate $\omega = 0$; b) angular rate $\omega \neq 0$

Following the analogy with the Coriolis mass flow rate sensor the possibility of using an oscillatory flow has been analyzed. The operation of an oscillatory fluid angular rate sensor is explained with the aid of Figure 10. Here, the pump is replaced with an oscillating piston or diaphragm driven with an electric driver. The tube deformation is measured by means of two motion-displacement detectors (MS1 and MS2). If the fluid of constant density is made to oscillate with a frequency Ω the fluid velocity changes according to

$$V_r = V_{r0} \cdot \sin(\Omega \cdot t) \quad (10)$$

The magnitude of the Coriolis force shall vary in the same way and according to the eq. (05), $dF_c = dm \cdot a_c = 2 \cdot (dx) \cdot \omega \cdot (V_r \cdot A \cdot \rho)$. If the angular velocity is equal to zero ($\omega = 0$) there is no Coriolis force and the "U" tube remains in the xy plane so that the motion detectors produce the signals of the same magnitude and the sensor output is equal to zero. In the presence of rotation ($\omega \neq 0$) the Coriolis force oscillates sinusoidally. At any instant of time at two equivalent positions (in outward and inward parts of the tube) the Coriolis forces are of equal magnitude but opposite in direction. These forces create an oscillating moment dM_x about x axis. According to the eq. (06) the oscillating moment magnitude is

$$dM_x = 2 \cdot (dx) \cdot \omega \cdot (V_r \cdot A \cdot \rho) \cdot (2 \cdot r) \quad (11)$$

and after integration

$$M_x = 4 \cdot M \cdot R' \cdot r \cdot \omega \quad (12)$$

If the "U" tube is permitted to deform the inertial moment is resisted by the elastic force moment M_e . Following the same way of development as for the circulatory angular rate sensor we obtain

$$\omega = \left(\frac{C_e}{4 \cdot M \cdot R' \cdot r} \right) \cdot \varphi \quad (13)$$

From the eq. (13) it is seen that the angular rate depends on the twist angle φ and the mass flow rate M . To avoid the simultaneous measurements of M (proportional to the piston velocity) and φ the alternative way of the time difference Δt measurement as applied in the mass flow rate sensor could be approached.

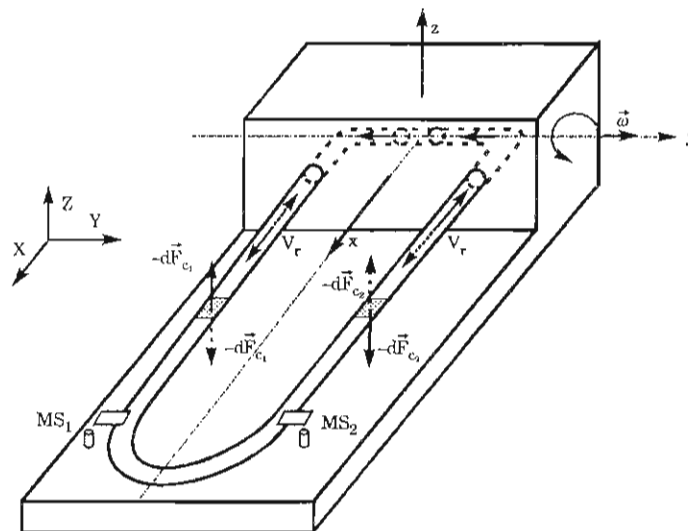


Figure 10. Oscillating fluid angular rate sensor

The proposed either circulatory or oscillatory angular rate sensors offer some potential advantages over the conventional rate gyros. The promising features of a proposed design are: there are no bearings and therefore no wear out, less sensitivity to shock, cannot be damaged by overcranking by building in limiters, shorter "spinning" time, smaller size and weight could also be expected.

5. CONCLUSIONS

In this paper a unified approach of using the Coriolis force on moving fluid for the purpose of the angular rate and the mass flow rate measurement is presented. The laminar angular rate sensor has become a recognized method for the angular rate measurement where the Coriolis force was used to deflect the free jet stream lines and to generate the output signal. In the Coriolis mass flow rate sensor fluid motion was restricted inside closed conduit and the external rotation (vibration) was imparted to generate the Coriolis force. By an approximate theoretical analyses was found that the circulatory fluid angular rate sensor and the oscillatory fluid angular rate sensor can be designed. The proposed solution offers some advantages over classical spring restrain gyros. The next phase of this research should be a distributed parameter modelling, experimental verification and the cost analysis.

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NOMENCLATURE

A	cross sectional area of the tube	R'	reduced length of the "U" tube
a_c	Coriolis acceleration	Re	Reynolds number
C_e	stiffness coefficient of the tube	V_r	fluid relative velocity
F_c	Coriolis force	V_{ro}	maximum fluid relative velocity
m	fluid mass	ω	angular rate of sensor rotation or vibration
M	mass flow rate through the tube, $\rho \cdot V_r \cdot A$	Ω	angular rate of the velocity change
M_x	Moment about x axis	φ	twist angle
r	half the distance between two legs of the "U" tube	ρ	fluid density
R	length of the "U" tube	$\times$	vector product

About the Interference Between Lifting Surfaces in BWT and BCT Configurations

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Abstract

The comparison of the interference effect of wing in the body-wing-tail (BWT) configuration or canard in body-canard-tail (BCT) configuration on the normal force of the tail, using undeveloped vortex sheet and fully developed vortex sheet, that rolled up into pair of concentrated vortices, will be presented in this paper. Upstream lifting surface in airplane or missile configuration generates on its trailing edge vortex sheet that significantly changes the flowfield at downstream lifting surface and therefore its aerodynamic characteristics. The determined shape of the vortex sheet developed behind the wing is similar to those presented in references. Interference effect on tail, for given airplane configuration example, differ from using the undeveloped vortex sheet and using the pair of concentrated vortices specially when tail is near the trailing edge of the wing and when the angle of attack of the wing is small.

1. INTRODUCTION

The wing in the body-wing-tail (BWT) configuration or canard in body-canard-tail (BCT) configuration changes the flowfield at tail, which would be very different if the wing or canard was not present. This change is represented with vortex sheet generated at trailing edge of the upstream lifting surface, that transforms into pair of concentrated vortices. References [1] and [2] use these concentrated vortices of fully rolled up vortex sheet in calculation of interference effect between lifting surfaces.

The flowfield behind the upstream lifting surface presented as undeveloped vortex sheet possibly has interference effect on downstream lifting surface different from one determined by methods presented in Ref. [1] and [2]. The emphasis of this paper is development of the vortex sheet from the lifting surface and determination of its interference effect on normal force of downstream lifting surface, and its comparison with interference effect calculated using the concentrated trailing vortex pair from Ref. [1] and [2].

Vortex sheet is dependent on change of the circulation distribution at trailing edge of the upstream lifting surface, and therefore has maximum strength near tips where the change of circulation is most peculiar.

The methods developed by Pitts et al. is still popular in preliminary estimation of aerodynamic performances and stability and control for conceptual design. It is a component build up method, Ref. [3] where the aerodynamic characteristics of isolated major airframe parts are calculated, adding the coefficient produced by component interference, like for example wing-tail interference. Estimates of interference effect of various components on each other from Ref. [1] is primarily based on slender body theory and its applicable for small angles of attack.

Determination of wing's vortex system is a difficult task. The roll up of the vortex sheet is influenced significantly by viscosity and other nonlinear effects difficult to model precisely.

2. VORTEX SHEET, GENERATION AND TRANSFORMATION

As mentioned earlier, determination of the vortex sheet is based on the determination of the circulation distribution along the wing. To be more precise, the change of the circulation $\Gamma(y)$ along the span, on the trailing edge, is required. Glauert's circulation model with Fourier representation from Ref. [4] is used in this paper with eight stations along one-half of the span.

The trailing vortex that is a part of the vortex sheet, generated on the trailing edge of the wing (or canard) at a spanwise station is equal in strength to the change of the circulation along the wing $\Gamma(y)$ at that station. For the width dy strength of the vortex is from [4] and [5]

$$d\Gamma = \left(\frac{d\Gamma}{dy} \right)_{wing} dy. \quad (1)$$

These vortices together with the bound vortex on the wing (Fig. 1) present the vortex system model of the lifting surface. Each vortex has its pair on the other half of the wing with the equal strength and opposite sense of rotation, so the vortex sheet has zero total strength.

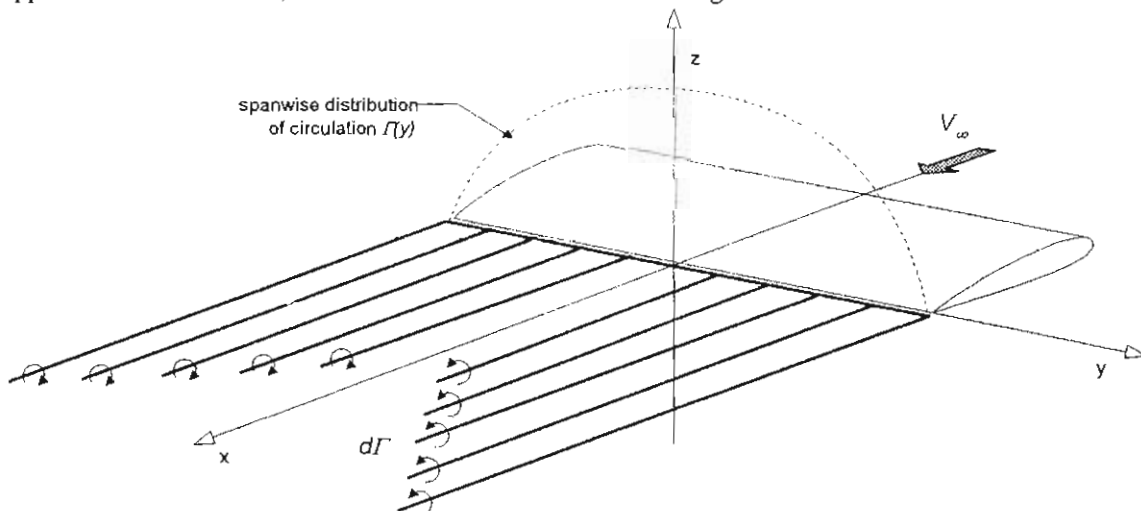


Figure 1. Vortex system of the wing

In this paper the wing is treated in the presence of the body. To satisfy the condition of the tangential crossflow on the body, Nielsen in Ref. [2] introduces image vortices inside the body for each external vortex. The image vortex Γ' , from the Fig. 2, has the same strength and opposite sense of rotation to the external vortex Γ . The location of the image vortex (y', z') for the position (y, z) of the external vortex, in the body axis, is given with relations

$$\begin{aligned} y' &= \frac{a^2}{s^2} y \\ z' &= \frac{a^2}{s^2} z, \end{aligned} \quad (2)$$

where a represents radius of the body as illustrated on the Fig.2, and where s is defined with $s = \sqrt{y^2 + z^2}$.

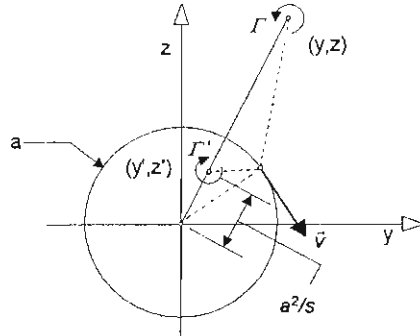


Figure 2. External vortex and its image vortex

In model given in this paper, vortex sheet for half wing is presented with the set of n vortices of constant circulation Γ_i . Every external vortex has its image vortex inside the body. Whole vortex sheet model of the trailing edge of the upstream lifting surface has $4 \times n$ vortices as illustrated in Fig. 3.

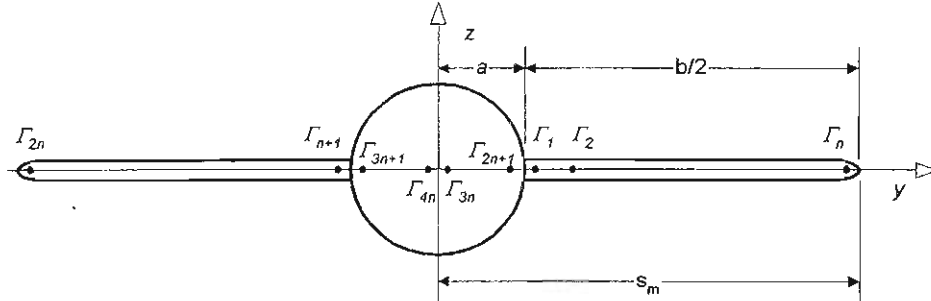


Figure 3. Model of the vortex system

Vortex paths for small angles of attack can be determined from the differential equation from Ref. [6]

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \quad (3)$$

The velocity for every vortex is influenced by two factors

- body crossflow,
- interaction between vortices.

Speed component due to the body crossflow (Fig. 4a) at a point (y, z) , in cylindrical coordinates are

$$\begin{aligned} \frac{V_r}{V_\infty} &= \alpha_T \left(\frac{z}{s} \cos \varphi + \frac{y}{s} \sin \varphi \right) \left(1 - \frac{a^2}{s^2} \right) + a', \\ \frac{V_\varphi}{V_\infty} &= \alpha_T \left(\frac{y}{s} \cos \varphi - \frac{z}{s} \sin \varphi \right) \left(1 + \frac{a^2}{s^2} \right), \end{aligned} \quad (4)$$

where α_T is total angle between the x axis and velocity. In this paper the angle of attack α and angle of sideslip β are taken through the total angle of attack α_T and the rolling angle φ . Angle α , i.e. $\alpha_T \cdot \cos \varphi$, influence on the circulation distribution and the angle of sideslip β influence on the vortex

paths. In the radial component from Eq. (4) the last member $a' = da/dx$ is introduced for the bodies with variable radius as a function of x . The transformation of the Eqs. (4) into the body axis results in relations

$$\begin{aligned}\frac{v}{V_\infty} &= \left(\frac{V_r}{V_\infty} y - \frac{V_\theta}{V_\infty} z \right) / s \\ \frac{w}{V_\infty} &= \left(\frac{V_r}{V_\infty} z + \frac{V_\theta}{V_\infty} y \right) / s.\end{aligned}\quad (5)$$

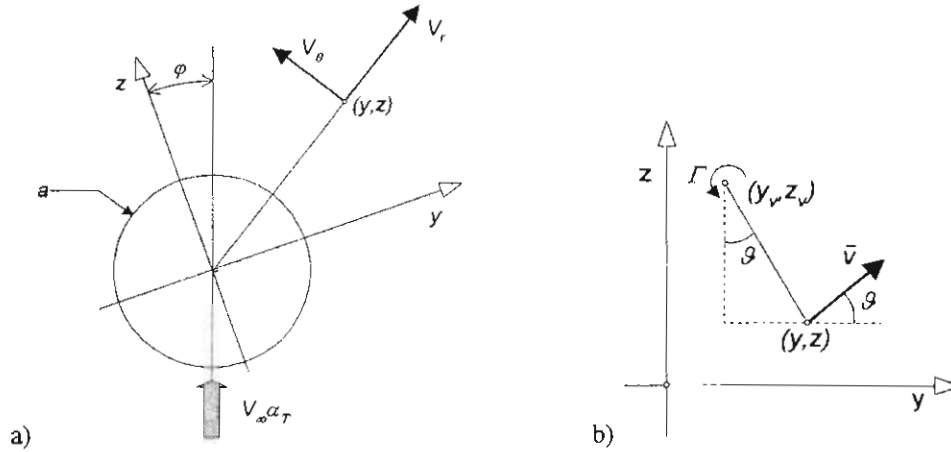


Figure 4. Velocity in the point (y, z) due to a) body crossflow; b) interaction between vortices

Vortex induced velocity components (Fig. 4b) are determined with relations, Ref. [7]

$$\begin{aligned}v &= V \cos \vartheta = V \frac{z_v - z}{\sqrt{(y_v - y)^2 + (z_v - z)^2}} = \frac{\Gamma}{2\pi} \frac{z_v - z}{(y_v - y)^2 + (z_v - z)^2} \\ w &= V \sin \vartheta = V \frac{y - y_v}{\sqrt{(y_v - y)^2 + (z_v - z)^2}} = \frac{\Gamma}{2\pi} \frac{y - y_v}{(y_v - y)^2 + (z_v - z)^2}.\end{aligned}\quad (6)$$

Induced velocity of a vortex has a singularity at its center. To remove this singularity vortex core is introduced, and velocity at vortex center is zero. Models used for vortex core, Ref. [8], range from simple cutoff distance (inside which induced velocity is zero), solid-body rotation model (linear velocity distribution from zero at center to value of induced velocity at specified radius) to diffusion core model (based on viscous decay of a laminar vortex). In vortex sheet model presented in this paper vortex core is not enclosed.

From the Eq. (3) two differential equations can be written as

$$\begin{aligned}\left(\frac{dy}{dx} \right)_j &= \frac{v_j}{V_\infty \cos \alpha} = f_{yj} \\ \left(\frac{dz}{dx} \right)_j &= \frac{w_j}{V_\infty \cos \alpha} = f_{zj}.\end{aligned}\quad (7)$$

The right-handed side of Eqs. (7) can be written using Eqs. (5) and (6)

$$\begin{aligned}f_{yj} &= \sum_{\substack{k=1, 4n \\ k \neq j, k \neq j+2n}} \frac{\Gamma_k}{2\pi V_\infty} \frac{z_k - z_j}{r_{kj}^2} + \frac{v_j}{V_\infty} \\ f_{zj} &= \sum_{\substack{k=1, 4n \\ k \neq j, k \neq j+2n}} \frac{\Gamma_k}{2\pi V_\infty} \frac{y_j - y_k}{r_{kj}^2} + \frac{w_j}{V_\infty},\end{aligned}\quad (8)$$

where

$$r_{kj}^2 = (y_k - y_j)^2 + (z_k - z_j)^2. \quad (9)$$

Index j denotes vortex whose path is being determined integrating Eqs. (7), and index k denotes all vortices that influence on the new position of j -th vortex, except for that vortex itself ($k=j$) and its image vortex ($k=j+2n$). Therefore vortex sheet model is defined with Eqs. (7), (8) with (5) and (6), together with (9). The new position of the image vortices can be evaluated using Eqs. (2).

Whole vortex sheet can be presented with vortex pair, i.e. center of gravity for each half wing, Ref. [2], and its lateral coordinate is

$$y_V = \frac{\sum_{i=1}^n \Gamma_i \cdot y_i}{\sum_{i=1}^n \Gamma_i}. \quad (10)$$

The start of the simulation is the trailing edge of the upstream lifting surface ($x=0$, Fig. 1). With numeric integration of Eqs. (7), new coordinates of the vortices, that make the vortex sheet, including image vortices, are calculated in every step with increment Δx . Grid of the vortices is determined with circulation distribution $\Gamma(y)$. More dense grid is obtained where $\Gamma(y)$ changes faster, that is at the wing tips.

3. LIFTING SURFACE - VORTEX INTERFERENCE

Vortex can produce significant changes in the aerodynamic performances of the lifting surface when passing close to it. Undeveloped or fully rolled up vortex sheet of the upstream lifting surface modificate the flowfield in the region of the downstream lifting surface, causing generally downwash field, and reducing the lift. When observing body in presence of tail, like in this model, image vortices of external vortices from the sheet should be considered.

Pitts et al. in [1] determined this effect of wing on the tail introducing the tail interference factor i_T . Interference effect described in Ref. [1] and [2] is based on the slender body theory with the restriction to small angles of attack.

If we consider vortex sheet from the wing at tail fully developed according to Ref. [1] and [2] it can be described with pair of external vortices of equal strength Γ and opposite sense of rotation, symmetrically placed with coordinates (y_V, z_V) and $(-y_V, z_V)$. Normal force on the tail due to the interference effect of this vortex pair and their image vortices is determined with relation

$$Z_{T(W)V} = q_\infty S_W C_{N\alpha} \frac{\Gamma}{2\pi V_\infty (s_m - a)} i_T, \quad (11)$$

where S_W is reference area (area of the wing), $C_{N\alpha}$ is coefficient of the normal force for tail and q_∞ is dynamic pressure. Nielsen in [2] states that the tail interference factor i_T is a ratio of two nondimensional quantities, ratio of the lift of tail due to the vortices and lift of tail alone, and nondimensional vortex strength. Values of i_T are given in Ref. [1] in sets of charts as a function of vortex coordinates $\bar{y}_V = y_V/s_t$ and $\bar{z}_V = z_V/s_t$, for various values of tail taper ratio λ_T and parameter $\bar{r} = a/s_t$, with s_t being the maximum lateral dimension of the tail. In model discussed in this paper i_T is determined numerically using the relations, Ref. [1]

$$i_T = -\frac{2}{1+\lambda} \left[U(\lambda_T, \bar{r}, \bar{y}_V, \bar{z}_V) - U(\lambda_T, \bar{r}, -\bar{y}_V, \bar{z}_V) + U(\lambda_T, \bar{r}, \bar{y}_V', \bar{z}_V') - U(\lambda_T, \bar{r}, -\bar{y}_V', \bar{z}_V') \right], \quad (12)$$

where the function $U(\lambda_T, \bar{r}, \bar{y}_V, \bar{z}_V)$ is determined with relation

$$U(\lambda_T, \bar{r}, \bar{y}_V, \bar{z}_V) = -\frac{1-\lambda_T \bar{r} - \bar{y}_V(1-\lambda_T)}{2(1-\bar{r})} \cdot \ln \frac{\bar{z}_V + (\bar{y}_V - 1)^2}{\bar{z}_V + (\bar{y}_V - \bar{r})^2} + \frac{1-\lambda_T}{1-\bar{r}} \left[1 - \bar{r} + \bar{z}_V \left(\arctg \frac{\bar{y}_V - 1}{\bar{z}_V} - \arctg \frac{\bar{y}_V - \bar{r}}{\bar{z}_V} \right) \right]. \quad (13)$$

The interference of the undeveloped vortex sheet, i.e. set of vortices on the tail can be determined by adding up the interference effect for each vortex pair using Eqs. (12) and (13). Normal force on tail due to the vortex sheet interference can be written as

$$Z_{T(W)VS} = q_{\infty} S_W C_{Na} \frac{1}{2\pi V_{\infty} (s_m - a)} \sum_{i=1}^n \Gamma_i \cdot i_{T_i} \quad (14)$$

We can use nondimensional quantity, the ratio of Eqs. (11) and (14) and after introducing the normal force coefficients it can be obtained

$$\frac{C_{NT(W)VS}}{C_{NT(W)V}} = \frac{\sum_{i=1}^n \Gamma_i \cdot i_{T_i}}{\Gamma \cdot i_T} \quad (15)$$

This normal force coefficient ratio can be used for comparison of interference effect on tail caused by undeveloped and fully developed vortex sheet.

4. SAMPLE RESULTS

The given vortex sheet model was applied to the simplified scheme of general aviation (GA) single engine airplane: BWT configuration with rectangular wing and tail, with $s_m/s_t = 2.5$ and $a/s_m = 0.11$, tail height with respect to wing $z_t = 0.5$ m, total angle of attack $\alpha = 2^\circ$, angle of bank $\varphi = 0$ and speed $V = 50.0$ m/s. The calculation was done with $n = 40$.

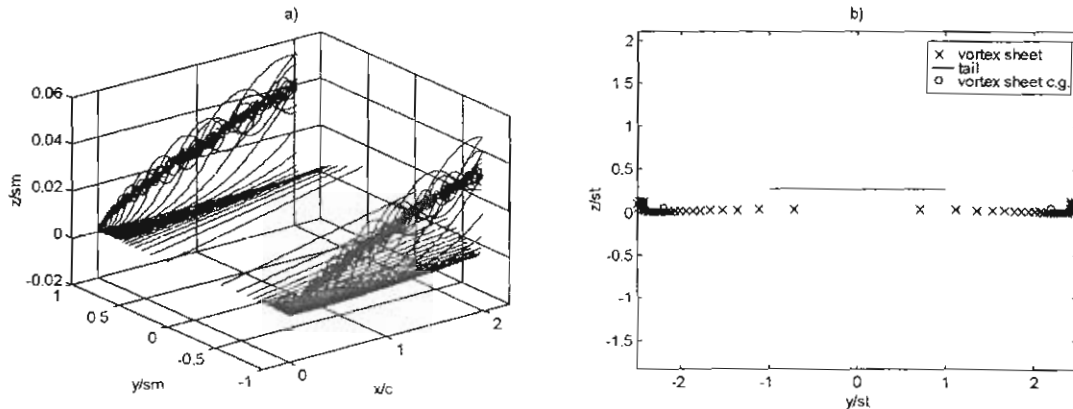


Figure 5. Vortex sheet development for GA airplane example: a) vortex paths, b) vortex sheet at $x/c=2$ with positions of its centers of gravity

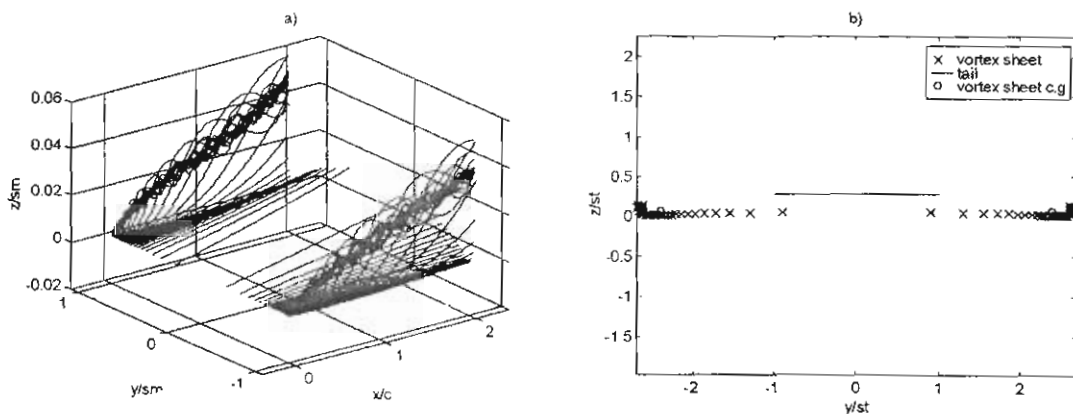


Figure 6. Vortex sheet development for GA airplane example with noncylindrical body ($a' \neq 0$): a) vortex paths, b) vortex sheet at $x/c=2$ with positions of its centers of gravity

Figure 5a) presents trajectories of each discrete vortex from the start point at x_b to the end point x_t . Vortices near wing tips very quickly form concentrated vortex pair. Lateral position of center of gravity of the vortex sheet, calculated with Eq. (10) practically does not change with x . But for the stronger influence of the body crossflow, i.e. greater ratio a/s_m or other than cylindrical body form, this lateral

motion is more significant. This position of vortex sheet center of gravity is not dependent on Mach number.

Figure 5b) illustrates vortex sheet at the end of simulation $x/c=2$ and its centers of gravity calculated with Eq. (10). Vortex sheet wake of the wing has similar shape as presented in Ref. [2] and [9].

Tail interference factor is determined at location of normal force of the tail, i.e. on distance x_t from the trailing edge of the wing. For GA airplane example normal force coefficient ratio, Eq. (15) is 1.0347.

This interference effect, i.e. i_T is greatly influenced by relative position of vortex pair to the tail tip. If the given example would be observed with $z_t=0$, ratio from Eq. (15) would be 1.0686. The shape of the body also influences the vortex sheet through $a' \neq 0$ in Eq. (4) and therefore influence the interference effect on tail. For GA airplane example with quadratic function of body radius $a(x)$, Fig.6, Eq. (15) results in 1.05.

Dependence of normal force coefficient ratio defined with Eq. (15) to the nondimensional distance from wing to tail x_t/c , and to the angle of attack α , i.e. α_T because of $\varphi=0$, for the presented example, is illustrated on the Fig 7.

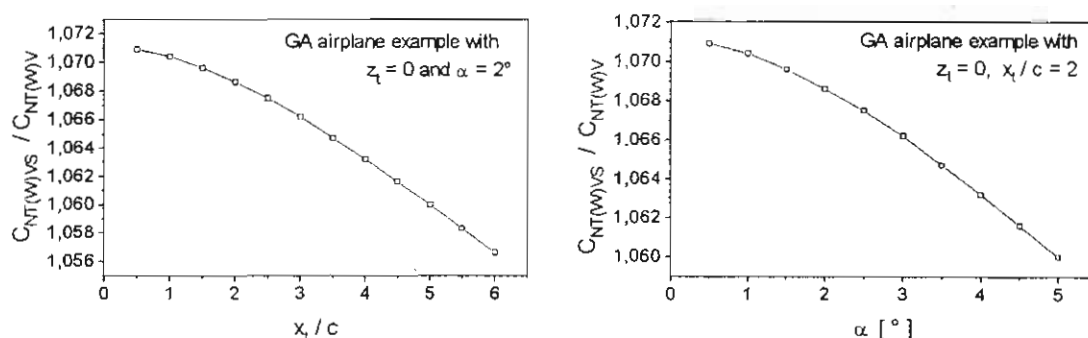


Figure 7. Dependence of normal force coefficient ratio from Eq. (15) to x_t/c and to α for GA example with $z_t=0$

5. CONCLUSION

The numerical model of vortex sheet presented gave good results in vortex sheet development from the trailing edge of the lifting surface. Determination of the grid of vortices in vortex sheet was based on Glauert's circulation distribution what is improvement to the slender body theory distribution used in Ref. [2]. The possible improvement of the model of vortex sheet development and its interference with downstream lifting surface would be introduction of the vortex core model.

It was determined with analysis of presented model that the interference effect of upstream lifting surface on normal force of the downstream lifting surface using undeveloped vortex sheet gave almost identical results as when the fully developed vortex sheet is used for aircraft configurations that puts concentrated trailing vortices away from the tail tips. For configurations where these concentrated vortices passes through region close to tail tips interference effect is very strong and normal force coefficient ratio from Eq. (15) could rise up to 1.5. Every other conclusion should be made for particular aircraft configuration. For aircraft example presented in this paper, significant change of the interference effect, i.e. ratio from Eq. (15) was obtained by putting the tail in the plane of the wing ($z_t=0$). And further for bringing the tail closer to the wing, normal force coefficient ratio increases, i.e. difference between interference effect of the undeveloped and fully developed vortex sheet increases. The same result is obtained with increasing the angle of attack at the wing.

Acknowledgments

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The panel method for subsonic aerodynamic flow

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Abstract

Potential flow about a given body can be generated by distributing singularities (sources, sinks, vortices) within its enclosure at specific locations such that the body surface becomes a streamline of the flow. Of practical interest is the pressure distribution at the surface of the body. This leads directly to the lift force on the body and provides pressure boundary condition for the equations governing boundary layer flow. In this paper we treat planar wing that has no camber. We consider aerodynamic lift coefficient over wing, effect of wing sweep on the spanwise loading and effect of taper ratio on the spanwise variation of the lift coefficient. This approach seems to be more economical, from the computational point of view, than methods that solve for the flowfield in the whole fluid volume.

1. Introduction

For flow at high Reynolds number the effects of viscosity are effectively confined to thin boundary layers and thin wakes. For this reason study of low speed aerodynamics will be limited to flow outside these regions where the flow is assumed inviscid and incompressible. In this region the fluid motion is completely described by the continuity equation and the Euler equations:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} &= 0 \\ \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} &= \vec{f} - \frac{\nabla p}{\rho}\end{aligned}\tag{1}$$

where $\vec{V}$ is velocity of fluid, ρ -density, p -pressure and $\vec{f}$ -body force. Suppose that body force is conservative and flow is steady.

For flows that are irrotational ($\text{curl} \vec{V} = 0$) it is useful to define a velocity potential Φ , such that $\vec{V} = \nabla \Phi$. The continuity equation then becomes

$$\nabla^2 \Phi = 0,\tag{2}$$

i.e. Laplace's equation. This class of flow (inviscid, incompressible and irrotational) is called potential flow. Laplace's equation for the velocity potential, along with the specification of zero normal velocity at the surface of an immersed body and specification of the velocity far from body completely determine the velocity distribution. The pressure can be directly obtained from Bernoulli equation for irrotational steady flow:

$$\frac{p}{\rho} + \frac{V^2}{2} = \text{constant} \quad (3)$$

The total force on an immersed body, calculated from solution Φ of the above problem, is equal to zero. To obtain lift, one must introduce additional impermeable surfaces, immersed in the flow region. These surfaces called wakes, originate at the sharp edges on surface of body, such as the leading or trailing edge of a wing. The wake sheet is an idealization of a thin layer present in a real viscous flow downstream of a wing.

2. Mathematical background

Consider the 3-D mathematical domain Ω and consider two arbitrary scalar fields Φ_1 and Φ_2 which are second-order continuous throughout Ω . Green's Second Theorem states:

$$\iiint_{\Omega} (\Phi_1 \nabla^2 \Phi_2 - \Phi_2 \nabla^2 \Phi_1) dV = \iint_S (\Phi_1 \nabla \Phi_2 - \Phi_2 \nabla \Phi_1) \cdot \bar{n} dS \quad (4)$$

where $\bar{n}$ is the unit outward normal vector at a point of surface S . For the idealized 3-D aerodynamic problem suppose that Φ_1 is replaced by the unknown velocity potential Φ satisfying $\nabla^2 \Phi = 0$ throughout Ω , and Φ_2 is replaced by quantity $1/r$ where, for any point Q (the source point) inside Ω , r is defined as to be the length of the vector $\bar{r}$ from Q to fixed point P (the observer point). The derivatives in vector operator ∇ are taken with respect to the coordinates of the source point Q .

Three cases may be considered: P lying outside Ω (r is never zero); P lying inside Ω ($1/r \rightarrow \infty$ as $Q \rightarrow P$); P lying on the boundary S of Ω .

In the case where P is outside Ω both integrands on the left-hand side of equations are always zero ($\nabla^2 \Phi = 0$ and $\nabla^2 \frac{1}{r} = 0$ throughout Ω) so that Eq. 4 reduces to:

$$\iint_S \left(\Phi \nabla \left(\frac{1}{r} \right) - \frac{1}{r} \nabla \Phi \right) \cdot \bar{n} dS = 0 \quad (5)$$

In the case where the observer point P is inside Ω it is necessary to exclude points Q in the immediate vicinity of P from domain of integration. This is done by surrounding P by small sphere of radius ϵ , surface Σ and volume V (Fig 1). Applying equation Eq. 4 to volume $(\Omega - V)$ with surface $(S + \Sigma)$. Again the left-hand side of Eq. 4 is zero. In the limit as $\epsilon \rightarrow 0$, the potential on Σ tends towards uniformity, its mean value being equal to the potential $\Phi(P)$, and it is easy to show that:

$$\iint_{\Sigma} \frac{1}{\epsilon} \nabla \Phi \cdot \bar{n} dS \rightarrow 0 \quad \text{and} \quad \iint_{\Sigma} \Phi \nabla \left(\frac{1}{r} \right) \cdot \bar{n} dS \rightarrow 4\pi \Phi(P). \quad (6)$$

Thus for points P inside Ω Eq. 4 reduce to:

$$4\pi \Phi(P) = \iint_S \frac{1}{r} \nabla \Phi \cdot \bar{n} dS - \iint_S \Phi \nabla \left(\frac{1}{r} \right) \cdot \bar{n} dS. \quad (7)$$

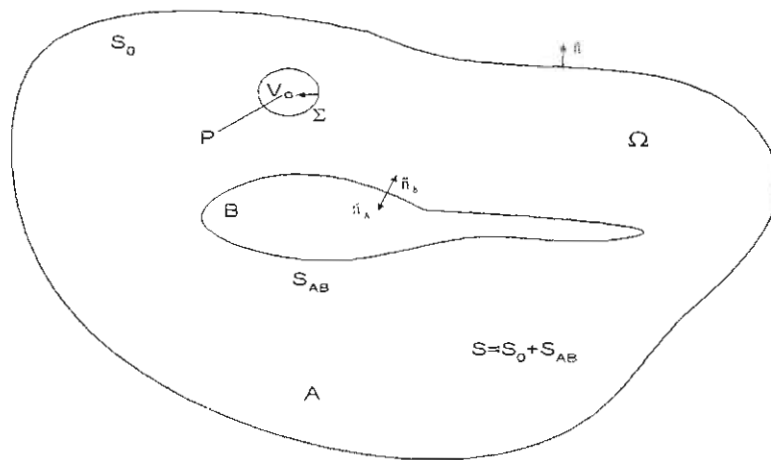


Figure 1. Potential flow problem

In the case where P lies on the boundary S of Ω a similar process is required. Then Eq. 7 reduces to:

$$A\Phi(p) = \iint_S \frac{1}{r} \nabla \Phi \cdot \bar{n} dS - \iint_S \Phi \nabla \frac{1}{r} \cdot \bar{n} dS \quad (8)$$

where A denotes the solid angle subtended at P by the infinitesimally small curved surface.

We shall consider a collection of sub-domains (in each of which a Laplacian velocity potential exists), which is theoretical modeling of wing (Fig 1.)

The complete equations may be generalized to the case where P lies in any one domains. Each domain has a closed boundary consisting of a number of connected portions, a portion interfacing two domains A and B being denoted by S_{AB} . If Φ_A and Φ_B denote the single-valued local potentials on the two arbitrary chosen side A and B of any one of these surfaces and $\bar{n}_A$ denotes local outward normal vector.

By integrating over all such surfaces we obtain:

$$4\pi\Phi(P) = \iint_{S_{AB}} \left[\frac{1}{r} (\nabla \Phi_A - \nabla \Phi_B) - (\Phi_A - \Phi_B) \nabla \left(\frac{1}{r} \right) \right] \cdot \bar{n}_A dS + \iint_{S_0} \left[\frac{1}{r} \nabla \Phi_1 - \Phi_1 \nabla \left(\frac{1}{r} \right) \right] \cdot \bar{n} dS \quad (9)$$

3. Interpretation in terms of surface singularities

A surface source distribution of variable density σ per unit area on an arbitrary surface S induces at an external point P a potential Φ_{P_s} and a velocity $\bar{V}_{P_s}$ given respectively by:

$$\begin{aligned} 4\pi\Phi_{P_s} &= \iint_S -\frac{\sigma}{r} dS \\ 4\pi\bar{V}_{P_s} &= \iint_S -\sigma \nabla_P \left(\frac{1}{r} \right) dS \end{aligned} \quad (10)$$

where r is the length of the vector $\vec{r}$ drawn to the observer point P from the variable point Q on S (Fig 2).

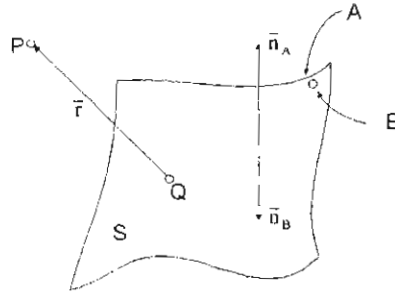


Figure 2.

Limiting analysis is required as $P \rightarrow Q$ on a smooth region of S . Such an analysis shows that singular contribution to $4\pi\bar{V}_P$ due to the infinitesimal portion of S in the immediate vicinity of P is equal to $2\pi\sigma\bar{n}_A$ on the side A of S and equal to $-2\pi\sigma\bar{n}_A$ on the side B of S . The jump $\bar{n}_A(\bar{V}_A - \bar{V}_B)$ in the normal velocity component in proceeding across S in the direction of $\bar{n}_A$ is thus equal to σ ; the potential is continuous across S .

A surface distribution of normal doublet of variable density μ per unit area on an arbitrary surface S , with the local unit vector $\bar{n}$ of the doublet axis (from negative to the positive end of the doublet) lying perpendicular to S , produces at an external point P a potential Φ_{P_d}

$$4\pi\Phi_{P_d} = - \iint_S \mu \bar{n} \cdot \nabla \left(\frac{1}{r} \right) dS \quad (11)$$

A limiting process is again required as $P \rightarrow Q$ on a smooth region of S . The discontinuity in Φ_{P_d} in proceeding across S in the direction of $\bar{n}$ is thus equal to $-\mu \bar{n}_A \cdot \bar{n}$. Normal component of the velocity is continuous across S while the tangential velocity jumps by an amount equal to the tangential derivative of the jump in potential.

By comparing Eq. 10 and Eq. 11 with Eq. 9 it can be seen that in Eq. 9 each term of the form :

$$\frac{1}{4\pi} \iint_{S_{AB}} \frac{1}{r} (\nabla \Phi_A - \nabla \Phi_B) \cdot \bar{n}_A dS \quad (12)$$

may be interpreted as the perturbation potential induced by a source distribution of density $\sigma = \bar{n}_A (\nabla \Phi_A - \nabla \Phi_B)$ on each surface portion S_{AB} and similarly each term of the form:

$$\frac{1}{4\pi} \iint_{S_{AB}} (\Phi_A - \Phi_B) \nabla \left(\frac{1}{r} \right) \cdot \bar{n}_A dS \quad (13)$$

may be interpreted as the perturbation potential induced by normal doublet distribution of density $\mu = (\Phi_A - \Phi_B)$ on the same surface.

Thus equation 9 may be written for point not lying on any of the surface S_{AB} :

$$4\pi\Phi_P = 4\pi\Phi_{P_{\infty}} + \iint_{S_{AB}} \left[\frac{\sigma}{r} - \mu \bar{n}_A \cdot \nabla \left(\frac{1}{r} \right) \right] dS \quad (14)$$

For points P lying on one of these surfaces the same equation may be used, provided the integral is defined to include the local singular contribution $-2\pi\mu_P \bar{n}_A \cdot \bar{n}$.

By applying the operator ∇_P to both sides of equation Eq. 14 gives:

$$4\pi V_P = 4\pi V_{P_{\infty}} + \iint_{S_{AB}} \left[\sigma \nabla_P \left(\frac{1}{r} \right) - \mu \nabla_P (\bar{n}_A \cdot \nabla \left(\frac{1}{r} \right)) \right] dS \quad (15)$$

The velocity field induced by the doublet distribution used in equation Eq. 14 is identical to that induced by a surface vorticity distribution on S with surface vector density $\vec{\gamma} = \nabla\mu \times \vec{n}$.

4. Steps toward constructing numerical solution

The first and one of the most important decisions are the type of singularity element. This includes the selection of source, doublet or vortex representation. Once the basic solution element is selected, the geometry of the problem has to be subdivided (or discretized), such that it will consist of those basic solution elements. In this grid generating process, the elements' corner points and collaction points are defined. The collaction points are points where the boundary condition, such as the zero normal flow to a solid surface, will be enforced. The true surface S is approximated by set of plane panels S_i , $i=1$ to N , so that complicated integral in Eq. 15 can replace with a sum of simple integral over panels. This surface integral over panel reduces to a set of line integrals along the edges of S_i . It is very important to realize that the grid does not have an effect on the solution. Good grid selection will converge to a certain solution when the density is increased.

On solid surface, the boundary condition $\vec{V}_n = 0$ is applied at the collaction point of each plane panel on the surface. By a specification of this boundary condition on N collaction points, N linear equations can be created. Now the equations are solved with standard matrix technique.

The solution of the matrix equation results in the singularity strengths and velocity field and secondary information can be computed now. The pressure will be computed by the Bernoulli equation, and the loads and aerodynamic coefficient by adding up the contributions of the element.

5. Results

For example consider a planar wing planform, where the leading, trailing and side edges are made of straight lines and wing has no camber. The three-dimensional thin lifting surface problem will be solved using the vortex ring elements. The thin-wing planform is divided into panels as shown in Fig. 3. The leading segment of the vortex ring is placed on the panel's quarter chord line and the collaction point is at the center of the three-quarter chord line. The normal vector $\vec{n}$ is defined at this point, too. A positive circulation, Γ is defined according to the right-hand rotation rule.

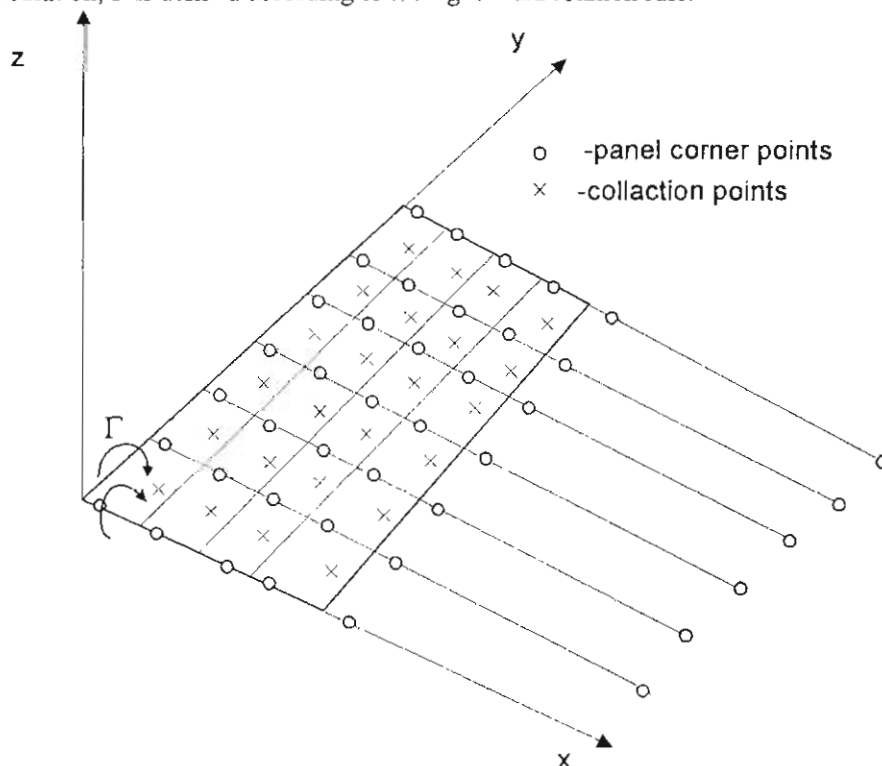


Figure 3.

From the numerical point of view these vortex ring elements are stored in rectangular arrays with index i and j . The velocity induced at arbitrary point $P(x,y,z)$ by adding the ring's four segments (Fig 4a). The velocity induced by constant strength vortex line segment of circulation Γ (Fig 4b) is:

$$\vec{V} = \frac{\Gamma}{4\pi} \left(\frac{\vec{r}_0 \cdot \vec{r}_1}{r_1} - \frac{\vec{r}_0 \cdot \vec{r}_2}{r_2} \right) \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|^3} \quad (16)$$

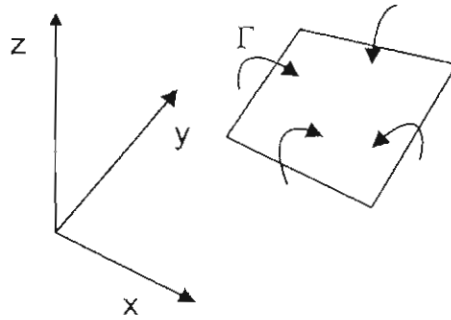


Figure 4a. Vortex ring.

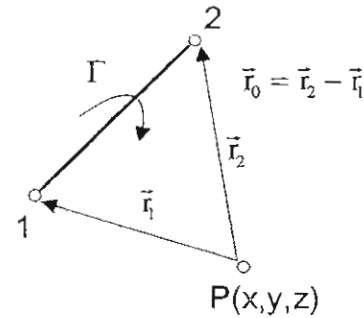


Figure 4b. Straight vortex segment.

On the every collection point we must evaluating the influence of the other vortex which resulted system of linear equation. The solution of the that system results circulation of panel (Γ). The lift of each bound vortex segment dL is obtained by using the Kutta-Joukowski theorem:

$$dL = \rho V_{\infty} \Gamma \quad (17)$$

The lift force is always perpendicular to the free stream velocity. The lift coefficient C_l is defined by :

$$C_l = \frac{dL}{\frac{1}{2} \rho V^2 S_l} \quad (18)$$

Wings have aspect ratio 5 and semi-span 8m. For this example the chord is divided equally into 20 panels and the semi-span is divided equally into 50 panels. Figure 5 shows ratio of local aerodynamics lift coefficient and total lift coefficient over a wing. Shape of this mesh agrees with theoretical prediction and results obtained with other methods.

The effect of leading-edge sweep on the spanwise loading is shown in Fig.6 for planar wing which have aspect ratio 5. Aft-swept wings will have more lift toward their tips while forward-swept wings will have larger loading near root.

Wing root bending can be reduced by tapering wing. The taper ratio λ is defined as the ratio of tip to root chords. Spanwise loading of an untwisted wing with various taper ratios is shown on Fig. 7. As was noted, the load is decreasing toward their tip but the local lift coefficient is increasing with a reduction in taper ratio. This means that the tip of such wing will stall first, an unfavorable behavior that can be corrected by twist.

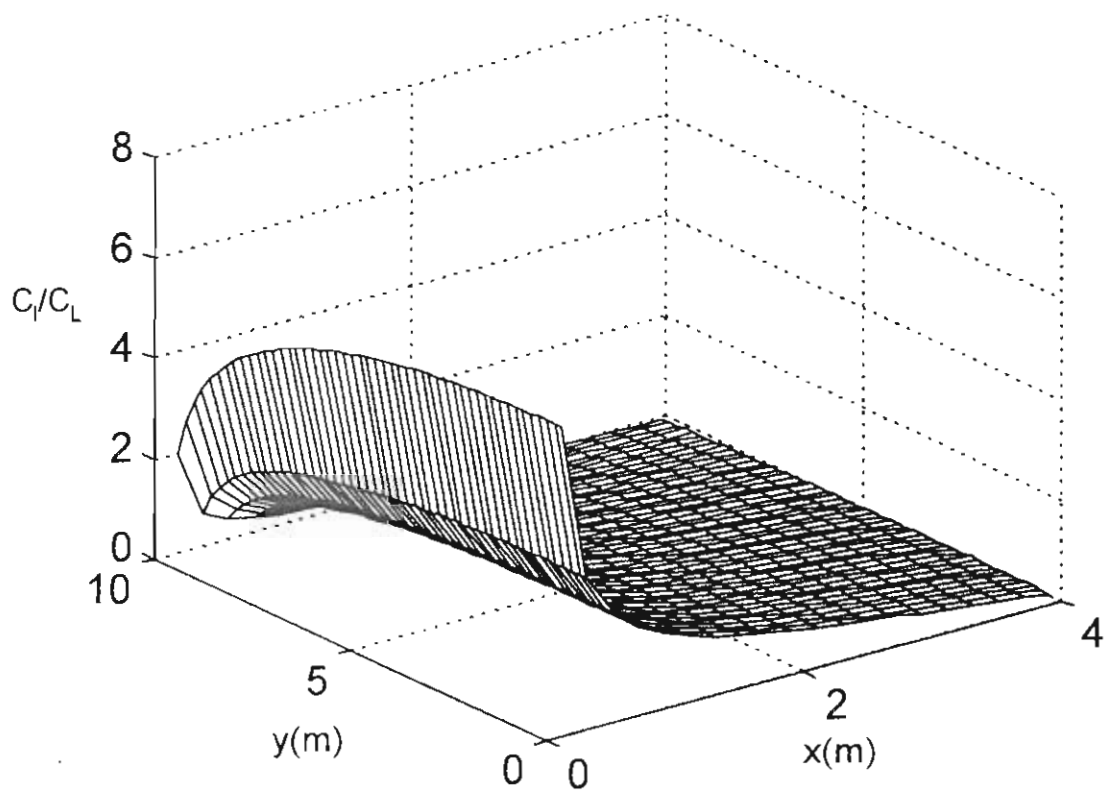


Figure 5. Aerodynamic lift coefficient over wing

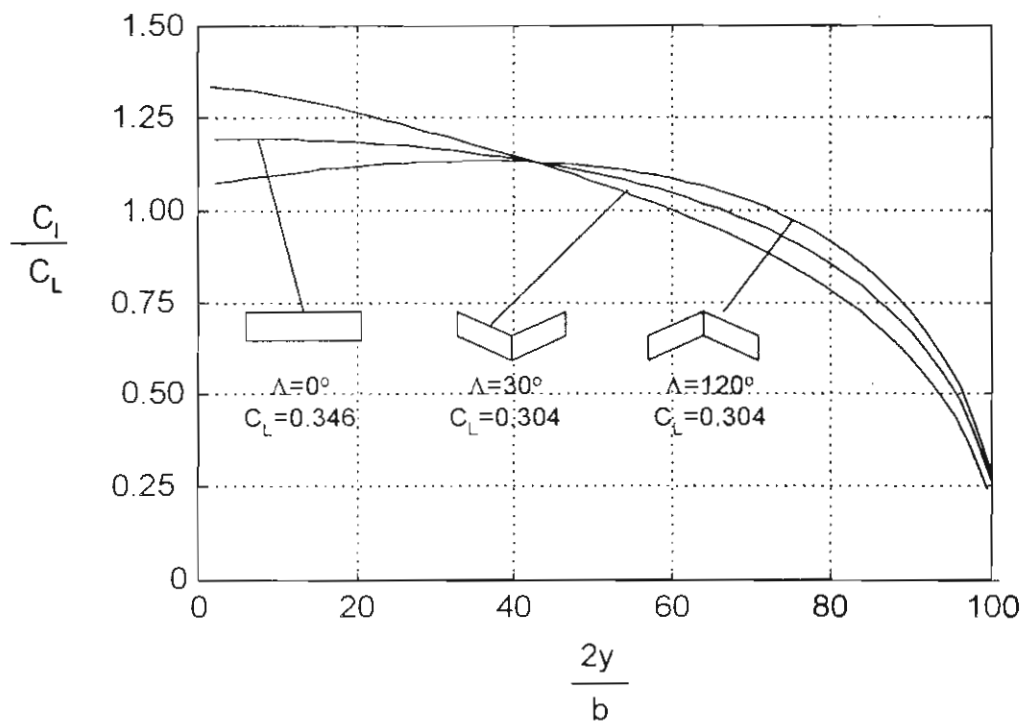


Figure 6. Effect of wing sweep on the spanwise loading of untapered wings.

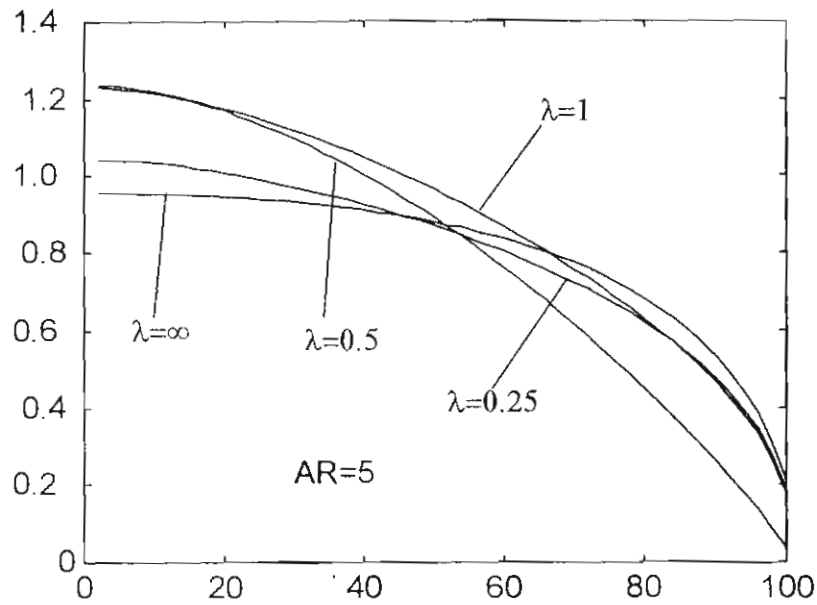


Figure 7. Effect of taper ratio on the spanwise variation of the lift coefficient.

6. Conclusions

It was demonstrated in the paper that solution of potential flow problem over wings can be obtained by distribution of elementary solution. Calculation was considered on very simple geometry of wing. Results obtained with panel method agree with result obtained with other method or analytically if it is possible. This calculation can be used for more complicated geometry such as wing-body combination. The improved methods use curved panel elements with varying distribution of singularities which reduce number of element used to represent body. It is possible to account for the viscosity effects such as displacement thickness and friction drag by using a boundary layer solution that can be matched with the potential flow solution.

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Numerical Analysis of Vehicles Aerodinamics Problem

Z. Žunič ¹, L. Škerget ²

Abstract

In this paper the comparison of three different $k-\varepsilon$ low-Reynolds turbulence models has been done. Lam-Bremhorst's, Nagano's and Fan-Lakshminarayana-Barnett's model was used. Reynolds equations for time averaged fluid flow variables are solved by high order finite volume method. All models have been tested against a range of measurements by Ruck and Makiola for backward facing step flow. The best model have been chosen and the flow field around bus and drag coefficient was calculated.

1 Governing Equations

Let's confine our attention to incompressible flow. The Reynolds averaged equations of motion (continuity and momentum) in conservation form are

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0, \quad (1)$$

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial}{\partial x_j} (\bar{u}_j \bar{u}_i) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \left((\mu + \mu_T) \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \right). \quad (2)$$

Turbulent viscosity μ_T is calculated as

$$\mu_T = \rho C_\mu f_\mu \frac{k^2}{\varepsilon}. \quad (3)$$

There are two variables in equation for turbulent viscosity, k being turbulent kinetic energy and ε is turbulent dissipation. We write different transport equations for turbulent kinetic energy

$$\rho \frac{\partial k}{\partial t} + \rho \bar{u}_j \frac{\partial k}{\partial x_j} = P - \rho \varepsilon - \rho D + \frac{\partial}{\partial x_j} \left(\left(\mu + \frac{\mu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right), \quad (4)$$

and for turbulent dissipation

$$\rho \frac{\partial \varepsilon}{\partial t} + \rho \bar{u}_j \frac{\partial \varepsilon}{\partial x_j} = C_{\varepsilon 1} f_1 \frac{\varepsilon}{k} P - C_{\varepsilon 2} f_2 \rho \frac{\varepsilon^2}{k} + \rho E + \frac{\partial}{\partial x_j} \left(\left(\mu + \frac{\mu_T}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right). \quad (5)$$

The production term P is defined as

$$P = \mu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j}. \quad (6)$$

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Equations (3-5) contain five empirical damping functions, f_μ , f_1 , f_2 , D and E that depend upon one or more of the following turbulent Reynolds numbers

$$Re_\tau = \frac{k^2}{\nu \varepsilon}, \quad R_y = \frac{y\sqrt{k}}{\nu}, \quad Re_\tau = \frac{y u_\tau}{\nu}. \quad (7)$$

The right choice of dumping functions is of main importance for the accuracy of computation. In the past, many researchers developed their own turbulence models and the only difference between models is choice of dumping functions and model constants.

2 Turbulent $k - \varepsilon$ low-Reynolds models

We used three different turbulence models for calculation, Lam-Bremhorst's [3], Nagano's [4] and Fan-Lakshminarayana-Barnett's [1] model. Dumping functions and model constants are given in Table 1.

model		C_μ	$C_{\varepsilon 1}$	$C_{\varepsilon 2}$	σ_k	σ_ε	ε_w
Lam-Bremhorst	LB	0.09	1.44	1.92	1.0	1.3	$\frac{\partial \varepsilon}{\partial y} = 0$
Nagano	NAG	0.09	1.45	1.9	1.0	1.3	$\varepsilon = 0$
Fan-Lakshminarayana-Barnett	FLB	0.09	1.39	1.8	1.0	1.3	$\frac{\partial \varepsilon}{\partial y} = 0$

LB	$f_\mu = (1 - e^{-0.0165 R_y})^2 (1 + \frac{20.5}{Re_\tau}),$ $f_1 = 1 + (\frac{0.05}{f_\mu})^3,$ $f_2 = 1 - e^{-Re_\tau^2},$ $D = 0,$ $E = 0$
NAG	$f_\mu = (1 - \exp(-\frac{Re_\tau}{26.5}))^2,$ $f_1 = 1,$ $f_2 = 1 - 0.3 \exp(-Re_\tau^2),$ $D = 2\nu(\frac{\partial \sqrt{k}}{\partial x_j})^2,$ $E = \nu \nu_\tau (1 - f_\mu)(\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k})^2$
FLB	$f_\mu = 0.4 \frac{f_w}{\sqrt{Re_\tau}} + (1 - 0.4 \frac{f_w}{\sqrt{Re_\tau}})(1 - e^{-R_y/42.63})^3,$ $f_1 = 1,$ $f_2 = (1 - e^{-(Re_\tau/6)^2}) f_w^2,$ $f_w = 1 - \exp(-\frac{\sqrt{R_y}}{2.30} + (\frac{\sqrt{R_y}}{2.30} - \frac{R_y}{8.89})(1 - e^{-R_y/20})^3),$ $D = 0,$ $E = 0$

Table 1: Dumping functions and turbulent model constants

3 Solving of Reynolds Equations

The Reynolds averaged equations with the artificial compressibility term added and with transport equations for turbulent kinetic energy and turbulent dissipation can be expressed

in generalized coordinates $\xi = \xi(x, z)$, $\zeta = \zeta(x, z)$ and dimensionless form as one vector equation

$$\frac{\partial Q}{\partial t} + \frac{\partial \mathcal{E}}{\partial \xi} + \frac{\partial \mathcal{G}}{\partial \zeta} = \frac{\partial \mathcal{E}_{vis}}{\partial \xi} + \frac{\partial \mathcal{G}_{vis}}{\partial \zeta} + S, \quad (8)$$

where

$$Q = JQ, \quad (9a)$$

$$\mathcal{E} = J \left(\frac{\partial \xi}{\partial x} E + \frac{\partial \xi}{\partial z} G \right), \quad (9b)$$

$$\mathcal{G} = J \left(\frac{\partial \zeta}{\partial x} E + \frac{\partial \zeta}{\partial z} G \right), \quad (9c)$$

$$\mathcal{E}_{vis} = J \left(\frac{\partial \xi}{\partial x} E_{vis} + \frac{\partial \xi}{\partial z} G_{vis} \right), \quad (9d)$$

$$\mathcal{G}_{vis} = J \left(\frac{\partial \zeta}{\partial x} E_{vis} + \frac{\partial \zeta}{\partial z} G_{vis} \right), \quad (9e)$$

$$S = JS. \quad (9f)$$

and

$$Q = \left\{ p/\beta, u, w, k, \varepsilon \right\}, \quad (10a)$$

$$E = \left\{ u, u^2 + p, uw, uk, u\varepsilon \right\}, \quad G = \left\{ w, uw, w^2 + p, wk, w\varepsilon \right\}, \quad (10b)$$

$$E_{vis} = \left\{ 0, \tilde{\tau}_{xx}, \tilde{\tau}_{xz}, \tilde{k}_x, \tilde{\varepsilon}_x \right\}, \quad G_{vis} = \left\{ 0, \tilde{\tau}_{xz}, \tilde{\tau}_{zz}, \tilde{k}_z, \tilde{\varepsilon}_z \right\}, \quad (10c)$$

$$S = \left\{ 0, 0, 0, P - \epsilon, C_{\epsilon 1} f_1 \frac{\epsilon}{k} P - C_{\epsilon 2} f_2 \frac{\epsilon^2}{k} \right\}, \quad (10d)$$

$$\tilde{\tau} = (\mu + \mu_T) \begin{bmatrix} 2 \frac{\partial u}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & 2 \frac{\partial w}{\partial z} \end{bmatrix}, \quad (10e)$$

$$\tilde{k} = \left(\mu + \frac{\mu_T}{\sigma_k} \right) \left\{ \frac{\partial k}{\partial x}, \frac{\partial k}{\partial z} \right\}, \quad (10f)$$

$$\tilde{\varepsilon} = \left(\mu + \frac{\mu_T}{\sigma_\varepsilon} \right) \left\{ \frac{\partial \varepsilon}{\partial x}, \frac{\partial \varepsilon}{\partial z} \right\}, \quad (10g)$$

$$J = \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \zeta} - \frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial \xi}. \quad (10h)$$

The equation 8 is solved using finite volume method. The characteristics based method (CBM) is used for the discretization of the convective fluxes. The CBM is based on a characteristic flux averaging of the inviscid fluxes according to the sign of the local eigenvalue of Riemann system. For discretization of the viscous fluxes an 'upwind'-type scheme for the cross derivatives and central discretization for the second-order derivatives were used. For the time integration the explicit fourth-order Runge-Kutta time stepping method was employed. The method was developed by Drikakis [2].

4 Examples

4.1 Turbulent flow in channel with a step

Different models described here have been tested on the case of the channel flow with a backwards facing step. That case is complex enough to show the drawbacks of the model and at the same time there are many experiments to compare with calculated results. The flow was calculated for Reynolds number $Re = 15000$. The Reynolds number was based on channel inlet height and maximum inlet velocity. Shape of the problem is shown in Figure 1.

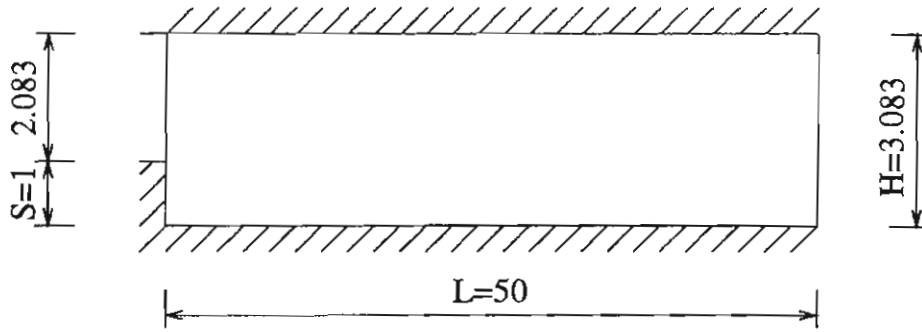


Figure 1: Backwards facing step: shape and boundary conditions

Boundary conditions:

- inlet: $\partial p / \partial n = 0$, $u = \text{measured}$, $w = 0$, $k = 0.01$, $\varepsilon = C_\mu \frac{k^{3/2}}{0.02}$,
- outlet: $p = 0$, $\partial u / \partial n = 0$, $\partial w / \partial n = 0$, $\partial k / \partial n = 0$, $\partial \varepsilon / \partial n = 0$,
- wall: $\partial p / \partial n = 0$, $u = 0$, $w = 0$, $k = 0$, ε depends on model.

Three different mesh sizes were used: 30×20 , 50×30 and 80×50 .

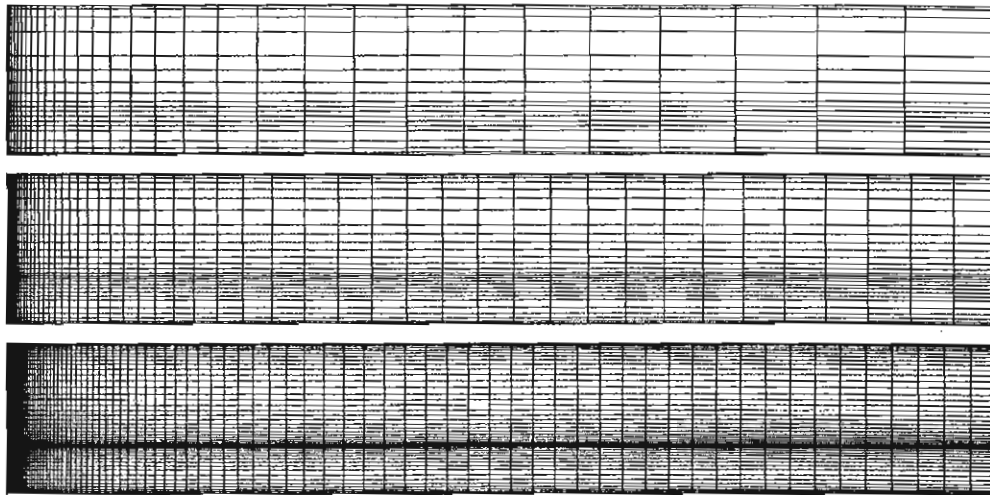


Figure 2: Domain discretization

Results were compared against measurements of Ruck and Makiola [5]. We investigated the accuracy of turbulence models.

In Figure 3 comparison of calculated velocity profiles, turbulent kinetic energy and turbulent dissipation profiles between three turbulence models is shown. Profiles were taken on different distances from the inlet ($X/S = 1, 2, 5$ and 30).

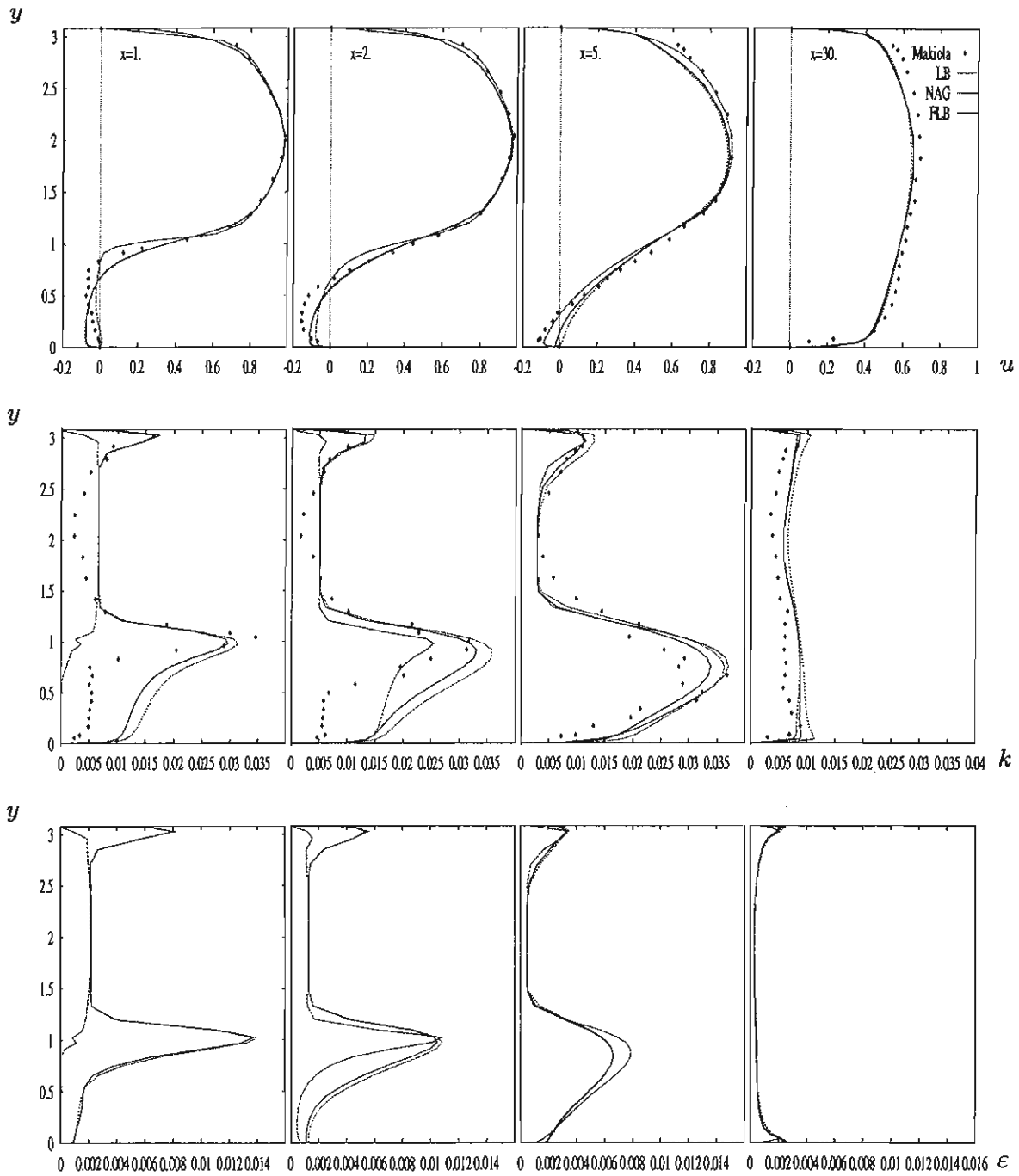


Figure 3: Comparison of profiles for different models, $Re = 15000$

Comparison of the results showed that Lam-Bremhorst and Fan-Lakshminarayana-Barnett model give similar results but the second is slightly better. Both models underestimate recirculation length. Nagano's model overestimates recirculation length and calculated profiles are in better agreement with measurements as both previous models, but the price for better results is longer computational time.

4.2 Turbulent flow past vehicle

To prove practical meaning of turbulence low-Reynolds models we calculated flow field around city traffic bus. In Figure 4 is depicted problem in nondimensional units. The height of the bus was used as the unit for nondimensionalization. The original height of the bus was $L = 2.41$ m.

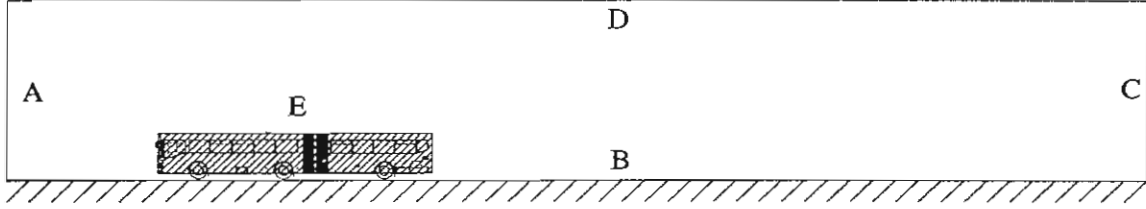


Figure 4: Bus: domain and boundary conditions

Boundary conditions:

- A: $\partial p / \partial n = 0$, $u = 1$, $w = 0$, $k = 0.01$, $\varepsilon = C_\mu \frac{k^{3/2}}{0.02}$,
- B: $\partial p / \partial n = 0$, $u = 1$, $w = 0$, $k = 0$, ε depend on model,
- C: $\partial p / \partial n = 0$, $\partial u / \partial n = 0$, $\partial w / \partial n = 0$, $\partial k / \partial n = 0$, $\partial \varepsilon / \partial n = 0$,
- D: $p = 0$, $\partial u / \partial n = 0$, $\partial w / \partial n = 0$, $\partial k / \partial n = 0$, $\partial \varepsilon / \partial n = 0$,
- E: $\partial p / \partial n = 0$, $u = 0$, $w = 0$, $k = 0$, ε depends on model.

Initial velocity values in domain were constant over domain and equal to the values on the inlet. Value of Reynolds number for the calculation was $Re = 15000$, based on the bus height and free stream velocity.

The mesh size was 50×30 , 80×50 and 120×80 points. The mesh was clustered near solid walls. Domain discretization is depicted in Figure 5.

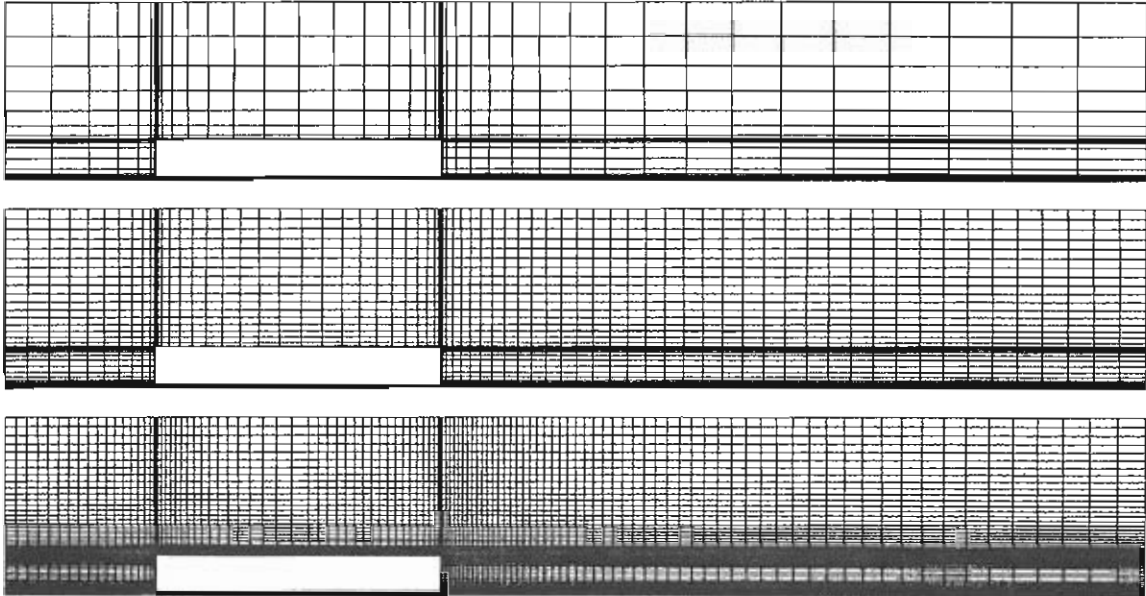


Figure 5: Domain discretization

In Figure 6 comparison of velocity profiles, turbulent kinetic energy and turbulent dissipation on different sections is shown. Results obtained with Fan-Lakshminarayana-Barnett model and program FIDAP were compared.

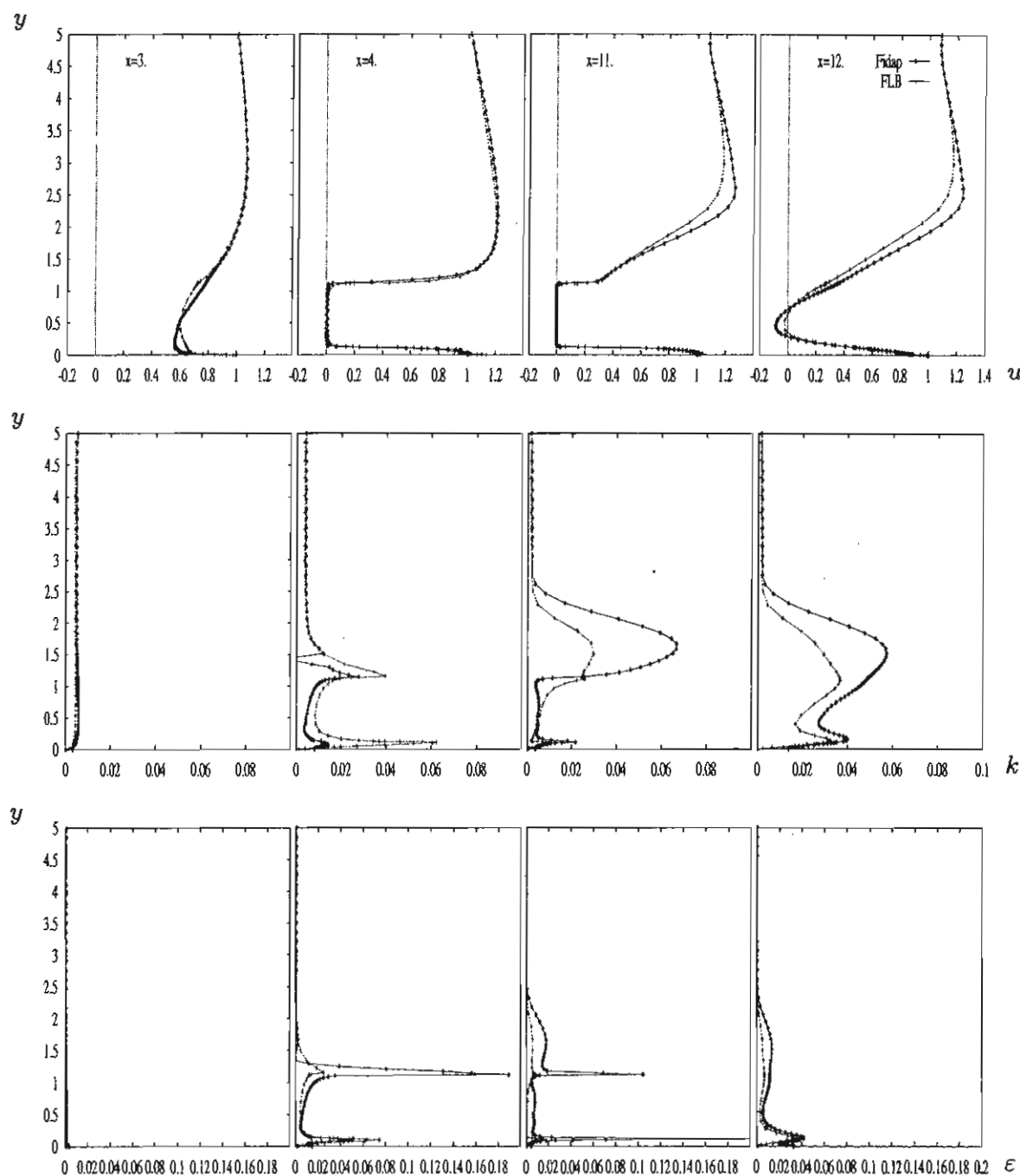


Figure 6: Comparison of profiles for different models, $Re = 15000$

The comparison showed that velocity profiles are in good agreement. Greater differences appeared in turbulent kinetic energy profiles and turbulent dissipation profiles. The greatest disagreement appear near solid walls. The reason for that is different approach to the modelling of turbulent quantities in wall region. FIDAP uses wall functions that give good results in areas without big pressure gradients and worse results where pressure gradients arise. Turbulence models with dumping functions calculate all of the turbulent quantities through the boundary layer and the results are adequately better.

In Table 2 the drag coefficients for three different meshes are presented. In the second column are results of program FIDAP and in the third column results of Fan-Lakshminarayana-Barnett model.

Mesh	FIDAP	FLB
50 × 30	1.82	1.74
80 × 50	1.64	1.69
120 × 80	1.63	1.65

Table 2: Drag coefficients, $Re = 15000$

5 Conclusions

The aim of contribution is the study of turbulent incompressible viscous fluid flow in channels and around obstacles. Reynolds equations have been closed by two-equation turbulence model. Dumping functions were used to account for solid walls effect. Three different turbulence models have been used: Lam-Bremhorst's, Nagano's and Fan-Lakshminarayana-Barnett's model. The system of partial differential equations has been solved by finite volume method. Models have been tested on the case of backward facing step flow and flow past vehicle.

In the case of flow over backward facing step comparison of the results showed that Lam-Bremhorst and Fan-Lakshminarayana-Barnett model give similar results but the second is slightly better. Both models underestimate recirculation length. Nagano's model overestimate recirculation length and calculated profiles are in better agreement with measurements as both previous models. The use of different parameters in dumping functions is the reason for that difference.

In the second case we calculated flow field around the city traffic bus. The measurements for that case don't exist so we compared our results with the results of program FIDAP. The comparison showed that velocity profiles are in good agreement. Greater differences appeared in turbulent kinetic energy profiles and turbulent dissipation profiles. In spite of that are the differences between drag coefficients quite small, less than 2%.

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Boundary - Domain Integral Formulation for the Numerical Simulation of the 3D Incompressible Viscous Fluid Flow

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Abstract

Three-dimensional numerical solution of fluid dynamics using boundary-domain integral method is considered. A velocity-vorticity formulation of the Navier-Stokes equations is adopted. The kinematic equation is written in its parabolic version.

1 Introduction

Bouyancy-driven fluid flow analysis in enclosures has many thermal engineering applications, such as heating and cooling of buildings, the cooling of electronic components and solar energy collectors. In the contemporary design of such devices computational fluid dynamics or better, numerical modelling and simulation of complex fluid flow are of great theoretical and practical importance.

In this paper boundary-domain integral method is presented for the solution of general fluid motion problem which is based on the velocity-vorticity formulation of the Navier-Stokes equations. Particular attention is given to formulate appropriate integral representations for all field functions based on the three-dimensional modified Helmholtz fundamental solution. A lot of attention is given also to find appropriate integration procedure. To optimize the position of Gauss points and to reduce the degree of singularity the polar transformation is applied. For the kinematic velocity equation the false transient approach is applied to increase the convergency of the proposed numerical algorithm. The large sparse system of nonlinear equations is solved with iterative solver. Results of computer code *BEEAS - modul BEMFLOW3D* - are compared with benchmark solutions.

2 Governing Equations in Velocity-Vorticity Formulation

The partial differential equations set governing the transport phenomena in incompressible fluid motion represents the basic conservation balances of mass, momentum

and energy

$$\frac{\partial v_j}{\partial x_j} = 0, \quad (1)$$

$$\rho_o \frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho_o f_i, \quad (2)$$

$$\rho_o c_p \frac{DT}{Dt} = -\frac{\partial q_j}{\partial x_j} \pm I_T + \Phi, \quad (3)$$

where v_i is the i -th instantaneous velocity component, x_i is the i -th coordinate, D/Dt represents the substantial or Stokes derivative, p is the pressure, τ_{ij} is the viscous stress tensor and f_i stands for the body force, e.g. the gravity g_i . The quantities ρ_o , c_p , and I_T are respectively the constant fluid mass density, specific isobaric heat capacity and heat source or sink. The variables T and q_j stand for temperature and heat flux. The term Φ is the Rayleigh viscous dissipation function.

With the vorticity vector ω_i representing the curl of the velocity field, e.g. written in the symbolic notation

$$\omega_i = e_{ijk} \frac{\partial v_k}{\partial x_j}, \quad \frac{\partial \omega_i}{\partial x_i} = 0, \quad (4)$$

the fluid motion computation scheme is partitioned into its kinematic and kinetic aspect. The kinetics is governed by the vorticity transport equation obtained as a curl of the momentum eq.(2), rendering the statement

$$\frac{D\omega_i}{Dt} = \frac{\partial \omega_j v_i}{\partial x_j} + \nu_o \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} + e_{ijk} g_k \frac{\partial F}{\partial x_j}, \quad (5)$$

describing the redistribution of the vorticity vector in fluid flow field.

Applying the curl operator directly to the vorticity definition and using continuity equation for incompressible fluid flow the kinematics can be formulated in the form of vector elliptic Poisson's equation for the velocity vector

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} + e_{ijk} \frac{\partial \omega_k}{\partial x_j} = 0. \quad (6)$$

The eq.(6) represents the kinematics of an incompressible fluid motion or the compatibility of the velocity and vorticity fields at a given point in space and time.

In a general fluid motion case the proper boundary condition associated to parabolic kinetic eq.(5) is the vorticity definition written for the boundary of the solution domain, e.g.

$$\omega_i = e_{ijk} \frac{\partial v_k}{\partial x_j} \quad \text{on } \Gamma. \quad (7)$$

The above statement is essential for the conservation of the solenoidality of the vorticity vector. It requires the solution of the coupled mass conservation problem. The boundary condition assigned to elliptic kinematic eq.(6) is the velocity vector prescribed on the boundary, e.g.

$$v_i = \bar{v}_i \quad \text{on } \Gamma. \quad (8)$$

The kinematics given by eq.(6) and the velocity boundary condition eq.(8) cannot assure a solenoidality of the velocity field for an arbitrary vorticity distribution, so this property may be fulfilled only by coupling kinetic and kinematic governing equations. Thus, the solenoidality constraints on velocity and vorticity fields require a coupled iterative solution of the nonlinear system given by eqs.(5) and (6), with the corresponding boundary conditions prescribed by eqs.(7) and (8).

To accelerate the convergency of the coupled velocity-vorticity iterative scheme, the false transient approach is used for the kinematic equation, thus in a solution scheme the parabolic eq.(5) and the following parabolized kinematic eq.(6)

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \frac{1}{\alpha} \frac{\partial v_i}{\partial t} + e_{ijk} \frac{\partial \omega_k}{\partial x_j} = 0, \quad (9)$$

where α is a relaxation parameter, have to be solved iteratively. It is obvious that the governing velocity equation is exactly satisfied only at the steady state ($t \rightarrow \infty$), when the time derivative vanishes.

3 Integral Representation of Velocity-Vorticity Formulation

The idea is to combine the finite difference method in time and boundary element method in space. One can consider that each component of the vorticity vector ω_i ($i = 1, 2, 3$) is governed by a nonhomogeneous modified Helmholtz eq.(13)

$$\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} - \beta \omega_i + b_i = 0 \quad \text{in } \Omega, \quad (10)$$

subject to the normal and essential boundary conditions. The initial condition is included beside convection, deformation and buoyancy source term in pseudo body force vector

$$b_i = \frac{1}{\nu_o} \frac{\partial}{\partial x_j} (-v_j \omega_i + \omega_j v_i + e_{ijk} g_k F) + \beta \omega_{i,F-1}, \quad (11)$$

yielding the integral representation for the spatial vorticity kinetics

$$c(\xi) \omega_i(\xi) + \int_{\Gamma} \omega_i \frac{\partial u^*}{\partial n} d\Gamma = \frac{1}{\nu_o} \int_{\Gamma} (\nu_o \frac{\partial \omega_i}{\partial n} - \omega_i v_n + v_i \omega_n + e_{ijk} n_j g_k F) u^* d\Gamma + \frac{1}{\nu_o} \int_{\Omega} (\omega_i v_j - v_i \omega_j - e_{ijk} g_k F) \frac{\partial u^*}{\partial x_j} d\Omega + \beta \int_{\Omega} \omega_{i,F-1} u^* d\Omega, \quad (12)$$

where u^* is the modified Helmholtz fundamental solution, i.e. the solution of equation

$$\frac{\partial^2 u^*}{\partial x_j \partial x_j} - \beta u^* + \delta(\xi, s) = 0 \quad (13)$$

and given for the space case by expression

$$u^* = \frac{\beta^{1/4}}{(8\pi^3 r)^{1/2}} K_{1/2}(\sqrt{\beta} r), \quad (14)$$

being $K_{1/2}$ the modified Bessel function of the second kind.

Recognizing that each component of the velocity vector $v_i (i = 1, 2, 3)$ obeys a non-homogeneous modified Helmholtz eq.(13)

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \beta v_i + b_i = 0 \quad \text{in } \Omega, \quad (15)$$

subject to Dirichlet and Neumann boundary conditions being now the pseudo body force vector equals to expression

$$b_i = e_{ijk} \frac{\partial \omega_k}{\partial x_j} + \beta v_{i,F-1}, \quad (16)$$

the following integral representation for the spatial kinematics can be written in a straightforward manner

$$\begin{aligned} c(\xi) v_i(\xi) + \int_{\Gamma} v_i \frac{\partial u^*}{\partial n} d\Gamma &= \int_{\Gamma} \left(\frac{\partial v_i}{\partial n} - e_{ijk} \omega_j n_k \right) u^* d\Gamma + \\ &\int_{\Omega} e_{ijk} \omega_j \frac{\partial u^*}{\partial x_k} d\Omega + \beta \int_{\Omega} v_{i,F-1} u^* d\Omega. \end{aligned} \quad (17)$$

4 Discretization and Numerical Solution

For the numerical approximate solution of various considered field functions, e.g. velocity, vorticity, temperature etc., the corresponding boundary-domain integral representation is written in discretized manner in which the integrals over the boundary and domain are approximated by a sum of integrals over E individual boundary elements and C internal cells, respectively. Next the variation of all field functions or their products within each boundary element or internal cell is approximated by the use of interpolation polynomials Φ or ϕ with respect to boundary or domain and nodal function values. Let the index n refers to the number of nodes in each boundary element or internal cell and also relates to the degree of the respective interpolation polynomials. It should be mentioned that in general the degree of interpolation polynomials on the boundary elements and internal cells can differ, making this numerical technique very flexible. In the discretized equations for the kinematics, vorticity kinetics and energy transport the integrals such as h , g , d_i , b etc. are involved, representing the integration over individual boundary element and internal cell respectively, i.e.

$$\begin{aligned} h_e^n &= \int_{\Gamma_e} \Phi^n \frac{\partial u^*}{\partial n} d\Gamma, \\ g_e^n &= \int_{\Gamma_e} \Phi^n u^* d\Gamma, \\ d_{ci}^n &= \int_{\Omega_c} \phi^n \frac{\partial u^*}{\partial x_i} d\Omega, \\ b_c^n &= \int_{\Omega_c} \phi^n u^* d\Omega \quad \text{etc.}, \end{aligned} \quad (18)$$

which are functions of the geometry and material properties.

In developed computer code *BEEAS - modul BEMFLOW3D* - 4-nodes linear boundary elements and constant internal cells are used for the computation of integrals mentioned above. To optimize the position of Gauss points and to reduce the degree of singularity the polar transformation is applied. The discretization of integral representations for velocity, vorticity and temperature is obtained as usual.

5 Test Examples

Some preliminary tests are studied first. One of them is numerical simulation of entry flow with $v_x = 1$, $v_y = 0$ and $v_z = 0$. In computational model mesh with $5 \times 1 \times 1$ internal cells is used (figure 1).

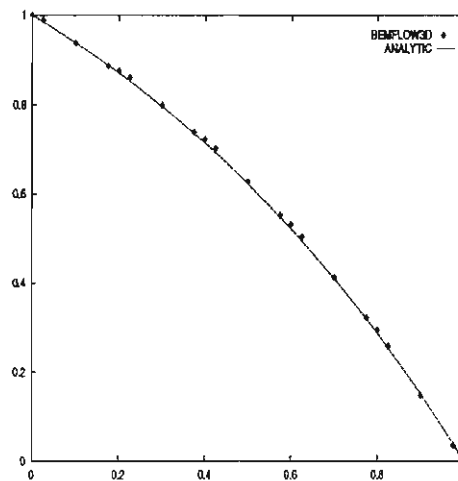


Figure 1: Entry flow, Steady state, $t = 0.1E + 16$, $Pe=1$

Couette flow - flow between two parallel plates - for Reynolds number value equals 10 is studied in the test example No. 2 (figure 2). While bottom plate is fixed, top plate is moving with $v_x = 10$, $v_y = 0$ and $v_z = 0$. All other fluid properties are in accordance with $Re = 10$. Mesh consists of $8 \times 2 \times 8$ internal cells.

The last test example, problem of natural convection, is often used for testing of the applied numerical scheme. Equations of motion and energy are coupled through the buoyancy term introducing additional nonlinearity in the nonlinear system of equations. The determination of the temperature field turns to be most delicate, since the numerical errors in the velocity field are magnified while determining temperatures, what may cause a nonconvergency of the procedure.

The evaluated three-dimensional case is shown in figure 3. The fluid motion is due to heating on the rear wall, while on the front wall a relative zero temperature was kept. The remained boundaries are assumed to behave ideally isolated. Rayleigh number value equals 1000. The mesh is of the same kind like in previous case.

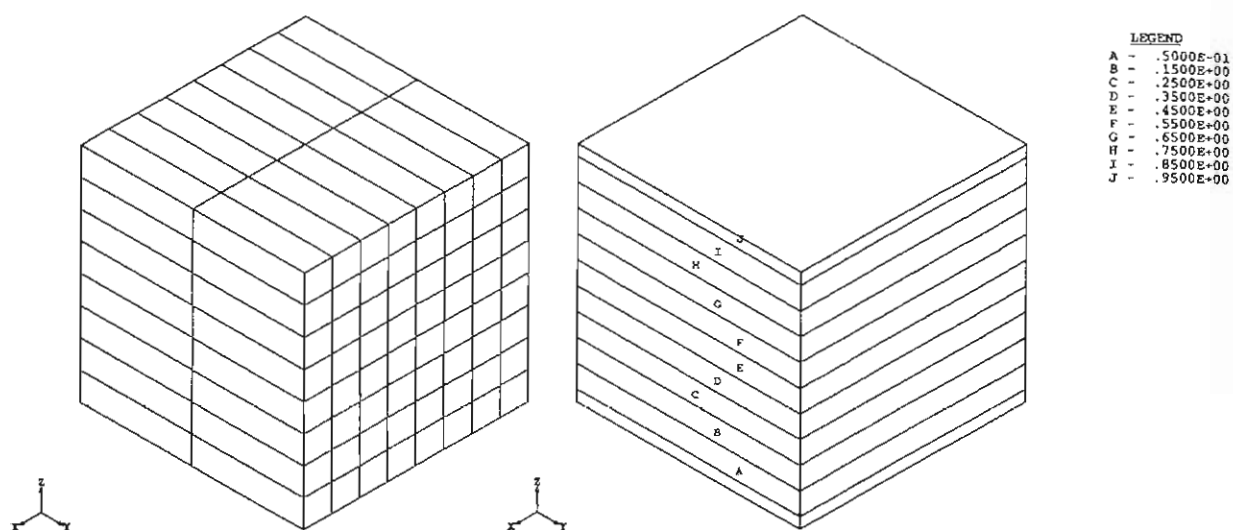


Figure 2: Couette flow: Discretization and speed contour plot, $Re=10$

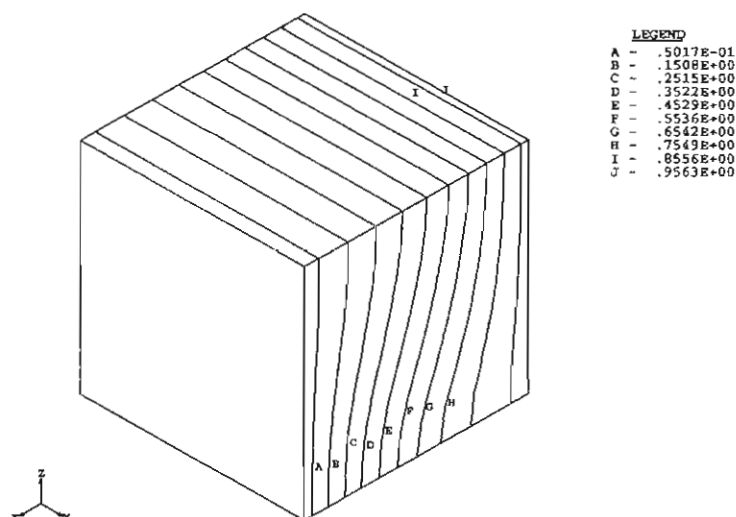


Figure 3: Natural convection: Temperature profile, $Ra=1000$

6 Conclusion

The boundary domain integral method is used to solve three-dimensional viscous fluid flow and energy transport problems. Owing to the spatial modified Helmholtz fundamental solution a part of the transport mechanism is transferred to the solution surface. This procedure should result in a very stable and accurate numerical algorithm. But, there are many difficulties during the evaluation of matrix coefficients. The integration process is not exact enough and it takes a great amount of computational time to compute the coefficients. Due to all mentioned above the development of the code is still in progress. Preliminary comparisons of *BEEAS - modul BEMFLOW3D* - results with analytical solutions show good agreements.

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Penalty Function Formulation of Incompressible Viscous Flows and FE Method

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Summary

This paper deals with the incompressible flow through various geometrical configurations. Basic features of the penalty method are described in the context of unsteady Navier Stokes equations. Numerical results of static and dynamical problems are presented and discussed to present the effectiveness of the method.

1. Introduction

The finite element method (FEM) is an established numerical technique that is being widely used in structural and fluid mechanics. The main advantage offered by FEM over other numerical methods is easy manoeuvrability of complex geometries and its ability to incorporate a variety of boundary conditions. In the last decade the FEM has been extensively used to evaluate problems in fluid mechanics (eg. [2], [3], [4], [5], [11]). With recent increases of computational power and a rise in available computer memory it became possible to analyse rather complex geometry at desktop level [11].

In this paper a description of the theory and development of an efficient and accurate method for solving incompressible Navier Stokes equations is given. The obtained results indicate that it is possible to implement FEM for the solution of complex 2D-problems even in a local computational environment.

2. Basic Equations

The constitutive equation of two dimensional Navier Stokes flow with constant viscosity and without buoyancy effects is given by

$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \frac{1}{\rho_0} \nabla p + \frac{1}{\text{Fr}} \mathbf{b} = 0 \quad (2)$$

where ρ_0 denotes the constant density, $\mathbf{u}$ the two dimensional velocity vector, $\mathbf{b}$ the body (gravity) force and p the pressure. Mass conservation is formulated by Eq.(1.) The Reynolds number is given by $\text{Re} = \frac{u_\infty l}{\nu_\infty}$, where the subscript ∞ denotes the ambient

velocity. The body force will be neglected in the further discussion, which devotes itself wholly to 2D constellations.

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Eq.(2) can be written as

$$A(u) = \frac{\partial u_i}{\partial x} + \frac{\partial}{\partial x_j} \left(u_i u_j + \frac{1}{\rho_0} p \delta_{ij} - \frac{1}{\text{Re}} \frac{\partial u_i}{\partial x_j} \right) = 0 \quad (3)$$

where $0 < (i,j) < 3$. The boundary conditions that will be taken into account in this part of the calculation are velocity fields, that are orthogonal to the boundary of the solution domain [1].

$$B(u) = a_i' \frac{\partial u_i}{\partial x_j} \hat{n}_j = 0 \quad (4)$$

The FE approximation of the fluid velocity and the pressure is given by

$$u_i^h(x_j, t) \equiv \sum_e u_i^e(x_j, t) = \sum_e \{N_k(x_j)\}^T \{q_i(t)\}_e \quad (5)$$

$$p^h(x_j, t) \equiv \sum_e p^e(x_j, t) = \sum_e \{N_r(x_j)\}^T \{P(t)\}_e \quad (6)$$

The vectors $\{N_k\}$, $\{N_r\}$ consist of the so called basis (form) functions. Form functions are dependent of the element type, they are polynomials of the space co-ordinates complete to the degree k and r , where $k > r$. The right hand sides of the eq.(5) and (6) indicate a summation over all elements.

By using the method of weighted residuals combined with the penalty function formulation [8] one obtains the finite element algorithm statement

$$\int_{R^2} \{N_k\} A(u_i^h) d\tau + \beta \int_{\partial R} \{N_k\} B(u_i^h) d\sigma + \alpha \int_{R^2} \frac{\partial}{\partial x_i} \{N_k\} \left(p^h + \lambda_h \frac{\partial u_j^h}{\partial x_j} \right) d\tau \equiv \{0\} \quad (7)$$

Using standard vector relations one obtains

$$\begin{aligned} & \overbrace{\int_{R^2} \{N_k\} \frac{\partial u_i^h}{\partial x} d\tau}^1 + \overbrace{\int_{\partial R} \{N_k\} \left(u_i^h u_j^h + \frac{1}{\rho_0} p^h \delta_{ij} - \frac{1}{\text{Re}} \frac{\partial u_i^h}{\partial x_j} \right) \cdot \hat{n}_j d\sigma}^2 - \overbrace{\int_{R^2} \frac{\partial}{\partial x_j} \{N_k\} \left(u_i^h u_j^h + \frac{1}{\rho_0} p^h \delta_{ij} - \frac{1}{\text{Re}} \frac{\partial u_i^h}{\partial x_j} \right) d\tau}^3 \\ & + \overbrace{\int_{\partial R} \{N_k\} \left(a_i' \frac{\partial u_i^h}{\partial x_j} \hat{n}_j \right) d\tau}^4 + \overbrace{\alpha \int_{R^2} \frac{\partial}{\partial x_i} \{N_k\} \left(p^h + \lambda_h \frac{\partial u_j^h}{\partial x_j} \right) d\tau}^5 \equiv \{0\} \end{aligned} \quad (8)$$

Defining α to be equal to $\alpha = \lambda_h / \rho_0$ the pressure gradient terms contained in the brackets 3 and 5 cancel each other. $\alpha \beta$ is set equal to λ_h / Re and hence the corresponding term in bracket 2 is compensated by the bracket term 4. Finally the no slip condition inherent in viscous flows leads to a cancellation of the rest surface integral in bracket 2. Thus eq.(8) can be rewritten as the penalty function formulation of the Navier-Stokes equation [6], [7] as follows

$$\int \{N_k\} \frac{\partial u_i^h}{\partial x} d\tau - \sum_{j=1}^2 \int \frac{\partial \{N_k\}}{\partial x_j} \left(u_i^h u_j^h - \frac{1}{\text{Re}} \frac{\partial u_i^h}{\partial x_j} \right) d\tau + \lambda_h \int \frac{\partial \{N_k\}}{\partial x_i} \sum_j \frac{\partial u_j^h}{\partial x_j} d\tau = 0 \quad (9)$$

Equation (9) can be expressed in matrix form as stated below

$$[C] \{u_i\}' + \{K\} + \{b\} = \{0\} \quad (10)$$

$\{K\}$ in Eq.(1) is a non-linear function of u as expressed in the second term of eq.(9).

The insertion of the equations (5) and (6) in equation (9) leads in the first instance to a semi-discrete formulation of the Navier Stokes equation. In the next step the full discretisation is achieved by a discretisation in the time domain, where it has been convenient to adopt the θ -implicit finite difference integration algorithm [4], which states that

$$\frac{\partial u_i}{\partial t} = f \Leftrightarrow \frac{u_i^{p+1} - u_i^p}{\Delta t} = \Theta f^p + (1 - \Theta) f^{p+1} \quad (11)$$

where p denotes the time step, with the choice $\Theta = \frac{1}{2}$ one obtains the trapezoidal rule. One then obtains the so called weak formulation of the Navier Stokes equation, which consists of two equations for the two velocity components:

$$F_i \Rightarrow \frac{\{N_k\}\{N_l\}(q_{l,i}^{p+1} - q_{l,i}^p)}{\Delta t} = \Theta \left[\frac{\partial \{N_k\}}{\partial x_1} \left(\{N_l\} q_{l,i}^p \{N_l\} q_{l,i}^p - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_1} q_{l,i}^p \right) + \frac{\partial \{N_k\}}{\partial x_2} \left(\{N_l\} q_{l,i}^p \{N_l\} q_{l,i}^p - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_2} q_{l,i}^p \right) - \lambda \frac{\partial \{N_k\}}{\partial x_1} \left(\frac{\partial \{N_l\}}{\partial x_1} q_{l,i}^p + \frac{\partial \{N_l\}}{\partial x_2} q_{l,i}^p \right) \right] + (1 - \Theta) \left[\frac{\partial \{N_k\}}{\partial x_1} \left(\{N_l\} q_{l,i}^{p+1} \{N_l\} q_{l,i}^{p+1} - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_1} q_{l,i}^{p+1} \right) + \frac{\partial \{N_k\}}{\partial x_2} \left(\{N_l\} q_{l,i}^{p+1} \{N_l\} q_{l,i}^{p+1} - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_2} q_{l,i}^{p+1} \right) - \lambda \frac{\partial \{N_k\}}{\partial x_1} \left(\frac{\partial \{N_l\}}{\partial x_1} q_{l,i}^{p+1} + \frac{\partial \{N_l\}}{\partial x_2} q_{l,i}^{p+1} \right) \right] \quad (11a)$$

The equations (11a) are written as components of a vector $\{F\}$. The adoption of the Newton Raphson method for the solution of non-linear systems of equations as represented by eq.(11) has been proved to be advantageous, since it is amenable to an easy implementation and leads to a rapidly converging solution. The formulation is here applied as follows:

$$[J(F)]^p \{\delta Q\}^{p+1} = -\{F\}^p \quad (12)$$

The solution is given by

$$\{Q\}^{p+1} = \{Q\}^p + \{\delta Q\}^{p+1} \quad (13)$$

where it is assumed that the solution for $t=0$ is known. The matrix $[J(F)]$ is the Jacobian matrix of the vector $\{F\}$ defined above, which can be directly evaluated by

$$[J(F)] = \frac{\partial \{F_i\}}{\partial \{u_j\}}$$

with $i,j=1,2$. One obtains the four elements of the Jacobian matrix

$$\begin{aligned} J_{11} &= \frac{\partial F_1}{\partial q^1} = \frac{\{N_k\}\{N_l\}}{\Delta t} - (1 - \Theta) \left[\frac{\partial \{N_k\}}{\partial x_1} \left(2\{N_l\} q_{l,i}^1 \{N_l\} - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_1} \right) + \frac{\partial \{N_k\}}{\partial x_2} \left(\{N_l\} \{N_l\} q_{l,i}^2 - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_2} \right) - \lambda \frac{\partial \{N_k\}}{\partial x_1} \frac{\partial \{N_l\}}{\partial x_1} \right] \\ J_{12} &= \frac{\partial F_1}{\partial q^2} = -(1 - \Theta) \left[\frac{\partial \{N_k\}}{\partial x_2} \left(\{N_l\} q_{l,i}^2 \{N_l\} \right) - \lambda \frac{\partial \{N_k\}}{\partial x_1} \frac{\partial \{N_l\}}{\partial x_2} \right] \\ J_{21} &= \frac{\partial F_2}{\partial q^1} = -(1 - \Theta) \left[\frac{\partial \{N_k\}}{\partial x_1} \left(\{N_l\} q_{l,i}^1 \{N_l\} \right) - \lambda \frac{\partial \{N_k\}}{\partial x_2} \frac{\partial \{N_l\}}{\partial x_1} \right] \\ J_{22} &= \frac{\partial F_2}{\partial q^2} = \frac{\{N_k\}\{N_l\}}{\Delta t} - (1 - \Theta) \left[\frac{\partial \{N_k\}}{\partial x_1} \left(\{N_l\} \{N_l\} q_{l,i}^1 - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_1} \right) + \frac{\partial \{N_k\}}{\partial x_2} \left(2\{N_l\} q_{l,i}^2 \{N_l\} - \frac{1}{\text{Re}} \frac{\partial \{N_l\}}{\partial x_2} \right) - \lambda \frac{\partial \{N_k\}}{\partial x_2} \frac{\partial \{N_l\}}{\partial x_2} \right] \end{aligned}$$

Initially the vector $\{F\}$ and Jacobian matrix of the above system have to be established for each element. Subsequently they are assembled together in the global domain. The equation system can then be solved by using standard procedures (Gaussian or Cholesky elimination schemes) to obtain $\{\delta Q\}^{p+1}$. The next time step calculation is then initiated with this value in eq.(13).

The penalty terms (terms containing λ) need to be integrated using a lower order integration scheme as compared to the terms containing velocity and pressure. Is the element basis given by polynoms of the degree k , the λ terms are integrated to the order of $k-1$. For example if bilinear elements are used, the penalty terms are integrated on only one point of the element. Since the integration is performed using the Gauss-Coats Eq.(14) integration formula [4] this can be done by using coefficients $n=1$, $H=2$. The other integration is performed with $n=2$, $H=1$ coefficients.

$$I = \int_{-1}^1 f(\xi) d\xi = \sum_1^n H_i f(\xi_i) \quad (14)$$

The list of basis functions and the node arrangement for various elements can be seen in the table.

Basis Functions		Node arrangement	
Velocity	Pressure	Velocity	Pressure
$\{N_1\}$	$\{N_0\}$		
$\{N_2\}$	$\{N_1\}$		
$\{N_1^+\}$	$\{N_0^+\}$		
$\{N_2^+\}$	$\{N_1^+\}$		

The reduced integration procedure was developed by [10] and extended for the problems in fluid dynamics by [8], [9].

3. Implementation

For studying Navier Stokes flows a computer program has been developed on the basis of Diffpack. Diffpack is a collection of C++ classes organised in a libraries for solving partial differential equations, which arise in various applications such as heat transfer, fluid flow etc.. The libraries are constructed for rapid prototyping of simulators for new problems using classes at high abstraction level. Low level classes are also offered for those who prefer to have absolute control over computational processes.

4. Flow configurations

In this paper an excerpt of the results of the FEM-based integration of the Navier Stokes equations for two novel geometrical configurations have been presented. The configuration in Fig.1 consists of an ensemble of three straight blades placed symmetrically around the centre of the geometry, such that the normals to the blade surfaces from the centre subtend an angle of 120° to each other. This configuration with a complexity, only seldom examined represents a 'snapshot' of a simplified rotor and can be considered as an initiation of future rotor calculations. It should however be noted, that the calculations for the stationary constellation do not have any relevance in rotating co-ordinate system. The determination of the flow field around this constellation was the object of a series of computer runs, in which the blade position was rotated trough an angle of 12° successively. The results in fig.1 consist of velocity contours extracted from computer runs for the different blade configurations already mentioned. The Reynolds number during all computer simulations was $Re=200$. These velocity contours correspond very closely with the details of the flow to be expected. Thus the first plot shows two symmetric areas of separation on the inner side of the leading blades leading to a sharp jet impinging on to the blade ahead. This jet however is split into two symmetrical effluxes which bypass this blade to the sides. As to be expected a stable dead water is also developed immediately ahead of the bypassed blade. In the next constellation one observers that the area of separation on the lower blade is successively washed away, and

the efflux out of the ensemble is biased towards the lower side. The events in the second half of the 120° - movement represent the reinstallation of the initial flow field.

The flow field which represents the basis of the vector contours in the fig.2 is wholly unsteady and results out of an integration of the Navier Stokes equations for a configuration of an inclined plate, which oscillates with a frequency corresponding to a Strouhals number $St \approx 1.2 \cdot 10^{-4}$. The ten contour plots show the variation of the flow field during one period, whereby the ten consecutive 'snapshots' are ordered row wise. The plots show an almost stationary recirculation area at the rear of the plate. As is to be expected the bifurcations of the incoming flow at the top of the plate is the faster one. The lower bifurcation stream has to accommodate a recirculation area at or near lower edge and is thus initially directed further to the bottom and regains the main flow direction only after bypassing the recirculating area at the lower tip. One clearly sees the effect of the oscillation especially in the various positions occupied in a cyclic manner by the recirculation area at the lower tip.

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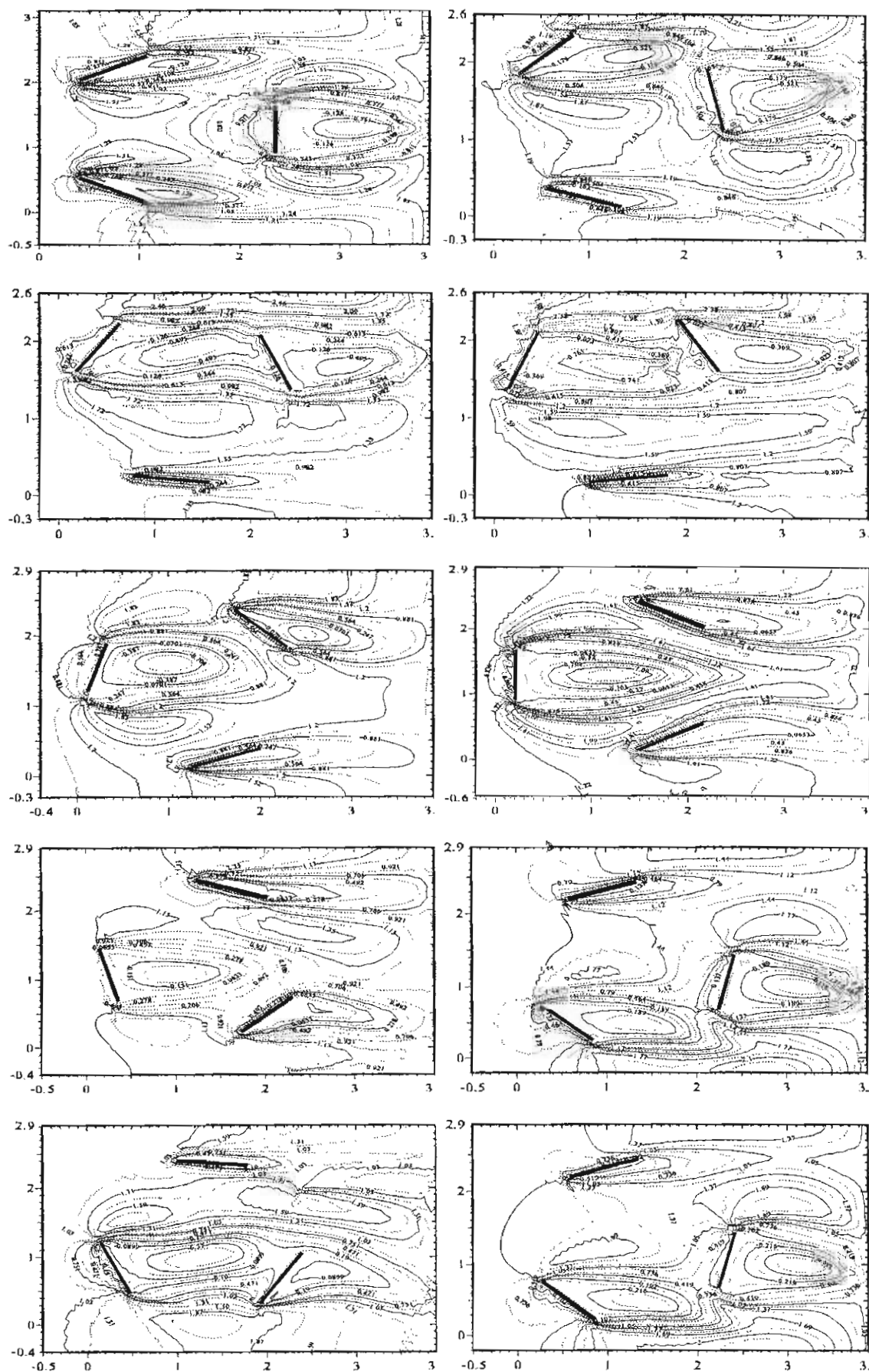


Fig.1

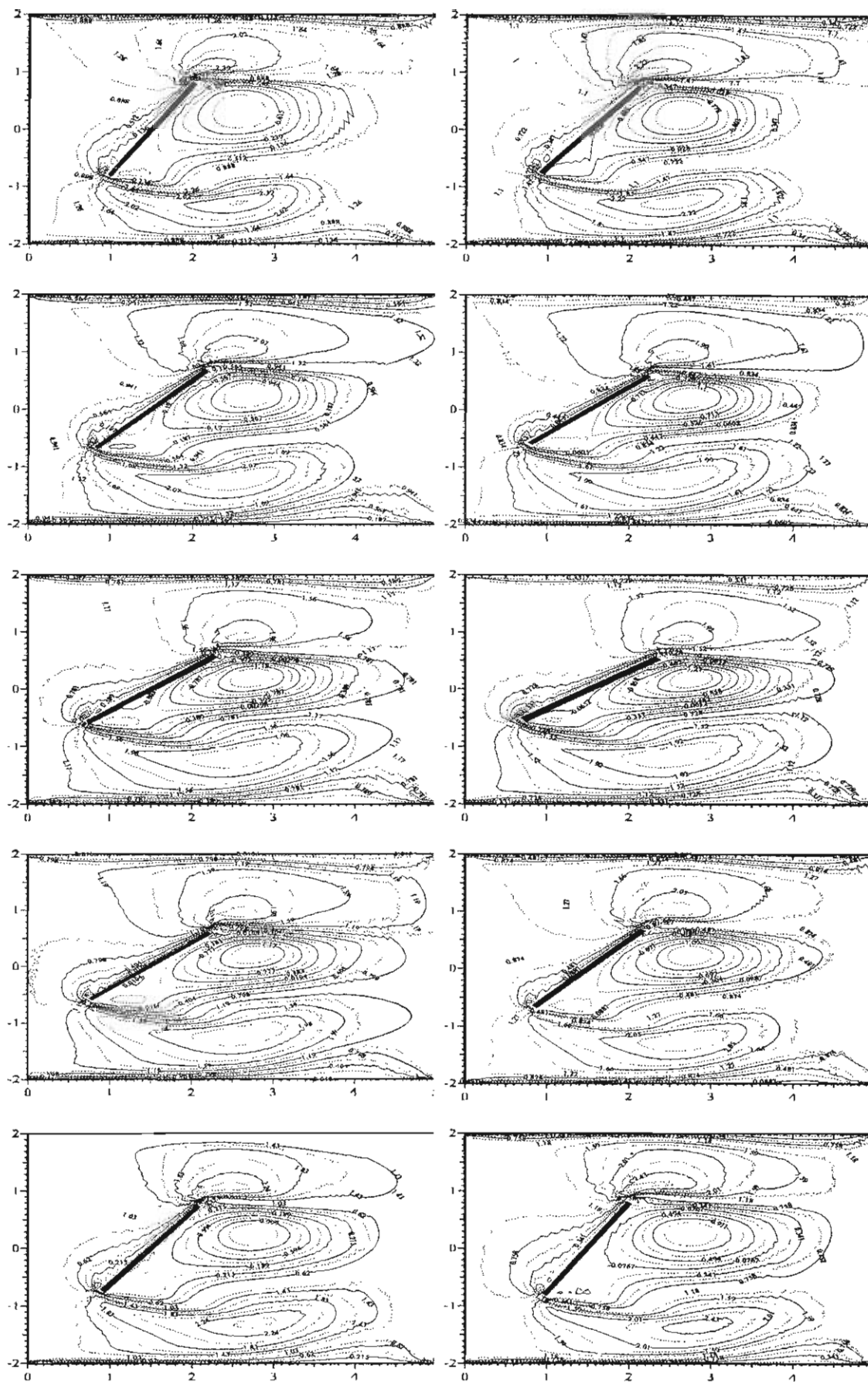


Fig.2

APPLICATION OF ADDITIVE CORECTION METHOD TO FLUID FLOW CALCULATIONS¹

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Summary

Two iterative procedures for solving algebraic equation systems that arise in two-dimensional fluid flow calculations when using a five-point scheme are proposed. The sensitivity of these procedures to the cell-aspect ratio and to the number of equations is examined. The proposed procedures are compared to the Strongly Implicit Procedure (SIP), and then applied to the laminar flow over a backward facing step. The obtained results demonstrate robustness of proposed methods as well as a significant improvement (with respect to SIP) in convergence rate of the SIMPLE algorithm, when large fluid flow problems are concerned. The proposed methods are simple and can be easily included in existing CFD codes.

Keywords: Computational Fluid Dynamics, Strongly Implicit Procedure, Additive Correction Multigrid

1. INTRODUCTION

The application of conventional finite-difference method or control-volume method to a fluid flow problem leads to a system of algebraic equations in the form

$$[A]\{\Phi\} = \{S\} \quad (1)$$

where the coefficient matrix $[A]$ depends on velocity field, diffusion coefficient and grid parameters, $\{\Phi\}$ is the vector of unknowns, and the right hand side vector $\{S\}$ represents the source term. A pentadiagonal coefficient matrix appears when structured grid and five-point scheme are used. In fluid flow problems, the coefficients are functions of unknowns, which imposes the iterative procedure. Therefore, direct methods for solving the system (1) are more time consuming than iterative ones, especially for large systems. That is the major motivation for development of iterative procedures that exploit the well-defined structure of matrix $[A]$.

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Standard iterative methods [1-2], like Line Gauss-Seidel (LGS, also known as Line-by-Line) method, are very efficient for the case of diagonally dominant coefficient matrix, which is typical for the momentum equation (especially after under-relaxation). However, they show a poor convergence rate when solving the discrete Poisson equation (pressure and pressure correction equations in SIMPLE and SIMPLER algorithm), especially on a non-uniform grid with large cell-aspect ratio. In such cases the incomplete decomposition methods offer certain improvement. The ones most frequently used are those that follow the Strongly Implicit Procedure (SIP) of Stone [3-5]. The SIP solver used in this paper is briefly described below.

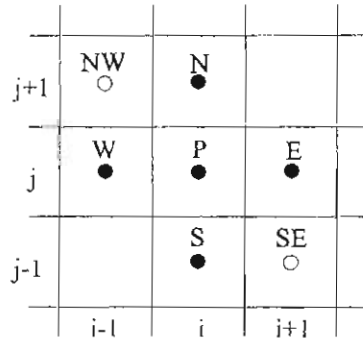


Fig. 1 Computational molecule

For the computation molecule shown in Fig. 1, matrix $[A]$ has five nonzero diagonals, and, according to the Stone's idea, it can be approximated by the product of two triangular matrices, as shown schematically in Fig. 2. The lower and the upper triangular matrix $[L]$ and $[U]$ have three diagonals at the same positions as matrix $[A]$ does. The main diagonal elements of matrix $[U]$

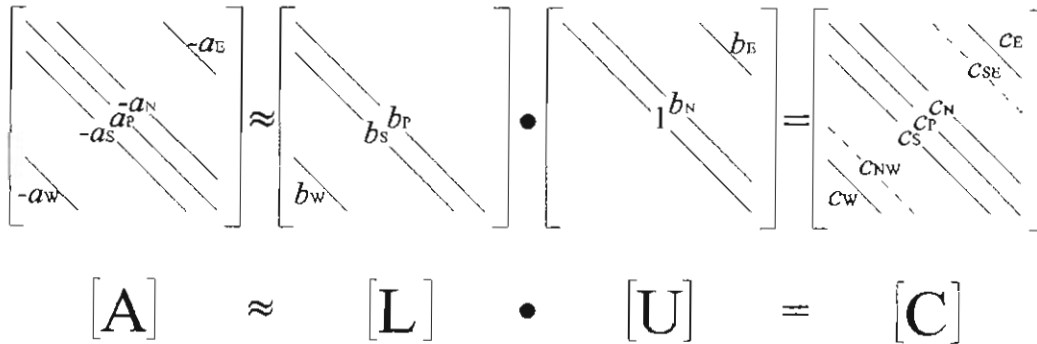


Fig. 2 Matrix decomposition

are all equal to unity. The coefficients of $[C] = [L][U]$ are defined as follows

$$\begin{aligned} c_P^{ij} &= b_P^{ij} + b_S^{ij} b_N^{i-1,j} + b_W^{ij} b_E^{i-1,j} & c_S^{ij} &= b_S^{ij} \\ c_E^{ij} &= b_P^{ij} b_E^{ij} & c_W^{ij} &= b_W^{ij} \\ c_{SE}^{ij} &= b_S^{ij} b_E^{i-1,j} & c_{NW}^{ij} &= b_W^{ij} b_N^{i-1,j} \end{aligned} \quad (2)$$

Matrix $[C]$ differs from $[A]$ in two extra diagonals which multiply Φ_{NW} and Φ_{SE} (see Fig. 1), so that the coefficients of $[L]$ and $[U]$ cannot be defined in a unique way. In order to make them unique and to minimize the influence of two extra diagonals of $[C]$, two additional relations are introduced

$$\begin{aligned} \Phi_{NW} &= \alpha(\Phi_N + \Phi_W - \Phi_P) \\ \Phi_{SE} &= \alpha(\Phi_S + \Phi_E - \Phi_P) \end{aligned} \quad (3)$$

where α is a parameter. From equations (2) and (3), it follows

$$\begin{aligned} b_s^{ij} &= \frac{-a_s^{ij}}{1 + \alpha b_E^{ij-1}} \\ b_w^{ij} &= \frac{-a_w^{ij}}{1 + \alpha b_N^{ij-1}} \\ b_p^{ij} &= a_p^{ij} - b_s^{ij} [b_N^{ij-1} - \alpha b_E^{ij-1}] - b_w^{ij} [b_E^{ij-1} - \alpha b_N^{ij-1}] \\ b_N^{ij} &= \frac{-a_N^{ij} - \alpha b_w^{ij} b_N^{ij-1}}{b_p^{ij}} \\ b_E^{ij} &= \frac{-a_E^{ij} - \alpha b_s^{ij} b_E^{ij-1}}{b_p^{ij}} \end{aligned} \quad (4)$$

Using these equations, for a given $[A]$ and a chosen value of parameter α , the elements of $[L]$ and $[U]$ are evaluated once and for all. In this paper the value $\alpha=0.92$ is applied. After decomposition, the following iterative scheme is introduced

$$[L][U]\{\Phi^{n+1}\} - \{\Phi^n\} = \{S\} - [A]\{\Phi^n\} \quad (5)$$

where the superscript n represents the iteration number. The foregoing system is efficiently solved using forward and back-substitution.

Recently, the additive correction and multigrid procedure have been proposed as remedies for the poor convergence rate of iterative methods. It is well known that the slow convergence rate of iterative methods is a consequence of poor low frequency error damping [6]. Because of this, in multigrid methods, the set of equations is solved on several coarser grids in which the low frequency errors behave like higher frequency errors and can easily be removed by standard iterative solvers. After that, the solution correction is prolonged from the coarser to a finer grid by using interpolation practice. In the additive correction method [7], the solution correction is common to all fine grid cells inside a coarse grid cell. The equations for additive corrections are derived by a simple addition of original equations for the fine grid cells.

Two iterative procedures for solving pentadiagonal algebraic equation systems are proposed in this paper. Both procedures are combinations of the described SIP solver and additive correction strategies. The sensitivity of the procedures to the cell-aspect ratio and grid non-uniformity has been investigated. The proposed procedures have been compared to the SIP solver and applied to solving the laminar flow over a backward facing step.

2. PROPOSED SOLUTION PROCEDURES

2.1 Method 1

The first method comprises two steps. In the first step, the additive correction is made for two levels of coarse grids, while in the second, the standard SIP procedure is applied. Fig.3 shows the finest grid and two levels of coarse grids.

If the finest grid contains NI times NJ cells, then the first coarse grid contains $NI \times 2$ cells, whereas the second one has $NJ \times 2$ cells. Odd numbered cells in the first coarse grid contain all the fine grid cells denoted by indices $j=1$ to $j=NJI$ in the corresponding row of cells, while even numbered cells contain those from $j=NJI+1$ to $j=NJ$. NJI has approximately half the value of NJ . To every fine grid Φ_{ij}^0 value within the k -th coarse grid cell, the same correction δ_k is added in the form

$$\Phi_{ij} = \Phi_{ij}^0 + \delta_k \quad (6)$$

where Φ_{ij} is the corrected value. If the difference equation for the finest cell denoted by indices ij has the form

$$-a_W^{ij}\Phi_{i-1,j} - a_S^{ij}\Phi_{i,j-1} + a_P^{ij}\Phi_{ij} - a_N^{ij}\Phi_{i,j+1} - a_E^{ij}\Phi_{i+1,j} = S_{ij} \quad (7)$$

then the equation for the correction δ_k is the sum of the above equations for all the cells within the k -th coarse cell, and is given in the form

$$-A_W^k\delta_{k-2} + A_P^k\delta_k - a_N^{j,NJ/1}\delta_{k+1} - A_E^k\delta_{k+2} = R_k \quad (8)$$

for odd numbered coarse grid cells, and

$$-A_W^k\delta_{k-2} + A_P^k\delta_k - a_S^{j,NJ/1+1}\delta_{k-1} - A_E^k\delta_{k+2} = R_k \quad (9)$$

for even numbered cells. In the above equations, coefficients A_W^k and A_E^k represent the sums of a_W^{ij} and a_E^{ij} coefficients for all the fine grid cells within the k -th coarse cell, respectively, whereas A_P^k is the sum of a_P^{ij} coefficients for the same fine grid cells less the sums of a_S^{ij} and a_N^{ij} coefficients belonging to the internal fine grid cell faces. R_k is the sum of corresponding S_{ij} terms. Equations (8) and (9) form a set of algebraic equations with five diagonal coefficient matrix (five adjoining diagonals) which can be solved efficiently by the direct Gauss elimination method. The additive correction procedure for the second coarse grid is completely analogous. An iteration step in this method contains the additive correction procedure for the first and the second coarse grid, followed by two sweeps of the SIP procedure.

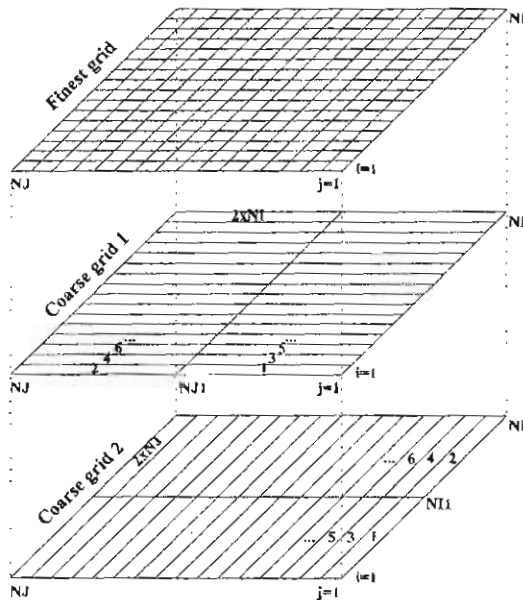


Fig. 3 Levels of grid for Method 1

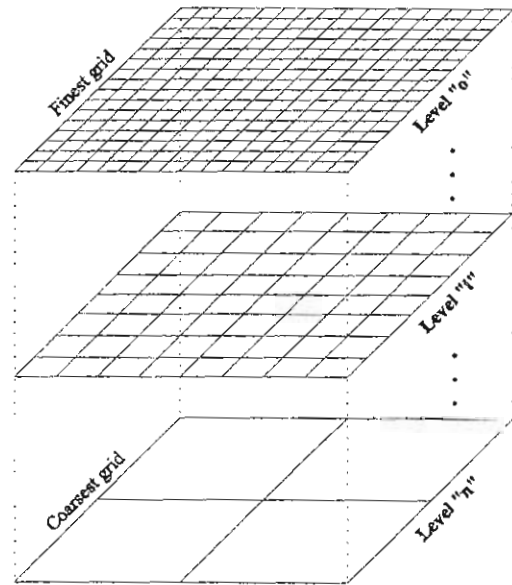


Fig. 4 Levels of grid for Method 2

2.2 Method 2

Fig. 4 shows the grid levels for the second method. At every level, the number of cells is halved in each direction. At all coarse grid levels, the equations for additive corrections are derived in the same way as in the first method, but now these equations form a sparse five diagonal coefficient matrix, so that they cannot be solved efficiently by a direct method. For solving the equations at each grid level, Method 1 is used. One iteration step of Method 2 consists of solving additive correction equations at all levels, starting from the coarsest grid and, finally, of solving the equations on the finest level using Method 1 again.

3. CONVERGENCE RATE INVESTIGATION

The procedures proposed in the previous section were applied to solving the general convection-diffusion equation in the form

$$u \frac{\partial \Phi}{\partial x} + v \frac{\partial \Phi}{\partial y} + \gamma \frac{\partial^2 \Phi}{\partial x^2} + \gamma \frac{\partial^2 \Phi}{\partial y^2} = S_0 - S_p \Phi \quad (10)$$

Here u and v are velocity field components, γ is the diffusion coefficient, and $S_0 - S_p \Phi$ is the source term in linearised representation. Convective terms are discretised using the upwind scheme and diffusion terms by applying the central differencing scheme. At the boundaries, Φ is set to zero. There are three cases of the above equation: (a) convection dominant case, (b) pure diffusion case ($u=v=S_p=0$), and (c) pressure equation case. In the pressure equation case, the coefficients at the main diagonal of matrix $[A]$ are equal to the sum of neighboring coefficients for all control volumes, except for the one through which the boundary condition for pressure is prescribed in the form $a_p=1$, $a_E=a_W=a_N=a_S=0$. In the paper, the above equation is considered as the model equation, so that the exact solution is prescribed in the following form (without consideration of its physical meaning)

$$\Phi_{exact} = 5 + 7x_0 + 9y_0 + 30x_0y_0 + 11 \sin\left(\frac{NI}{3} \frac{\pi}{2} x_0\right) + 6 \cos\left(\frac{NI}{3} y_0 \pi\right) \quad (11)$$

In this equation $x_0=x/x_{max}$ and $y_0=y/y_{max}$, where x_{max} and y_{max} denote the dimensions of calculation domain. If the exact solution is given, and $[A]$ and S_p are known, the source term S_0 is obtained easily. In all the cases, the solution procedure started with the uniform field $\Phi=10$. During the iteration process, both maximum and average absolute difference between the exact Φ and calculated Φ were monitored. The iteration process was terminated after both the errors had fallen below 10^{-5} . The problem was solved using rectangular grids with different cell-aspect ratios, as well as on uniform and monotonically non-uniform grids. All solvers show the best convergence rates when convection dominated equations and/or diagonally dominant cases ($S_p > 0$) are solved. Since, the poorest convergence rate occurs when solving the pressure equation, the rest of the paper is devoted to the pressure equation case.

Equation (10) has been solved on three different grids and each grid has three different numbers of control volumes: i.e. 100x100, 200x200 and 300x300 cells. The first grid is uniform, with the cell-aspect ratio equal to unity (computational domain has dimensions $x_{max}=20$, $y_{max}=20$). The second grid is a uniform grid with the cell-aspect ratio 10 ($x_{max}=200$, $y_{max}=20$), and the third grid is non-uniform, with a variable cell aspect ratio. The computational domain is the same for the third and the second grid, while the smallest cell is ten times smaller than the uniform cell from the second grid. In both directions, cell dimensions increase by geometric progression in such a way that x_{max} and y_{max} are defined by the given number of cells, so that the largest cells are denoted by indices $i=NI$ and $j=NJ$. Fig. 5 to 7 show the results obtained for the three grids with 200x200 cells. Results on 100x100 and 300x300 grids are similar. It is visible that Method 2 shows the fastest convergence rate and is almost insensitive to grid properties. Method 1 is most efficient on a uniform grid with cell-aspect ratio equal to one. It is interesting to note that this method is more efficient on a non-uniform grid with variable cell-aspect ratio than on a uniform grid with constant cell-aspect ratio. After the initial reduction in high frequency errors, SIP solver shows a slow convergence rate, especially on a non-uniform grid. Table 1 shows the times necessary to reduce the average error to the level of 10^{-5} on the first uniform grid, for different numbers of cells.

Grid	100x100	200x200	300x300
Method 1	9.66	102.42	425.57
Method 2	5.05	34.52	110.04

Table 1: The computer times required to reduce average error below 10^{-5} (in seconds).

If it is assumed that the required computer time is proportional to N^b , then the exponent b is about 1.72 in Method 1 and above 1.4 in Method 2.

4. APPLICATION TO A FLUID FLOW CALCULATION

The proposed iterative methods and the SIP solver were implemented in a computer code [8] for incompressible turbulent flow calculations. The laminar flow over a backward facing step with a uniform inlet velocity profile was chosen as a test example. A uniform 230 times 115 control volume computational grid was defined in a rectangular calculation domain. The SIMPLE algorithm was selected for pressure velocity coupling, and the underrelaxation factors $\alpha_u=0.911$ for velocity equations and $\alpha_p=0.1$ for the pressure equation were chosen. The routines for solving difference equations were called in the manner that one sweep for velocity equations was followed by four sweeps for pressure correction equation.

Fig. 7 shows the residual norm for the mass conservation equation as a function of the number of outer iterations. It is obvious that the application of SIP solver results in a rather oscillatory norm and a slow convergence rate. As expected, the fastest convergence rate has been obtained by Method 2, whereas Method 1 is the most efficient one. That can be explained by the fact that one sweep in Method 2 consumes more time than in Method 1, and the reduction in number of outer iterations of Method 2 is not sufficient to compensate for this difference.

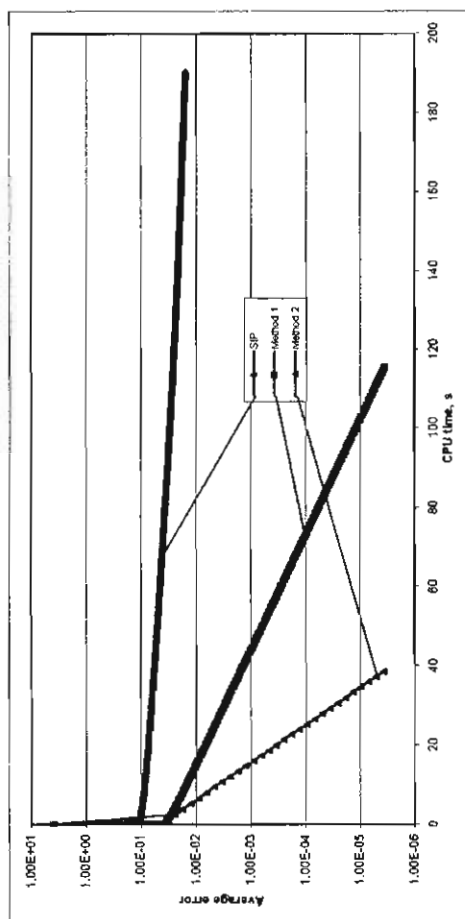


Fig. 5: Average error as a function of consumed computer time for the uniform grid with cell-aspect ratio equal to 1

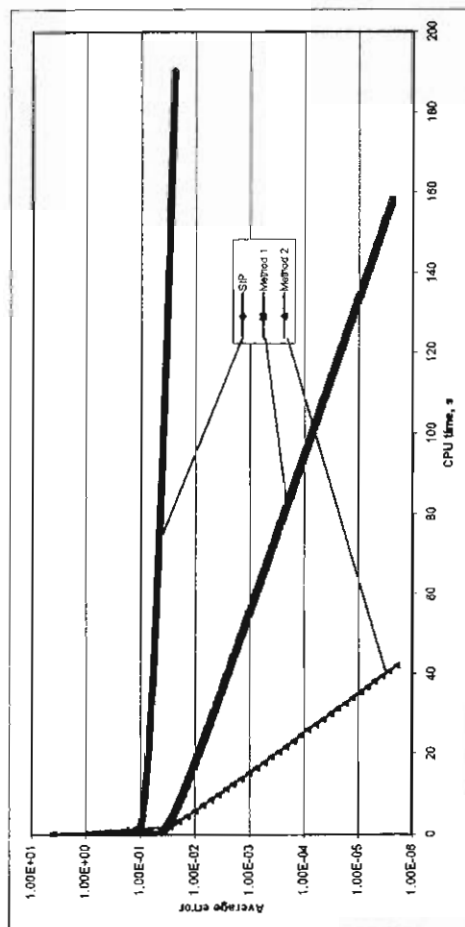


Fig. 6: Average error as a function of consumed computer time for the uniform grid with cell-aspect ratio equal to 10

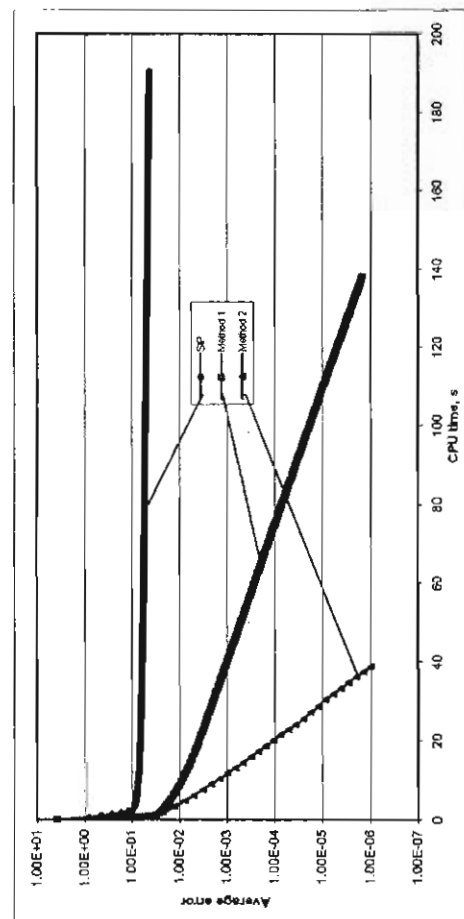


Fig. 7: Average error as a function of consumed computer time for the non-uniform grid with variable cell-aspect ratio

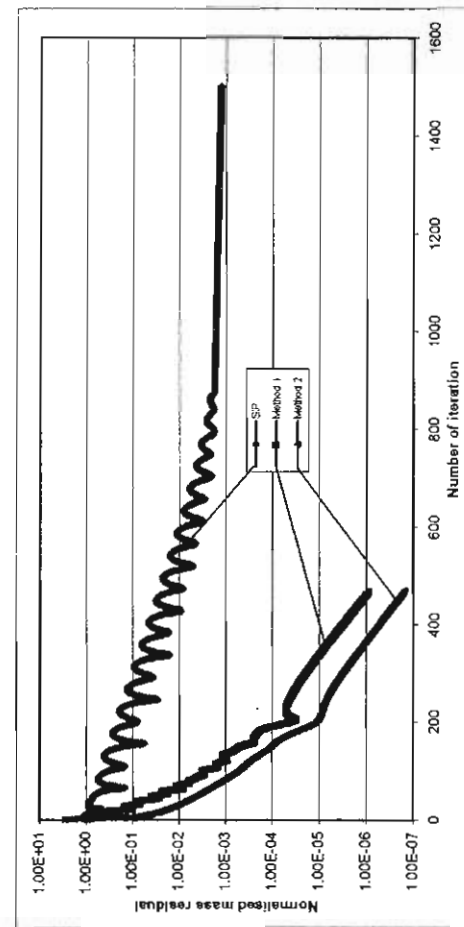


Fig. 8: Normalised mass residual as a function of the number of iteration

5. DISCUSSION AND CONCLUSIONS

Two new procedures have been presented for solving the algebraic equations that arise in control volume representation of typical equations of Computational Fluid Dynamics (CFD). It has been found that the solution of Poisson pressure equation requires more computer time than other equations appearing in CFD. The new methods have been applied for solving large systems of equations on uniform and non-uniform grids with various cell-aspect ratios and, finally, within the SIMPLE algorithm, for solving a specific laminar flow problem. Based on the performed tests, the following conclusions can be drawn regarding the proposed procedures:

- 1) Method 2 is quite appropriate for solving large systems of algebraic equations (the required computer time varies with the number of equations to the power of 1.4).
- 2) The proposed procedures are almost insensitive to the grid non-uniformity and cell-aspect ratio (especially Method 2).
- 3) Application of the method to CFD codes using SIMPLE algorithm improves significantly the convergence rate of the overall iteration process, when compared to the SIP solver. Both methods require similar number of iterations (this number is seemingly determined by the SIMPLE algorithm). However, Method 1 shows better efficiency, because its sweep requires less computer time than a sweep in Method 2. With other algorithms, like SIMPLER or PISO, Method 2 would probably demonstrate advantages over Method 1.
- 4) The proposed procedures are simple and can easily be included in any CFD code by replacing the existing equation solver.

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AN APPLICATION OF THE METHOD OF CHARACTERISTICS ON FLOW AROUND RECTANGULAR CYLINDER¹

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Abstract:

This paper examines an application of the method of characteristics developed for unsteady two-dimensional viscous flow. This method has direct relevance in many practical applications because of its great stability, high accuracy and low computer time consumption. Flow around a rectangular cylinder has been calculated using the developed method, in the range of Reynolds numbers from 100 to 1600. The computed stream lines, base pressure, drag force and Strouhal number show good agreement with experimental results.

Keywords: Computational Fluid Dynamics, Method of Characteristics, Separated Flow, Karman Vortex, Rectangular Cylinder.

1. INTRODUCTION

Unsteady separated flows around bluff bodies, such as circular cylinders, rectangular cylinders and flat plates, are directly relevant in many engineering constructions: e.g., chimneys, suspension bridges, road vehicles, and heat exchangers. The flow around a cylinder with rectangular cross section has been extensively studied through both experiments and numerical computations in aerodynamics and hydrodynamics. Some interesting phenomena regarding the flow around a rectangular cylinder have been determined through experiments. It has been found that aerodynamic characteristics such as lift and drag force, base pressure, and the Strouhal number are strongly dependent on the Reynolds number and the shape of the model cross section. Experimental fluid dynamics provides an effective means of simulating real flows, whereas computational fluid dynamics provides an alternative cost-effective means for fluid flow simulations. The method of characteristics is currently not so popular as the finite difference method, the finite element method or the finite volume method. Because of its theoretical complexity in multi dimensional cases, it is not used extensively, but it, nevertheless, possesses some advantages: greater stability, bigger accuracy and lower computer time consumption than the finite difference method.

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2. MATHEMATICAL MODEL AND COMPUTATIONAL METHOD

The governing equations for the unsteady compressible fluid flows at low Mach numbers [1] are the continuity equation the momentum equation and the equation for sound speed

$$\begin{aligned}\frac{\partial p}{\partial t} + v_i \frac{\partial p}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} &= 0 \\ \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\mu}{\rho} \frac{\partial^2 v_i}{\partial x_j^2} - f_i &= 0\end{aligned}\quad (1)$$

$$c^2 = dp/d\rho = \text{const.}$$

where

v_i - velocity

p - pressure

f_i - mass forces

μ - viscosity coefficient

ρ - density

c - sound speed

For two-dimensional flows, after neglecting mass forces, the system (1) reduces to

$$\begin{aligned}\frac{\partial p}{\partial t} + U \frac{\partial p}{\partial x} + V \frac{\partial p}{\partial y} + c^2 \rho \frac{\partial U}{\partial x} + c^2 \rho \frac{\partial V}{\partial y} &= 0 \\ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= I_1 \\ \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} &= I_2\end{aligned}\quad (2)$$

Here U stands for the v_1 , V stands for the v_2 and

$$I_1 = \frac{\mu}{\rho} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \quad (3)$$

$$I_2 = \frac{\mu}{\rho} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \quad (4)$$

Viscous terms I_1 and I_2 are much smaller than inertial forces terms (in case when the Reynolds number is much greater than the reciprocal value of the Strouhal number) so the terms I_1 and I_2 can be treated implicitly through the source terms without losing accuracy. The system (2) is then considered as a set of differential equations of hyperbolic type that can be rewritten in terms of characteristic variables. Following the procedure given in [2] and [3] the three eigenvalues of Jacobian matrix of system (2) have been found in the form:

$$\begin{aligned}
\lambda_1 &= \vec{v} \cdot \vec{n} \\
\lambda_2 &= \vec{v} \cdot \vec{n} + c \\
\lambda_3 &= \vec{v} \cdot \vec{n} - c
\end{aligned} \tag{5}$$

Where λ is the eigenvalue and $\vec{n}$ is the unit normal to the characteristic surface. The envelope of all the characteristic surfaces that correspond to arbitrary vector $\vec{n}$ forms the Mach conoid [2] as shown in Fig. 1.

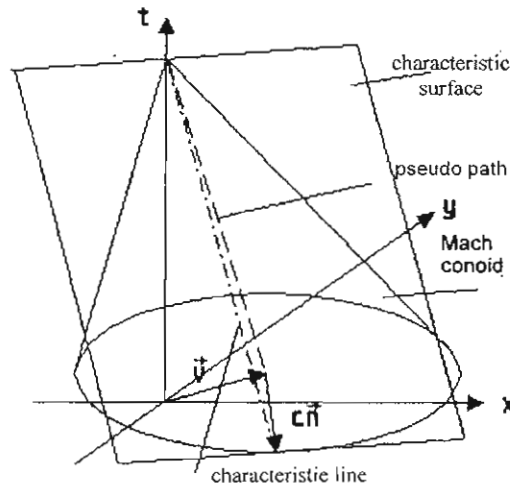


Fig.1: Mach conoid

For chosen eigenvalues (λ_2 or λ_3) and a chosen characteristic surface (with the normal in the direction of the x or y base vector), the characteristic variable w_1 , w_2 , w_3 , and w_4 [3] are defined by the following equations.

$$\begin{aligned}
\frac{D_x^- w_1}{Dt} &= \frac{\partial p}{\partial t} + (U - c) \frac{\partial p}{\partial x} - \rho c \left(\frac{\partial U}{\partial t} + (U - c) \frac{\partial U}{\partial x} \right) = -\rho c \left(I_1 - V \frac{\partial U}{\partial y} + c \frac{\partial V}{\partial y} + \frac{V}{\rho c} \frac{\partial p}{\partial y} \right) \\
\frac{D_y^- w_2}{Dt} &= \frac{\partial p}{\partial t} + (V - c) \frac{\partial p}{\partial y} - \rho c \left(\frac{\partial V}{\partial t} + (V - c) \frac{\partial V}{\partial y} \right) = -\rho c \left(I_2 - U \frac{\partial V}{\partial x} + c \frac{\partial U}{\partial x} + \frac{U}{\rho c} \frac{\partial p}{\partial x} \right) \\
\frac{D_x^+ w_3}{Dt} &= \frac{\partial p}{\partial t} + (U + c) \frac{\partial p}{\partial x} + \rho c \left(\frac{\partial U}{\partial t} + (U + c) \frac{\partial U}{\partial x} \right) = \rho c \left(I_1 - V \frac{\partial U}{\partial y} - c \frac{\partial V}{\partial y} - \frac{V}{\rho c} \frac{\partial p}{\partial y} \right) \\
\frac{D_y^+ w_4}{Dt} &= \frac{\partial p}{\partial t} + (V + c) \frac{\partial p}{\partial y} + \rho c \left(\frac{\partial V}{\partial t} + (V + c) \frac{\partial V}{\partial y} \right) = \rho c \left(I_2 - U \frac{\partial V}{\partial x} - c \frac{\partial U}{\partial x} - \frac{U}{\rho c} \frac{\partial p}{\partial x} \right)
\end{aligned} \tag{6}$$

It is visible that above equations directly follow from original system (2). Each equation of the system (6) is valid along the one of characteristic lines C_1^- , C_2^- , C_1^+ , and C_2^+ shown in Fig. 2. System (6) can be written more concisely as

$$\begin{aligned}
\frac{D_x^- w_1}{Dt} &= \frac{D_x^- p}{Dt} - \rho c \frac{D_x^- U}{Dt} = S_1^- \quad \text{along line } C_1^- \\
\frac{D_y^- w_2}{Dt} &= \frac{D_y^- p}{Dt} - \rho c \frac{D_y^- V}{Dt} = S_2^- \quad \text{along line } C_2^- \\
\frac{D_x^+ w_3}{Dt} &= \frac{D_x^+ p}{Dt} + \rho c \frac{D_x^+ U}{Dt} = S_1^+ \quad \text{along line } C_1^+ \\
\frac{D_y^+ w_4}{Dt} &= \frac{D_y^+ p}{Dt} + \rho c \frac{D_y^+ V}{Dt} = S_2^+ \quad \text{along line } C_2^+
\end{aligned} \tag{7}$$

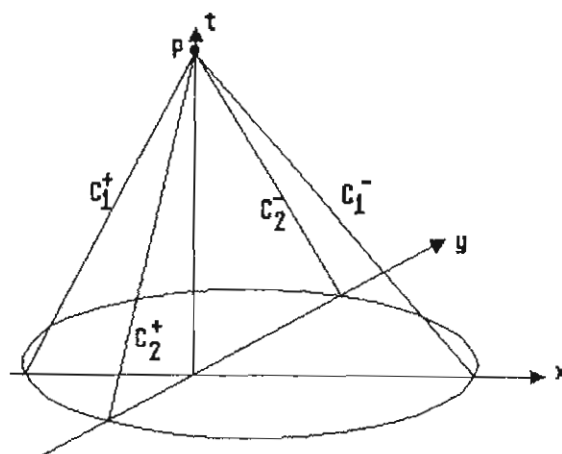


Fig. 2: Characteristic lines in coordinate directions

It is obvious from (7) that the characteristic variables w_1 , w_2 , w_3 , and w_4 change their value along the characteristics C_1^- , C_2^- , C_1^+ , and C_2^+ only due to influence of the source terms S_1^- , S_2^- , S_1^+ , and S_2^+ , respectively.

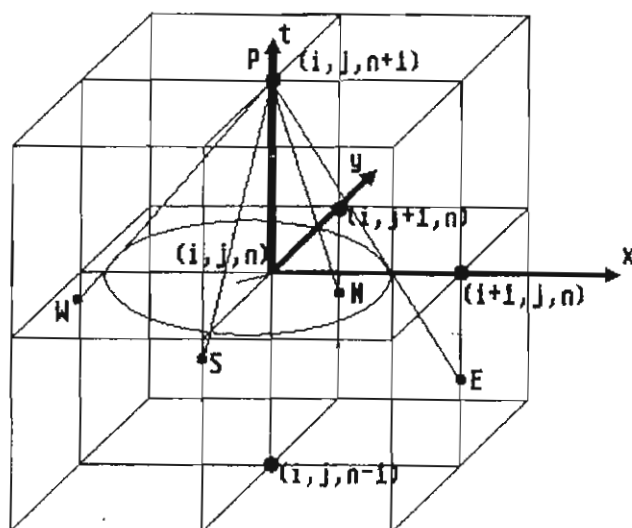


Fig. 3: Interpolation scheme

For the time-space grid defined in Fig. 3, the system (7) can be discretised [4] into the following system of algebraic equations

$$\begin{aligned} p_P - p_E - \rho c(U_P - U_E) &= S_1^- \Delta t \\ p_P - p_N - \rho c(V_P - V_N) &= S_2^- \Delta t \\ p_P - p_W + \rho c(U_P - U_W) &= S_1^+ \Delta t \\ p_P - p_S + \rho c(V_P - V_S) &= S_2^+ \Delta t \end{aligned} \quad (8)$$

The unknown values in these equations (pressure p_P , velocities U_P and V_P) at time $(n+1)\Delta t$ can be determined explicitly.

$$\begin{aligned} p_P &= 0.25[p_W + p_E + p_S + p_N + \rho c(U_E - U_W + V_N - V_S) + \Delta t(S_1^+ + S_2^+ + S_1^- + S_2^-)] \\ U_P &= 0.5[U_E + U_W + (p_W - p_E)/\rho c + \Delta t(S_1^+ - S_1^-)] \\ V_P &= 0.5[V_N + V_S + (p_S - p_N)/\rho c + \Delta t(S_2^+ - S_2^-)] \end{aligned} \quad (9)$$

The values of velocity, pressure, and source terms at points E, W, N, S are linearly interpolated from the values at time $n\Delta t$ and time $(n-1)\Delta t$.

3. RESULTS

An application of the method of characteristics is illustrated through the calculation of two-dimensional fluid flows around a rectangular cylinder at Mach numbers $M=U/c$ below 0.1. Over the cylinder surface the no-slip boundary conditions ($U = 0, V = 0$) are imposed. The flow at the upstream inlet is undisturbed and uniform. At the downstream exit, zero gradient boundary conditions are applied, while the side boundaries are planes of symmetry. The flow was calculated on a grid consisting of 200×1200 uniform cells, with cell-aspect ratio equal to one.

The typical representative examples of calculated isobars, vorticity isolines and streamlines (at the Reynolds number 400, and sufficiently after an instantaneous start from the rest) are shown in Fig. 4 to 6. The time variations of drag and lift coefficients are given in Fig. 7. In Fig 8, the separation of Karman vortices behind the cylinder (within the rectangular area denoted by the frame in Fig. 6) is illustrated for the period from 151 to 158 seconds after an instantaneous start.

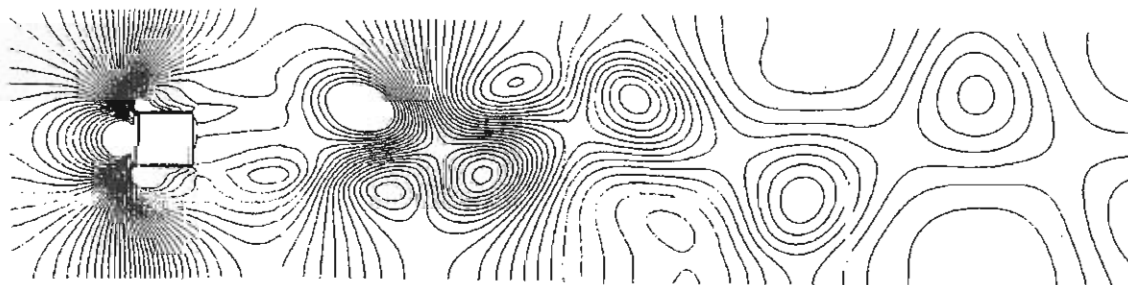


Fig. 4: Isobars

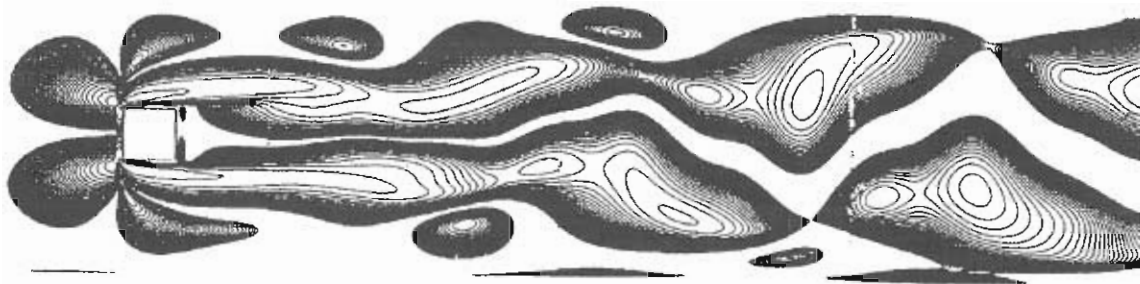


Fig. 5: Vorticity isolines

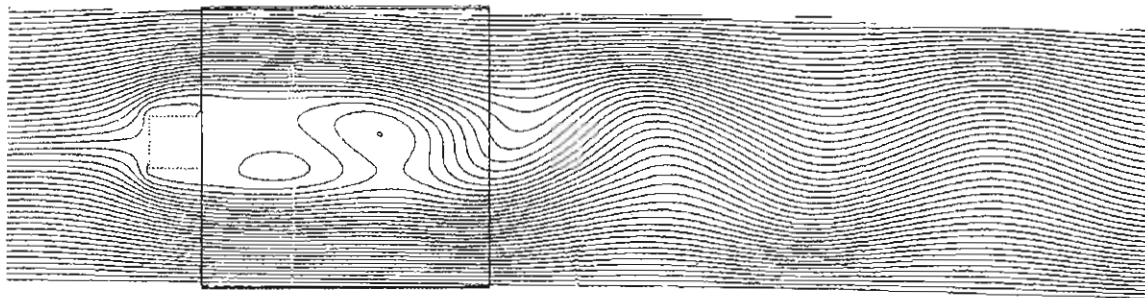


Fig. 6: Streamlines

The Strouhal number was calculated on the basis of the period of lift force oscillations, the linear dimension of the cylinder, and the inlet velocity. In the range of the Reynolds numbers between 100 and 1200, the Strouhal number has nearly constant value $St=0.14$. These calculations show good agreement with experimental results and the calculations of Okajima [5] (Fig. 9).

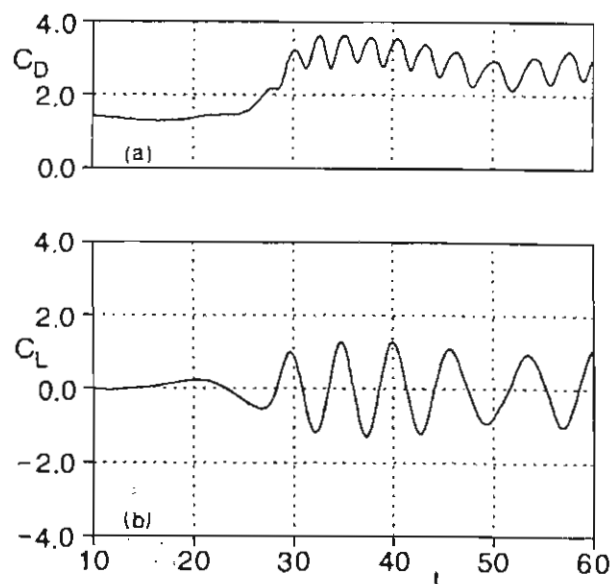


Fig. 7: Drag and lift force coefficients

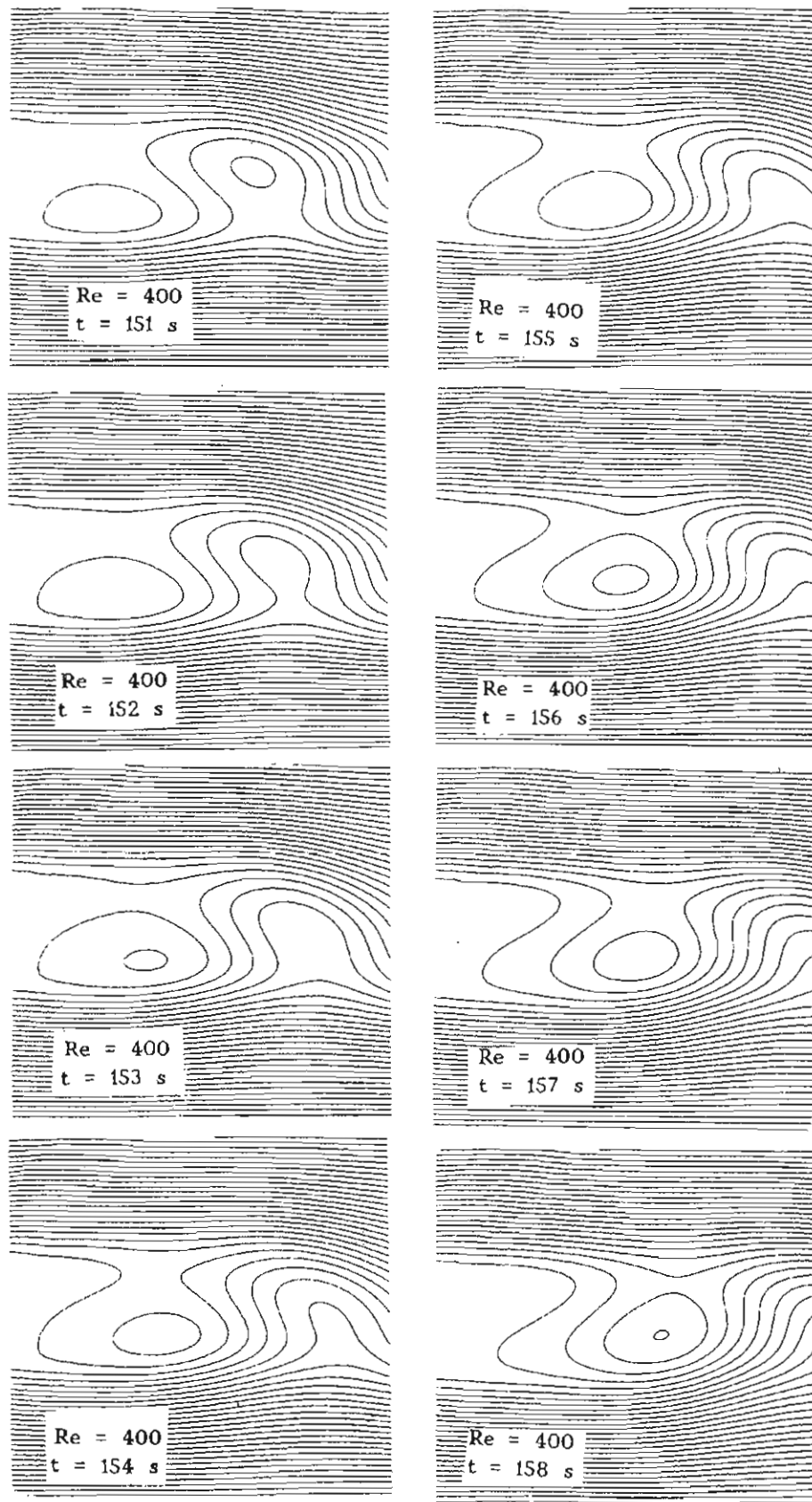


Fig. 8: Streamlines behind the rectangular cylinder

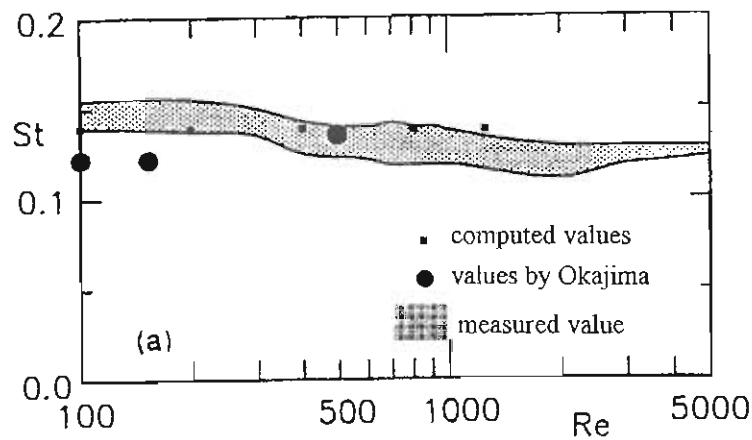


Fig. 9: Comparison of the computed and experimental values of the Strouhal number

4. CONCLUSION

In this paper, the method of characteristics, very popular for solving unsteady one-dimensional flows, was applied for solving the unsteady flow around the rectangular cylinder. Calculations have been performed for a range of Reynolds numbers, and obtained results are in good agreement with similar calculations of other authors and with experimental results. The developed method demonstrates great stability and low computer time consumption so that the method deserves to be used more frequently in the future.

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Shape optimization of the reversible hydro-power plant tunnel barrier

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Abstract

The optimization of the shape of the reversible hydro-power plant tunnel barrier is worked out for four different simple shapes, from vertical plate to streamlined shape. The objective function to be minimized is flow losses. They are calculated using the results of flow simulation by FLUENT commercial software. It is shown, that, by adequate shape optimization, the flow losses can be halved.

1. INTRODUCTION

The paper presents results of the optimization of the tunnel barrier shape of reversible hydro-power plant. Barrier of the given height should be placed at short distance in front of the tunnel entrance to ensure reservoir from complete drainage. Its shape should be optimized for minimum fluid flow losses in both working conditions: pump and turbine. The fluid flow simulation is done using commercial software FLUENT for the set of four different barrier shapes. The software is based on finite element implementation of turbulent $k-\epsilon$ Reynolds averaged Navier-Stokes equations, capable of modeling two- and three-dimensional flows.

The objective functional for the shape optimization is defined as energy losses of the flow between the control volume inflow and outflow cross-sections. The losses are calculated using flow simulation results.

Four different barrier shapes were considered. Barrier no. 0 represents flow without any barrier, barrier no.1 is vertical flat plate, barrier no.2 is slightly modified barrier no.1 with curved tip in the circular arc form with 0.8m radius. Barrier no.3 is constructed from barrier no. 2 by inserting rectangular section of 2m length. Finally barrier no. 4 is most streamlined consisting of tip curvature of 1.8m radius, its center displaced one meter downstream from the step. The shapes of the barriers are given in Fig. These barrier shapes were chosen as the optimizing set mostly because of their low production costs.

The ability of software to model jet flows is tested. The comparison between experimental velocity distribution for square 3-D wall jet [3] and our pump flow at the tunnel entrance shows good agreement (see Fig. 2). In the lateral (horizontal) direction velocity distribution obeys Gauss curve and in vertical law-of-wall curve combined with radial wall jet profile.

Two- and three-dimensional test cases showed little difference as far as flow losses trend calculations are considered. The actual optimization is therefore done using two-dimensional model.

2. MATHEMATICAL MODEL

The partial differential equations governing incompressible viscous turbulent steady fluid flow, known as continuity equation, Reynolds equations (k-ε turbulent model), are [1]:

$$\frac{\partial}{\partial x_i}(\rho v_i) = 0,$$

$$\frac{\partial(\rho v_i v_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i} \left(\mu \left[\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right] \right) + \frac{\partial}{\partial x_i} \left(\rho \frac{2}{3} k \delta_{ij} + \mu_t \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right) + \rho f_j$$

In the equations given above Einstein convention of summation over repeated indexes is accepted, δ_{ij} is Kronecker symbol, μ dynamic viscosity of the fluid, ρ density, p pressure, v_i and f_i i-th velocity and force density component.

The transport equations for the kinetic energy of turbulence k , and the dissipation rate ε are

$$\frac{\partial(\rho v_j k)}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\mu_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right) + G_k - \rho \varepsilon,$$

$$\frac{\partial(\rho v_j \varepsilon)}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\mu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_j} \right) + C_1 \frac{\varepsilon}{k} G_k - C_2 \rho \frac{\varepsilon^2}{k}.$$

G_k , the kinetic energy production and μ_t , turbulent viscosity are defined as:

$$G_k = \mu_t \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \cdot \frac{\partial v_j}{\partial x_i}, \quad \mu_t = \rho C_\mu \frac{k^2}{\varepsilon}$$

with the constants having values

$$C_1=1.44, C_2=1.92, C_\mu=0.09, \sigma_k=1.0, \sigma_\varepsilon=1.3,$$

The boundary conditions are chosen as follows: at inflow boundary given velocity profile; at outflow developed flow conditions, i.e. zero gradients; at wall no slip conditions. Special attention is given to free surface boundary. Considering small free surface waves to reservoir depth ratio the waves are not simulated. Only zero normal velocity with no restriction to tangential velocity to the undisturbed free surface is enforced. The domain is 100m long in the stream direction, giving 50 tunnel diameters, the length big enough for the flow to reach fully developed regime.

The objective functional for the shape optimization is defined as energy losses of the flow between the control volume inflow and outflow cross-sections:

$$e = e_i - e_o,$$

where the energy flux through given cross-section A is defined as:

$$e_A = \int_A \rho \left(\frac{|v|^2}{2} + \frac{p}{\rho} + gz \right) v_n dA,$$

g being gravitational acceleration, z height, v velocity and v_n normal velocity component.

3. ANALYSIS OF RESULTS

The streamlines for four characteristic barrier for pump regime are given in Fig. 3. From the dimensions of the dominant vortex in recirculating flows one can predict the flow losses. The numerical values are given in tabular form.

Table 1. gives the calculated values of the total mechanical energy, for one meter of lateral dimension, at inflow and outflow cross-section, the difference between the two, that is the flow losses and its percentage of inflow mechanical energy, for pump and turbine working conditions respectively. The flow losses calculation is described in previous paragraph.

For the pump operation, the least losses, 8.394 kW/m, or 0.89%, are calculated for barrier no. 4., while the highest, 12.593 kW/m, or 1.34%, gives vertical flat plate barrier (no.1). One can conclude that the losses can be almost halved by suitable barrier shape. The losses for the flow without barrier are 10.829 kW/m, or 1.15%, even more then for the case with barrier no.4, what is somewhat surprising. The explanation is in high velocity gradients of the jet along the bottom of the reservoir for the case without barrier.

Table 1. Calculated flow losses for different barrier shapes and working regimes

Pump regime

	$E_{infl}(W)$	$E_{outfl}(W)$	$\Delta E (W)$	$\Delta E (\%)$
Without barrier	940708	929879	10829	1.15
Vertical barrier (no.1)	941916	929323	12593	1.34
Shaped barrier (no.2)	940805	928457	12348	1.31
Shaped barrier (no.3)	938786	929865	8921	0.95
Shaped barrier (no.4)	938283	929889	8394	0.89

Turbine regime

	$E_{infl}(W)$	$E_{outfl}(W)$	$\Delta E (W)$	$\Delta E (\%)$
Without barrier	932713	929592	3121	0.33
Vertical barrier (no.1)	930816	923180	7636	0.82
Shaped barrier (no.4)	932550	929099	3451	0.37

For the turbine working conditions the flow losses are approximately four times smaller for the case without barrier then for the pump regime, amounting to 3.121 kW/m, or 0.33% of inflow energy. That is due smaller velocities and their gradients. For vertical barrier the losses are more than double: 7.636 kW/m, or 0.82%, while for barrier no.4 they are 3.451 kW/m, or 0.37%.

5. CONCLUSION

The flow of the water for turbine regime resembles very much that of the jet flow in the reservoir. The comparison of the simulation results for two-dimensional and three-dimensional flows for the cases without and with the vertical barrier for both pump and turbine working regimes show good agreement with the experimentally observed flow patterns of the jets [2,3].

Also, the comparison leads to the conclusion that for the shape optimization it is sufficient to model only two-dimensional flows giving good prognosis of the trends, more then of the actual values of the losses . The two-dimensional model is more convenient considering CPU time and memory usage. Finally, it is shown that it is pump and not turbine regime, which is dominant for the amount of flow losses.

The results of flow simulation showed that it is sufficient to analyze only four different barrier shapes, from the starting vertical flat plate to the streamlined barrier shape (barrier no. 4) with the high tip radius of 1.8m , with the center located 1m downstream of the step.

The criteria for the optimization of the shape of the barrier is the flow losses for the two-dimensional flow with unit lateral dimension between inflow and outflow cross-sections.

Final conclusion derived from the flow simulation results and loss calculations qualifies barrier no.4 as optimal, its losses ,8.394 kW/m, being almost half the losses of the vertical barrier: 12.593 kW/m.

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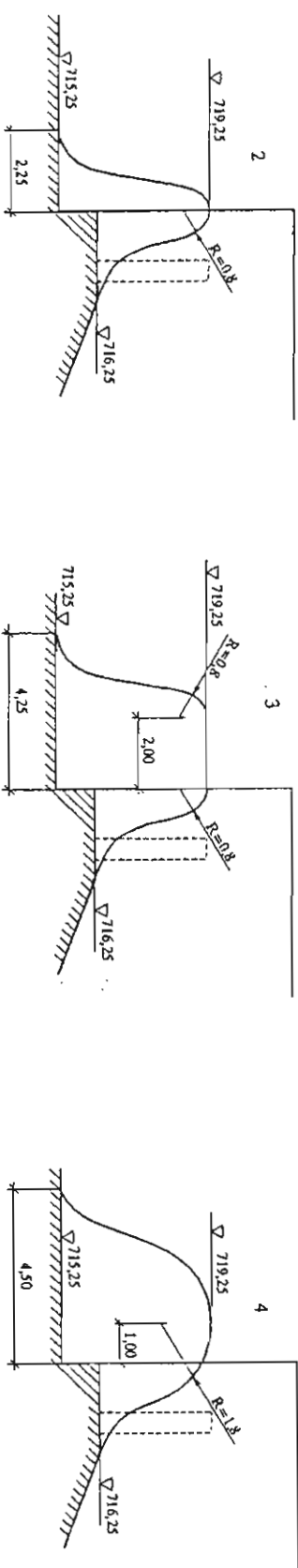
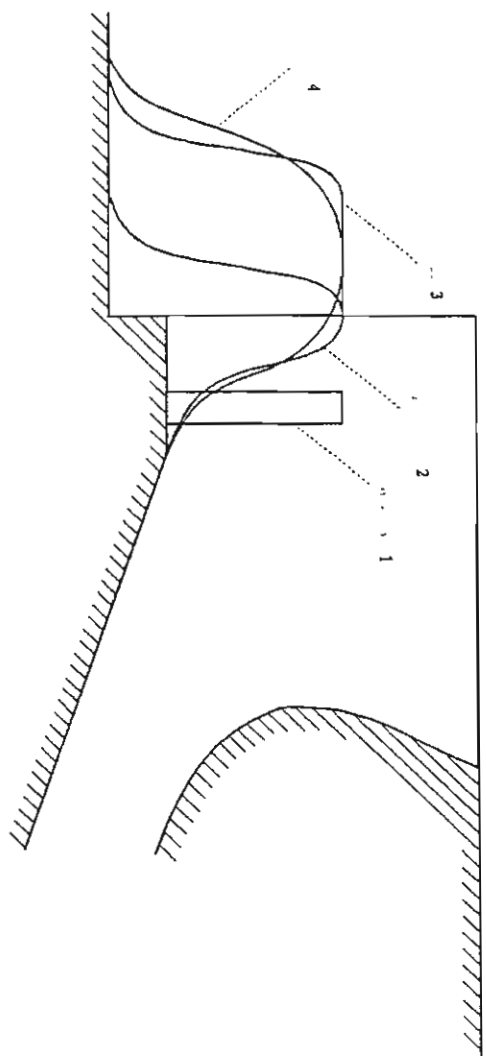


Figure 1. Different barrier shapes and their placement

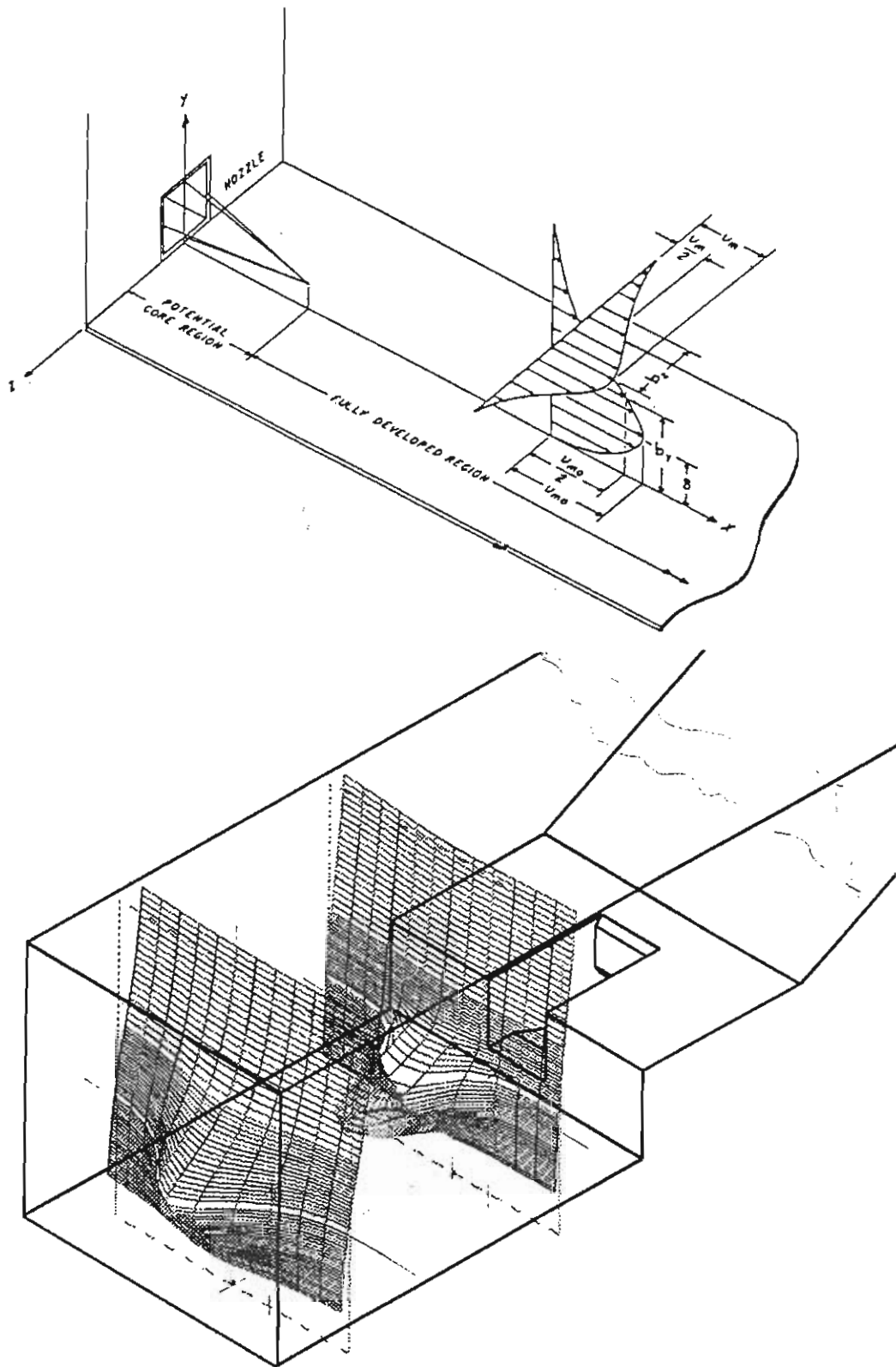


Figure 2. Experimental velocity profiles for wall jet [3] and simulated for the pump flow at tunnel entrance

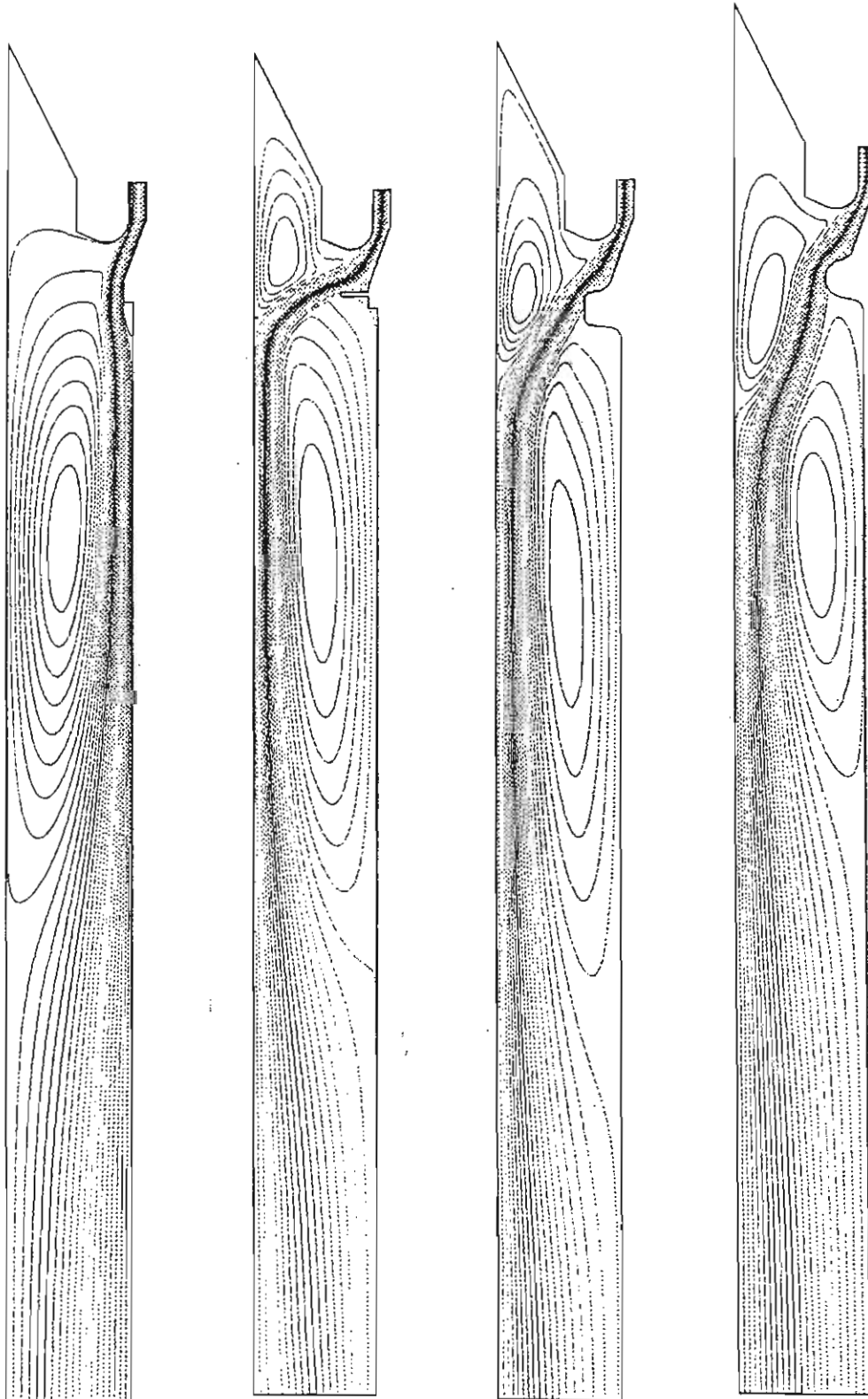


Figure 3. Simulated streamlines for different barrier shapes

Second order interaction of water waves with arrays of vertical circular cylinders

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Abstract

The paper deals with the second order interactions of an incident monochromatic wave field with an array of vertical circular cylinders standing on the sea-bottom. The assumptions of potential flow are adopted. The method of solution is combination of the linear (first-order) interaction method and the semi-analytical solution for second-order single cylinder case. The first interest of this work is that, due to the high precision of semi-analytical approach, obtained results can be used as a benchmark for numerical methods. Another interest lies in a relatively small requirements for CPU time so that the method can be used as an engineering tool in the initial stage of the design of some off-shore platforms where the second order effects play an important role (springing-ringing problems). In fact, multiple column structures are very common in off-shore engineering (TLP, Semi-sub, GBS, ...) and idealised representation of these structures as an array of complete cylinders can sometimes be sufficient.

1 Introduction

We place us in the usual frame of potential theory where the viscous effects and the effects of flow rotationality are neglected so that the governing equation in the fluid becomes Laplace equation for the potential $\Phi(x, y, z, t)$. The velocity field is found as a gradient of the potential $\vec{v} = \nabla \Phi$, and pressure is calculated by well known Bernoulli equation. The periodicity of solution is also assumed so that the time dependence can be factorised. In the case of monochromatic incident wave field, this means that the first order quantities at frequency ω will produce the second order effects at frequencies 0 and 2ω . Thus, if we assume the following perturbation series for the potential :

$$\Phi = \epsilon \phi^{(1)}(\mathbf{x}, t) + \epsilon^2 \phi^{(2)}(\mathbf{x}, t) + \dots \quad (1)$$

the first and the second order potentials $\phi^{(1)}$ and $\phi^{(2)}$ (and similarly all other quantities) will have the following form :

$$\epsilon \phi^{(1)}(\mathbf{x}, t) = \Re\{\varphi^{(1)}(\mathbf{x})e^{-i\omega t}\} \quad (2)$$

$$\epsilon^2 \phi^{(2)}(\mathbf{x}, t) = \bar{\varphi}^{(2)}(\mathbf{x}) + \Re\{\varphi^{(2)}(\mathbf{x})e^{-2i\omega t}\} \quad (3)$$

with $\mathbf{x} = (x, y, z)$.

After introducing the perturbation series (1) in the original nonlinear boundary value problem (BVP) we can obtain corresponding BVP-s for different potentials. The details of derivation are well known and will not be repeated here. We just recall final expressions for the free surface condition which is the main difficulty in the resolution of the problems of this type (here we are interested only in the periodic components so that the potential $\bar{\varphi}^{(2)}$ will not be treated) :

$$-\nu \varphi^{(1)} + \frac{\partial \varphi^{(1)}}{\partial z} = 0 \quad (4)$$

$$-4\nu \varphi^{(2)} + \frac{\partial \varphi^{(2)}}{\partial z} = \frac{i\omega}{g} [\nabla \varphi^{(1)} \nabla \varphi^{(1)} - \frac{1}{2} \varphi^{(1)} (\frac{\partial^2 \varphi^{(1)}}{\partial z^2} - \nu \frac{\partial \varphi^{(1)}}{\partial z})] \quad (5)$$

where $\nu = \omega^2/g$ is the infinite depth wave number and g is gravity.

Beside this condition the potential should also satisfy Laplace equation in the fluid domain, no flow condition on the fixed boundaries and the radiation condition at infinity.

In both cases the potential φ is further decomposed into incident and diffracted parts ($\varphi = \varphi_I + \varphi_D$). Incident potential at first two order corresponds to the plane sinusoidal wave and is easily evaluated :

$$\varphi_I^{(1)} = -\frac{igA}{\omega} f_0^{(1)}(z) e^{ik_0(x \cos \beta + y \sin \beta)} = -\frac{igA}{\omega} f_0^{(1)}(z) \sum_{m=-\infty}^{\infty} e^{im(\pi/2-\beta)} J_m(k_0 r) e^{im\theta} \quad (6)$$

$$\begin{aligned} \varphi_I^{(2)} &= \frac{3i\omega\nu A^2}{2 \sinh^2 k_0 H} f_I^{(2)}(z) e^{2ik_0(x \cos \beta + y \sin \beta)} \\ &= \frac{3i\omega\nu A^2}{2 \sinh^2 k_0 H} f_I^{(2)}(z) \sum_{m=-\infty}^{\infty} e^{im(\pi/2-\beta)} J_m(2k_0 r) e^{im\theta} \end{aligned} \quad (7)$$

where A is amplitude of the incident wave, β is the incident angle, H is the waterdepth, k_0 is the finite depth wavenumber ($k_0 \tanh k_0 H = \nu$), and :

$$f_0^{(1)}(z) = \frac{\cosh k_0(z+H)}{\cosh k_0 H}, \quad f_I^{(2)}(z) = -\frac{\cosh 2k_0(z+H)}{4\nu \sinh^2 k_0 H} \quad (8)$$

2 First order solution

All notations correspond to the configuration presented on the figure 1. First order diffraction solution was presented in [3] and we just recall the basic steps. The diffraction potential $\varphi_D^{(1)}$ is first assumed in its general form :

$$\varphi_D^{(1)} = f_0^{(1)}(z) \sum_{k=1}^N \sum_{m=-\infty}^{\infty} A_m^k Z_{m0}^{(1)k} H_m(k_0 r_k) e^{im\theta_k} \quad (9)$$

where $Z_{m0}^{(1)k} = J'_m(k_0 a_k) / H'_m(k_0 a_k)$.

The use of the Graff's theorem for Bessel functions permits the application of the boundary conditions on each cylinder ($\partial \varphi_D^{(1)} / \partial n = -\partial \varphi_I^{(1)} / \partial n$) and the interaction coefficients A_m^k are found as a solution of the following linear system of equations :

$$A_m^k + \sum_{j \neq k}^N \sum_{n=-\infty}^{\infty} A_n^j Z_{n0}^{(1)j} e^{i(n-m)\alpha_{jk}} H_{n-m}(k_0 R_{jk}) = -I_k^{(1)} e^{im(\pi/2-\beta)} \quad (10)$$

where $I_k^{(1)} = e^{ik_0[X_k \cos \beta + Y_k \sin \beta]}$ is the phase correction accounting for the position (X_k, Y_k) of each of N cylinders.

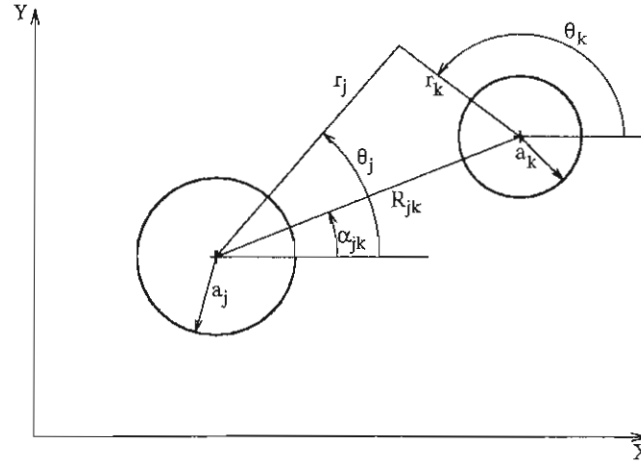


Figure 1: Basic configuration.

3 Second order solution

For the sake of clarity the second order diffraction potential $\varphi_D^{(2)}$ is further decomposed in two parts $\varphi_D^{(2)} = \varphi_{DI}^{(2)} + \varphi_{DD}^{(2)}$. The potential $\varphi_{DI}^{(2)}$ is chosen to satisfy the homogeneous condition on the free surface and nonhomogeneous condition on the cylinders, while the potential $\varphi_{DD}^{(2)}$ satisfies nonhomogeneous condition on the free surface and homogeneous one on the cylinders.

3.1 Potential $\varphi_{DI}^{(2)}$

Due to the homogeneous condition on the free surface, this potential satisfies the similar BVP as the first order potential $\varphi_D^{(1)}$, so that the same method can be used. The only difference is that the solution for the potential $\varphi_{DI}^{(2)}$, in addition to the propagative part associated with the Hankel functions H_m , will also include evanescent (local) wave components associated with the modified Bessel functions K_m .

First we divide the total potential $\varphi_{DI}^{(2)}$ into two parts $\varphi_{DI}^{(2)} = \hat{\varphi}_{DI}^{(2)} + \tilde{\varphi}_{DI}^{(2)}$. The potential $\hat{\varphi}_{DI}^{(2)}$ corresponds to the diffraction of the incident wave by each cylinder separately, while the potential $\tilde{\varphi}_{DI}^{(2)}$ accounts for interaction effects. The solution for $\hat{\varphi}_{DI}^{(2)}$ can easily be found in the form of eigenfunctions expansion :

$$\hat{\varphi}_{DI}^{(2)} = \sum_{k=1}^N \sum_{m=-\infty}^{\infty} \left[f_0^{(2)}(z) \gamma_{m0}^k H_m(\kappa_0 r_k) + \sum_{l=1}^{\infty} f_l^{(2)}(z) \gamma_{ml}^k K_m(\kappa_l r_k) \right] e^{im\theta_k} \quad (11)$$

where κ_l are solution of $4\nu = \kappa_0 \tanh \kappa_0 H = -\kappa_l \tanh \kappa_l H$ and $f_0^{(2)}(z)$ and $f_l^{(2)}(z)$ are the eigenfunctions in z direction :

$$f_0^{(2)}(z) = \frac{\cosh \kappa_0(z+H)}{\cosh \kappa_0 H} \quad f_l^{(2)}(z) = \frac{\cos \kappa_l(z+H)}{\cos \kappa_l H} \quad (12)$$

The unknown coefficients γ_{mi}^k are found after applying the body boundary condition for each cylinder $\partial\hat{\varphi}_{DI}^{(2)}/\partial n = -\partial\varphi_I^{(2)}/\partial n$. We obtain :

$$\gamma_{m0}^k = -\frac{3i\omega\nu A^2}{2\sinh^2 k_0 H} I_k^{(2)} \frac{4k_0 C_0^{(2)}}{\kappa_0(4k_0^2 - \kappa_0^2)} e^{im(\pi/2-\beta)} \frac{J'_m(2k_0 a_k)}{H'_m(\kappa_0 a_k)} \quad (13)$$

$$\gamma_{ml}^k = -\frac{3i\omega\nu A^2}{2\sinh^2 k_0 H} I_k^{(2)} \frac{4k_0 C_l^{(2)}}{\kappa_l(4k_0^2 - \kappa_l^2)} e^{im(\pi/2-\beta)} \frac{J'_m(2k_0 a_k)}{K'_m(\kappa_l a_k)} \quad (14)$$

where $I_k^{(2)} = e^{2ik_0(X_k \cos \beta + Y_k \sin \beta)}$ and $C_i^{(2)} = [2 \int_{-H}^0 f_i^{(2)}(z) f_i^{(2)}(z) dz]^{-1}$.

Now we write the eigenfunctions expansion for the "interaction" potential $\hat{\varphi}_{DI}^{(2)}$:

$$\hat{\varphi}_{DI}^{(2)} = \sum_{k=1}^N \sum_{m=-\infty}^{\infty} \left[f_0^{(2)}(z) D_{m0}^k Z_{m0}^{(2)k} H_m(\kappa_0 r_k) + \sum_{l=1}^{\infty} f_l^{(2)}(z) D_{ml}^k Z_{ml}^{(2)k} K_m(\kappa_l r_k) \right] e^{im\theta_k} \quad (15)$$

with $Z_{m0}^{(2)k} = J'_m(\kappa_0 a_k)/H'_m(\kappa_0 a_k)$, $Z_{ml}^{(2)k} = I'_m(\kappa_l a_k)/K'_m(\kappa_l a_k)$.

According to the above discussions, the boundary condition on each cylinder becomes :

$$\frac{\partial}{\partial r_k} \sum_{j=1, N} (\hat{\varphi}_{DI}^{(2)j} + \hat{\varphi}_{DI}^{(2)j}) \Big|_{r_k=a_k} = \sum_{m=-\infty}^{\infty} \left[f_0^{(2)}(z) \kappa_0 \gamma_{m0}^k H'_m(\kappa_0 a_k) + \sum_{l=1}^{\infty} f_l^{(2)}(z) \kappa_l \gamma_{ml}^k K'_m(\kappa_l a_k) \right] e^{im\theta_k} \quad (16)$$

Exploiting the orthogonality of the eigenfunctions $f_l^{(2)}(z)$ and using the Graff's theorem for Bessel functions we obtain the following systems of equations for the unknown coefficients D_{mi}^k :

$$D_{m0}^k + \sum_{j \neq k} \sum_{n=-\infty}^{\infty} D_{n0}^j Z_{n0}^{(2)j} e^{i(n-m)\alpha_{jk}} H_{n-m}(\kappa_0 R_{jk}) = - \sum_{j \neq k} \sum_{n=-\infty}^{\infty} \gamma_{n0}^j e^{i(n-m)\alpha_{jk}} H_{n-m}(\kappa_0 R_{jk}) \quad (17)$$

$$D_{ml}^k + (-1)^m \sum_{j \neq k} \sum_{n=-\infty}^{\infty} D_{nl}^j Z_{nl}^{(2)j} e^{i(n-m)\alpha_{jk}} K_{n-m}(\kappa_l R_{jk}) = -(-1)^m \sum_{j \neq k} \sum_{n=-\infty}^{\infty} \gamma_{nl}^j e^{i(n-m)\alpha_{jk}} K_{n-m}(\kappa_l R_{jk}) \quad (18)$$

3.2 Potential $\varphi_{DD}^{(2)}$

This part of the potential is the most difficult to calculate because of the nonhomogeneous condition on the free surface. We start by writing the Green's theorem for one point outside of the fluid domain :

$$\int \int_{S_B} \varphi_{DD}^{(2)} \frac{\partial G}{\partial n} = \int \int_{S_F} G Q_D^{(2)} dS \quad (19)$$

$Q_D^{(2)}$ is the right hand term in the free surface condition (3) where the contributions from the direct product of incident potentials have been removed. In the case of bottom mounted circular cylinders it can be written in the form :

$$Q_D^{(2)} = \frac{i\omega}{2g} (3\nu^2 - k_0^2) (\varphi_D^{(1)} \varphi_D^{(1)} + 2\varphi_I^{(1)} \varphi_D^{(1)}) + \frac{i\omega}{g} (\nabla \varphi_D^{(1)} \nabla \varphi_D^{(1)} + 2\nabla \varphi_I^{(1)} \nabla \varphi_D^{(1)}) \quad (20)$$

The Green function $G(r_k, \theta_k, z; \rho_k, \vartheta_k, \zeta)$ is written in its eigenfunctions expansion form :

$$G = \sum_{m=-\infty}^{\infty} \left\{ -\frac{i}{2} C_0^{(2)} \left(\frac{H_m(\kappa_0 r_k) J_m(\kappa_0 \rho_k)}{J_m(\kappa_0 r_k) H_m(\kappa_0 \rho_k)} \right) f_0^{(2)}(z) f_0^{(2)}(\zeta) \right. \\ \left. - \frac{1}{\pi} \sum_{n=1}^{\infty} C_n^{(2)} \left(\frac{K_m(\kappa_n r_k) I_m(\kappa_n \rho_k)}{I_m(\kappa_n r_k) K_m(\kappa_n \rho_k)} \right) f_n^{(2)}(z) f_n^{(2)}(\zeta) \right\} e^{im(\theta_k - \vartheta_k)} \quad ; \quad \begin{cases} r_k > \rho_k \\ r_k < \rho_k \end{cases} \quad (21)$$

The potential $\varphi_{DD}^{(2)}$ on the k -th cylinder is also assumed in the eigenfunctions expansion form :

$$\varphi_{DD}^{(2)k}(a_k, \theta_k, z) = \sum_{m=-\infty}^{\infty} \left[B_{m0}^k f_0^{(2)}(z) + \sum_{l=1}^{\infty} B_{ml}^k f_l^{(2)}(z) \right] e^{im\theta_k} \quad (22)$$

After writing the Green's theorem (19) for one point inside the cylinder $k \rightarrow (r_k = a_k - \delta, 0 < \delta \leq a_k)$, carrying out the integration by ζ , using the orthogonality of the functions $f_n^{(2)}(z)$, using the Graff's addition theorem for Bessel functions, exploiting the orthogonality of the functions $e^{im\theta}$ and rearranging the different terms we obtain the following systems of equations for the unknown coefficients B_{ml}^k :

$$B_{m0}^k + \sum_{j \neq k} \frac{a_j}{a_k} \sum_{n=-\infty}^{\infty} B_{n0}^j \frac{J'_n(\kappa_0 a_j)}{H'_m(\kappa_0 a_j)} H_{n-m}(\kappa_0 R_{jk}) e^{i(n-m)\alpha_{jk}} =$$

$$- \frac{C_0^{(2)}}{\pi a_k \kappa_0 H'_m(\kappa_0 a_k)} \int \int_{S_F} H_m(\kappa_0 \rho_k) e^{-im\vartheta_k} Q_D^{(2)}(\rho_k, \vartheta_k) dS \quad (23)$$

$$B_{ml}^k + (-1)^m \sum_{j \neq k} \frac{a_j}{a_k} \sum_{n=-\infty}^{\infty} B_{nl}^j \frac{I'_l(\kappa_l a_j)}{K'_m(\kappa_l a_j)} K_{n-m}(\kappa_l R_{jk}) e^{i(n-m)\alpha_{jk}} =$$

$$- \frac{C_n^{(2)}}{\pi a_k \kappa_l K'_m(\kappa_l a_k)} \int \int_{S_F} K_m(\kappa_l \rho_k) e^{-im\vartheta_k} Q_D^{(2)}(\rho_k, \vartheta_k) dS \quad (24)$$

The main difficulty in solving these systems of equations is the evaluation of the free surface integrals which should exclude the cylinder surfaces. In order to minimize the 2D integration the following procedure can be applied for 4 equally spaced cylinders :

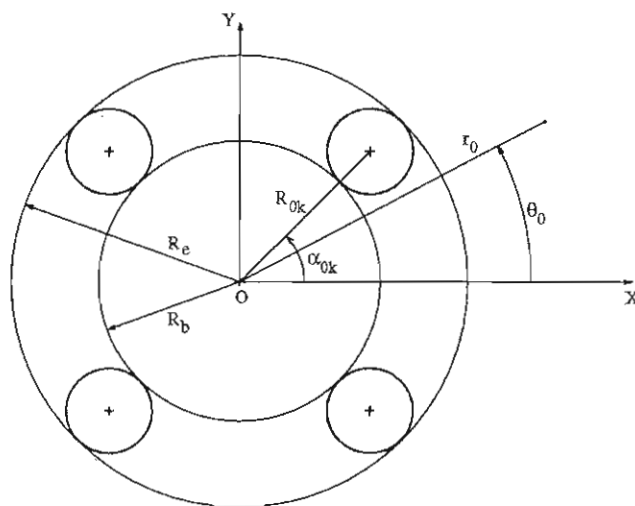


Figure 2: Different regions for integration.

In the inner ($r_0 \leq R_b$) and outer ($r_0 \geq R_e$) regions we can use the Graff's addition theorem and write the first order diffraction potential $\varphi_D^{(1)}$ in terms of the global coordinates (r_0, θ_0) [figure 2] :

$$\varphi_D^{(1)} = f_0^{(1)}(z) \sum_{m=-\infty}^{\infty} \left\{ \sum_{k=1}^N \sum_{n=-\infty}^{\infty} A_n^k Z_{n0}^{(1)k} H_{m-n}(k_0 R_{0k}) e^{i(n-m)\alpha_{0k}} \right\} J_m(k_0 r_0) e^{im\theta_0} \quad ; \quad (r_0 \leq R_b)$$

$$\varphi_D^{(1)} = f_0^{(1)}(z) \sum_{m=-\infty}^{\infty} \left\{ \sum_{k=1}^N \sum_{n=-\infty}^{\infty} A_n^k Z_{n0}^{(1)k} I_{m-n}(k_0 R_{0k}) e^{i(n-m)\alpha_{0k}} \right\} H_m(k_0 r_0) e^{im\theta_0} \quad ; \quad (r_0 \geq R_e)$$

so that the forcing function $Q_D^{(2)}$ can be easily written in the form of the Fourier series expansion :

$$Q_D^{(2)} = \sum_{m=-\infty}^{\infty} Q_{Dm}^{(2)}(r_0) e^{im\theta_0} \quad (25)$$

At the same time we use the Graff's theorem to develop the term $H_m(\kappa_0 \rho_k) e^{-im\vartheta_k}$ as follows :

$$\begin{aligned} H_m(\kappa_0 \rho_k) e^{-im\vartheta_k} &= \sum_{n=-\infty}^{\infty} H_{n-m}(\kappa_0 R_{0k}) e^{i(n-m)\alpha_{0k}} J_n(\kappa_0 \rho_0) e^{-in\theta_0} \\ &= \sum_{n=-\infty}^{\infty} \alpha_{mn}^{k0} J_n(\kappa_0 \rho_0) e^{-in\theta_0} \quad , \quad (\rho_0 \leq R_{0k}) \end{aligned} \quad (26)$$

$$\begin{aligned} H_m(\kappa_0 \rho_k) e^{-im\vartheta_k} &= \sum_{n=-\infty}^{\infty} J_{n-m}(\kappa_0 R_{0k}) e^{i(n-m)\alpha_{0k}} H_n(\kappa_0 \rho_0) e^{-in\theta_0} \\ &= \sum_{n=-\infty}^{\infty} \beta_{mn}^{k0} H_n(\kappa_0 \rho_0) e^{-in\theta_0} \quad , \quad (\rho_0 \geq R_{0k}) \end{aligned} \quad (27)$$

The free surface integral is now divided into three parts :

$$\iint_{S_F} = \int_0^{2\pi} \int_0^{R_b} + \iint_{S_d} + \int_0^{2\pi} \int_{R_e}^{\infty} \quad (28)$$

where S_d is the surface between R_b and R_e without cylinder surfaces.

In the first and third integral, the integration by ϑ_0 can be carried out explicitly (similar procedure applies for evanescent parts of the solution) :

$$\int_0^{2\pi} \int_0^{R_b} H_m(\kappa_0 \rho_k) e^{-im\vartheta_k} Q_D^{(2)} \rho_0 d\rho_0 d\theta_0 = 2\pi \sum_{n=-\infty}^{\infty} \alpha_{mn}^{k0} \int_0^{R_b} J_n(\kappa_0 \rho_0) Q_{Dn}^{(2)}(\rho_0) \rho_0 d\rho_0 \quad (29)$$

$$\int_0^{2\pi} \int_{R_e}^{\infty} H_m(\kappa_0 \rho_k) e^{-im\vartheta_k} Q_D^{(2)} \rho_0 d\rho_0 d\theta_0 = 2\pi \sum_{n=-\infty}^{\infty} \beta_{mn}^{k0} \int_{R_e}^{\infty} H_n(\kappa_0 \rho_0) Q_{Dn}^{(2)}(\rho_0) \rho_0 d\rho_0 \quad (30)$$

In this way we reduced the 2D integration to 1D and, in the same time, the most difficult integral (from R_e to ∞) is put in the same form as in the single cylinder case and the same method can be used for its evaluation [2]. On the other hand the integration in the inner and outer regions becomes same for all cylinders which is very important from numerical point of view. Only "real" 2D integration is needed over the surface S_d , and since this surface is relatively small, this can be done with relatively small computational effort.

4 Numerical results and discussions

In order to validate the present approach we chose the exemple of two cylinders for which the numerical results were presented in [1]. Each cylinder has the radius a , water depth is $3a$ and wave incidence $\beta = 0$. Positions of the cylinders are $(X_1, Y_1) = (2a, 0)$, $(X_2, Y_2) = (-2a, 0)$. On the figure 3 we present vertical distribution of $|\varphi_D^{(2)}|$ for 4 points : $T1 = (3a, 0)$, $T2 = (-3a, 0)$, $T3 = (a, 0)$, $T4 = (-a, 0)$. Lines represent semi-analytical results obtained here, and marks represent the numerical results from [1]. The agreement between the results is very good.

We present now some results for an array of 4 cylinders. Each cylinder has the radius a , water depth is $10a$ and wave incidence is $\beta = 0$. Positions of the cylinders are $(X_1, Y_1) = (3.04a, 3.04a)$, $(X_2, Y_2) = (-3.04a, 3.04a)$, $(X_3, Y_3) = (-3.04a, -3.04a)$, $(X_4, Y_4) = (3.04a, -3.04a)$ which correspond to the dimensions of the Snorre TLP platform. On the figure 4 second order free surface elevation on the cylinders 1 and 2 is presented. This elevation is calculated from the following expression :

$$\eta^{(2)} = \frac{2i\omega}{g} \varphi^{(2)} - \frac{1}{4g} \nabla \varphi^{(1)} \nabla \varphi^{(1)} - \frac{\nu^2}{2g} \varphi^{(1)} \varphi^{(1)} \quad (31)$$

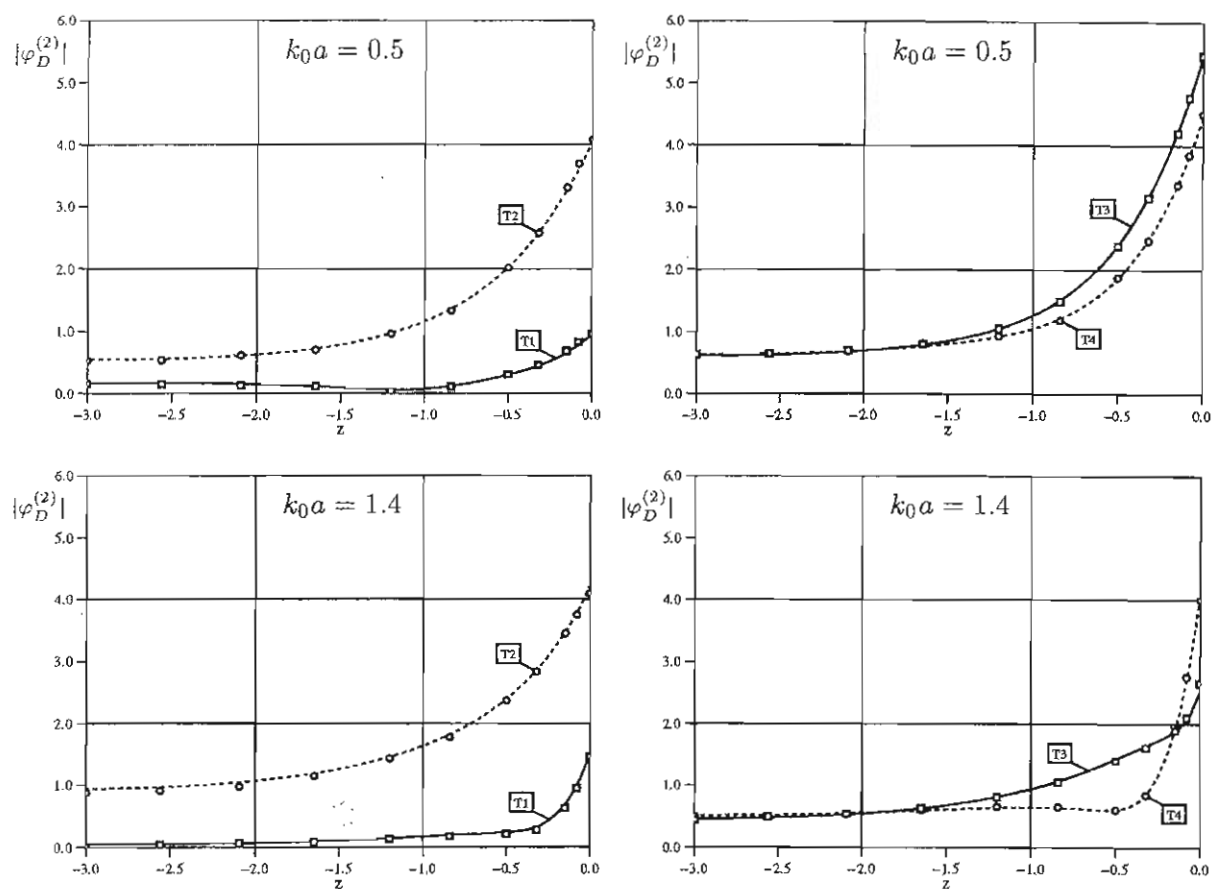


Figure 3: Vertical distribution of $|\varphi_D^{(2)}|$ on two vertical cylinders

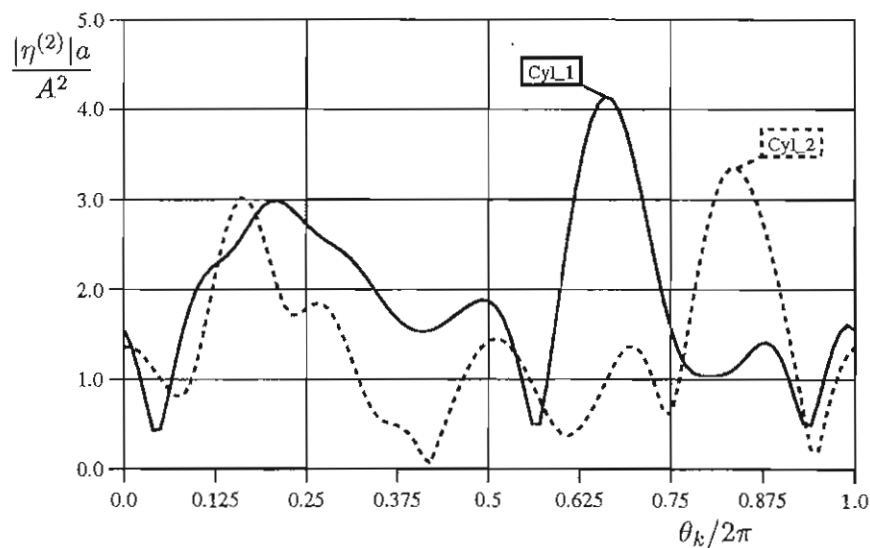


Figure 4: Second order free surface elevation for an array of 4 cylinders. ($k_0 a = 1.1$)

Finally, on the figure 5 the results for the second order surge force are shown. They are calculated using the following expression :

$$\mathbf{F}_x^{(2)} = \int_{S_{B0}} (2i\omega \varrho \varphi^{(2)} - \frac{1}{4} \varrho \nabla \varphi^{(1)} \nabla \varphi^{(1)}) n_x dS + \frac{1}{4} \varrho g \int_{C_{B0}} \eta^{(1)} \eta^{(1)} n_x dC \quad (32)$$

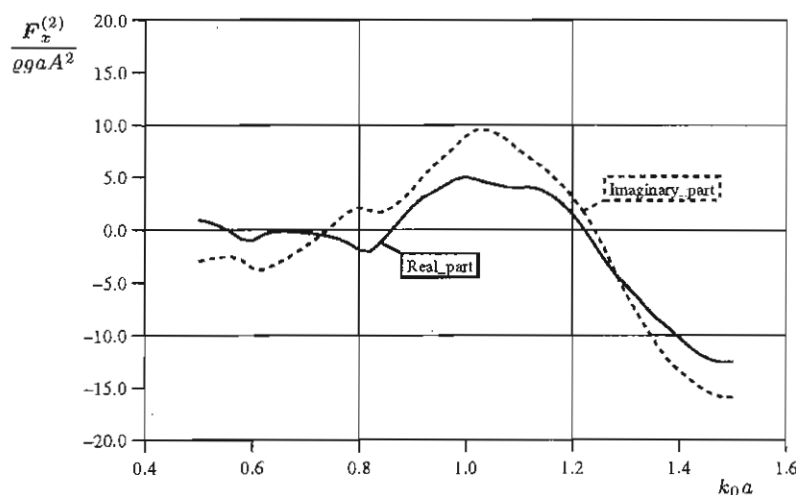


Figure 5: Second order surge force for an array of 4 cylinders.

5 Conclusion

Semi-analytical method for second order wave diffraction for an array of vertical bottom-mounted circular cylinders is presented. The main purpose of the paper was to validate the proposed approach, which is done by comparing obtained results with the numerical ones from the literature. To illustrate the robustness of the method some results for 4 cylinders in "Snorre" configuration are also presented. The principal advantages of the method are its high precision and relatively small CPU time requirements.

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Nonlinear model of random wave load on horizontal members of marine structures

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Abstract

In this paper the authors present a developed analytical procedure for predicting the wave load on horizontal cylindrical members of marine structures to the action of random waves, representing the nonlinear drag force by cubic approximation. Using Morison's equation, the spectra of the horizontal and vertical component of wave force on horizontal members are defined by the spectral analysis method. A numerical example is included to demonstrate the potential of the analytical model and to quantify the effects of nonlinear drag force on the structural load.

1 Introduction

The in-line wave forces acting on a slender member of marine structure consist of two parts; drag and inertia. In the case when the wave length is much larger than the characteristic structural diameters, the drag forces can be significant. Since the drag forces, as normally defined through Morison's equation, are nonlinear, it is difficult to obtain an analytical solution, particularly when sea state is random.

With some assumptions and additional statistical approximations, one can easily study the above problem. The same is applied in this paper, which deals with the nonlinear drag force along the lines given by Borgman.^{1,2} In fact, an attempt is made to exemplify and quantify the effects of nonlinear drag force on the second-order statistics of structural wave loading.

2 Assumptions

(a) In the following analysis the waves are considered in deep water. The linear (Airy) wave theory is employed to relate surface wave parameters and wave motions. The sea surface elevation is assumed to be an ergodic, normally (Gaussian) distributed random process of zero mean and uni-directional.

(b) It is assumed that on the analysed section the structure member is laid horizontally; that it is of a constant outside diameter; rigid; fixed; slender (in the sense that its characteristic dimension - diameter is relatively small in relation to wave length); that it is located in the horizontal co-ordinate $x = 0$; and that waves arrive perpendicularly to its axis.

(c) For the evaluation of wave force, Morison's equation³ is used in this study. That is, wave force, $F(z, t)$, is considered to be consisting of two parts; the drag component, nonlinearly related to water particle velocity; and the inertia component, linearly proportional to water particle acceleration.

Horizontal component, $F_x(z, t)$, and vertical component, $F_z(z, t)$, of this force per unit length

on the horizontal member of the marine structure, Fig. 1, can be expressed as:

$$F_x(z,t) = F_{Dx}(z,t) + F_{Ix}(z,t) = u_D v_x(z,t) |v_x(z,t)| + u_I a_x(z,t) \quad (1)$$

$$F_z(z,t) = F_{Dz}(z,t) + F_{Iz}(z,t) = u_D v_z(z,t) |v_z(z,t)| + u_I a_z(z,t) \quad (2)$$

where:

$$u_D = 0.50 C_D \rho D \quad (3)$$

$$u_I = 0.25 C_I \rho \pi D^2 \quad (4)$$

In the above equations $F_{Dx}(z,t)$ and $F_{Ix}(z,t)$ = horizontal components of drag and inertia force, respectively; $F_{Dz}(z,t)$ and $F_{Iz}(z,t)$ = vertical components of drag and inertia force, respectively; C_D and C_I = drag and inertia coefficients, respectively; ρ = water density; D = outside structure member diameter; x and z = horizontal and vertical co-ordinates, i.e. horizontal and vertical axes, respectively; $v_x(z,t)$ and $v_z(z,t)$ = horizontal and vertical components of water particle velocity, respectively; $a_x(z,t)$ and $a_z(z,t)$ = horizontal and vertical components of water particle acceleration, respectively; and t = time.

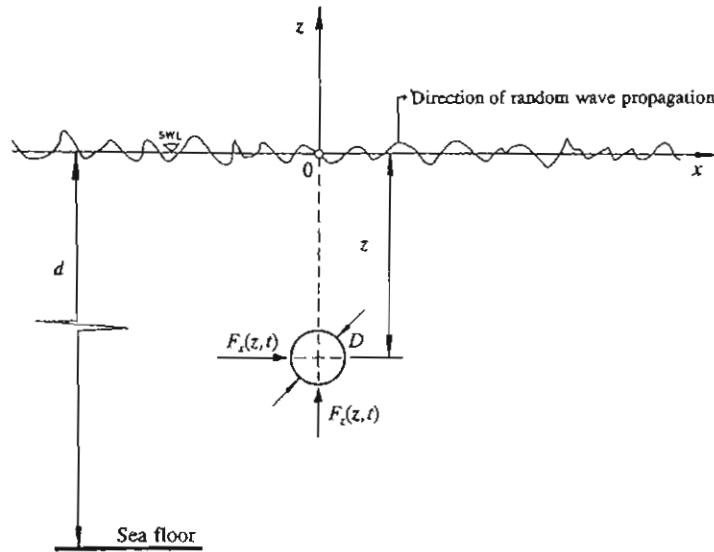


Figure 1: Definition drawing of horizontal members wave load

3 Review of wave load analysis

For the case of random waves action, Borgman^{1,2} obtained the theoretical autocorrelation function, $R_{FF}(z, \tau)$, for the horizontal force, $F(z,t) = F_x(z,t)$, on a rigid and vertical cylinder per unit length, as:

$$R_{FF}(z, \tau) = u_D^2 \sigma_{v_x}^4(z) G \left[\frac{R_{v_x v_x}(z, \tau)}{\sigma_{v_x}^2(z)} \right] + u_I^2 R_{a_x a_x}(z, \tau) \quad (5)$$

where:

$$G(r) = \frac{(4r^2 + 2) \arcsin r + 6r \sqrt{1 - r^2}}{\pi} \quad (6)$$

$$r = \frac{R_{v_x v_x}(z, \tau)}{\sigma_{v_x}^2(z)} \quad (7)$$

$$\sigma_{v_x}(z) = \left[\int_0^\infty S_{v_x v_x}(z, \omega) d\omega \right]^{1/2} \quad (8)$$

$$S_{v_x v_x}(z, \omega) = \left(\omega \frac{\cosh[k(z+d)]}{\sinh kd} \right)^2 S_{\eta\eta}(\omega) \quad (9)$$

In the above equations $\sigma_{v_x}(z)$ = standard deviation of $v_x(z, t)$; $R_{v_x v_x}(z, \tau)$ and $R_{a_x a_x}(z, \tau)$ = autocorrelation functions of $v_x(z, t)$ and $a_x(z, t)$, respectively; τ = time lag in autocorrelation functions; $S_{v_x v_x}(z, \omega)$ = spectrum of $v_x(z, t)$; $S_{\eta\eta}(\omega)$ = uni-directional wave spectrum; ω = wave frequency; k = wave number; and d = water depth.

The wave number is related to wave frequency by:

$$\omega^2 = gk \tanh kd \quad (10)$$

where g = acceleration of gravity.

The function $G(r)$ can be expanded in a power series in r as follows:

$$G(r) = \frac{1}{\pi} \left(8r + \frac{4r^3}{3} + \frac{r^5}{15} + \frac{r^7}{70} + \frac{5r^9}{1008} + \dots \right) \quad (11)$$

This series converges quite rapidly for $0 \leq r \leq 1$.

At $r = 1$, the linear approximation:

$$G_1(r) = \frac{8r}{\pi} \quad (12)$$

differs from $G(r)$ by 15 %.

The cubic approximation:

$$G_3(r) = \frac{1}{\pi} \left(8r + \frac{4r^3}{3} \right) \quad (13)$$

differs from $G(r)$ at $r = 1$ by only 1.1 %. That means, if the drag contribution is approximated by the two first terms in the series, the maximum error is only of the order of 1 %.

Hence, Eq. (5) may be recast as follows with good accuracy for most engineering problems:

$$R_{FF}(z, \tau) = \frac{u_D^2 \sigma_{v_x}^4(z)}{\pi} \left[\frac{8R_{v_x v_x}(z, \tau)}{\sigma_{v_x}^2(z)} + \frac{4R_{v_x v_x}^3(z, \tau)}{3\sigma_{v_x}^6(z)} \right] + u_I^2 R_{a_x a_x}(z, \tau) \quad (14)$$

The Fourier transform of $R_{FF}(z, \tau)$ gives the spectral density of $F(z, t)$ as:

$$S_{FF}(z, \omega) = \frac{u_D^2 \sigma_{v_x}^4(z)}{\pi} \left\{ \frac{8 S_{v_x v_x}(z, \omega)}{\sigma_{v_x}^2(z)} + \frac{4 [S_{v_x v_x}(z, \omega)]^{*3}}{3 \sigma_{v_x}^6(z)} \right\} + u_I^2 S_{a_x a_x}(z, \omega) \quad (15)$$

where the three-fold convolution of $S_{v_x v_x}(z, \omega)$ with itself is given as:

$$\begin{aligned} [S_{v_x v_x}(z, \omega)]^{*3} &= S_{v_x v_x}(z, \omega) * S_{v_x v_x}(z, \omega) * S_{v_x v_x}(z, \omega) = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [S_{v_x v_x}(z, \omega'') S_{v_x v_x}(z, \omega' - \omega'') d\omega''] S_{v_x v_x}(z, \omega - \omega') d\omega' \end{aligned} \quad (16)$$

and the spectrum of $a_x(z, t)$ is defined as:

$$S_{a_x a_x}(z, \omega) = \left(\omega^2 \frac{\cosh[k(z+d)]}{\sinh kd} \right)^2 S_{\eta\eta}(\omega) \quad (17)$$

4 Nonlinear model of random wave load on horizontal members

As shown by Vuković & Kuspilić⁴ and by analogy with Eqs. (1), (2) and (15), it follows that the load spectra on a horizontal cylindrical member of a marine structure due to the horizontal component, $S_{F_x F_x}(z, \omega)$, and vertical component, $S_{F_z F_z}(z, \omega)$, of random wave force are defined by the equations:

$$S_{F_x F_x}(z, \omega) = \frac{u_D^2 \sigma_{v_x}^4(z)}{\pi} \left\{ \frac{8 S_{v_x v_x}(z, \omega)}{\sigma_{v_x}^2(z)} + \frac{4 [S_{v_x v_x}(z, \omega)]^{*3}}{3 \sigma_{v_x}^6(z)} \right\} + u_I^2 S_{a_x a_x}(z, \omega) \quad (18)$$

$$S_{F_z F_z}(z, \omega) = \frac{u_D^2 \sigma_{v_z}^4(z)}{\pi} \left\{ \frac{8 S_{v_z v_z}(z, \omega)}{\sigma_{v_z}^2(z)} + \frac{4 [S_{v_z v_z}(z, \omega)]^{*3}}{3 \sigma_{v_z}^6(z)} \right\} + u_I^2 S_{a_z a_z}(z, \omega) \quad (19)$$

where:

$$\sigma_{v_x}(z) = \left[\int_0^{\infty} S_{v_x v_x}(z, \omega) d\omega \right]^{1/2} \quad (20)$$

$$S_{v_x v_x}(z, \omega) = \left(\omega \frac{\sinh[k(z+d)]}{\sinh kd} \right)^2 S_{\eta\eta}(\omega) \quad (21)$$

$$\begin{aligned} [S_{v_x v_x}(z, \omega)]^{*3} &= S_{v_x v_x}(z, \omega) * S_{v_x v_x}(z, \omega) * S_{v_x v_x}(z, \omega) = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [S_{v_x v_x}(z, \omega'') S_{v_x v_x}(z, \omega' - \omega'') d\omega''] S_{v_x v_x}(z, \omega - \omega') d\omega' \end{aligned} \quad (22)$$

$$S_{a_x a_x}(z, \omega) = \left(\omega^2 \frac{\sinh[k(z+d)]}{\sinh kd} \right)^2 S_{\eta\eta}(\omega) \quad (23)$$

In the above equations $\sigma_{v_x}(z)$ = standard deviation of $v_x(z, t)$; $S_{v_x v_x}(z, \omega)$ and $S_{a_x a_x}(z, \omega)$ = spectra of $v_x(z, t)$ and $a_x(z, t)$, respectively; and $[S_{v_x v_x}(z, \omega)]^{*3}$ = three-fold convolution of $S_{v_x v_x}(z, \omega)$ with itself.

Substituting the velocity and acceleration spectrum functions from Eqs. (9), (17), (21) and (23) into Eqs. (18) and (19), it follows that:

$$S_{F_x F_x}(z, \omega) = \frac{8}{\pi} \left\{ \omega u_D \sigma_{v_x}(z) \frac{\cosh[k(z+d)]}{\sinh kd} \right\}^2 S_{\eta\eta}(\omega) + \frac{4u_D^2 [S_{v_x v_x}(z, \omega)]^{*3}}{3\pi \sigma_{v_x}^2(z)} + \left\{ \omega^2 u_I \frac{\cosh[k(z+d)]}{\sinh kd} \right\}^2 S_{\eta\eta}(\omega) \quad (24)$$

$$S_{F_z F_z}(z, \omega) = \frac{8}{\pi} \left\{ \omega u_D \sigma_{v_z}(z) \frac{\sinh[k(z+d)]}{\sinh kd} \right\}^2 S_{\eta\eta}(\omega) + \frac{4u_D^2 [S_{v_z v_z}(z, \omega)]^{*3}}{3\pi \sigma_{v_z}^2(z)} + \left\{ \omega^2 u_I \frac{\sinh[k(z+d)]}{\sinh kd} \right\}^2 S_{\eta\eta}(\omega) \quad (25)$$

One of the main difficulties in practical application of Eqs. (24) and (25) lies in the evaluation of the convolution integrals, Eqs. (16) and (22), which is a rather time consuming numerical operation.

5 Numerical example

In order to illustrate the previous analysis, a numerical example of random wave load on a horizontal cylindrical member of a marine structure is presented in the text below.

It is assumed that the water depth at the member location, $d = 75.0$ m; water density, $\rho = 1025.0$ kg/m³; acceleration of gravity $g = 9.81$ m/s²; vertical co-ordinates of member horizontal axis, $z = -10.0, -20.0$ and -30.0 m (Fig. 1); member outside diameter, $D = 0.8$ m; and the values of drag and inertia coefficients, $C_D = 1.0$ and $C_I = 2.0$, respectively.

For the wave spectrum we will use Tabain's⁵ wave spectrum applicable for the Adriatic Sea conditions, for the significant wave height, $H_s = 3.6$ m.

The so selected random wave climate parameters and the diameter of a hypothetical member grants the drag loading regime.

For the purpose of evaluation of the convolution integrals given by Eqs. (16) and (22), an effective computation algorithm is developed by means of the fast Fourier transformation technique.

The calculation results are shown in Fig. 2 (a) to 2 (c). Fig. 2 (a) represents Tabain's wave spectrum. The horizontal force spectra, $S_{F_x F_x}(z, \omega)$, from Fig. 2 (b), and vertical force spectra, $S_{F_z F_z}(z, \omega)$, from Fig. 2 (c), are calculated for different member depths according to Eqs. (24) and (25), respectively.

It can be clearly seen from Fig. 2 (b) and 2 (c) that the influence of the drag force nonlinear term is considerable, both for the increase of the spectra peak values and for the increase of the total area below the spectra. Besides, the occurrence of spectra secondary peaks is evident in the region of higher frequencies, which can in the case of a certain type of structures cause a significantly higher displacement response that cannot be defined by the application of linear model.

This points to the need of taking into account the drag force nonlinear term while calculating the response of marine structures located in the drag loading regime.

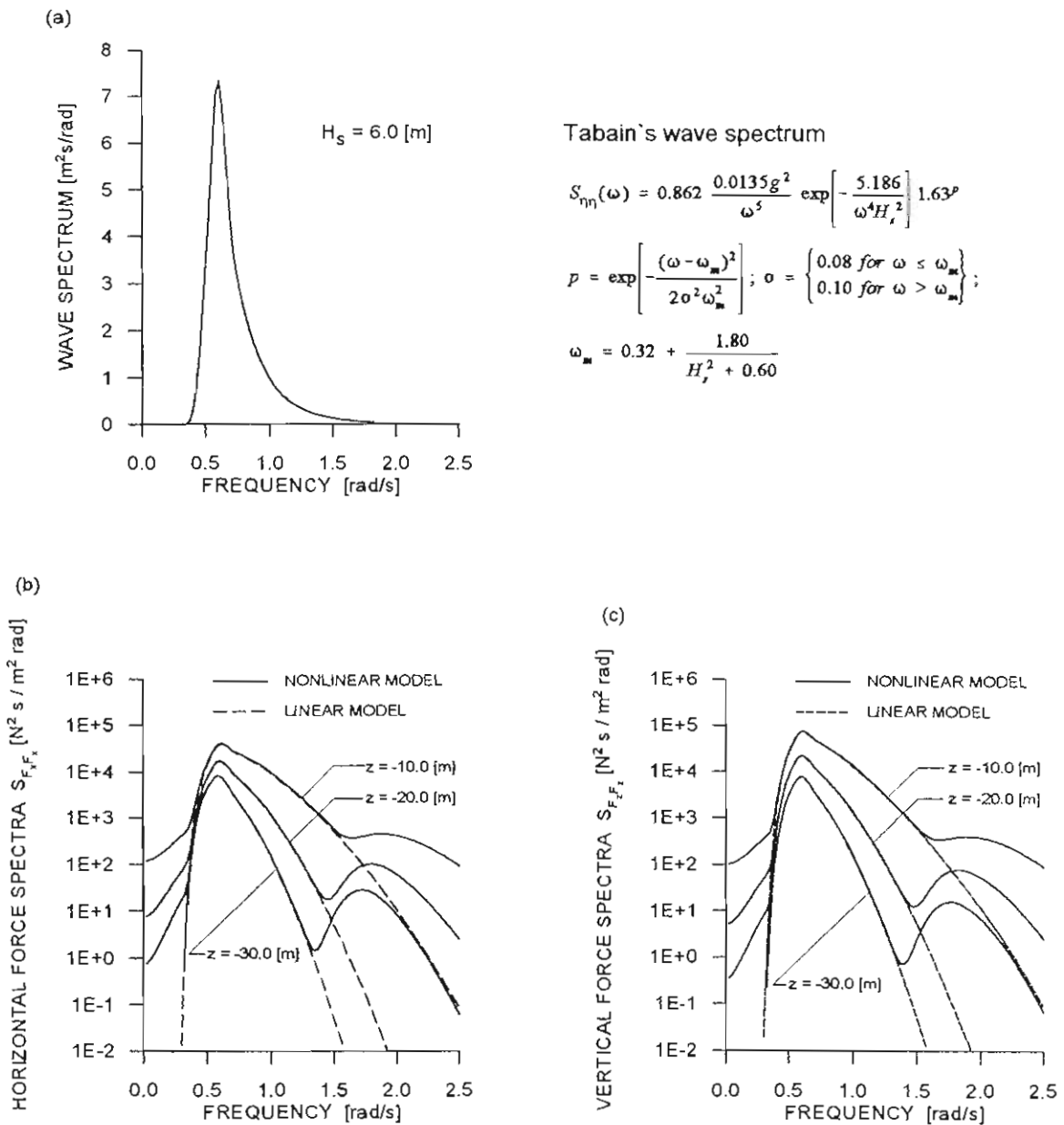


Figure 2: Wave spectrum and random wave load spectra for a hypothetical horizontal member of marine structures

6 Conclusion

A nonlinear random wave load effects on horizontal cylindrical members of marine structures are presented by applying the spectral analysis method. The performed analyses and the results of the numerical example point to the considerable influence of the drag force nonlinear term in the calculation of the wave load of horizontal members located in the drag loading regime. It is therefore justified to use in engineering problems for such cases a nonlinear model of random wave load.

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FLOW IN COLLAPSIBLE TUBES

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ABSTRACT

This study used elastic modes of thick walled physiological flow conditions. The mode exhibited flow choking when stenoses were greater than 80% by diameter. The choked flow rates were less than the normal peak systolic flow in human carotid arteries indicating the physiologic range for this phenomenon. Maximum velocities at the throat of the phantoms were approximately 6m/s, consistent with those measured by Duplex scanning in human carotids. The dynamic pressure conditions contributed to increasing the severity of the elastic stenosis at high velocities. Hemodynamic compression of severe stenoses may play an important mechanical role in the rupture of carotid plaques.

INTRODUCTION

Vessel collapse is a topic of great interest in biomechanical engineering, since most of the fluid conducting vessels are elastic in nature. The collapse of a vessel occurs when the transmural pressure, defined as the difference between the intraluminal pressure and the external tube pressure, falls below a certain value. These examples demonstrate that the phenomena of collapsible tube does exist in the compliant stenosis in this study illustrates some differences in comparison with rigid stenosis studies performed by others. The models by Sieges (1991) have described hemodynamic studies through rigid asymmetric stenoses. Their model illustrated flow limitation but collapse was not easily seen.

The Judd and Mates (1989) model also illustrates non-linear resistance's but did not address the collapse question. Binns and Ku (1989) collapse and paradoxical collapse using a rigid stenosis. Powell (1991) used rigid funnel shaped stenoses within a latex tube and found slightly higher choked flow rates at stenoses greater than 80%. As the stenosis progresses, flow is limited by losses produced by flow separation and turbulence downstream of the stenosis. In elastic arteries, the stenosis can be dynamic and increase in severity with higher flow (Fung, 1981).

Collapse in stenoses has been predicted by several one-dimensional theoretical models. Differences are expected because of the three-dimensionality of the true flow during collapse and non-uniform nature of the manufactured tubes. Downing and Ku (1997) predicted that severity of stenosis would be the dominant factor versus variations in tube stiffness, frictional loss, and pulsatile flow. The goals of this study were to create an experimental model that more closely approximated the arterial disease situation where the entire stenosis is compliant and the stenosis wall was not rigidly constrained. The model could then be used to quantify conditions in which pressure drops were predominantly due to separation or due to flow choking from collapse. The study also provides experimental data for the verification of theoretical models of arterial stenotic hemodynamics.

THEORETICAL MODEL

High-grade arterial stenosis may exhibit usual hemodynamic properties due to the low Bernoulli pressures which may develop at the throat. Blood flow through these compliant stenoses in the carotid and coronary arteries is complicated with three-dimensional, unsteady, viscous and turbulent characteristics. Analysis of fluid circulation into tube collapse is complex due to three-dimensional deforming of tube geometry, interaction between fluid and model and presence of elastic jumps. Shapiro (1977) proved that differential equation describing the tube which may collapse are analogue to flows with gas dynamic and open free surface flows. For description of one-dimensional flow in tube collapse the following equation of continuity is used

$$\frac{\partial A}{\partial t} + \frac{\partial (A u)}{\partial x} = 0 \quad (1)$$

and the Euler's equation

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \quad (2)$$

where p is fluid pressure inside the tube, $A(t,x)$ is the cross-sectional area and u is fluid velocity.

Variables defining flow through collapsible tube are transmural pressure $p - p_e$, the cross-sectionally averaged velocity, u , and the cross-sectional area $A(t,x)$.

For collapsible tube the cross-sectional area $A(t,x)$ is function of time and axial position. Nondimensional form of tube law is obtained by defining nondimensional ratio of surface

$$\alpha = \frac{A(t,x)}{A_{00} \lambda_A(x)} \quad (3)$$

A_{00} is the tube cross-sectional area at zero transmural pressure

$$\lambda_A = \frac{A_0(x)}{A_{00}} \text{ stenosis shape function}$$

and nondimensional transmural pressure

$$\lambda_K = \frac{Kp_0(x)}{Kp_{00}} \quad (4)$$

$Kp_0(x)$ nominal tube stiffness variation

Kp_{00} nominal tube stiffness

Nominal tube stiffness defined by Fung (1981) with

$$Kp_0 = -\frac{E h^3}{12(1-\nu^2) R^3} \quad (5)$$

E is the Young's modulus of the vessel,

h is the vessel wall thickness,

ν is the Poisson's ratio,

R is the mean tube wall radius.

Varying parameter λ_K from 1 to 1000 nominal tube stiffness variation $Kp_0(x)$ on complete area of stenosis can be numerically calculated, while stenosis geometry influence the tube stiffness.

It is difficult to determine exact analytical relation which presents curve pressure/surface through wide range of pressure which is present with physiological condition. In order to form behavior of the tube wall throughout complete ratio of transmural pressure, Elad et al. in 1987 and 1989 showed mathematical model through the law of tube to get

$$P = A^{n_1} - A^{-n_2} \quad (6)$$

P nondimensional pressure

A nondimensional area

n_1 tube law parameter governing positive pressure response

n_2 tube law parameter governing negative pressure response.

This equation models ratio pressure/area for A approaching zero to values much greater than 1. In the area of negative pressure, modulus of elasticity becomes very small as the tube collapses on itself. However in higher values of negative pressure tube collapse is stiff since contacts of opposite walls occur. Stenotic part of the tube has converging and diverging areas so losses on separation will not occur until gradient of opposite pressure has not appeared, diverging areas so losses on separation will not occur until gradient of opposite pressure does occur, diverging by section or elastic jump.

Elastic jump may be characterised as sudden widening of the part of the tube and then slowing of fluid circulation occurred. For collapsible tube distractions in the tube are disposed along the tube as wavy area. Wave velocity on elastic part of collapsible tube is given

$$c^2 = \frac{A}{\rho} \frac{\partial(p - p_e)}{\partial A} \quad (7)$$

A tube cross-sectional area on the part of tube

ρ density

p_e permanent outer pressure.

EXPERIMENTAL METHODS

Experimental setup and model

A drawing of the general experimental system is shown in Figure 1. The reservoirs consisted of partitioned chambers, which allowed one half of the chamber to contain a constant height of fluid. This study was designed to investigate the effect of stenosis on wall motion and collapse in a PVA tube model, Figure 2.

The axial location of wall collapse was different depending on the degree of stenosis. Stenoses of 65%, 70%, 75%, 80%, 85% and 90% diameter narrowing were produced in this fashion. All stenoses were placed 20mm downstream from the inlet end of the experimental tubing. Inner diameter of PVA tube is 8mm and outer diameter is 10mm. Initial length of PVA tube is 115mm and thickness is 1mm. The PVA tube used in these experiments had an elastic modulus (measured by pressure-diameter test) of 10 MPa over the relevant range of pressures used and a Poisson's ratio of 0.5.

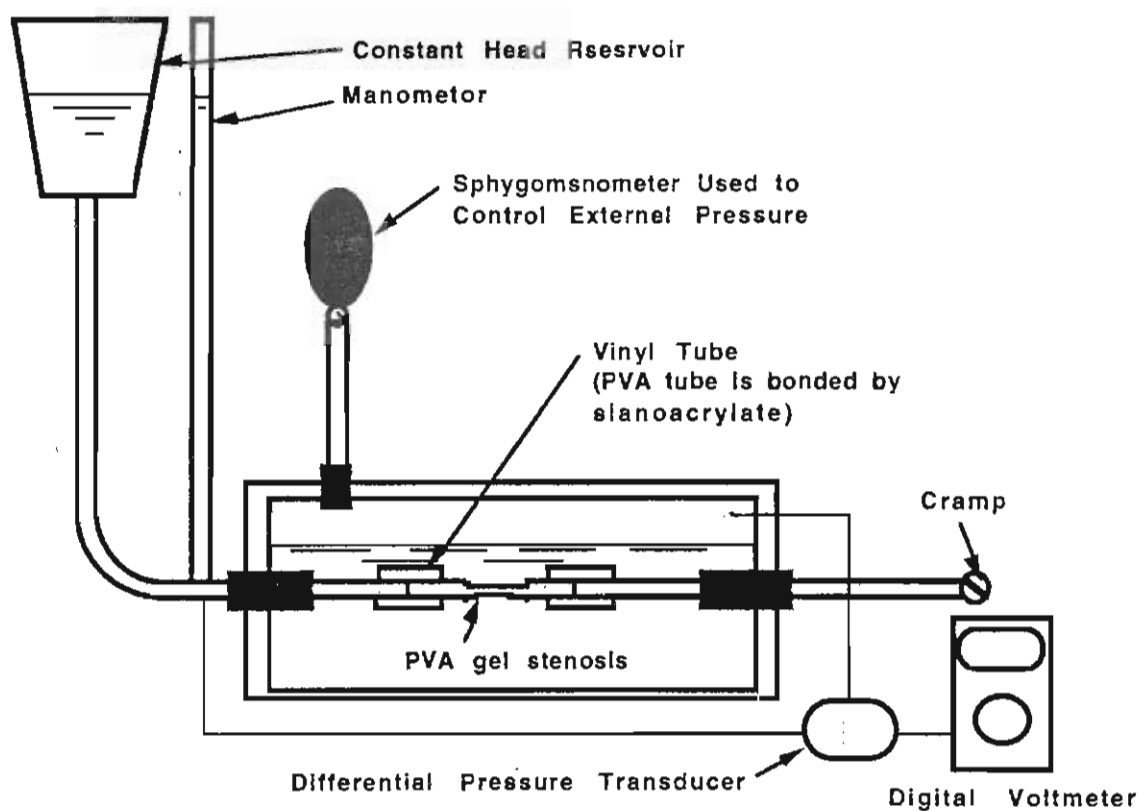


Figure 1. Schematic diagram of the steady flow system

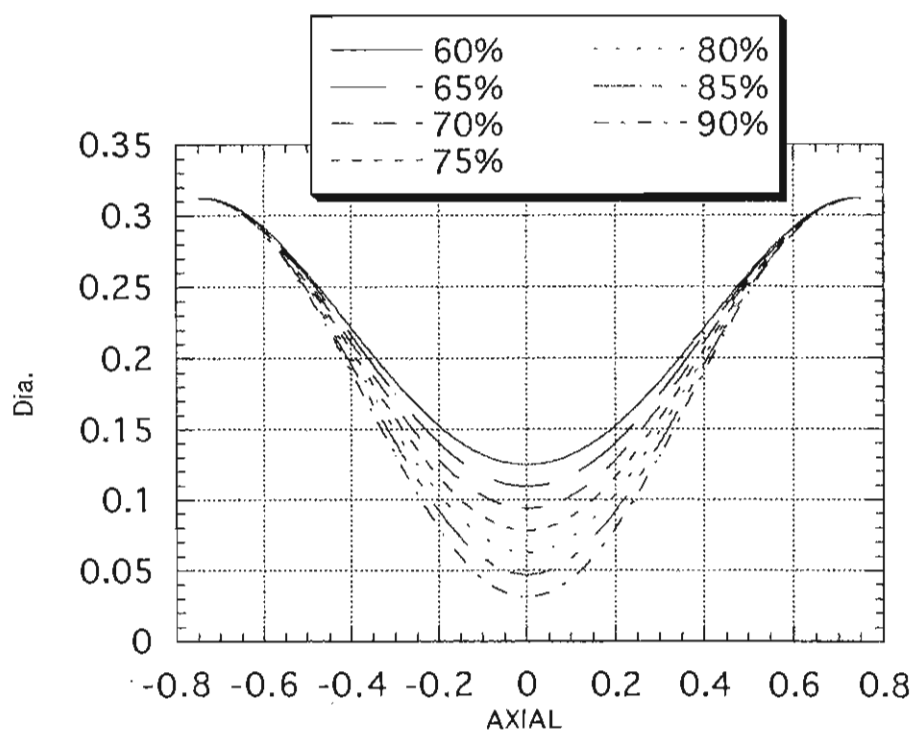


Figure 2. Diagram of the stenosis (axial (inch.) and dia (inch))

Fluids

The working fluid used was liquid water at room temperature with a kinematic viscosity of $1 \text{ cm}^2/\text{s}$.

Pressure-flow experiment

Steady flow was perfused through the silicone models of stenosis. A stenosis model was mounted on rigid pipettes-its length was measured unstretched-and placed in the test chamber, which consisted from a Starling box as shown. At this point, tap water was poured into the constant head reservoir to bring the internal tube pressure to approximately 100mmHg. Then, an external pressure of 150mmHg was applied to the tube, leading to a transmural pressure of -50mmHg. The transmural pressure was then varied from -50mmHg to 100mmHg by decreasingly the external pressure. The transmural pressure was measured by an Omega differential pressure transducer model (Omega # PX26 - 005 GV). Calibration tests revealed an accuracy of within 2mmHg. Flow rate was monitored using an ultrasound transit-time flowmeter (Transonics T101) with an 8mm in-line probe. Accuracy was within 1%.

RESULTS AND DISCUSSION

This study was designed to define the effects of stenosis on the production of arterial wall collapse using a PVA tube model. Collapse of the vessel was obvious and distinct when the pressure surrounding the tube was raised. Figure 3 illustrates relationship between mean flow rate and pressure drop across the stenosis and relationship between true stenosis severity diameter and pressure drop across the stenosis. Proximal pressure is $p_1 = 100 \text{ mmHg}$.

Flow in human carotid was simulated by PVA rubber tubing to model the compliant artery because the elastic characteristics of this tube are similar to normal arteries under physiologic pressures. Collapse of the vessel was obvious and distinct when the pressure surrounding the tube was raised. The proximal location of collapse beyond high-grade stenoses indicates that the lowest static pressure within the experimental tubing is at, or just downstream from, the stenosis. Collapse of the normally circular wall is a buckling phenomenon in which the external pressure exceeds the internal pressure. We have found some physiologic conditions in which a local stenosis produces wall collapse in a PVA tube model. Further experimental work on the effect of stenosis on the in vivo collapse of human arteries needs the pulsatile fluid flow analyse.

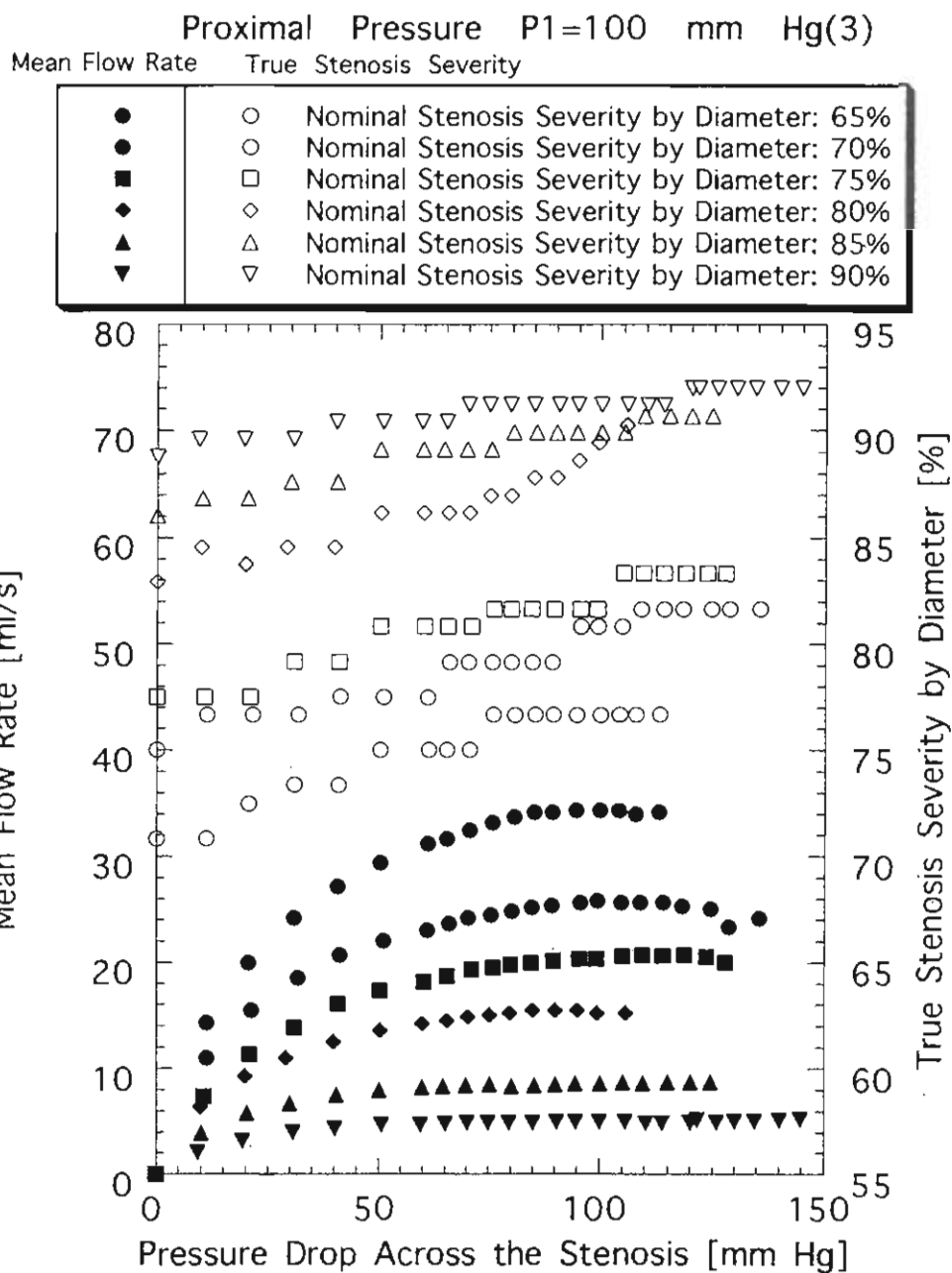


Figure 3. Graf of mean flow rate and true stenosis severity by diameter vs. pressure dropp across the stenosis.

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AN INTEGRAL REPRESENTATION OF FLOWS NEAR THE SOUND SPEED

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Abstract

The present paper is concerned with the *Tricomi-equation*

$$\eta\psi_{\vartheta\vartheta} + \psi_{\eta\eta} = 0.$$

This differential equation of mixed type is transformed in a formal-hyperbolic equation in the complex. The solutions of this equation are calculated by an integral operator. For this we consider a transformation for simplifying the differential equation of the kernel which can be represented in closed form.

The integral operator also provides a way for studying some properties of the solutions. These results ψ can be given as well for the subsonic and supersonic zone as for their transonic line $\eta = 0$ which corresponds to the sound speed. Some particular solutions of the *Tricomi-equation* which are already known are special cases of this solution.

AMS(MOS) Subject Classification. 35C15

1 Introduction

Transitional problems are problems of flow, at which subsonic and supersonic currents appear. Such a problem we have for instance for a wing at subsonic flight near the sound speed because in this case a supersonic zone will appear at the upperside of the wing. Searched are profiles, at which the non-viscous and compressible flow is almost shock free. Plane flows of compressible mediums, which velocities of flow are lower as well as higher than the sound speed can be described by partial differential equations of second order. But it is very difficult to solve such equations because they change their type by passing the *Mach cone*. In the subsonic area they are of elliptical type. In the supersonic region hyperbolic equations must be discussed. It is very difficult to solve such equations. Mostly we need a simplification of them. They become linear if we transfer to the plane of velocity by introducing the components of velocity of flow as independent variables. But we also obtain disadvantages in this transition. Even in the plane of velocity we only can get solutions in special cases and we have to accept further simplifications for solving.

We shall consider one of the simplest examples of differential equations of mixed type. That is the *Tricomi-equation*

$$\eta\psi_{\vartheta\vartheta} + \psi_{\eta\eta} = 0, \quad \psi = \psi(\eta, \vartheta) \tag{1}$$

which is of elliptical resp. hyperbolic type for $\eta > 0$ (infrasonic zone) resp. $\eta < 0$ (ultrasonic zone). This equation describes currents which possess velocities near the sound speed. In gasdynamical applications ψ can be the function of the flow and ϑ represents the angle of the flow. η results from a reversible and unique transformation of the velocity of the current. $\psi = \text{const.}$ expresses streamlines. Stagnation points correspond to the value $\eta = +\infty$.

2 A derivation of the Tricomi-equation

In this chapter we give a very simple derivation of (1) by using fundamental equations.

The theorem of conservation of mass implies the equation of continuity

$$\rho_t + \text{div}(\rho v) = 0,$$

where t is the time, ρ is the density, and v is the velocity. In the stationary case $\rho_t = 0$. If the flow is two-dimensional and vortex-free, then a velocity potential ϕ exists. So we obtain by $v = (v_1, v_2) = \text{grad } \phi$

$$(\rho \phi_{x_1})_{x_1} + (\rho \phi_{x_2})_{x_2} = 0.$$

This is the condition of integrability of the system

$$\rho \phi_{x_1} = \psi_{x_2}, \quad \rho \phi_{x_2} = -\psi_{x_1}. \quad (2)$$

If there exist no external forces, then the second Newton law implies that ρ is a function of $q = |v|$. So (2) is not linear. But it can be made linear by using q and $\vartheta = \arctan(v_2/v_1)$. It means that we transfer to the hodographic plane, i. e. v_1, v_2 - plane. In this way equation (2) becomes

$$\phi_q = \frac{M^2 - 1}{\rho q} \psi_\vartheta, \quad \phi_\vartheta = \frac{q}{\rho} \psi_q.$$

$M = q/a$ represents the *Mach-number* with $a = \sqrt{dp/d\rho}$ as the local sound speed and p is the pressure. By defining η via $d\eta/dq = -\rho/q$, we transfer from q into η . Furthermore, we eliminate ϕ by differentiation. So we obtain the *Chaplygin-equation*

$$K \psi_{\vartheta\vartheta} + \psi_{\eta\eta} = 0 \quad \text{with} \quad K = \frac{(1 - M^2)}{\rho^2}.$$

Near at the stream line $|\eta|$ is *small*. Then we obtain $K \sim b\eta$, $b = \text{constant}$. By using a suitable linear transformation of independent variables we receive $K \sim \eta$. This implies the Tricomi-equation (1) which can be seen as an approximation equation for the hodograph with flows near the sound speed.

3 The solution of the Tricomi-equation

First of all we want to transform the Tricomi-equation into an elliptical one. Furthermore, this equation will be transformed into a formally hyperbolic equation which possesses a complex form. This equation we shall solve by an integral operator. The real and imaginary part of this integral operator are solutions of (1).

To obtain the elliptical form of (1) we have to employ

$$x = \frac{2}{3} \eta^{3/2}, \quad y = \vartheta, \quad u(x, y) = \eta^\beta \psi(\eta, \vartheta), \quad \beta \in \mathbb{R}.$$

If we substitute this into (1) and divide the new equation by $\eta^{\beta-1}$ the Tricomi-equation becomes

$$u_{xx} + u_{yy} + \left(-2\beta + \frac{1}{2}\right) \eta^{-3/2} u_x + \beta(\beta + 1) \eta^{-3} u = 0.$$

The requirement $\beta = 1/4$ yields the selfadjoint elliptical equation

$$\Delta u + c(x, y) u = 0, \quad c = \frac{k}{x^2}, \quad k = \frac{5}{36}. \quad (3)$$

Remark 1: If we have subsonic resp. supersonic speed the variable x is a real one resp. an imaginary one.

Now we assume that $c(x, y)$ is any *arbitrary* analytic function of the real variables x, y for $(x, y) \in D \subset \mathbb{R}^2$ (resp. $\mathbb{C}$), where D denotes a subdomain of $\mathbb{R}^2$. This means that for each $(x_0, y_0) \in D$ exists a rectangle $R(x_0, y_0) := \{x_0 - h_1 \leq x \leq x_0 + h_1, y_0 - h_2 \leq y \leq y_0 + h_2\}$ in which the coefficient $c(x, y)$ of equation (3) has a convergent Taylor series expansion: $c(x, y) = \sum_{m \geq 0} \sum_{n \geq 0} c_{m,n} (x - x_0)^m (y - y_0)^n$. However, this means that $c(x, y)$ converges for complex values of x, y contained in the bidisk $(z_0; r) \equiv \{z \mid |z - z_0| \leq \sqrt{h_1^2 + h_2^2}\}$, i. e., $c(x, y)$ has a holomorphic extension to a complex neighborhood of $D \times \overline{D}$. $\overline{D}$ denotes the mirror domain of D , that is, $\overline{D} = \{z \mid \bar{z} \in D\}$. If D is a simple connected domain of $\mathbb{R}^2$ and $c(x, y)$ has a holomorphic extension, then D is called a fundamental domain of (3) (see [7]).

The basic idea for deriving a formally hyperbolic equation from the real elliptic equation (3) with two independent real variables x, y is the following one: We assume x, y to be independent *complex* coordinates, i. e. $x = x_1 + ix_2, y = y_1 + iy_2, i^2 = -1$. Then we may continue u and c analytically into a complex domain. Now we introduce the new complex variables

$$z = x + iy \quad \text{and} \quad \zeta = x - iy$$

which are independent. Note that $\zeta = \bar{z}$ (the conjugate) if and only if x and y are real). Furthermore, we introduce the differential operators of Wirtinger

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \zeta} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

(the Laplace operator $\Delta = 4\partial^2/\partial z\partial\zeta$). The substitution of $x, y, \partial/\partial x$ and $\partial/\partial y$ into (3) implies that this equation becomes

$$Lw = w_{z\zeta} + C(z, \zeta)w = 0, \quad w = w(z, \zeta) \quad \text{with} \quad (4)$$

$$C(z, \zeta) = \frac{1}{4}c\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = \frac{k}{(z+\zeta)^2}, \quad w(z, \zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right). \quad (5)$$

Note: If the coefficient $c(z, \bar{z})$ is a real function of the real variables x, y then $C(z, \bar{z})$ is a real function too.

Further on we will treat $Lw = 0$ in $D \times \bar{D}$. After this extension of (3) into (4) and the solution of (4) in the complex we will make the restriction $\zeta = \bar{z}$ for obtaining the real solutions of (3) resp. (1). If we have the solutions of (4), make the restriction $\zeta = \bar{z}$, we obtain real solutions of (3) by $\text{Re}\{w(z, \bar{z})\}$ and $\text{Im}\{w(z, \bar{z})\}$.

We shall consider the integral operator suggested by M. Eichler [2]. It is defined by

$$(Tf)(z, \zeta) := \int_a^z E(z, \zeta, s)f(s)ds - f(z). \quad (6)$$

where Γ is an arc in D connecting the arbitrary fixed point a and z . The holomorphic function E is called the kernel, must be determined and depends only on C , whereas the arbitrary holomorphic function $f(z)$ is independent of C and E . By formal substitution of (6) into (4) Eichler has shown that $w(z, \zeta) = Tf$ is a solution of (4) for any arbitrary function $f(z)$ if

$$LE = 0 \quad \text{for all } s \in \Gamma \text{ on } D \times \bar{D} \times \Gamma \text{ and} \quad (7)$$

$$E_\zeta(z, \zeta, s) = C(z, \zeta) \quad \text{with } s = z. \quad (8)$$

Remark 2: A second solution of the differential equation (4) for any arbitrary holomorphic function $C(z, \zeta)$ can be represented by the operator

$$(T_1g)(z, \zeta) := \int_b^\zeta F(z, \zeta, s)g(s)ds - g(\zeta).$$

The solution $w = T_1g$ is independent of $w = Tf$. Here Λ is an arc in $\bar{D}$ which connects the arbitrary fixed point b with ζ . $g(\zeta)$ can be any arbitrary holomorphic function defined in $\bar{D}$. In this case the condition (8) becomes

$$F_z(z, \zeta, s) = C(z, \zeta) \quad \text{with } s = \zeta,$$

whereas the conditional equation (7) stays the same: $LF = 0$ for $s \in \Lambda$ on $D \times \bar{D} \times \Lambda$, e. g. $F \equiv E$. If solutions of a differential equation which is not a self adjoint one are searched, $F \neq E$. But the condition $F_z(z, \zeta, \zeta) = C(z, \zeta)$ does not change.

In our case even $E_\zeta(z, \zeta, z) = F_z(z, \zeta, \zeta) = C(z, \zeta)$ holds, because $C(z, \zeta) = C(\zeta, z)$; (see (5)).

The two conditions (7) and (8) for the kernel become much simpler, if we set

$$E(z, \zeta, s) = \frac{2}{z+\zeta} \frac{dp(r)}{dr}, \quad r = \frac{2s-z+\zeta}{z+\zeta}.$$

By substituting this set-up into (7) this conditional equation becomes the ordinary differential equation

$$(1 - r^2)p''' - 4rp'' - (2 + k)p' = 0.$$

The integration of this equation yields that $p(r)$ has to fulfill the Legendre-equation

$$(1 - r^2)p'' - 2rp' + n(n + 1)p = 0, \quad \text{with} \quad n = -1/6. \quad (9)$$

So $p(r)$ represents the Legendre-function of the first kind $P_{-1/6}(r)$. Equation (8) with $C(z, \zeta) = k/(z + \zeta)^2$ (see (5)) simplifies to the boundary condition

$$p(1) = -\frac{k}{2} = -\frac{5}{72}. \quad (10)$$

Observe, that the use of the integral operator T_1 with the equations $LF = 0$ and $F_z(z, \zeta, \zeta) = k(z + \zeta)^{-2}$ for the kernel F , the transformation $F(z, \zeta, s) = 2(z + \zeta)^{-1}p'(r)$ but with the variable $r = (2s + z - \zeta)(z + \zeta)^{-1}$ also implies equations (9) and (10).

Remark 3: The Legendre-function of the 2. kind $Q_{-1/6}(r)$ is also a solution of (9) but it has a singularity in $r = 1$. So it does not satisfy the condition equation (10).

With the kernel $E(z, \zeta, r) = \frac{2}{z + \zeta} P'_{-1/6}(r)$ the representation of solutions of (4) is given by

$$w(z, \zeta) = \frac{2}{z + \zeta} \int_a^z \frac{d}{ds} P_{-\frac{1}{6}}(s) f(s) ds - f(z). \quad (11)$$

Now we introduce the new variable of integration t

$$s = \frac{1}{2}(z + \zeta)t + \frac{1}{2}(z - \zeta) \quad \text{with} \quad \text{real} \quad t: 1 \leq t \leq \infty.$$

If $a \rightarrow \infty$ — the kernel $E = P'_{-1/6}(r) = 0$ with $a = \infty$ — our integral (11) exists with $a = \infty$ by determining

$$|f(s)| < c_1 |s|^{\frac{1}{6} - \varepsilon} \quad \varepsilon > 0, \quad c_1 > 0. \quad (12)$$

After partiell integration and interchanging the limits of the integral, (11) becomes

$$w(z, \zeta) = \int_1^\infty P_{-\frac{1}{6}}(t) \frac{d}{dt} f\left(\frac{1}{2}(z + \zeta)t + \frac{1}{2}(z - \zeta)\right) dt. \quad (13)$$

So a solution of the Tricomi-equation can be represented by

$$\psi(\eta, \vartheta) = \eta^{-\frac{1}{4}} u(z, \bar{z}) = \int_1^\infty P_{-\frac{1}{6}}(t) \frac{d}{dt} f(tx + iy) dt \quad \text{with} \quad x = \frac{2}{3}\eta^{3/2}, \quad y = \vartheta. \quad (14)$$

resp. by real- and imaginary part of (14). Substitution of different functions f yields single particular solutions of (1).

Henceforth we denote by

$$\psi = \psi[f(z)]$$

a shorter manner of writing for formula (14).

If $f(z)$ generates ψ then (14) implies that ψ_{ϑ} can be constructed by $if'(z)$. That means

$$\frac{\partial}{\partial \vartheta} \psi[f(z)] = \psi[if'(z)],$$

where the sign ' means the derivation with respect to $tx + i\vartheta$. Notice, that the behavior of the decay of $|f(z)|$ which we demanded in (12) improves by the differentiation.

If $\psi = \psi[f(z)]$ is a solution of $Lw = 0$, then

$$\psi = \psi[f(\zeta)]$$

also satisfies (4); (see Remark 2). This can be deduced from the symmetry of $Lw = 0$ in z and ζ which is equivalent to the fact, that the equations (1) and (3) don't change if we replace ϑ by $-\vartheta$.

The Tricomi-equation (1) only contains ϑ in the derivation but does not possess coefficients which are functions of ϑ . So we can imply that with the holomorphic function $f(z)$ also with $f(z + z_0)$, $z_0 = \text{constant}$, a solution

$$\psi = \psi[f(z + z_0)]$$

can be generated.

Remark 4: We also can represent solutions of $Lw = 0$ if we use the integral operator of *St. Bergman* [1], which is defined by

$$(Bf)(z, \zeta) := \int_{\Upsilon}^{-1} e(z, \zeta, s) f\left(\frac{z}{2}(1 - s^2)\right) \frac{ds}{\sqrt{1 - s^2}}.$$

where the rectifiable path of integration Υ joining $s = -1$ to $s = 1$ lies in the disk $D_0 = \{s \mid |s| \leq 1\} \subset \mathbb{C}$ and usually omits the origin. (If $s^{-1}e_{\zeta}$ is continuous for $s = 0$ then Υ can pass through $s = 0$.) $f(z)$ is an arbitrary holomorphic function in D which is independent of e and the coefficients of $Lw = 0$. Bergman has shown that $w = Bf$ is a solution of $Lw = 0$ if the kernel e satisfies the equation

$$(1 - s^2)e_{\zeta s} - \frac{1}{s}e_{\zeta} + 2zsLe = 0$$

and fulfills the conditions (I): $(1 - s^2)^{1/2}e_{\zeta} \rightarrow 0$ as $s \rightarrow \pm 1$ (uniformly on $D \times \overline{D}$), and (II): $z^{-1}s^{-1}e_{\zeta}$ is a continuous function on $D \times \overline{D} \times \Upsilon$.

If we set $X^2 = s^2z/(z + \zeta)$ we can calculate e and obtain the closed forms

$$\sqrt{1 - X^2}e(z, \zeta, s) = C_1 (X + \sqrt{X^2 - 1})^{2\alpha-1} + C_2 (X - \sqrt{X^2 - 1})^{2\alpha-1} \quad \text{if } 2\alpha \neq 1,$$

$$\sqrt{1 - X^2}e(z, \zeta, s) = C_1 + C_2 \arcsin x \quad \text{if } 2\alpha = 1.$$

This Bergman operator can be preferred if 2α is an integer (incompressible axialsymmetric flows and problems of the torsion) because in the case of $2\alpha \neq 1$ we obtain the *Tschebyschew-polynoms*:

$$\sqrt{1 - X^2}e(z, \zeta, s) = A_1 T_{2\alpha-1}^*(X) + A_2 U_{2\alpha-1}(X).$$

4 The solution $\psi(\eta, \vartheta)$ on the sonic line $\eta = 0$

Of special interest in connection with partial differential equations of mixed type is the case where the function values of the equation are searched on the *transonic line*, i. e., the curve which separates the part of the plane in which the equation is elliptic from this domain in which the equation is hyperbolic. In the Tricomi-equation the transition line is given by $\eta = 0$. This line corresponds to the sound speed. For the moment we only obtain undetermined values of (14) since this formula has a singularity at $\eta = 0$. Eichler [2] investigated the behaviour of axial-symmetric potential functions at their singularities. Using this method we obtain (see also [1], p. 121)

$$\psi(0, \vartheta) = \frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi} \Gamma\left(\frac{5}{6}\right)} \int_0^{\infty} f'(\tau + iy) \frac{d\tau}{(3\tau)^{\frac{1}{12}}} \quad \text{with} \quad y = \vartheta.$$

In a similar way we get the values of the derivation on the sonic line:

$$\psi_{\eta}(0, \vartheta) = \frac{\Gamma\left(-\frac{1}{3}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{6}\right)} \int_0^{\infty} f'(\tau + iy) \frac{d\tau}{(3\tau)^{\frac{5}{12}}} \quad \text{with} \quad y = \vartheta.$$

In these two relations Γ represents the *Gamma Function*, also called the *Euler's Integral of 2. Kind*, which is defined by $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$, $\text{Re}\{z\} > 0$. With these results we can continue the solutions ψ from the elliptic domain resp. the subsonic region $\eta > 0$ to the hyperbolic region which presents the supersonic zone $\eta < 0$.

5 Several particular solutions

The result (13) resp. (14) is significant in two different ways: First, by substituting different functions f we obtain particular solutions of the Tricomi-equation which already were considered by *Carrier*, *Chaplygin*, *Darboux*, *Ehlers*, *Guderley*, *Tamada* and *Tomotika*. Secondly, Eichler's integral operator provides a way of applying function theoretic methods and results to the study of those solutions. The rate of success of this approach depends on the choice and construction of suitable kernels. And we obtained a simple one. The use of special classes of functions f , for instance polynoms, meromorphic or algebraic functions, etc. delivers the investigation of general properties of solutions. This second significance is also valid for the Bergman operator (see *Remark 4*) and other integral operators (see [6]).

The particular solutions of *Guderley* can be constructed by

$$f(z) = z^{\frac{2}{3}n + \frac{1}{6}} \quad n < 0;$$

((12) yields $n < 0$). If $n = -1/4$ then $f(z) = \text{const.}$ and $\psi \equiv 0$. This trivial solution corresponds to the stagnation point ($\eta = +\infty$) which can be seen by (14). The solution $\psi[f(z^{2n/3 + 1/6})]$ with $n = -1/4$ can be gained with $f(z) = \ln z$, but does not vanish if $\eta < \infty$.

The particular solutions of *Darboux* can be gotten by a transformation of *Guderley's* solution, therefore by powers and the logarithmic function of z .

The results of *Tomotika* resp. *Tamada* we can obtain if we replace y by $y + iy_0$ ($y = \vartheta$, $y_0 = \vartheta_0$) in Guderley's and Darboux's solution, that is

$$f(z) = (z - \vartheta_0)^{2\mu}, \quad \mu < \frac{1}{12} \quad \text{resp.} \quad f(z) = \ln(z - \vartheta_0).$$

The substitution of

$$f(z) = \exp\left(-\frac{m\pi}{\vartheta_0} z\right), \quad \operatorname{Re}\left\{\frac{m}{\vartheta_0} z\right\} \geq 0$$

into (14) yields the particular solutions of *Tschaplgin*. This function f fulfills the requirement (12).

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ON BEHAVIOUR AND STABILITY OF SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. *The present paper deals with the behaviour, approximation and stability of solutions of linear differential equations with functional coefficients. The behaviour of solutions in corresponding neighbourhoods of an arbitrary integral curve is considered. The concept of a f -stability and F -instability are introduced. The qualitative analysis theory and the topological retraction method are used.*

1. INTRODUCTION

Let us consider the system of equations

$$x' = A(t)x + b(t), \quad (1)$$

where $t \in I = (a, \infty)$, $a \in \mathbf{R}$; $n \geq 2$; the matrix-function $A(t) = (a_{ij}(t))_{i,j=1,\dots,n}$ and the vector-function $b(t) = (b_1(t), b_2(t), \dots, b_n(t))^T$ are real and continuous on interval I . The system (1) with initial value in $\Omega = \mathbf{R}^n \times I$ has unique integral curve.

The system (1) has been studied by various authors, e.g., [1-3,6,7]. We shall study (1) with respect to the behaviour, approximation as well as stability in the sense of known and following definitions of stability.

Definition 1 *Consider the system*

$$x' = \Psi(x, t), \quad (2)$$

$\Psi : \Omega \rightarrow \mathbf{R}^n$ is real and continuous. Let $f \in C^1([\bar{a}, \infty), \mathbf{R}_+^n)$ decreasing function, $f_i(\bar{a}) = 1$, $i = 1, \dots, n$. The solution φ of the system (2) is called f -stable (stable with the function of stability f) on interval I if for every $\varepsilon_i > 0$ and $t_0 \in I$ there are $\delta_i > 0$ ($i = 1, \dots, n$) such that for every solution x of (2) for which

$$|x_i(t_0) - \varphi_i(t_0)| \leq \delta_i, \quad i = 1, \dots, n \quad (3)$$

the inequalities

$$|x_i(t) - \varphi_i(t)| < \varepsilon_i f_i(t - t_0 + \bar{a}), \quad i = 1, \dots, n, \quad \forall t \geq t_0 \quad (4)$$

hold.

Stability with function f is equivalent to stability with any its translation.

If definition 1 is valid for every solution φ from certain class $S(I)$ of solutions of (2) defined on I , then the class of solutions $S(I)$ is called f -stable. If definition 1 is valid for every solution φ of (2), then the system (2) is called f -stable.

If definition 1 holds only for solutions φ and x from $S(I)$, then the class of solutions $S(I)$ is called f -auto-stable (the class of solutions $S(I)$ is stable with respect to itself).

If condition (4) valid only for $i = m_1, \dots, m_k$, $1 \leq k < n$, then solution φ (class of solutions $S(I)$ or system (2)) is called f -stable (f -auto-stable) on interval I along coordinates $x_{m_1}, x_{m_2}, \dots, x_{m_k}$.

Definition 2 Let $F \in C^1([\bar{a}, \infty), \mathbf{R}_+^n)$ increasing function tending to infinity as $t \rightarrow \infty$, $F_i(\bar{a}) = 1$, $i = 1, \dots, n$. Solution φ of (2) is called F -instable (instable with the function of instability F) on interval I if for every $\varepsilon_i > 0$ and $t_0 \in I$ there are $\delta_i > 0$ ($i = 1, \dots, n$) such that for every solution x of (2) for which (3) is valid the inequalities

$$|x_i(t) - \varphi_i(t)| \geq |x_i(t_0) - \varphi_i(t_0)| F_i(t - t_0 + \bar{a}), \quad i = 1, \dots, n, \quad \forall t \geq t_0$$

hold.

In the case $f_i(t) \equiv 1$ ($F_i(t) \equiv 1$), $i = 1, \dots, n$, we have stability (instability) according to Liapunov.

2. PRELIMINARIES

In this paper we shall consider the behaviour of integral curves $(x(t), t)$, $t \in I$ of (1) with respect to the sets

$$\begin{aligned} \omega &= \left\{ (x, t) \in \Omega : \sum_{i=1}^n \frac{1}{\alpha_i^2(t)} [x_i - \varphi_i(t)]^2 < 1 \right\} \quad \text{and} \\ \sigma &= \{ (x, t) \in \Omega : |x_i - \varphi_i(t)| < \alpha_i(t), \quad i = 1, \dots, n \}, \end{aligned}$$

where $\alpha_i \in C^1(I, \mathbf{R}^+)$, $\varphi_i(t) \in C^1(I, \mathbf{R}_+)$, $i = 1, \dots, n$, and $(\varphi(t), t)$, $t \in I$, is an arbitrary integral curve of (1).

The boundary surfaces of ω and σ are, respectively,

$$\begin{aligned} W &= \left\{ (x, t) \in \bar{\omega} : H(x, t) := \sum_{i=1}^n \beta_i^2(t) X_i^2(x, t) - 1 = 0 \right\}, \\ W_i^k &= \left\{ (x, t) \in \bar{\sigma} : H_i^k(x, t) := (-1)^k X_i(x, t) - \alpha_i(t) = 0 \right\}, \quad k = 1, 2, \\ \beta_i(t) &= \frac{1}{\alpha_i(t)}, \quad X_i(x, t) = x_i - \varphi_i(t), \quad i = 1, \dots, n. \end{aligned}$$

To obtain our results we need the following results concerning the applicability of the qualitative analysis theory and the retraction method (see, e.g., [2, 3, 6, 7]).

Let us denote the tangent vector field to an integral curve $(x(t), t)$ of (1) by X . The vectors ∇H and ∇H_i^k are the external normals on surfaces W and W_i^k , respectively. We have

$$\begin{aligned} X(x, t) &= \left(\sum_{j=1}^n a_{1j} x_j + b_1, \dots, \sum_{j=1}^n a_{nj} x_j + b_n, 1 \right), \\ \frac{1}{2} \nabla H(x, t) &= \left(\beta_1^2 X_1, \dots, \beta_n^2 X_n, \sum_{i=1}^n (\beta_i \beta_i' X_i^2 - \beta_i^2 \varphi_i' X_i) \right), \\ (-1)^k \nabla H_i^k(x, t) &= (\delta_{1i}, \dots, \delta_{ni}, -\varphi_i' - (-1)^k \alpha_i'), \quad i = 1, \dots, n; \quad k = 1, 2, \end{aligned}$$

where δ_{mi} is Kronecker delta.

By means of scalar products $P = \left(\frac{1}{2} \nabla H, X \right)$ on W and $P_i^k = \left((-1)^k \nabla H_i^k, X \right)$ on W_i^k we shall establish the existence and behaviour of integral curves of (1) with respect to the sets ω and σ , respectively.

Let us designate with $S^p(I)$, $p \in \{0, 1, \dots, n\}$ a class of solutions $x(t)$ of (1) defined on I which depends on p parameters. We shall simply say that class of

solutions $S^p(I)$ belongs to the set η ($\eta = \omega$ or $\eta = \sigma$) if graphs of functions in $S^p(I)$ are contained in η . In that case we shall write $S^p(I) \subset \eta$. For $p = 0$ we have the notation $S^0(I)$, which means there exists at least one solution $x(t)$ on I of (1) whose graph belongs to the set η .

The results of this paper are based on the following Lemmas (see [9, 10]).

Lemma 1 *If, for the system (1), the scalar product $P < 0$ on W ($P_i^k < 0$ on W_i^k , $i = 1, \dots, n$), then the system (1) has a class of solutions $S^n(I)$ belonging to the set ω for every $t \in I$, i.e., $S^n(I) \subset \omega$ ($S^n(I) \subset \sigma$).*

Lemma 2 *If, for the system (1), the scalar product $P > 0$ on W ($P_i^k > 0$ on W_i^k , $i = 1, \dots, n$), then the system (1) has at least one solution on I whose graph belongs to the set ω for every $t \in I$, i.e., $S^0(I) \subset \omega$ ($S^0(I) \subset \sigma$).*

Lemma 3 *If, for the system (1), the scalar product $P_i^k < 0$ on W_i^k for $i = n_1, \dots, n_p$, and $P_i^k > 0$ on W_i^k for $i = n_{p+1}, \dots, n_n$, then the system (1) has a class of solutions $S^p(I)$ belonging to the set σ for every $t \in I$, i.e., $S^p(I) \subset \sigma$, $p \in \{0, 1, \dots, n\}$.*

We shall use the notations: $\sum_i = \sum_{i=1}^n$, $\sum_{j \neq i} = \sum_{j=1, j \neq i}^n$, $\sum_{i,j} = \sum_{i=1}^n \sum_{j=1}^n$.

3. ON BEHAVIOUR OF SOLUTIONS

In this section we shall consider the behaviour of solutions of system (1) in ω and σ neighbourhoods of an arbitrary integral curve $(\varphi(t), t)$, $t \in I$. Let

$$A_{ii}(t) = \left[a_{ii}(t) - \frac{\alpha'_i(t)}{\alpha_i(t)} \right] \frac{1}{\alpha_i^2(t)}, \quad \bar{A}_{ii}(t) = a_{ii}(t) - \frac{\alpha'_i(t)}{\alpha_i(t)}, \quad i = 1, \dots, n,$$

$$A_{ij}(t) = \frac{1}{2} \left[\frac{a_{ij}(t)}{\alpha_i^2(t)} + \frac{a_{ji}(t)}{\alpha_j^2(t)} \right], \quad \bar{A}_{ij}(t) = \frac{\alpha_i(t) \alpha_j(t)}{2} \left[\frac{a_{ij}(t)}{\alpha_i^2(t)} + \frac{a_{ji}(t)}{\alpha_j^2(t)} \right], \quad j \neq i.$$

Theorem 1 *Let either*

$$B_{ij}(t) = A_{ij}(t) \quad \text{or} \quad B_{ij}(t) = \bar{A}_{ij}(t), \quad i, j = 1, \dots, n \quad (B_{ij} = B_{ji}).$$

(i) *If coefficients of system (1) satisfy on I either the conditions*

$$\sum_{j \neq i} |B_{ij}(t)| < -B_{ii}(t), \quad i = 1, \dots, n \quad \text{or} \quad (5)$$

$$(-1)^k \begin{vmatrix} B_{11}(t) & \dots & B_{1k}(t) \\ \vdots & & \vdots \\ B_{k1}(t) & \dots & B_{kk}(t) \end{vmatrix} > 0, \quad k = 1, \dots, n, \quad (6)$$

then the system (1) has a class of solutions $S^n(I) \subset \omega$.

(ii) *If coefficients of system (1) satisfy on I either the conditions*

$$\sum_{j \neq i} |B_{ij}(t)| < B_{ii}(t), \quad i = 1, \dots, n \quad \text{or} \quad (7)$$

$$\begin{vmatrix} B_{11}(t) & \dots & B_{1k}(t) \\ \vdots & & \vdots \\ B_{k1}(t) & \dots & B_{kk}(t) \end{vmatrix} > 0, \quad k = 1, \dots, n, \quad (8)$$

then the system (1) has at least one solution on I whose graph belongs to the set ω .

Proof. Let

$$X_i = x_i - \varphi_i(t), \text{ when } B_{ij} = A_{ij}; \quad X_i = \beta_i(t) [x_i - \varphi_i(t)], \text{ when } B_{ij} = \bar{A}_{ij}.$$

For the scalar product P we have

$$\begin{aligned} P(x, t) &= \sum_i \left[\beta_i^2(x_i - \varphi_i) \left(\sum_j a_{ij} x_j + f_i \right) \right] + \sum_i \left[\beta_i \beta'_i (x_i - \varphi_i)^2 - \beta_i^2 \varphi'_i (x_i - \varphi_i) \right] \\ &= \sum_i \left\{ \beta_i^2(x_i - \varphi_i) \left[\sum_j a_{ij} (x_j - \varphi_j) + \sum_j a_{ij} \varphi_j + f_i - \varphi'_i \right] \right\} + \sum_i \left[\beta_i \beta'_i (x_i - \varphi_i)^2 \right] \\ &= \sum_i \left(\beta_i^2 a_{ii} + \beta_i \beta'_i \right) (x_i - \varphi_i)^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left(\beta_i^2 a_{ij} + \beta_j^2 a_{ji} \right) (x_i - \varphi_i) (x_j - \varphi_j) \\ &= \sum_i \left(a_{ii} + \frac{\beta'_i}{\beta_i} \right) \beta_i^2 (x_i - \varphi_i)^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^n \left(\beta_i^2 a_{ij} + \beta_j^2 a_{ji} \right) (x_i - \varphi_i) (x_j - \varphi_j) \\ &= \sum_i B_{ii} X_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^n 2B_{ij} X_i X_j. \end{aligned}$$

Now, using that $ab \leq \frac{1}{2}(a^2 + b^2)$, $a, b \in \mathbf{R}$, and the conditions of the Theorem 1 successively, the following estimates for P on W are valid, respectively:

$$P(x, t) \leq \sum_i B_{ii} X_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^n |B_{ij}| (X_i^2 + X_j^2) = \sum_i \left(B_{ii} + \sum_{j \neq i} |B_{ij}(t)| \right) X_i^2 < 0,$$

$$P(x, t) = \sum_{i,j} B_{ij} X_i X_j < 0, \quad (9)$$

$$P(x, t) \geq \sum_i B_{ii} X_i^2 - \sum_{i=1}^{n-1} \sum_{j=i+1}^n |B_{ij}| (X_i^2 + X_j^2) = \sum_i \left(B_{ii} - \sum_{j=1, j \neq i}^n |B_{ij}(t)| \right) X_i^2 > 0,$$

$$P(x, t) = \sum_{i,j} B_{ij} X_i X_j > 0. \quad (10)$$

In (9) and (10) P is a quadratic symmetric form and according to the Sylvester criterion the conditions (6) and (8) grant the inequalities (9) and (10), respectively. This, according to the Lemmas 1 and 2, confirms the statements of the Theorem 1. $\square$

Theorem 2 Let $\alpha_i \in C^1(I, \mathbf{R}_+)$. If, on I ,

$$\frac{1}{\alpha_i(t)} \sum_{j \neq i} |a_{ij}(t)| \alpha_j(t) < -a_{ii}(t) + \frac{\alpha'_i(t)}{\alpha_i(t)}, \quad i = n_1, \dots, n_p, \quad (11)$$

$$\frac{1}{\alpha_i(t)} \sum_{j \neq i} |a_{ij}(t)| \alpha_j(t) < a_{ii}(t) - \frac{\alpha'_i(t)}{\alpha_i(t)}, \quad i = n_{p+1}, \dots, n_n, \quad (12)$$

then the system (1) has a class of solutions $S^p(I) \subset \sigma$, $p \in \{0, 1, \dots, n\}$.

Proof. Here we consider behaviour of solutions of (1) with respect to the set σ and we have

$$\begin{aligned} P_i^k(x, t) &= (\nabla H_i^k, X) = (-1)^k \left[\sum_j a_{ij} (x_j - \varphi_j) + \sum_j a_{ij} \varphi_j + f_i - \varphi_i' \right] - \alpha_i' \\ &= (-1)^k \sum_j a_{ij} (x_j - \varphi_j) - \alpha_i' \\ &= a_{ii} \alpha_i - \alpha_i' + (-1)^k \sum_{j \neq i} a_{ij} (x_j - \varphi_j), \quad i = 1, \dots, n; \quad k = 1, 2, \\ P_i^k(x, t) &\leq a_{ii} \alpha_i - \alpha_i' + \sum_{j \neq i} |a_{ij}| \alpha_j < 0, \quad i = n_1, \dots, n_p, \\ P_i^k(x, t) &\geq a_{ii} \alpha_i - \alpha_i' - \sum_{j \neq i} |a_{ij}| \alpha_j > 0, \quad i = n_{p+1}, \dots, n_n, \end{aligned}$$

and this, according to Lemma 3, confirms the statements of the Theorem 3. $\square$

4. ON STABILITY

In this section we shall give corresponding statements with respect to stability.

Theorem 3 (i) If coefficients of the system (1) satisfy any of conditions (5), (6), (11) ($\forall i = 1, \dots, n$) for bounded function α on I , then the system (1) is stable on I (stable according to Liapunov), and if function α tend to zero as $t \rightarrow \infty$, then the system (1) is asymptotic stable, and if function α is decreasing function, then the system (1) is α -stable on I .

(ii) If coefficients of (1) satisfy any of conditions (7), (8), (12) ($\forall i = 1, \dots, n$) for increasing function α , then the system (1) is instable on I (instable according to Liapunov), and if function α tend to infinity as $t \rightarrow \infty$, then the system (1) is α -instable on I .

Proof. The statements of this theorem follow from Theorems 1 and 2, the property that if conditions of the Theorems 1 and 2 are valid for some function α they are also valid for function $C\alpha$, where $C > 0$ is arbitrary constant and the fact that α_i is the width of sets ω and σ in the direction of x_i -axis. $\square$

Theorem 4 Let the function $\alpha(t) = (r_1, \dots, r_n)$, $0 < r_i = \text{const}$, $i = 1, \dots, n$. If the conditions of the Theorems 1 and 2 are true, then:

- (i) If any of conditions (5), (6), (11) ($\forall i = 1, \dots, n$) is valid, then the system (1) is stable on I and asymptotic stable;
- (ii) If any of conditions (7), (8), (12) ($\forall i = 1, \dots, n$) is valid, then the system (1) is instable on I and all solutions of (1) except φ are unbounded.
- (iii) If (11) and (12) ($1 < p < n$) are satisfied, then p -parameter class of solutions $S^p(I)$ of (1) is auto-stable on I and the system (1) is stable on I and asymptotic stable along coordinates $x_{n_1}, \dots, x_{n_p}$, and $n-p$ -parameter class of solutions $S^{n-p}(I)$ is auto-instable on I and (1) is instable on I along coordinates $x_{n_{p+1}}, \dots, x_{n_n}$.

Proof. The stability and instability of (1) follow from Theorem 3. It is need to note that there exist monotonic functions $\lambda, \Lambda \in C^1(I, \mathbf{R}_+)$ such that

$$\lambda(t) \rightarrow 0, \quad \frac{\lambda'(t)}{\lambda(t)} \rightarrow 0, \quad \Lambda(t) \rightarrow \infty, \quad \frac{\Lambda'(t)}{\Lambda(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (13)$$

and that for sufficiently large t the conditions (5), (6) and (11), which are valid for α , are also valid for $\alpha\lambda$; the condition (7), (8) and (12), which are valid for α , are also valid for $\alpha\Lambda$. $\square$

Theorem 5 (i) If the conditions (11) and (12) ($1 < p < n$) hold for decreasing function α , then p -parameter class of solutions $S^p(I)$ of (1) is α -auto-stable on I and system (1) is α -stable on I along coordinates $x_{n_1}, \dots, x_{n_p}$.

(ii) If the conditions (11) and (12) ($1 < p < n$) hold for decreasing function α and if there exists a increasing function $\rho \in C^1(I, \mathbf{R}_+)$ such that

$$\eta_i(t) := \alpha_i(t) \rho(t) \rightarrow \infty \quad \text{as } t \rightarrow \infty, \quad i = 1, \dots, n$$

so that (11) and

$$\frac{1}{\eta_i(t)} \sum_{j \neq i} |a_{ij}(t)| \eta_j(t) < a_{ii}(t) - \frac{\eta'_i(t)}{\eta_i(t)}, \quad i = n_{p+1}, \dots, n_n \quad (14)$$

hold on I , then the class of solutions $S^p(I)$ of the system (1) is α -auto-stable on I , and the class of solutions $S^{n-p}(I)$ is η -auto-unstable on I .

Proof. The statement (i) follows from the Theorem 2 and definition 1. In the case (ii) it is need to note that relation (14) implies relation (12). Indeed, multiplying (14) by $\frac{1}{\rho}$ and applying relation $\frac{\eta'_i}{\rho} \geq \frac{\eta'_i}{\rho} - \frac{\eta_i \rho'}{\rho^2} = \alpha'_i$ we obtain (12) from (14). Hence, according to (i), system (1) has an α -auto-stable class of solutions $S^p(I)$. On the other hand, by multiplying (11) by ρ and taking $\alpha'_i \rho \leq \alpha'_i \rho + \alpha_i \rho' = \eta'_i$ we have

$$\frac{1}{\eta_i(t)} \sum_{j \neq i} |a_{ij}(t)| \eta_j(t) < -a_{ii}(t) + \frac{\eta'_i(t)}{\eta_i(t)}, \quad i = n_1, \dots, n_p.$$

This relation together with (14) grants that class of solutions $S^{n-p}(I)$, for a sufficiently large t , is found outside set

$$\omega_\eta = \{(x, t) \in \Omega : |x_i - \varphi_i(t)| < \eta_i(t), \quad i = 1, \dots, n\}.$$

Hence, class of solutions $S^{n-p}(I)$ is η -auto-unstable. $\square$

Using the Theorems 1 and 2, for particular choice of function α , we can give, for example, the following corollaries.

Corollary 1 Let coefficients of the system (1) satisfy any of the conditions

$$\sum_{j \neq i} \left| \frac{a_{ij}(t)}{r_i^2} + \frac{a_{ji}(t)}{r_j^2} \right| < \frac{2}{r_i^2} |a_{ii}(t)|, \quad \sum_{j \neq i} \frac{1}{r_j} \left| r_j^2 a_{ij}(t) + r_i^2 a_{ji}(t) \right| < 2r_i |a_{ii}(t)|, \\ \sum_{j \neq i} r_j |a_{ij}(t)| < r_i |a_{ii}(t)| \quad \forall i = 1, \dots, n \quad \text{on } I, \quad r_i \in \mathbf{R}_+. \quad (15)$$

(i) If $a_{ii}(t) < 0$ on I , $i = 1, \dots, n$, then (1) is stable on I and asymptotic stable.

(ii) If $a_{ii}(t) > 0$ on I , $i = 1, \dots, n$, then (1) is unstable on I and all solutions of (1) except φ are unbounded.

(iii) Let the inequalities (15) hold. If $a_{ii}(t) < 0$ for $i = n_1, \dots, n_p$ and $a_{ii}(t) > 0$ for $i = n_{p+1}, \dots, n_n$ ($1 < p < n$) on I , then class of solutions $S^p(I)$ of (1) is auto-stable on I and asymptotic auto-stable, and class of solutions $S^{n-p}(I)$ of (1) is unstable on I and unbounded.

Corollary 2 Let $C, r_i \in \mathbf{R}_+, i = 1, \dots, n, a > 1$. If any of the conditions

$$\frac{r_i^2}{2} \sum_{j \neq i} \left| \frac{a_{ij}(t)}{r_i^2} + \frac{a_{ji}(t)}{r_j^2} \right| < \zeta_i(t), \quad \frac{1}{2r_i} \sum_{j \neq i} \frac{1}{r_j} |r_j^2 a_{ij}(t) + r_i^2 a_{ji}(t)| < \zeta_i(t),$$

$$\frac{1}{r_i} \sum_{j \neq i} r_j |a_{ij}(t)| < \zeta_i(t) \quad \forall i = 1, \dots, n \text{ on } I, \quad \zeta_i(t) = -a_{ii}(t) + \frac{C}{t \ln t (\ln t + C)}$$

holds, then the system (1) is stable on I and asymptotic stable.

In case of Corollary 1 and 2 we have, respectively,

$$\alpha(t) = (r_1, r_2, \dots, r_n), \quad \alpha(t) = \frac{\bar{C} \ln t}{\ln t + C} (r_1, r_2, \dots, r_n),$$

where $\bar{C} > 0$ is arbitrary constant. In view of Corollary 2 it is evident that system (1) can be stable also when coefficients a_{ii} are positive.

Example 1 The following system is stable on I ($a > 1$) and asymptotic stable

$$(t^3 - t) x'_1 = x_1 + tx_2 + b_1(t), \quad (t^3 - t) x'_1 = tx_1 + x_2 + b_2(t), \quad b_1, b_2 \in C(I).$$

5. APPLICATIONS

Let us consider now the system (1) and the systems

$$x' = Ax + b(t), \quad (16)$$

$$x' = [A + D(t)]x + b(t), \quad (17)$$

$$x' = A(t)x + b(t) + q(x, t), \quad (18)$$

where $A = (a_{ij})_{i,j=1,\dots,n}$ is a constant matrix, $D(t) = (d_{ij}(t))_{i,j=1,\dots,n}$, $D(t) \in C(I)$, $q(x, t) = (q_1(x, t), \dots, q_n(x, t))^T$, $q(x, t) \in C^1(\mu, \mathbf{R})$, $\mu = \bar{\omega}$ or $\mu = \bar{\sigma}$ for some integral curve $(\varphi(t), t)$, $t \in I$, of (1).

Theorem 6 Let $D(t) \rightarrow 0$ as $t \rightarrow \infty$. If the system (16) is α_r -stable on I , where $\alpha_r(t) = (r_1, \dots, r_n)$, $r_i \in \mathbf{R}_+$, then the system (17) is asymptotic stable.

Proof. According to this statement, for the system (16), corresponding conditions of the Theorems 1 and 2 are valid with $\alpha = \alpha_r$. Because $D(t) \rightarrow 0$ as $t \rightarrow \infty$, there exists function λ , with property (13), such that corresponding conditions of the Theorems 1 and 2 are valid also with $\alpha = \alpha_r \lambda$ for sufficiently large t .

Example 2 Let $n = 3$ and $A = \begin{pmatrix} -1 & 4 & -2 \\ -1 & -1 & 2 \\ 1 & -5 & -2 \end{pmatrix}$. Corresponding system (17) is asymptotic stable. Here is $r_1 = r_3 = C > 0$, $r_2 = 2C$.

Theorem 7 Let $q(x, t) \rightarrow 0$ as $t \rightarrow \infty$, $(x, t) \in \mu$. If for (1) the Theorems 1 and 2 hold with $\alpha(t) = \alpha_r(t)$, then the Theorems 1 and 2 hold also for the system (18) for sufficiently large t on $\mu = \bar{\omega}$ (for Theorem 1) or $\mu = \bar{\sigma}$ (for Theorem 2).

For the proof of Theorem 7 it is valid similarly observation like in the proof of Theorem 6. Further, we can note that given studying of (1) can be applied in studying of (18).

Example 3 Consider the well-known continuous predator-prey model of an ecosystem introduced by Volterra ([5])

$$\frac{dN_1}{dt} = \rho N_1 - \sigma N_1^2 - \theta N_1 N_2, \quad \frac{dN_2}{dt} = -\mu N_2 + \gamma N_1 N_2, \quad (19)$$

where $N_1(t)$, $N_2(t)$ represent, respectively, the prey and predator at time t ; $\rho, \sigma, \theta, \mu, \gamma \in \mathbf{R}^+$.

Theorem 8 Let $\eta, m \in \mathbf{R}_+$ such that

$$(\sigma + \theta + \gamma)\eta < \rho + \mu \quad \text{and} \quad (\sigma + \theta)\eta - \rho < m < \mu - \gamma\eta.$$

The system (19) has an one-parameter class of solutions $(N_1(t), N_2(t))$ satisfying the conditions

$$|N_1(t)| < \eta e^{-mt}, \quad |N_2(t)| < \eta e^{-mt} \quad \forall t \geq 0.$$

Example 4 Let us consider the Lane-Emden equation ([4])

$$\Delta u + |x|^s u^p = 0, \quad x \in \mathbf{R}^n, \quad n > 2, \quad p > 1. \quad (20)$$

This equation arises in many areas of applied mathematics, e.g., the scalar field equation for the standing wave of nonlinear Schrödinger and Klein-Gorden equations, the Matukuma equation describing the dynamics of globular cluster of stars.

Theorem 9 Let $r = |x|$, $\theta, \rho, m \in \mathbf{R}_+$ such that

$$0 < \theta \leq 1, \quad m = (n-2)p - s - 2 > n-2, \quad \theta \rho^{1-p} m(m-n+2) > 1.$$

Then (20) has at least one positive radial solution $u(r)$ satisfying the conditions

$$-\theta \rho r^{-m} < u(r) - \rho r^{-n+2} < 0, \quad 0 < u'(r) + \rho(n-2)r^{1-n} < \theta m \rho r^{-m-1} \quad \forall r \geq 1.$$

Since we are interested in radial solutions $u = u(r)$, then (20) gives the equation

$$u'' + \frac{n-1}{r} u' + r^s u^p = 0 \quad \text{or system} \quad u' = v, \quad v' = -\frac{n-1}{r} v - r^s u^p.$$

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INVERSE DYNAMICS AS A METHOD FOR LINEARISATION OF NONLINEAR SYSTEMS

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Abstract

Due to difficulties arising when dealing with nonlinear systems it is common practice to analyze their linear counterparts. One of the algorithyms to reach linear properties is by means of using inverse dynamics, the task which can be utilized using feedback linearisation. In contrary to Jacobian linearisation which takes into account first derivates only, this algorithym takes into account derivates of higher order too. That's the reason of improved accuracy in rendering linear properties. Method has been used to construct command line of sight guidance for guided missile.

1. INTRODUCTION

Nonlinear systems are very common in engineering practice. However, due to difficulties in handling with nonlinearities, it has been common practice dealing with their linear representations. The most widely used algorithm for linearisation is so-called Jacobean linearisation. The drawback of this kind of representation is the narrow dynamic regime which can be represented with acceptable error, which is often unacceptable.

Another method which utilizes system's inverse dynamics will be presented. This algorithm is based on feedback linearization, which can be considered as a nonlinear counterpart to the pole placement method for linear systems.

2. INPUT-OUTPUT LINEARIZATION OF NONLINEAR SYSTEMS

Consider (SISO) nonlinear system that can be presented with following equations:

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y(t) &= h(x(t))\end{aligned}\tag{1}$$

Functions $f(x):R^n \rightarrow R^n$, $g(x):R^n \rightarrow R^n$ and $h(x):R^n \rightarrow R^1$ are C^∞ functions of state vector x .

The problem can be stated as follows:

given the system (1), find the integer p (called relative degree) and smooth functions $\alpha(x)$, $\beta(x)$ which form state feedback:

$$u = \alpha(t) + \beta(t)v\tag{2}$$

such that the feedback modified system

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))\alpha(x(t)) + g(x(t))\beta(x(t))v(t) \\ y(t) &= h(x(t))\end{aligned}\tag{3}$$

has the property that the p -th derivative of the output is directly connected to the input by:

$$y^{(p)}(t) = v(t)\tag{4}$$

System defined by equation (4) represents chain of p integrators. Therefore, it will render properties of linear systems, as shown in the following picture.

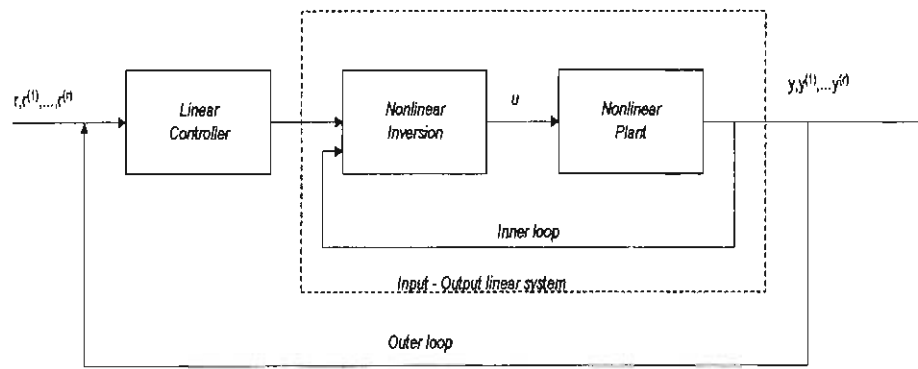


Figure 1. Schematic representation of input - output linearisation

The feedback law needed to achieve this property can be found using a following algorithm:

- differentiate output of the system until the input is found in the expression. Solve it for input u ;
- form the control law for the feedback modified system using equation (4).

Consider, for example, reference trajectory asymptotic tracking. Error could be found, which is the difference between the output and the reference input (r). If the system is to asymptotically track reference trajectory, the following equation must be satisfied:

$$(y-r)^{(p)} + \alpha_{p-1}(y-r)^{(p-1)} + \dots + \alpha_1(y-r)^{(1)} + \alpha_0(y-r) = 0 \quad (5)$$

which can be solved for input v :

$$v = r^{(p)} - \{ \alpha_{p-1}(y-r)^{(p-1)} + \dots + \alpha_0(y-r) \} \quad (6)$$

If this input is included in the first of the equations (3), resulting system will exhibit properties of the linear systems.

For MIMO systems, algorithm is essentially the same, with some differences: functions $\alpha(X)$ and $\beta(X)$ are matrix functions in general case, relative degree p is not scalar but vector. Then matrix $\beta(X)$ must be invertible to make feedback linearisation working.

The problem of stability of such linearised system has not been discussed till now. It can be shown(ref. [1]) that this algorithm cancels zeros of the original system. In other words linearised system will be stable if the system is minimum phased one.

3. COMMAND LINE OF SIGHT GUIDANCE EVALUATION

This example represents using of the feedback linearisation technique to formulate control law for a guided missile. The missile is guided from the ground control center, and it is guided to be as near as possible to the command line of sight (CLOS) to the target. Dynamics of the missile and autopilot have been neglected. The angle of attack has been neglected too. With these assumptions missile can be presented with a mass point modified model. The output of the system is the distance from the missile to the command line of sight. State variables are $[x, y, z, v_x, v_y, v_z, \psi, \theta]^T$. Coordinates and velocities are presented in fixed coordinate system. Angle ψ is the angle between projection of velocity V_{xy} and x axis, θ is the angle between velocity V and plane xOy . Inputs to system are accelerations a_y, a_z . Acceleration a_x has not been considered input, because it has been assumed that the throttle couldn't be regulated.

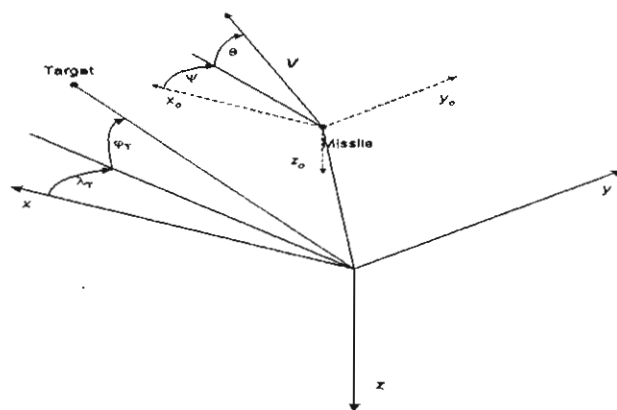


Figure 2. Coordinate systems orientation and angles used in CLOS guidance

Accelerations $\ddot{x}, \ddot{y}, \ddot{z}$ are related to a_x, a_y, a_z as follows:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = [L_y(\theta) * L_z(\psi)]^T * \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \quad (7)$$

Matrixes L_y, L_z represent coordinate transformations. Angles θ, ψ can be found using following equations:

$$\begin{aligned} \dot{\psi} * V \cos(\theta) &= a_y \\ \dot{\theta} * V &= -a_z - g \cdot \cos(\theta) \end{aligned} \quad (8)$$

Error can be presented as y, z coordinates of missile in locator's coordinate system, with R_p representing distance to the missile along the line of sight:

$$\begin{bmatrix} R_p \\ e_1 \\ e_2 \end{bmatrix} = L_y(\varphi_T) * L_z(\lambda_T) * \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (9)$$

or in expanded form:

$$\begin{aligned} e_1 &= -\sin(\lambda_T) * x + \cos(\lambda_T) * y \\ e_2 &= \sin(\varphi_T) * \cos(\lambda_T) * x + \sin(\varphi_T) * \sin(\lambda_T) * y + \cos(\varphi_T) * z \end{aligned} \quad (10)$$

Complete system can now be represented in the form of equations (1).

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} &= \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} \\ \begin{bmatrix} \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} &= [L_z(x_7) \cdot L_y(x_8)]^T \cdot \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \\ \begin{bmatrix} \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} &= \begin{bmatrix} \frac{a_y}{\cos(x_8)} \\ -a_z - g \cdot \cos(x_8) \end{bmatrix} \cdot \frac{1}{\sqrt{(x_4^2 + x_5^2 + x_6^2)}} \\ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} &= \begin{bmatrix} -\sin(\lambda_T) & \cos(\lambda_T) & 0 \\ \sin(\varphi_T) * \cos(\lambda_T) & \sin(\varphi_T) * \sin(\lambda_T) & \cos(\varphi_T) \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned} \quad (11)$$

To find feedback linearisation it is necessary to differentiate e_1 and e_2 twice until input is found in expression. After collecting terms including input $u = [a_y, a_z]^T$, the matrix differential equation can be presented:

$$\ddot{e} = \begin{bmatrix} \ddot{e}_1 \\ \ddot{e}_2 \end{bmatrix} = E(X(t)) + D(X(t)) * u \quad (12)$$

$E(X(t))$ is 2 by 1 matrix, $D(X(t))$ is 2 by 2 matrix and $X(t)$ is the state vector. Solving this equation for u (recalling equation (4)) gives:

$$u = [D(X(t))]^{-1} * (v - E(X(t))) \quad (13)$$

In order to make feedback linearization working, it is necessary for matrix $D(X(t))$ being invertible.

Aim of this system is tracking error to zero asymptotically, so the reference trajectory in equation (6) equals to zero too. Then the reference input can be presented in the following form:

$$\begin{aligned} v_1 &= \ddot{e}_1 = -\alpha_{11}\dot{e}_1 - \alpha_{10}e_1 \\ v_2 &= \ddot{e}_2 = -\alpha_{21}\dot{e}_2 - \alpha_{20}e_2 \end{aligned} \quad (14)$$

Note that the second derivate of the output is not only directly connected to the input, but is uncoupled too. Coefficients α_{ij} , ($i, j = 1:2$) specify dynamic properties of error (output of the system) in the same way as for the linear systems.

As an illustration, a program has been written in **MATLAB**, including nonlinear dynamics described by equations (11), which was used for simulation of CLOS guiding. Starting velocity of the missile was 80 m/s along x axis. Angles $\lambda = 3^\circ$, $\varphi = 5^\circ$ represent starting angle error (in other words for $t=0$, missile is on the line of sight, but it's velocity isn't pointed correctly). State vector for $t=0$ is $X = [0, 0, 0, 80, 0, 0, 0, 0]^T$.

Settling time for the error is chosen to be $t_s = 5s$, and natural frequency $\omega_n = 6.28$ rad/s (properties usually connected to linear systems), resulting in coefficients $\alpha_{11} = \alpha_{21} = 1.84$, $\alpha_{10} = \alpha_{20} = 40.28$. Results are shown in the figure 3.

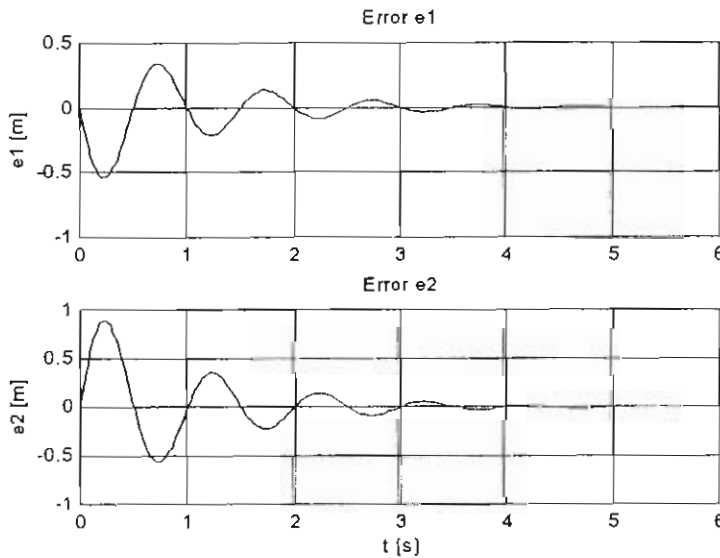


Figure 3. Dynamics of error corresponding to $\alpha_{11} = \alpha_{21} = 1.84$, $\alpha_{10} = \alpha_{20} = 40.28$

If α_{21} is set to zero, there will be no damping in error e_2 . Resulting dynamics will present classical sinusoidal motion:

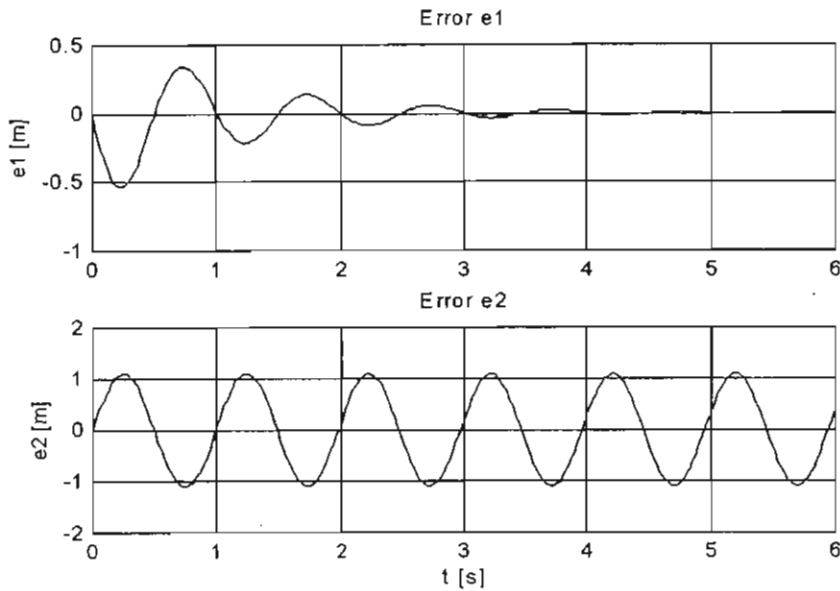


Figure 4. Dynamics of error corresponding to $\alpha_{11} = 1.84$, $\alpha_{21} = 0.0$, $\alpha_{10} = \alpha_{20} = 40.28$

It should be noted that diagrams on figures 3 and 4 are obtained by using nonlinear system described by equations (11) and guidance law formed by equations (14), which was included in equation (13) to create input (u) for original nonlinear system. The resulting motion obeys rules of linear systems, despite the nonlinear structure of equations (11). On the other hand the example presented is the ideal one, because in the practise mathematical model never describes exactly. Next figure represents results obtained by applying same mathematical model and procedure to complete 6DOF model of rotating, impulse controlled missile.

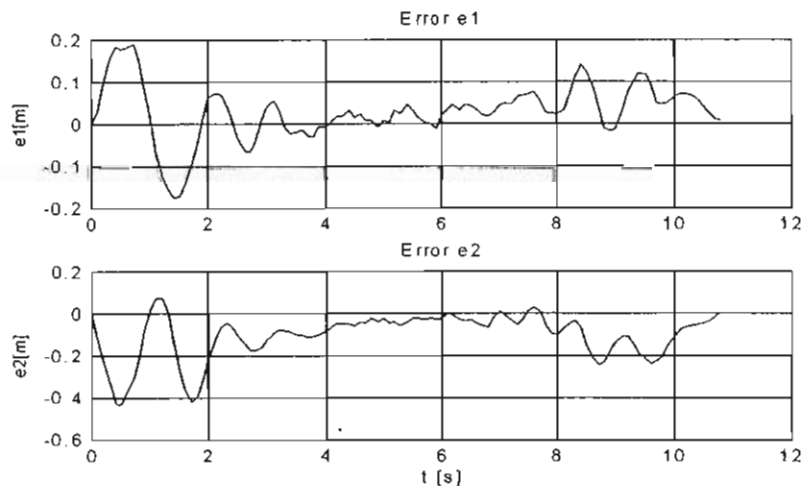


Figure 5. Rotating, impulse steered missile described by 6DOF model. Dynamics of error corresponding to $\alpha_{11} = \alpha_{21} = 1.84$, $\alpha_{10} = \alpha_{20} = 40.28$

4. CONCLUSION

An alternative approach to reach linear properties of the dynamic systems has been presented. Control law has been formulated using rules for linear systems, which has been transformed into nonlinear one. The procedure shown is nonlinear counterpart to pole placement method for linear systems (ref. [2]). The method is quite simple from teoretical point of view, but it's not been so widely used because of amount of computation required for realisation of algorithm.

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THE METHOD OF INTRAABDOMINAL FORCE DETERMINATION IN HUMAN'S BODY MODEL

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SUMMARY

This work deals with the simple method for estimating of abdominal resultant force in man depending on his age and statistical distribution. Many biomechanical analyses usually start with the formation of the biomechanical model of the human body. In this paragraph the method for the human figure drawing is combined with the simple method for estimating the abdominal pressure depending of their age and individual distribution of the anthropomeasures. The analysis and the measurements were carried out only for adult males, because this work is done as the basic preparation in establishing the methodology for the analysis of the working ability of the males. Recorded forces are very good tools for future study of modelling the biomechanical models in work analysis.

1. INTRODUCTION

Precise analysis of the human figure partially in motion in space is a very complex task. The reasons for this complexity are numerous and particularly important in the formation of biomechanical models. This fact incited us to the elaboration of this work.

In the existing literature available, there are lots of data on parameters of the human body and data from works of Grandjean (1980), Bernstein (1974), Kaleps and collaborators (1984), Donskij and Zatscijorskij (1979), Božičević J. (1986), Keros (1977), Rudan (1978) and Muftić (1984, 1988) are particularly distinguished. In the mentioned sources, values of so called static anthropomeasures are predominantly pointed out. Linear dimensions of body parts, namely lengths of the body and its parts, are in them as well. Besides, data on the extent of some typical motions of body parts also appear in this literature and are taken as kinematic anthropomeasures.

Mentioned results are usually presented on charts accompanied with drawings provided with signs to which individual figures relate. The charts are provided either with mean or limiting values presenting the highest and the lowest values or are distributed into percentile groups (usually three or five groups). Disregarding the clarity, the use of these results is relatively unsuccessful when one wants to include the figure of the acting subject into the drawn-in work place to serve for ergonomical analysis.

The above difficulty has motivated this work with clear goal to establish a method of drawing those characteristic sizes of human body which could determine its special configuration together with the characteristic point of its parts which are necessary for the biomechanical models.

Several investigations have shown that the stress imposed on the trunk in various handling tasks induce a pressure rise in the abdominal cavity (Morris, Davis, Stubbs, Troup, Muftić).

This pressure acting upwards on the diaphragm, downwards on the pelvic floor, frontally on the stomach muscles and on the back side on the spine and lumbar muscles. Also this pressure relieving some part of the stress applied to the spine. According to Morris this pneumatic mechanism could sustain about 30% of calculate compressive force.

In our opinion the projective area of diaphragm and the projective area of the pelvic floor are of the same value, which should depend on statistical anthropometrical distribution as for males as well for females.

In such case the value of resultant intraabdominal force instead of pressure could be sometimes more practical and even precise in the biomechanical analysis.

In that direction in the present analysis we tried to determine relations between average cross-section of the abdomen in the function of human height.

2. METHODS AND MATERIALS

To reach the given goal we have availed ourselves to the literature proposal that the analysis of anthropomeasures is done by help of eight head length canon adds to the special geometrical construction so called harmonical circle. Using this idea we draw into the net on Figure 1., that makes the limits of normal man's contour, where the characteristic points A, B, C, D, ...M represent either positions of joints or contour edges. Coordinates of the points are determined as functions of the total height of the man and its eights, i.e. the lengths of head (HL). The following group of data was available which were determined from the relation of the drawn man according to the photography.

The general results of the mentioned harmonical analysis are given as the function of the standing height $h = 2R$ of the regarded subject and are shown in Table 1.

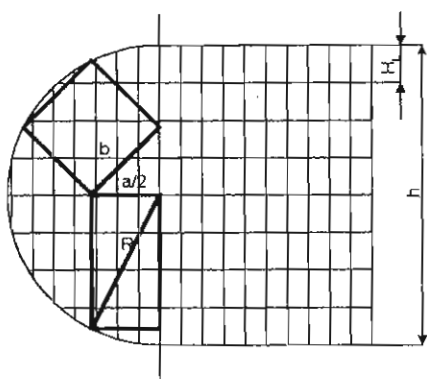


Figure 1

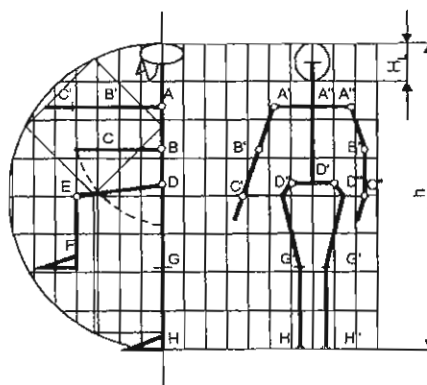


Figure 2

Table 1. The lengths of human body segments as the function of human heights.

length of arm = $(25/64)h$	length of upper arm = $(5/32)h$
length of lower arm = $h/8$	length of hand = $(7/64)h$
length of leg = $(17/32)h$	length of upper leg = $(9/32)h$
length of lower leg = $(7/32)h$	length of foot = $(1/32)h$

With such relations the next step is to draw the corresponding links between the mentioned characteristic points A, B, C, ... etc. This has determined a kind of an auxiliary skeleton presented in Figure 2. Upon the determined "skeleton", as the next step, one can draw-in the contour of the subject, but usually it is not necessary.

For a selected height and mass of the tested persons which corresponds to the highest value of the height of 181 cm and mass of 86 kg, the measuring of the characteristic lengths was performed (e.g. of the upper leg, foot etc.). The obtained values, quoted in Table 1, can easily be compared with Kroemer's subject values of standing examinee in erect posture.

According to the results of Mairiaux intraabdominal pressure IAP (in kPa) was related to the lumbar moment M_L (in Nm) at the level of lumbar spine L4/L5 by regressive equation:

$$IAP = 0,079 M_L - 1,127, \text{ kPa} \quad (1)$$

with the coefficient of correlation $r = 0,75$. On such way IAP measurements may be used as an index of spinal stress in real life tasks. We doubted to upper formula, because at moment $M_L = 0$, IAP should be even negative, or moment must have some value to make IAP always positive. It was reason that we repeated the measuring of IAP for known lumbar level L4/L5 where we performed 32 males in a series of 96 pulling the rope task.

Anthropometric values between the human height and the dimensions of the intraabdominal pressure we determined by mean of well known canon of eight head lengths, as it is shown in Figures 1 and 2. So determined anthropometric values correspond well to the responsible results in literature e.g. to the results of Kroemer (7).

3. RESULTS

The construction of the idealized geometrical construction of the abdominal cross-section according to the Fig. 3 is defined by mean of the harmonical circle and Jurčević et coll. (4) established the statistical function between area of cross section of abdomen plotted to height of the examinee as follows:

$$A = (86,5 \text{ to } 104,72) 10^{-4} h^2, \text{ m}^2 \quad (2)$$

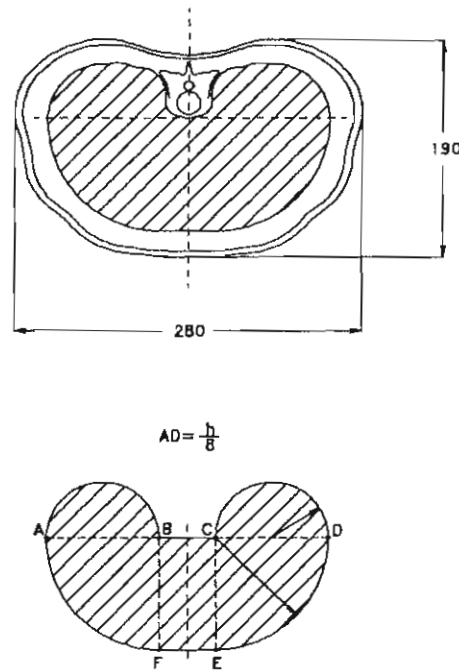


Figure 3. Cross-section through abdomen

From this formula it is easy to calculate relevant values of area for five percentiles groups for males as well females. According to determined area of the abdominal cross sections we calculated the three different groups of cross sections for three different heights of subjects.

Table 2. Calculated values of the abdominal cross sections magnitude as the function of the individual heights of examinee

	For h = 160 cm	for h = 172 cm	for h = 184 cm
Area in m ²	0,0227	0,0307	0,0352
Area in m ²	0,0247	0,0284	0,0325
Area in m ²	0,0266	0,0263	0,0300

The lumbar moment is determined through a biomechanical model deriving data from the subject anthropometry and photographically recorded postures. From these measurements is created the 3-D diagram in the Fig.4 which presents relations of statistical distribution of the heights, age and intraabdominal force. Intraabdominal forces for this diagram were calculated from the values of reduced moments on L4/L5 point, because to this value of moment there is corresponding IAP which could be represent with the adequate force. Using the regression formula of Donskij and Zatscijorskij (2) we determined the elements of the biomechanical model which were used in the calculation of the respective lumbar moments.

4. CONCLUSION

On the basis of the biomechanical elements presented, several conclusions can be drawn:

- Objective evaluation of the mechanical strain of the workers, depending on their posture could be done by means of the very simple method of calculating of the intraabdominal pressure i.e. intraabdominal force which must be in the equilibrium to the lumbar moment caused by the weight of segmental body parts and actual outside loading;

- Abdominal resultant force, which is calculate by the multiplying of the corresponded abdominal pressure and calculated area of the cross-section of the examine, for the three different body postures, have the maximal values for the subjects at about 32 years of the age.

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Cancellous bone micromechanics

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Abstract

The study of architecture-mechanics relations of the cancellous bone two aspects should be included, connectivity and anisotropy. The connectivity are described as topological measure over structure. Anisotropy of the microstructure are described across some fabric tensors. Mechanical anisotropy can be resolved by use large-scale finite element analysis. The availability of 3-D reconstruction open door for the truly description geometry of the microstructure. The elementary examples are included in order to illustrate theory.

1 Introduction

Biological structure are result of genuine design of the nature over millennia. The nature teach as to design structure for multifunctional purposes on the best possible way. From engineering point of view, the bone is a heterogeneous, porous structure, hierarchically ordered multifunctional composite with complex microstructure optimized for withstanding functional loads. The bone continually adapts to its mechanical environment and for cancellous bone this adaptation results in varying trabecular architecture. The study of architecture-mechanics relations of cancellous bone, two aspects should be included; connectivity and anisotropy.

2 Connectivity

Cancellous bone is a continuous three-dimensional (3D) network of bars and plates with varying densities and orientations. For two-phase (bone and marrow) structure, the topological properties of interest are the number of connected components of the two phases and the connectivity, which is same for the both phases. Many researches [1], have accepted that cancellous bone is a multiply connected structure without any isolated piece. The Euler number χ , is the key to all determinations of connectivity. The 3-D body may be decomposed into convex bodies $\mathfrak{R} = \bigcup_i A_i$, with defined a binary function $\mathfrak{I}$ as follow

$$\mathfrak{I}(A) = \begin{cases} 1 & A \notin \emptyset \\ 0 & A \in \emptyset \end{cases} \quad (1)$$

For particular decomposition of the body $\mathfrak{R}$, denoted by $(\mathfrak{R}, A)$, then the Euler number has been written as

$$\chi(\mathcal{R}, A) = \sum_1 \mathfrak{I}(A_1) - \sum_2 \mathfrak{I}(A_1 \cap A_2) + \sum_3 \mathfrak{I}(A_1 \cap A_2 \cap A_3) \quad (2)$$

where $\sum_n$ denote a summation over all n - permutations of the convex bodies $A_1, A_2, \dots, A_n$. The Euler number is independent of the particular decomposition into convex bodies. Another topological set of parameters are Betti numbers which provide a more detailed description of structure than Euler number [2]. Recently developed 3-D imaging technique made possible true quantification of the cancellous bone architecture. The morphology of the specimen had been digitized in a serial sectioning procedure. After segmentation of the images, the specimen architecture had been numerically reconstructed in a 3-D voxel data set. The very high numbers of voxels in 3-D reconstruction's give a risk of wrong marking voxels as bone, or the marrow phase. For any tessellation of parallelepipeds in 3-D images, the Euler characteristic of a structure in the entire image has the form

$$\chi(\mathcal{R}) = \sum_E (V) - \sum_E (F) + \sum_E (E) - \sum_E (C) \quad (3)$$

where V is volume, and sum $\sum_E$ is taken over all faces F , all edges E , and all corners C . A poor relationship between connectivity and density (Fig. 1) determined experimentally [2], suggests that connectivity should be considered an independent parameter in bone structure description. There is also a theoretical foundation about dependence other material characteristic (modules, strength) on connectivity [3].

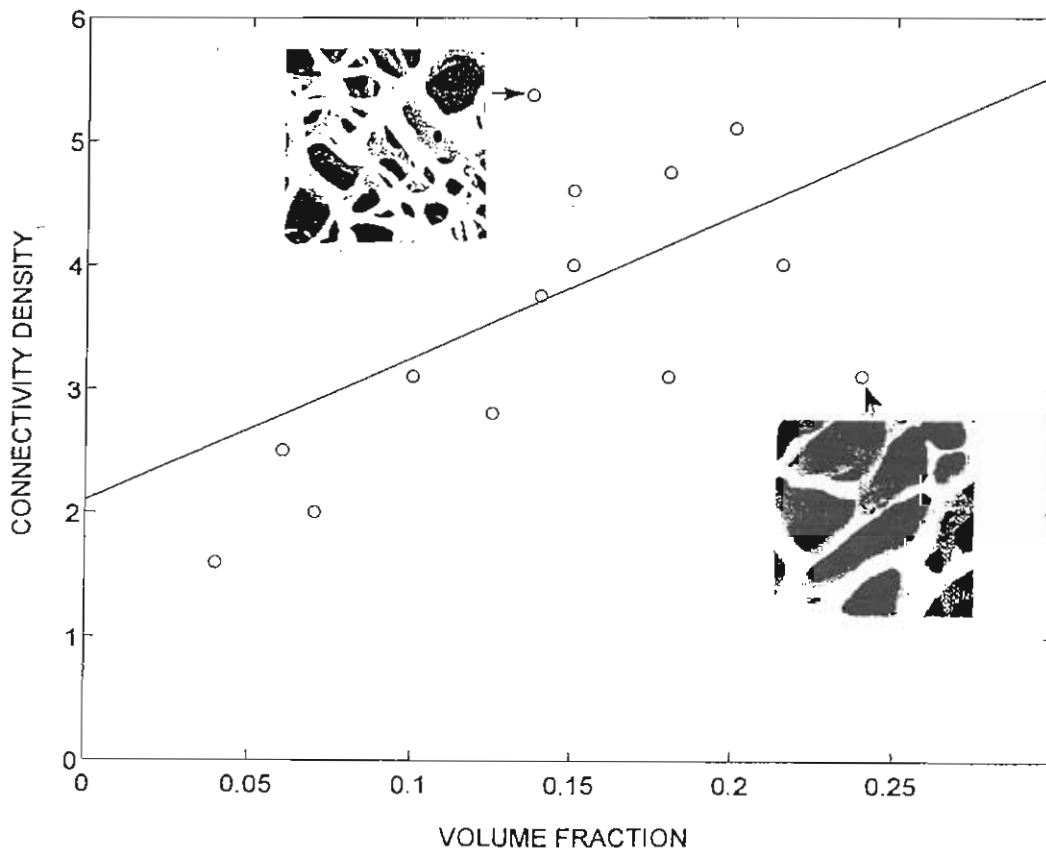


Fig. 1 Relationship between connectivity density and volume fraction for cancellous bone specimens

3 Fabric Anisotropy

Cowin[4] introduce the term fabric tensor (second order tensor) in bone mechanics as a description of the local anisotropy of a material's microstructure. Different fabric measures had been proposed for constitutive relations (Figure 2). The mean intercept length method [5], quantities the anisotropy of trabecular surface. The numbers of intersections between test lines and structure interfaces are counted as a function of orientation ϕ . Another method for describing architectural anisotropy was introduced with the volume orientation method [6]. For any point within bone the local volume orientation was defined as the orientation of the longest intercept through the point. This method does not depend on trabecular surfaces but on the typical distribution of trabecular volume around a typical point in bone. Recently , a variation of the volume orientation method known as the star volume distribution method has been introduced [6]. Generally speaking the morphological property can be characterized by an orientation distribution function

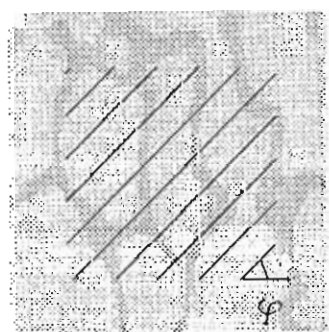
$$f(N) \geq 0 \quad (4)$$

$$N = \vec{n} \otimes \vec{n}$$

by unit vector $\vec{n}$ is specified the orientation. The function of the microstructure property can be expanded in a convergent Fourier series

$$f(N) = g \cdot 1 + G : F_1(N) + H : F_2(N) \quad (5)$$

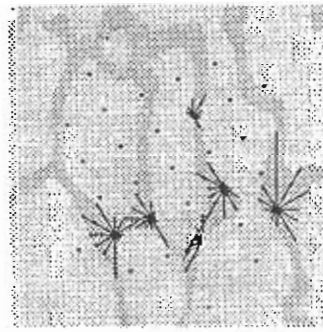
where $1, F_1(N), F_2(N) + \dots$ are even ranked tensorial basic functions and g, G and H are corresponding even ranked tensorial coefficients, called fabric tensors.



MEAN INTERCEPT
LENGHT



VOLUME ORIENTATION



STAR VOLUME
DISTRIBUTION

Fig. 2 The fabric anisotropy measures

In most applications, the first and second terms provide a sufficient description of material anisotropy (second order approximation) The hypothesis that anisotropy of the elastic constitutive law is independent of the size or physical units of the microstructure property f give the homogeneity relation for the elasticity tensor. Elasticity tensor C_{ijkl} exhibits the symmetry associated with the fabric tensors $\{g, G\}$. In the principal reference frame of G, the elasticity tensor becomes (orthotropic form);

$$\begin{aligned} C = & (\lambda + 2 \mu) \rho^{2l} m_i^{2k} (G_i \otimes G_i) \\ & + \lambda \rho^{2l} m_i^k m_j^k (G_i \otimes G_j + G_j \otimes G_i) \\ & + 2\mu \rho^{2l} m_i^k m_j^k (G_i \oplus G_j + G_j \oplus G_i) \end{aligned} \quad (6)$$

$$m_i = g + g_i$$

$$G_i = \vec{g}_i \otimes \bar{g}_i$$

where g_i and $\bar{g}_i$ are the eigenvalues and unit orthogonal eigenvectors of G, respectively. The ρ and k are structural density and accompanied exponent, while μ, λ are Lamé like constants .

The measured data (orthotropic elastic moduli, structural density and fabric components) [7] from specimen's bovine femora a fitted into equation (6) using nonlinear regression analysis. On figure 3 graphically presented relationship between Young module in principal direction, fabric component and structural density. It is easy evident that fabric and elastic principle directions are aligned.

A finite number of terms in the Furrier expansion of the orientation distribution function imply a restriction on material symmetry. For an example, a cubic symmetry requires a fourth rank tensor in the expansion of the microstructure property. In order to improve relationship between elasticity tensor and microstructures it is necessary include influence some metric parameters that describe connectivity.

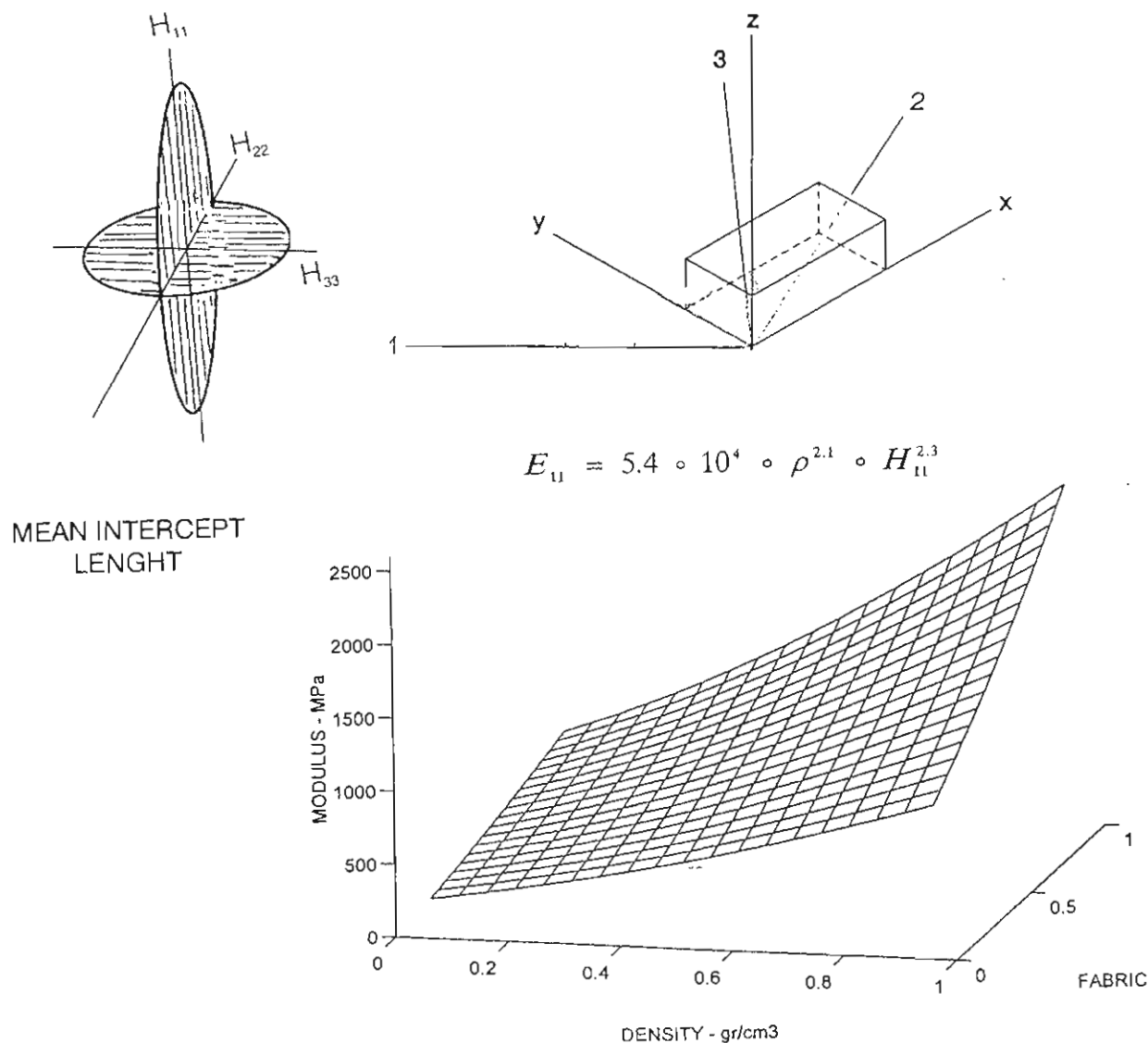


Fig. 3 The visualization of coincidence between mechanical and architectural main directions and dependence module on fabric and density

4 Mechanical Anisotropy

The three-dimensional reconstruction's (serial sectioning, X-ray tomographic, NMR scanning) in combination with recently developed methods for large-scale finite element analysis allow calculations elastic properties on continuum level [8]. The voxel data set was used as the geometry's for the microstructure finite element models. The fourth-rank effective stiffness tensor $\bar{C}_{ijkl}$ is obtained from representative volume concept

$$\bar{C}_{ijkl} = \frac{1}{V} \int C_{ijmn} \cdot M_{mnkl} dV \quad (7)$$

where V is the volume of the representative volume element, M_{ijkl} is strain localization tensor [8]. The best orthotropic representation and elastic constants are find by optimization procedure. In adaptive bone remodeling theory the state of the microstructure may be are characterized by the two sets coupled rate-of-change equations [9]

$$\begin{aligned}\frac{d\xi}{dt} &= \Xi(C_{ijkl}, \varepsilon_{ij}, \xi) \\ \frac{dC_{ijkl}}{dt} &= \Theta(C_{ijkl}, \varepsilon_{ij}, \xi)\end{aligned}\quad (8)$$

where ξ are set internal variables (density, connectivity, damage,charge density..), and ε_{ij} is strain tensor. Solution above equations give the patterns of fabric and mechanical anisotropy across structure.

5 Conclusion

Bone as complex structure need more precise material description then other simple structure. The property function must be included higher terms in Fourier series. There are few open questions ; which architectural measure are most important for cancellous bone. Topological model of the microstructure must be computationally oriented. 3-D imaging give a new horizon in the study mechanical properties of the bone on continuum level.

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COMPUTER AIDED DESIGN OF TENSILE-COMPRESSIVE LOAD CELL

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ABSTRACT

Computer aided design (CAD) has become an everyday routine in industry. Computer calculation of mechanic and dynamic problems of solids enables faster and chipper development of products and on the other hand provides the designer a good position in the competitive struggle.

Present paper deals with the finite element strength calculations of bending measuring system for the fretting test machine in the course of development process. The test machine was made within Copernicus project - Improvement of Strength properties at Aluminium Components by Manufacturing Procedures¹. The main aim was to show the design process and the role of computer simulations in assurance of load cell quality. We will expose the problems of assurance of homogenous deformation zone what is the basic condition for measurement accuracy.

¹ Co-ordinator of the project: Fraunhofer Institut, Labor für Betriebsfestigkeit (LBF), Darmstadt (D).

Partners:- Faculty of Mechanical Engineering, Laboratory for Evaluation of Structures and Components, Ljubljana (SLO),

- Slovakian Academy of Science (SK),

- Skoda Vyzkum Ltd., Plzn (CZ).

1. BACKGROUND

In fretting tests special condition (pressure, friction, strain, stress...) must be ensured. To define stress condition caused by the bending moment in the specimen the stress must be measured in some region in the specimen. The measuring system used in measuring stress caused by bending moment in the case of simple bending problem was useless because of the fixed clamps (Fig. 1.1).

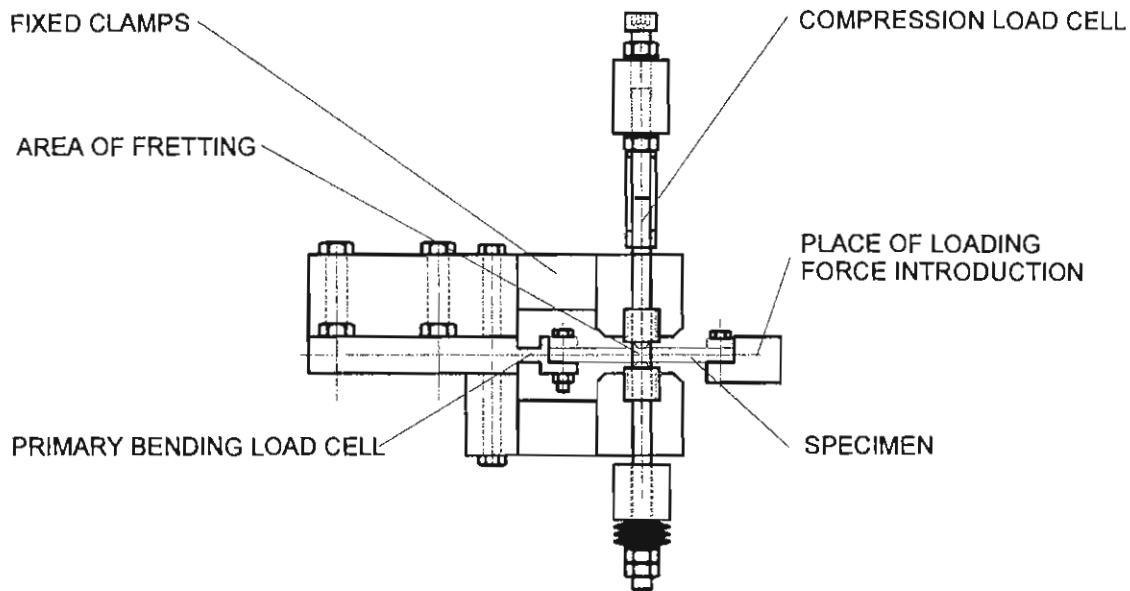


Fig. 1.1: Detail of the test machine

Using a simple strain-stress theory, an analytical analysis of the bending moment along the specimen can be made (Fig. 1.2). There is no bending moment in the region of the primary bending measuring system.

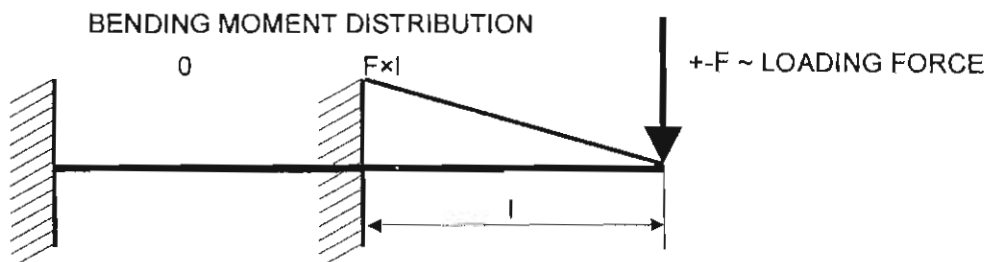


Fig. 1.2: A simple strain-stress analysis

Because of the specific condition on the test machine, the new measuring system has to be design. While designing the measuring system, some condition had to be carried out:

- the precision of the measuring system,
- the sensitivity of the measuring system,
- the linearity of the measuring system,
- the geometric limit of the test machine...

For the applied type of measuring (with strain gauges) the condition of homogenous deformation zone must be assured. This is the basic condition for a good measurement.

2. DESIGN OF LOAD CELL WITH MODELS FOR NUMERICAL COMPUTATIONS

The design of the bending measuring system was developed in two directions. The first idea was to make some design correction in the clamps in order to convert them to bending load cells. The second idea was to design a new part for measuring the load force on the end of the specimen. Both ideas were developed in the same time.

2.1. THE CLAMPS AS A BENDING LOAD CELLS

2.1.1. BASIC DESIGN

In the aim to make deformation measuring possible in the clamps, the rigidity of the construction had to be reduced. The reduction of the rigidity had to be done in such way, that greater deformation was obtained in the measuring area with no significant enlargement of deformation in vertical direction. For this reason, an elliptical hole was planed to be cut out in the clamp to achieve the desired reduction in rigidity. In this way, greater deformation could be obtained in the horizontal direction with smaller deformation in vertical direction (Fig. 2.1.).

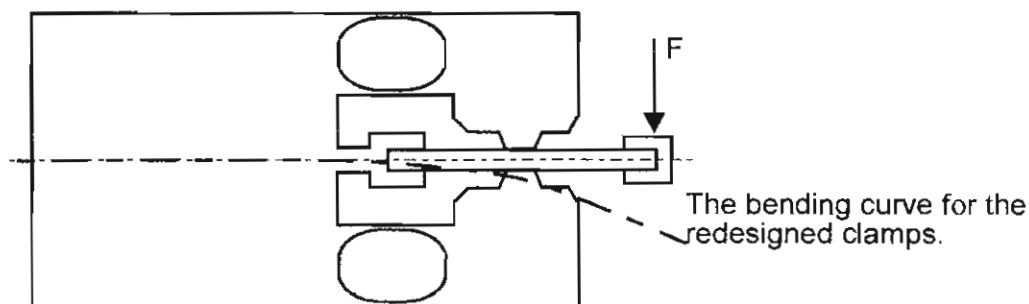


Fig. 2.1: The reduction of the rigidity

The problem in this design consisted in the amount of the deformation needed for measurement with strain gauges. In order to achieve the desired measurement region the deformation must be large enough. This means that the construction rigidity must be reduced what represent a problem in sense of assurance of contact conditions.

2.1.2. NUMERICAL SIMULATION

FEM (finite element method) models were made for different types of holes (Fig. 2.2 and Fig. 2.3 show only one type of hole). They allow us to study the strain and stress conditions in the clamps for different load cases (up and down load for different magnitude of force).

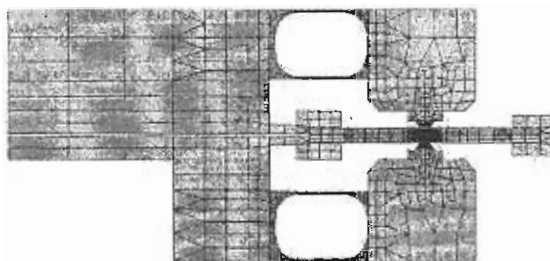


Fig. 2.2: FEM model

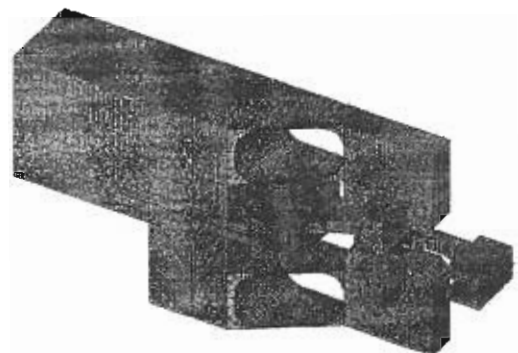


Fig. 2.3: FEM model

Fig. 2.4 shows the results of numerical computation for one direction and magnitude of force. The results present the distribution of stress and strain in the clamps with the elliptical hole. In contrast to the expected, the deformation of the clamps was quite big, but that was not the main problem. This deformation is the consequence of the elliptical hole, which produced very thin wall of the clamps. Such thin wall was needed to achieve enough deformation in the area of the measurement.

The main problem of this type of bending load cell was that there was no homogenous deformation (stress) zone in the measuring area (Fig. 2.5). As mentioned above homogenous deformation zone in the measuring area is the basic condition for good measuring accuracy.

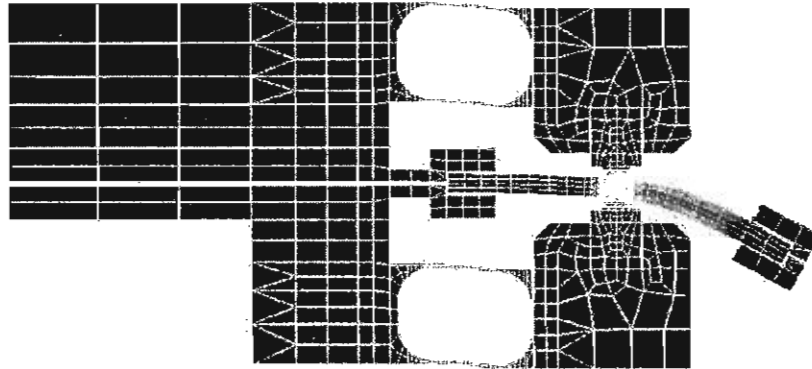


Fig. 2.4: Stress and strain results in the clamps with the elliptical hole



Fig. 2.5: Stress and strain result in the measurement area

2.1.3. CONCLUSION

In the previous paragraphs, the finite element strength calculation was shown only for one type of hole in the clamps. The results were not satisfactory and showed that this type of load cell (modified clamps) could not be applied for accurate measurements of bending moment.

In the case where enough deformation in horizontal direction for measuring with strain gauges was achieved, there was no homogenous deformation zone in the measuring area. The system lost its rigidity and became useless. In the other case, with a small hole, enough rigidity was achieved, but the deformations in the measuring area were too small to ensure accurate measurement.

The finite element strength calculations have shown that this type of bending measuring system could not be used for the measurements.

2.2. TENSILE-COMPRESSIVE LOAD CELL

As mentioned at the beginning, a tensile-compressive load cell for measuring the load force at the end of the specimen was also developed. If the bending force and the geometry of the specimen are known, the calculation of the stress in different region of the specimen is possible.

2.2.1. BASIC DESIGN

While designing the measuring system for load force, attention must be paid on different items. A tensile-compression load cell must be designed in a way to satisfy next requests:

- high measuring accuracy (homogenous deformation area, sensitivity, linearity, ...),
- small mass and moment of inertia,
- geometric limitation caused by existing test machine,
- appropriate geometry for placing strain gauges,
- maximal stress in the cell must be less than the fatigue stress...

In this aim, the tensile-compression load cell was designed (Fig. 2.6), which considered all the previous requests. In the first place, the mass and moment of inertia must be carried out. The extra mass in the place of load force represents an additional load on the specimen. This inertia force is uncontrolled and represents some noise in the test machine.

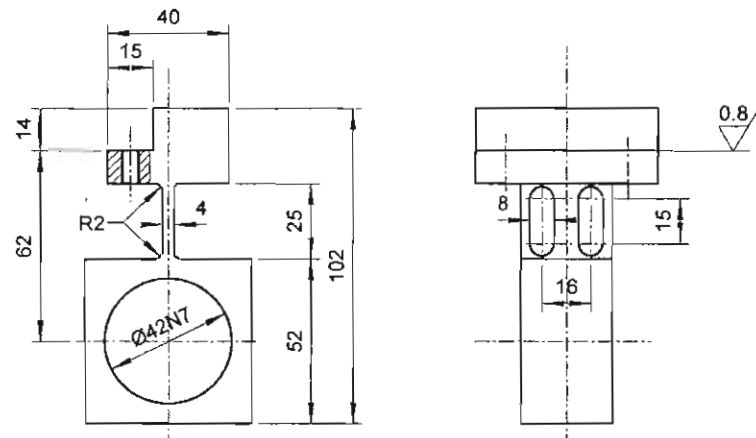


Fig. 2.6: The tensile-compression load cell

2.2.2. NUMERICAL SIMULATION

In order to get some more precise results the finite element strength calculation was made. In Fig. 2.7 the FEM mesh and boundary condition are shown.

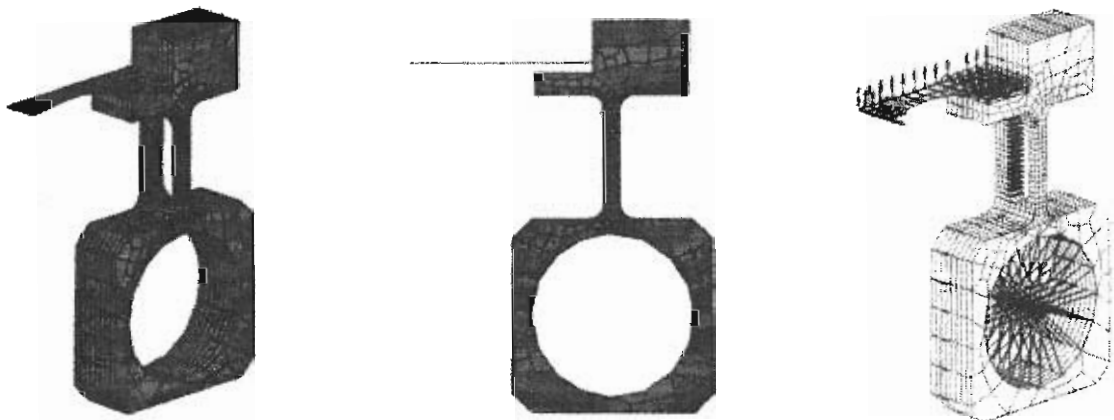


Fig. 2.7: FEM mesh and boundary conditions for the tensile-compression load cell

The tensile-compressive load cell was calculated for two kinds of loading (force up and force down). The results for the finite element strength calculation for two different kinds of loading (Fig. 2.8, Fig. 2.9 and Fig. 2.10) showed the vast area of homogenous deformation in the measuring places. This kind of stress (deformation) distribution allows us to make accurate measurements.

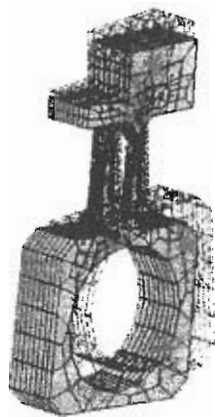


Fig. 2.8: Load force down

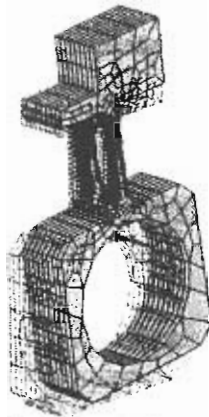


Fig. 2.9: Load force up



Fig. 2.10: Stress results

2.2.3. CONCLUSION

Based on numerical computation mentioned above the decision was to make measurement of bending moment in specimen with tensile-compressive load cell of the form pictured in Fig. 2.6. It is now working on the Copernicus test machine (Fig. 2.11).

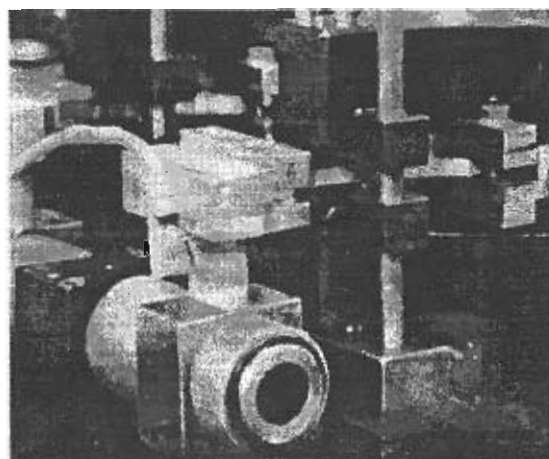


Fig. 2.11: The tensile-compression load cell on the Copernicus test machine

In the exploitation there was no major difference observed about the computation model. Until now there was no damage on the cell. The tensile-compression load cell shows a good sensitivity and linearity in the vast measuring range.

Some problems occur with inertia forces and extra stress in the specimen caused by them. However, such kinds of problems were expected and we are working on to solve them as soon as possible.

3. SUMMARY

Concerning the finite element strength analysis our main aim was to show the role of the computer added design in the work of modern engineer. These analysis enable us to compare the different versions of construction. It is rather easy to define and design some simple construction, but it is quite difficult to design a good complex construction without using of the CAD support.

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INSTRUMENTED IMPACT TESTING SYSTEM DEVELOPED IN SHIPYARD SPLIT

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Abstract

After the two years of teamwork cooperation an old Charpy tester has been successfully transformed to instrumented impact testing machine. Basic approach to hardware and software subsystems, is given briefly. Output performances in refer to the requirements of ESIS [2] standard are considered and some extra possibilities of software solutions are illustrated by the output graphs. Some advantages of this method of material testing and the plans for future experimental research are mentioned.

1. Introduction

The idea of introducing instrumented impact testing in Brodosplit Shipyard Laboratory, is alive since MESUCORA 85, specialized exhibition of laboratory and measuring equipment which took place in Paris. Namely, impact toughness of metals and their welds is of the great importance in building ships and offshore structures. Impact toughness, determined by figure read on the dialing scale of the Charpy tester, in some occasions has been additionally supported by visual examination of surface fractured. These analyses were made by laboratory staff and classification societies' surveyors as well. But, additional interesting informations, which the diagram of the Charpy impact test could give, was the ever present challenge. In the meantime, Shipyard invested in strain gauge measurement system. Knowledge and experience in this field, together with the engagement of the team expert in the field of electronic research, provided conditions necessary for starting the project [1]. Results of this project were implemented to the Charpy tester in the sense of hardware and software construction, so an advanced Instrumented Impact Testing System has been developed. Upgrading of this system continued in the year 1997. by wider research team in which two experts in the field of mechanical engineering from FESB-Split and Croatian Register of Shipping were included. Now, special efforts are given to the ESIS standard requirements for the Charpy-V impact test on metallic materials. Because of importance of this standard and its implementation to this Impact Testing System, short description of ESIS standard requirements is given in the next section.

1.1 ESIS standard (basic) requirements

ESIS standard establishes the requirements for performing and recommendations for evaluating instrumented Charpy-V impact test on metallic materials. The test can be carried out with a pendulum impact testing machine according to EN 10 045-2 which is additionally instrumented to determine force-time or force-displacement curves.

The designations applicable to this standard are as indicated in Table 1.

Table 1. Symbols and designations

<i>Symbol</i>	<i>Designation</i>	<i>Unit</i>
fg	output frequency limit	Hz
Fa	force at the end of unstable crack propagation (arrest)	N
Fgy	yield force at onset of plastic collapse of ligament	N
Fm	maximum force in course of force-displacement curve	N
Fiu	force at initiation of unstable crack propagation	N
KV	absorbed energy as defined in EN 10 045-1	J
m	total mass of moving tup	kg
Sa	displacement at end of unstable crack propagation (arrest)	m
Sgy	displacement at onset of plastic collapse of ligament	m
Sm	displacement at maximum force	m
Siu	displacement at initiation of unstable crack propagation	m
St	displacement at end of force-displacement curve	m
t _r	signal rise time	s
v ₀	tup impact velocity	ms ⁻¹
Wa	partial impact energy from F = 0 to F = Fa	J
Wm	partial impact energy from F = 0 to F = Fm	J
Wt	total impact energy calculated from area under complete force-displacement curve to F = 0.02Fm	J
Wiu	partial impact energy from F = 0 to F = Fiu	J

Force measurement:

- Force measurement is usually achieved by using two active and two passive resistance strain gauges,
- The force measuring system shall have an upper frequency bound of at least 100 kHz (-3db) which corresponds to a rise time t_r of 3.5μs. Each component is therefore required to exhibit a frequency bound of at least 100 kHz,
- The instrumentation of the tup should be arranged to give the required nominal force range (10 to 40 kN for all steel types).

Displacement measurement

Displacement is normally determined from force-time measurement. The force-time relationship measured on the tup is proportional to the acceleration characteristic. Given an assumed rigid tup of mass **m**, the initial impact velocity **v₀**, and the time **t** following

the beginning of the deformation at t_0 , the test piece bending displacement is calculated by double numerical integration :

$$v(t) = v_0 - \frac{1}{m} \cdot \int_{t_0}^t F(t) dt$$

$$s(t) = \int_{t_0}^t v(t) dt$$

Test procedure

The test is performed in the same way as the notch impact test according to EN 10 045-1. In addition, the force-displacement curve is determined and evaluated in respect to the characteristic phases of the deformation and fracture stages.

2. Basic system description

Our impact testing machine is a pendulum type - Charpy tester (Amsler) which covers the range from 0 to 300 J. It has been instrumented on the basis of the research and development project [1]. This machine is calibrated and it is used continuously in Mechanical laboratory of Sector Testing in Brodosplit Shipyard, for ordinary Charpy impact tests.

Hardware and software subsystems of this Instrumented Impact Charpy Tester are described in the next sections.

2.1 Hardware subsystem

Hardware subsystem of Instrumented Impact Charpy Tester is realised by well known cascade chain: Instrumented Tup - Signal Conditioner Amplifier - AD Converter - Personal Computer.

Instrumented Tup

ISO/DIN - TUP LR type tup is used with two active and two passive resistance strain gauges type HBM DY11 in Wheatstone bridge configuration. This unit generates voltage output signal (few hundreds of μV to few tens of mV) on dependence of a force.

Signal Conditioner Amplifier

This is a very important unit in cascade, constructed to meet the following requirements:

- Generation of very stabile DC power supply for the Wheatstone bridge circuit in instrumented tup,
- Effective temperature drift and noise compensation to value less then 0.5% of the full scale output ($\pm 10V$),
- Upper frequency bound to 150 kHz,
- Voltage gain to 500 (flat to 100kHz),

- Generation of output signal according to input requirement of AD Convertor (DAS50).

This unit was made by ourselves.

AD Convertor

Keithley Metrabyte Data Acquisition Board DAS50 is used. It is an extremely high speed 12 bit analog-to-digital convertor interface board designed to provide 1 Mhz conversion rate, and up to 1 Megaword of on-board memory, making it ideally suited for many high speed data acquisition applications, with its own internal track and hold amplifier.

Personal Computer

DTK PC Pentium 166 Mhz / 32 MB RAM, HDD WD 2.1 GB with graphics accelerator on VGA card is used.

2.2 Software subsystem

Software is based on "ASYST " (Scientific Language For DOS), "DAS50 Utility Software " and "External Metrabyte DAS50 Driver" (Rev. 1.00) and they were used to create the application program in accordance to numerical algorithm of our own.

The acquisition results are transferred from the DAS50 on-board memory to Hard Disc of the PC to the specially defined and organized file templates. Opening these file templates and managing the data on the precisely defined algorithm, it is possible to get a lot of useful informations about the signal acquisition.

Program is menu oriented and by simple pushing the corresponding function keys on the PC keyboard, user is able to get the different types of analysis : Transient response, Impact response, Standard response and Signal comparing.

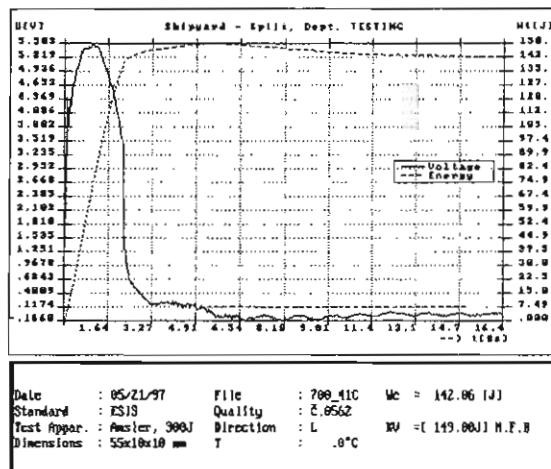
Transient response (Fig. 1) is able to determine four different graphs in time domain:

- V - t, E - t
- F - t, E - t
- F - t, V - t
- v - t, E - t

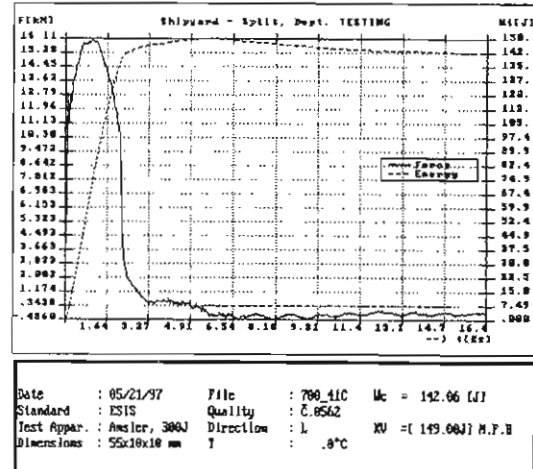
Effective graph zooming (Fig. 2) can be used for the analysis of the respective parts in output signals in arbitrarily selected bounds of the time interval. It is also possible to plot constant time line, which is parallel to time axis, to point out the function value $F = 0.02 \text{ Fm}$.

Impact response This type of analysis has the same possibilities and Output Graphs as the Time Response with the only difference in automatic selecting of time boundaries to the impact integration interval.

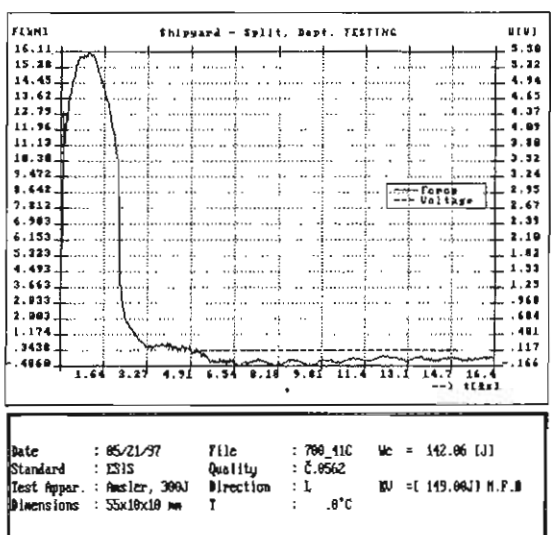
Standard response (Fig.3) All of ESIS standard requirements are integrated and presented by Standard Response graph. All characteristic function parameters, according to Table 1., are calculated and presented in output graph of Force and Energy functions in dependence of the time and displacement variables.



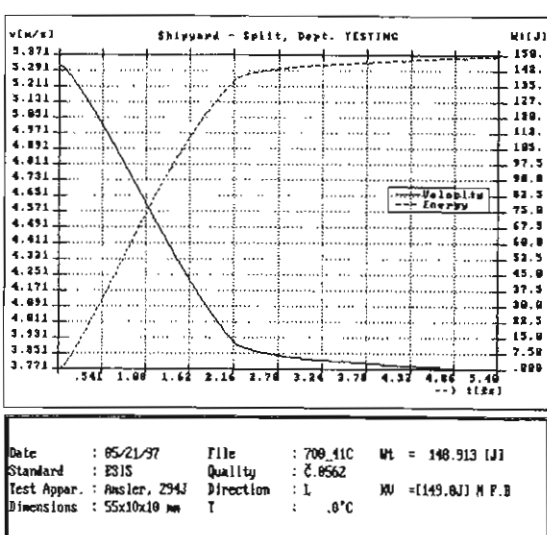
a) Graph : $V-t, E-t$



b) Graph : $F-t, E-t$



c) Graph : $F-t, V-t$



d) Graph : $v-t, E-t$

Fig. 1 Different Transient responses

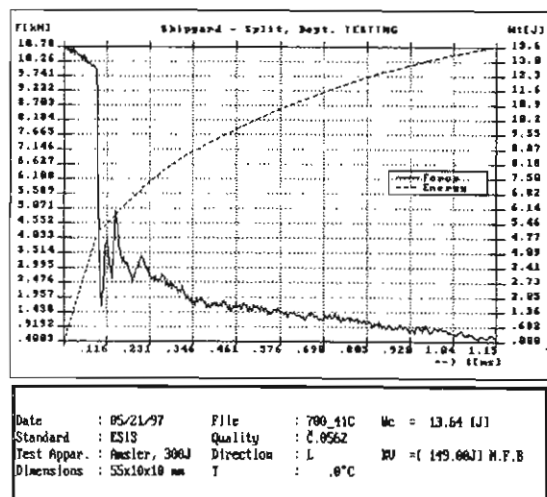
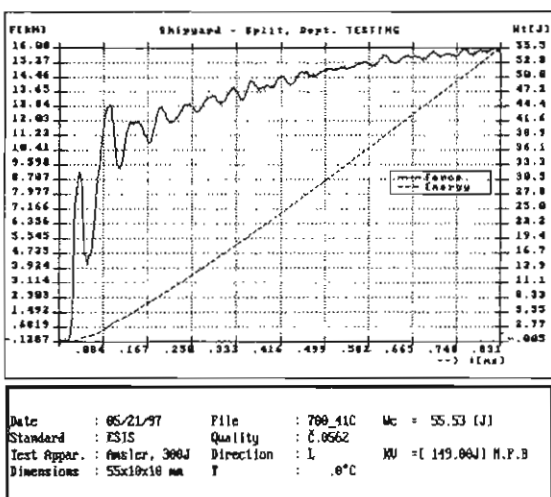


Fig. 2. Graph zooming

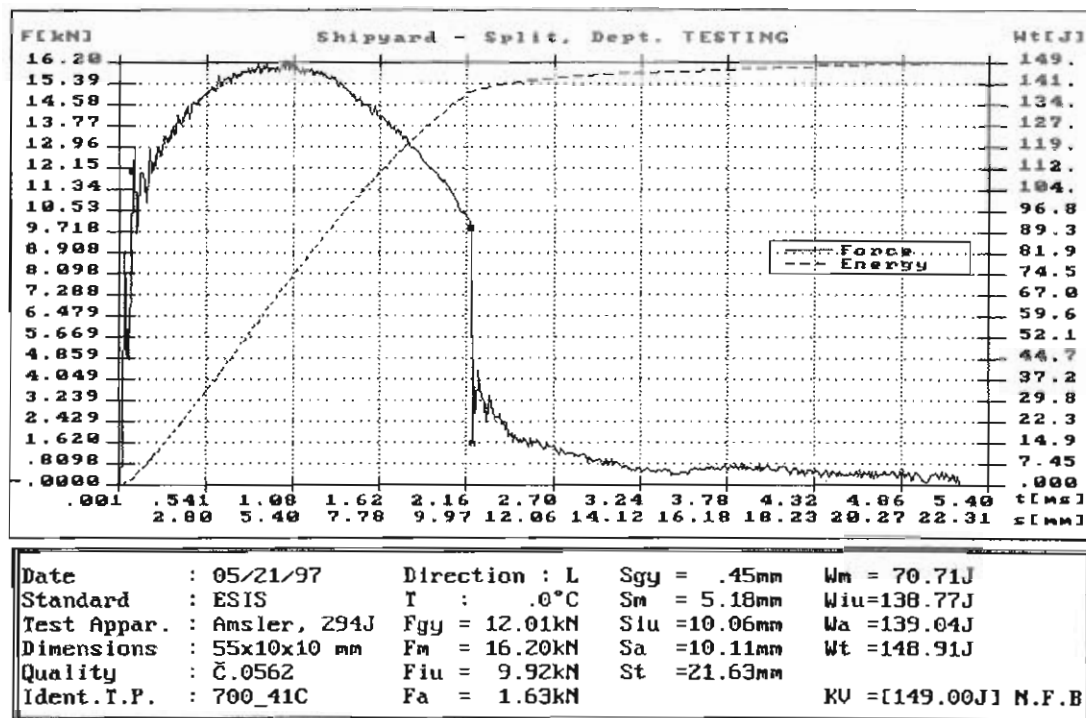


Fig. 3. Standard response (ref. to Table 1.)

Signal comparing can be used for the analysis of different sample plots (up to 12 signals) in the same XY axis boundaries. The user is also able to select the automatic scale selection (boundaries are determined to the maximal signal requirements, Fig. 4), or the user may define boundaries which are the best suited to his own opinion. It is also possible to plot different signals on the same graph (up to 6 signals) in the modes of automatic or manual scale selection (Fig.5., 6.). Any signal may be shifted (left or right) on this graph and put in arbitrary position.

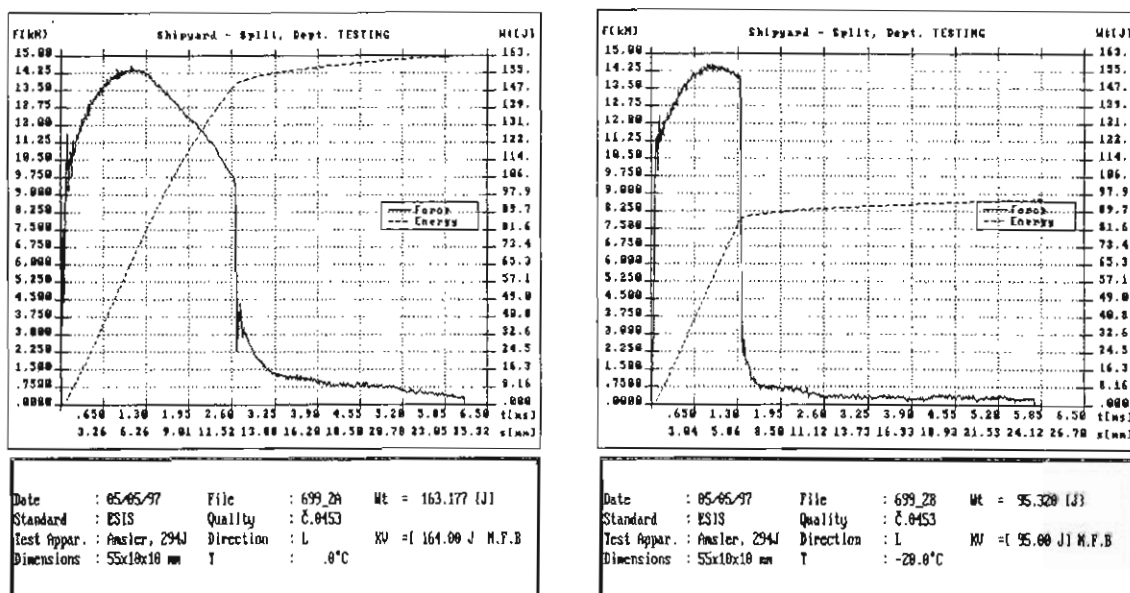


Fig. 4. Impact response comparing in the same XY axis boundaries

3. Some advantages of this method of material testing

The instrumented Charpy impact test extends the conventional test by enabling the force-time and force-displacement behaviour of a Charpy V-notch specimen to be measured during impact. The area under the force-displacement record corresponds to the energy measured in the conventional test. When impact energies of two materials are similar, the force-displacement records provide additional information which can differentiate between a material that fractures at a high force and small displacement and one that fractures at a low force and high displacement (Fig. 5.). The force-displacement record is usually considered partitioned (Table 1., Fig. 3.) in such a way as to provide additional information about fracture behaviour of the specimen (Fig. 6.).

Instrumented impact testing is still a developing science [3] and gives us a lot of room for scientific work. Due to the fact that Shipyard Mechanical Laboratory carries out testing of a reasonable number of impact test samples as the support of ordinary production process in Shipyard and Diesel Engine Factory, there is a good opportunity for providing an adequate number of files to be analysed without additional cost. Further activities in this field would deal with a large field of welded specimens, as well as precracked specimens for determination of fracture mechanic parameters.

For the time being, the Rules for the classification and technical supervision of sea going ships of the Croatian Register of Shipping, Part 25 - Metallic materials comprise the requirements based upon the conventional procedure for determination of impact toughness. It is expected that the results obtained by impact testing system described, will improve the present state-of-the-art and will be introduced into Rules.

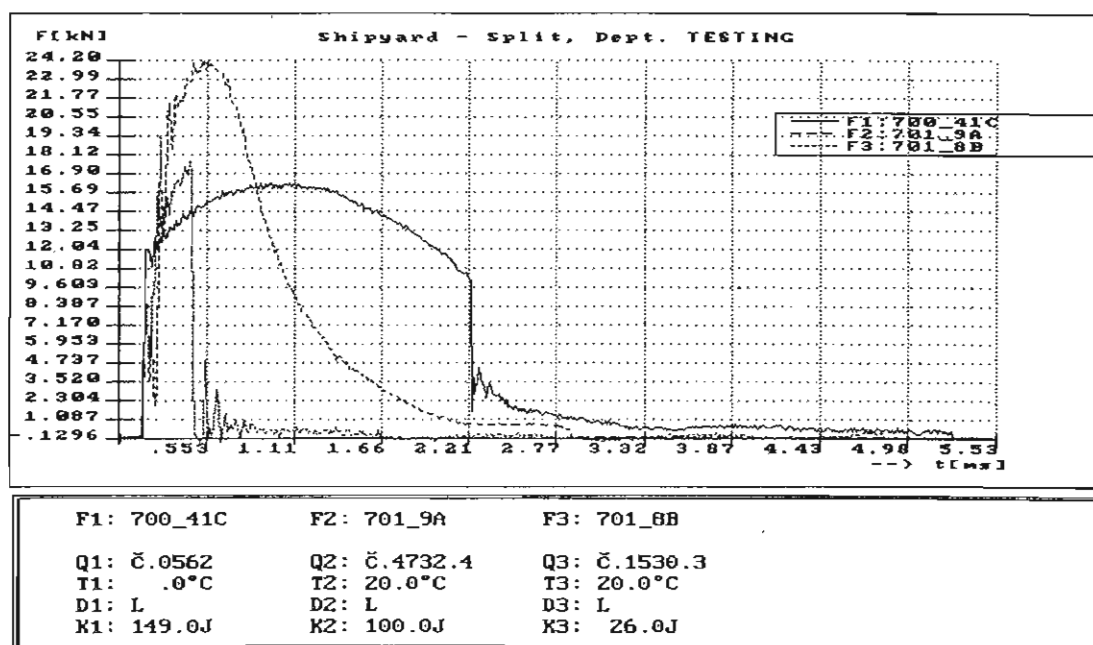


Fig. 5. $F(t)$ graphs for different steels: Č.0562 (St 52 - 3 U), Č.4732.4 (42 CrMo4 V) and Č.1530.3 (C 45 N). The form of each particular diagram is typical for materials stated, so it is possible to identify the material quality on the basis of the diagram shape.

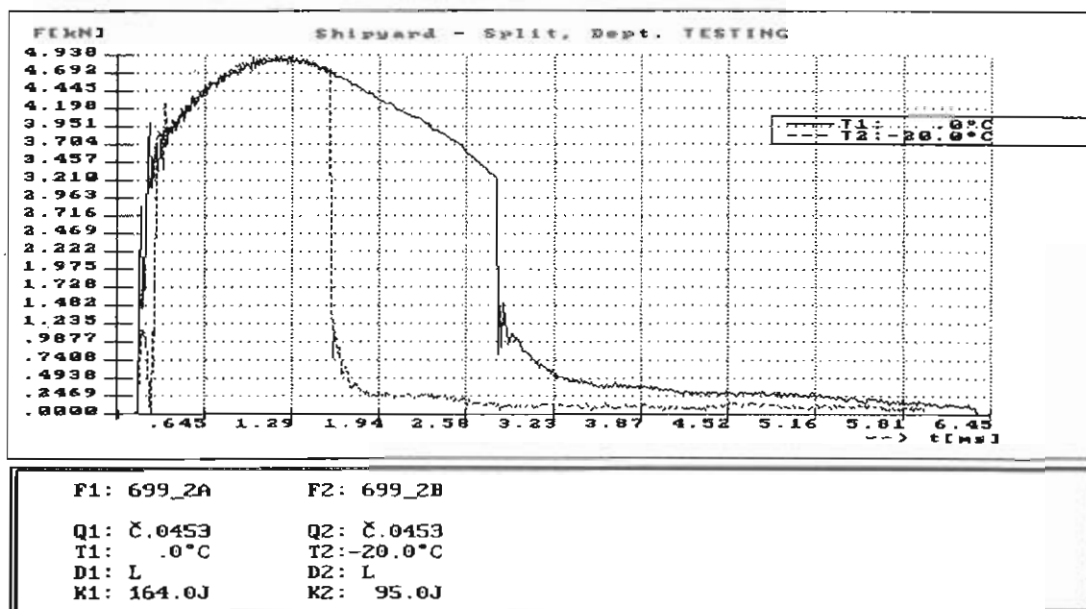


Fig. 6. Two $F(t)$ signals of completely the same material Č.0453 (St 44 - 3 N). The only difference is the test temperature. At the test temperature of -20°C , part of stable crack propagation is shortened significantly if compared to diagram at 0°C .

4. Conclusions

- A good agreement with common used impact tests and literature respecting, has been shown by the very first results of this Instrumented Impact Charpy Tester.
- Instrumented Impact Testing System, developed and constructed in Shipyard Split, is able to provide useful informations for solving some specific production and maintenance problems. Also, it is the tool of meaningful importance in R&D activities in the field of mechanical engineering.
- In the near future, the upgrading of software solution described will be realised, using the Windows platform, Microsoft Visual Basic and VTX Macro Modul from Keithley - Metrabyte.

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Tribological Performance of Epoxy Resin Filled by Abrasive

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ABSTRACT

The principal function of friction material is to convert kinetic energy to heat and reduce the relative movement between the friction material and the part to which it is engaged. In order to achieve these objectives, friction coefficient should be as high as possible with a minimum wear on the contact surfaces. Articles such as bags and containers fabricated from cellulose material and used for the packaging of foodstuffs and chemicals possess the disadvantage of slipping and sliding against each other. In such applications, the tendency to slip and/or slide is undesirable and harmful. In order to overcome this difficulty, the moving surfaces of the articles and/or the conveyors should be treated with a slip resistant coating.

This work discusses the tribological performance of slip resistant material made of epoxy resin filled by abrasive grains like silicon oxide, aluminium oxide and silicon carbide of different particle size and concentration. The friction and wear of the tested materials sliding against steel counterface were investigated.

The test results show that addition of silicon oxide of (0 - 90) μm , (300 - 450) μm particle size to epoxy decreased friction coefficient while silicon oxide of (90 - 150) μm , (150 - 300) μm particle size increased friction coefficient. Generally, wear resistance of epoxy filled by silicon oxide was improved. Epoxy transfer to the counterface decreased as the silicon oxide content increased. The best wear resistance was observed for epoxy filled by silicon carbide and/or aluminium oxide at concentration of 40 wt. %.

Static coefficient of friction increased with increasing silicon oxide content to 50 wt. % for particle size of (90 - 150) μm and (150 - 300) μm . As for epoxy filled by silicon oxide of (300 - 450) μm , aluminium oxide of (0 - 10) μm and silicon carbide of (0 - 90) μm , considerable reduction in static coefficient of friction was observed.

INTRODUCTION

Epoxy resins are extensively used in moulded glass fibre reinforced composites in automotive industry, [1,2]. Carbon fibre was used as solid lubricant in epoxy laminates, [3,4]. Friction and wear of epoxy resins composites reinforced by different types of fibre materials were investigated, [5-7]. It was observed for graphite fibre, kevlar fibre and glass fibre composites that the lowest wear and friction were obtained for fibre oriented normal to the sliding surface. Epoxy resins are used in a number of tribological applications such as automotive and chemical industries. They are applied as bearing materials in a cast form filled by

graphite or molybdenum disulphide and as a thin film lining of filled epoxy applied to the bearing surface, [8].

Epoxy resin composites were used to extend the service of the railway brake shoe, [9]. The brake shoe body was adhered to the metallic back and formed from a mouldable composition of epoxy resin, graphite in a concentration of about 20 wt. % and an abrasive substance in a quantity capable of diminishing the lubricity exhibited by the graphite present in the composition.

In some applications, the tendency to slip and/or slide may be undesirable and in many cases harmful. Attempts have been made to overcome this difficulty, [10]. Epoxy resins were used as an adhesive to treat the surfaces of containers and conveyors by coatings of silica and alumina powder of fine particle size within the range of 4 to 120 millimicrons as abrasive material to increase friction coefficient, [11,12]. The addition of pulverized rubber to an adhesive can present a resilient coating material which has inherent skid resistance and high friction surface when applied to a substrate, [13]. The rubber particles remain tightly bonded in the dried coating and are not scuffed out with wear.

In the present work, the friction and wear of epoxy resins filled by silicon oxide (SiO_2), aluminium oxide (Al_2O_3) and silicon carbide (SiC) powders of different particle size were investigated at different values of applied load.

KEY WORDS

Friction, wear, epoxy resins, silicon oxide, aluminium oxide, silicon carbide, static coefficient of friction.

EXPERIMENTAL

Epoxy resins were Araldite resin type Kemapoxy 150 A and hardener type Kemapoxy 150 B. The filling materials used were silicon oxide (SiO_2) of (0 - 90), (90 - 150), (150 - 300) and (300 - 450) μm particle size, aluminium oxide (Al_2O_3) of (0 - 10) μm particle size and silicon carbide (SiC) of (0 - 90) μm particle size. The filling materials were added to epoxy resins in concentrations of 10, 20, 30, 40 and 50 % by weight. The test specimens were prepared by mechanically mixing the epoxy and the powders in a container and pouring in a mold. The specimens were in form of rod of 20 mm long and of square cross-sectional area of $4 \times 4 \text{ mm}^2$. The friction surface of the test specimens was ground by an emery paper of silicon carbide of 500 grade before the test. The counterface was carbon steel (St. 52.11) disc of 150 VHN and of surface roughness of $0.4 \mu\text{m}$ (R_a). The counterface, in form of hollow disc of 50 and 100 mm inner and outer diameter respectively and thickness of 2 mm, was fastened to the rotating disc of the pin-on-disc wear tester, Fig. 1. The test specimen was held in the specimen holder which was attached to the loading lever through two thin spring steel sheets to facilitate the measurement of friction force. Loads were applied by weights. Wear of the test specimen and counterface was measured by the weight difference before and after the test using an electric balance of $\pm 0.1 \text{ mg}$ accuracy. Wear tests were carried out at load of 6.0 N for 120 seconds. Dynamic and static coefficient of friction were determined at load values of 4.0, 6.0, 8.0, 10.0 and 12.0 N. The sliding velocity was 2.5 m/s. Static coefficient of friction was determined considering the maximum value of friction torque measured just after sliding of test specimen on the counterface at the beginning of the motion.

RESULTS AND DISCUSSION

The friction and wear of test specimens made of epoxy resin and filled by silicon oxide of different particle size are shown in Figs. 2-9. Epoxy resin filled by silicon oxide of (0 - 90) μm particle size showed an increase in friction as the content of silicon oxide increased to 10 wt. % then slightly decreased with increasing silicon oxide content, Fig. 2. Generally friction coefficient decreased with increasing applied load. Wear of test specimen and steel counterface is illustrated in Fig. 3, where wear of test specimens slightly decreased with increasing silicon oxide content. Counterface showed an increase in its weight after test confirming the transfer of epoxy resin from test specimen to the surface of counterface. Epoxy transfer decreased with increasing silicon oxide content.

Friction coefficient of epoxy filled by silicon oxide of (90 - 150) μm decreased down to minimum value then increased with increasing silicon oxide content, Fig. 4. Slight increase was observed for wear of test specimen with increasing silicon oxide content, Fig. 5. Epoxy transfer to the surface of counterface decreased with increasing silicon oxide content. Minimum epoxy transfer to counterface was observed at 50 wt. % silicon oxide content.

Increasing the particle size of silicon oxide to be (150 - 300) μm caused considerable increase in friction coefficient, Fig. 6. Wear of test specimen decreased with increasing silicon oxide content, Fig. 7. Counterface surface was adhered by a layer of epoxy transferred from the test specimen. This transfer decreased with increasing silicon oxide content.

As for epoxy filled by silicon oxide of (300 - 450) μm particle size, friction coefficient increased slightly up to maximum value then decreased with increasing silicon oxide content, Fig. 8, at applied load of 4.0 N. For higher load values, no significant change was observed. Wear of test specimen decreased to minimum value at 20 wt. % of silicon oxide content then increased with increasing silicon oxide content, Fig. 9. It seems that, as the particle size of silicon oxide increases the cohesion between epoxy and the particles decreases. Epoxy transfer to counterface surface decreased down to minimum value at 30 wt. % of silicon oxide content then increased with increasing silicon oxide, Fig. 10. The increase of epoxy transfer at 40, 50 wt. % of silicon oxide content could be as a result of the weak cohesion of test specimen.

The effect of addition of aluminium oxide to epoxy on friction and wear is shown in Fig. 10. Slight increase of friction coefficient was observed as aluminium oxide content increased. Wear of test specimen and epoxy transfer to the counterface decreased as aluminium oxide content increased, Fig. 11.

Silicon carbide as filling material in epoxy caused slight increase in friction coefficient, Fig. 12. Wear of test specimen significantly decreased to minimum at 40 wt. % silicon carbide content, Fig. 13, while epoxy transfer to counterface decreased as silicon carbide content increased.

Static coefficient of friction was measured for the materials tested in this work to investigate their antiskid property. For epoxy filled by silicon oxide of (0 - 90) μm , it can be seen that static coefficient of friction decreased down to minimum value then increased with increasing silicon oxide content, Fig. 14. At the highest value of load (12.0 N) static coefficient of friction remained constant.

Test specimens filled by silicon oxide of (90 - 150) μm and (150 - 300) μm particle size showed the same trend and the same values of static coefficient of friction, Figs. 15, 16. As for epoxy filled by silicon oxide of (300 - 450) μm , static coefficient of friction slightly decreased with increasing silicon oxide content, Fig. 17, where the lowest values were observed at 50 wt. % of silicon oxide content. The effect of addition of aluminium oxide (Al_2O_3) and silicon carbide (SiC) as filling material is shown in Figs. 18 & 19 respectively, where static coefficient of friction decreased with increasing the content of aluminium oxide and/or silicon carbide.

CONCLUSIONS

On the bases of the experimental results the followings can be concluded:

1. Addition of silicon oxide to epoxy resin increased wear resistance and decreased epoxy transfer to the steel counterface.
2. The improvement in wear resistance was accompanied by a friction decrease for silicon oxide of (0 - 90) μm particle size and friction increase for silicon oxide of (90 - 150) μm , (150 - 300) μm particle size.
3. No improvement was observed for wear resistance of epoxy filled by silicon oxide of (300 - 450) μm particle size due to the weak cohesion between epoxy and the filling particles.
4. Addition of aluminium oxide and silicon carbide at concentration of 40 wt. % displayed the best wear resistance.
5. Static coefficient of friction increased with increasing silicon oxide content to 50 wt. % for particle size of (90 - 150) μm and (150 - 300) μm .
6. Considerable reduction in static coefficient of friction was observed for epoxy filled by silicon oxide of (300 - 450) μm , aluminium oxide and silicon carbide.

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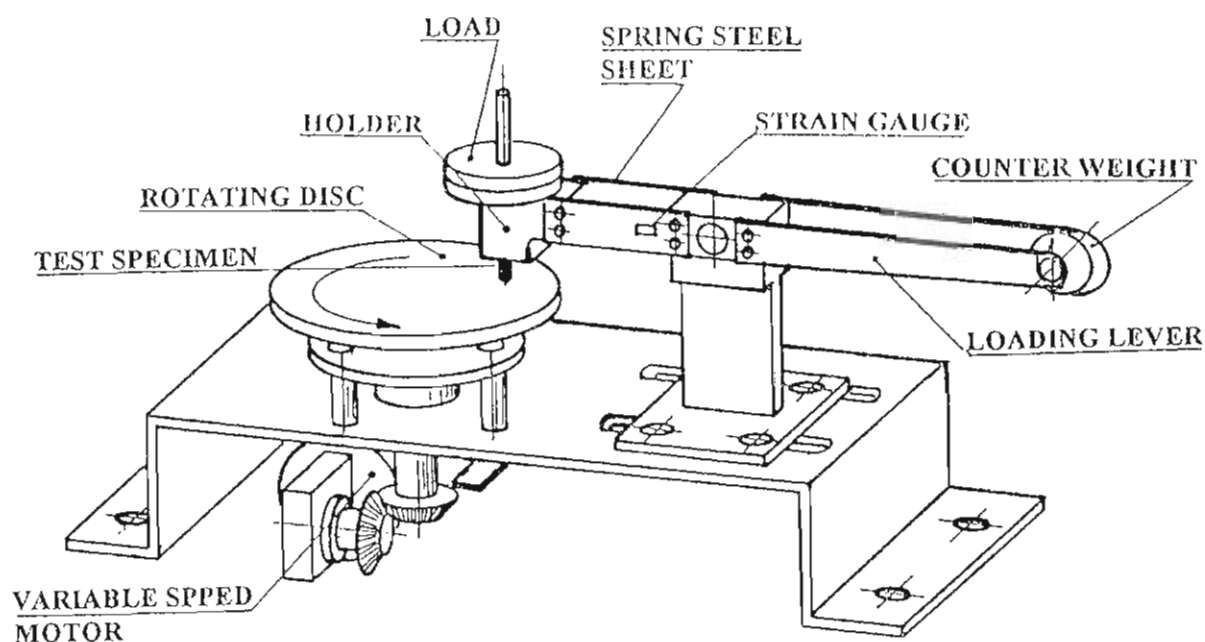


Fig. 1 Layout of the pin on disc wear tester.

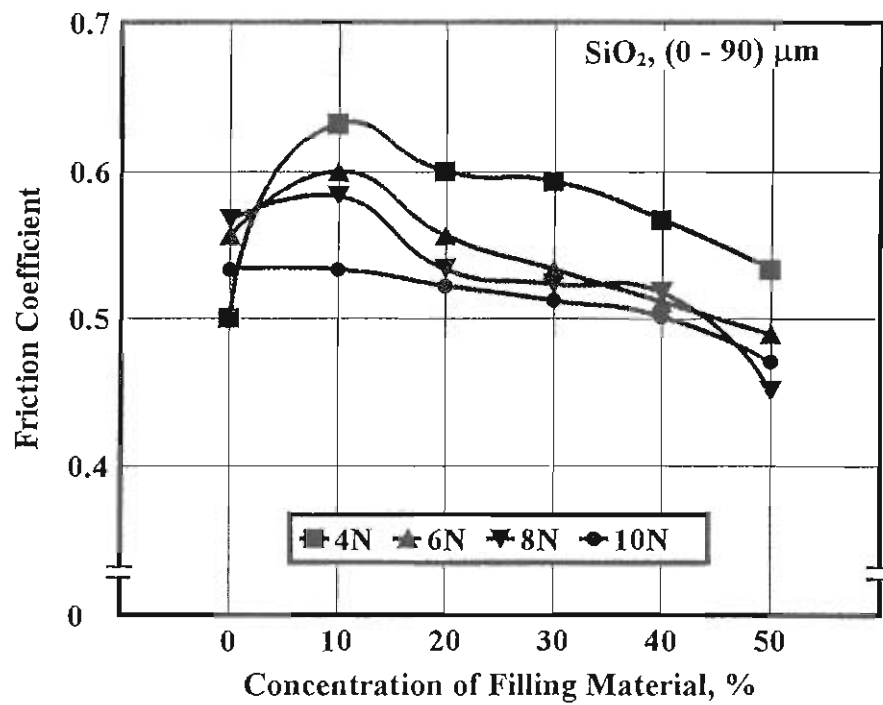


Fig. 2 Friction coefficient of epoxy resin filled by silicon oxide of (0 - 90) μm particle size.

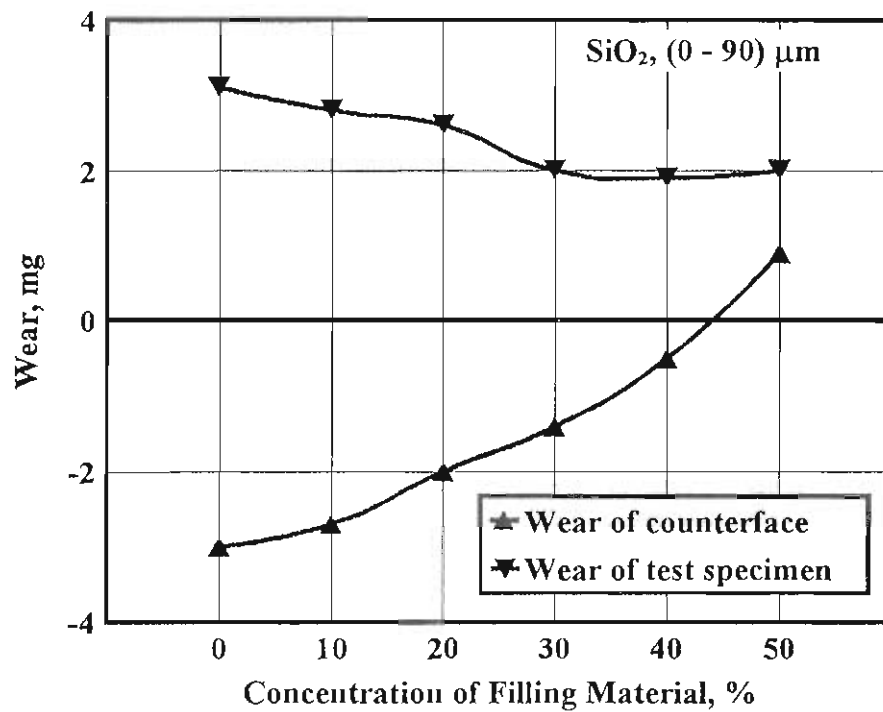


Fig. 3 Wear of epoxy resin test specimens filled by silicon oxide of (0 - 90) μm particle size at applied load of 6.0 N.

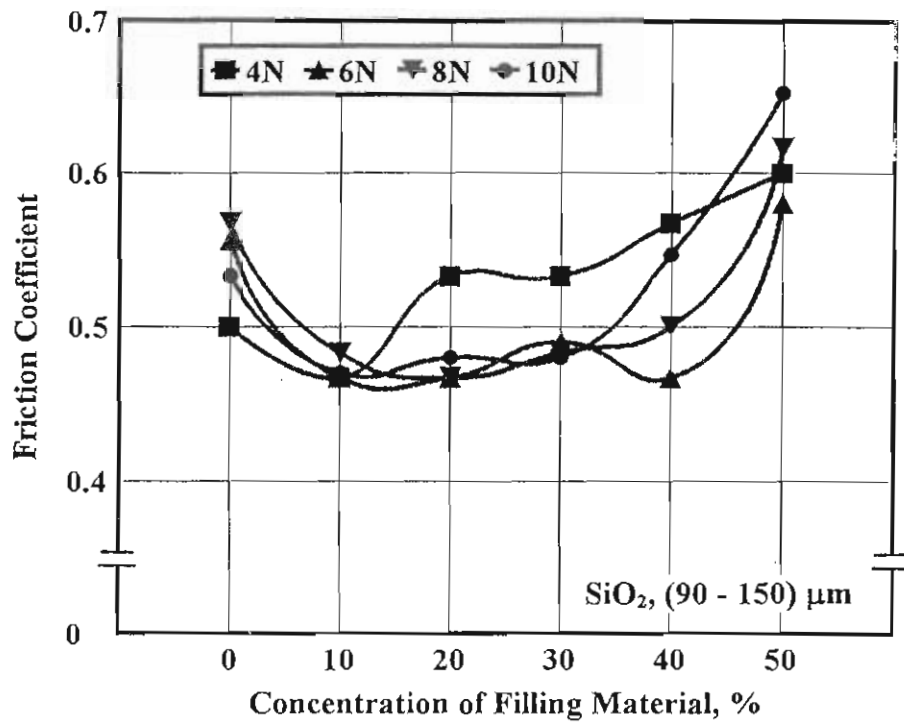


Fig. 4 Friction coefficient of epoxy resin filled by silicon oxide of (90 - 150) μm particle size.

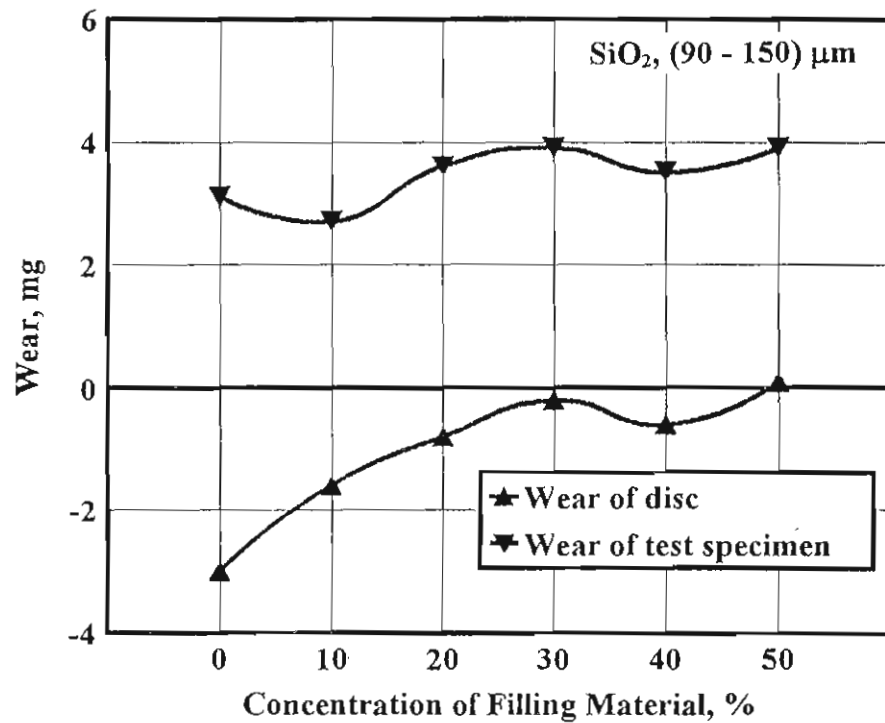


Fig. 5 Wear of epoxy resin test specimens filled by silicon oxide of (90 - 150) μm particle size at applied load of 6.0 N.

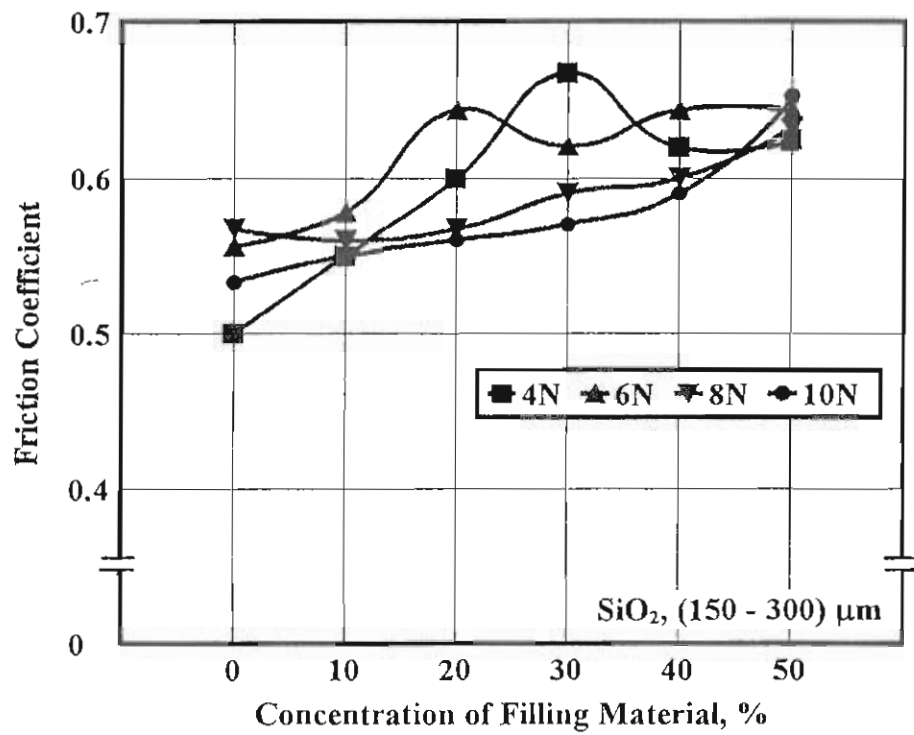


Fig. 6 Friction coefficient of epoxy resin filled by silicon oxide of (150 - 300) μm particle size.

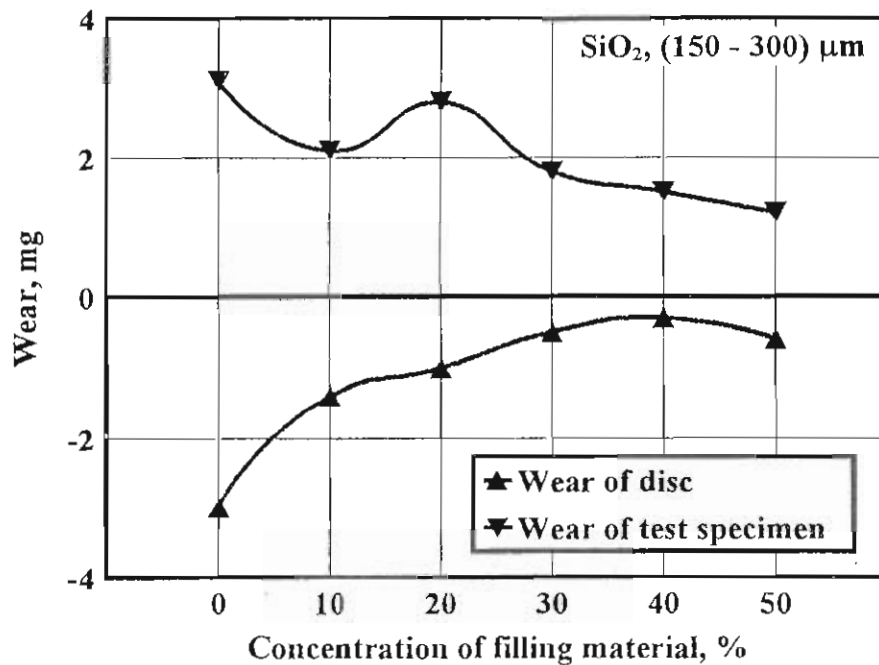


Fig. 7 Wear of epoxy resin test specimens filled by silicon oxide of (150 - 300) μm particle size at applied load of 6.0 N.

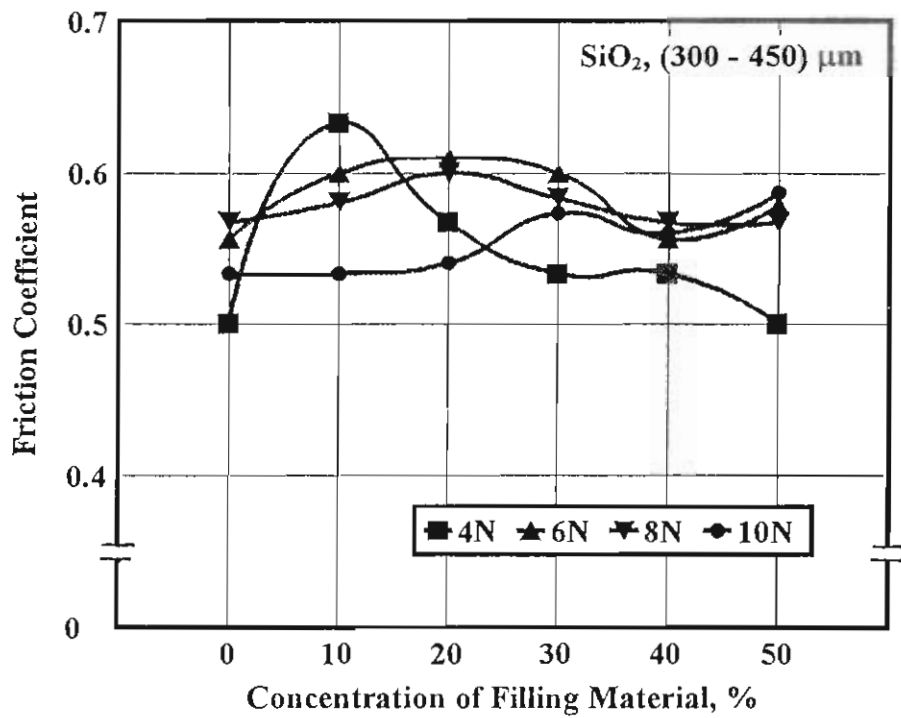


Fig. 8 Friction coefficient of epoxy resin filled by silicon oxide of (300 - 450) μm particle size.

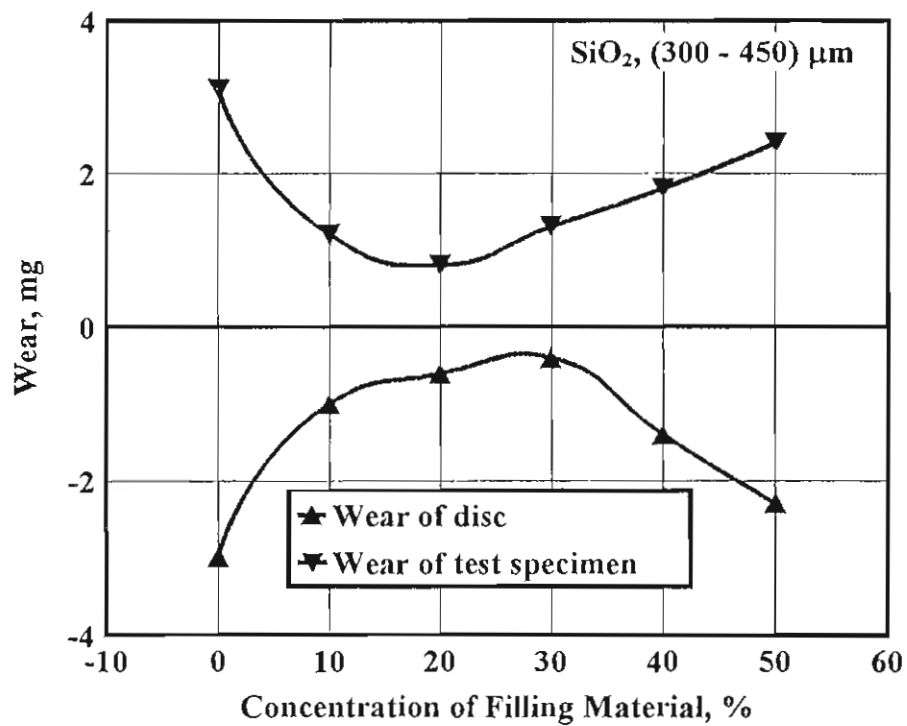


Fig. 9 Wear of epoxy resin test specimens filled by silicon oxide of (300 - 450) μm particle size at applied load of 6.0 N.

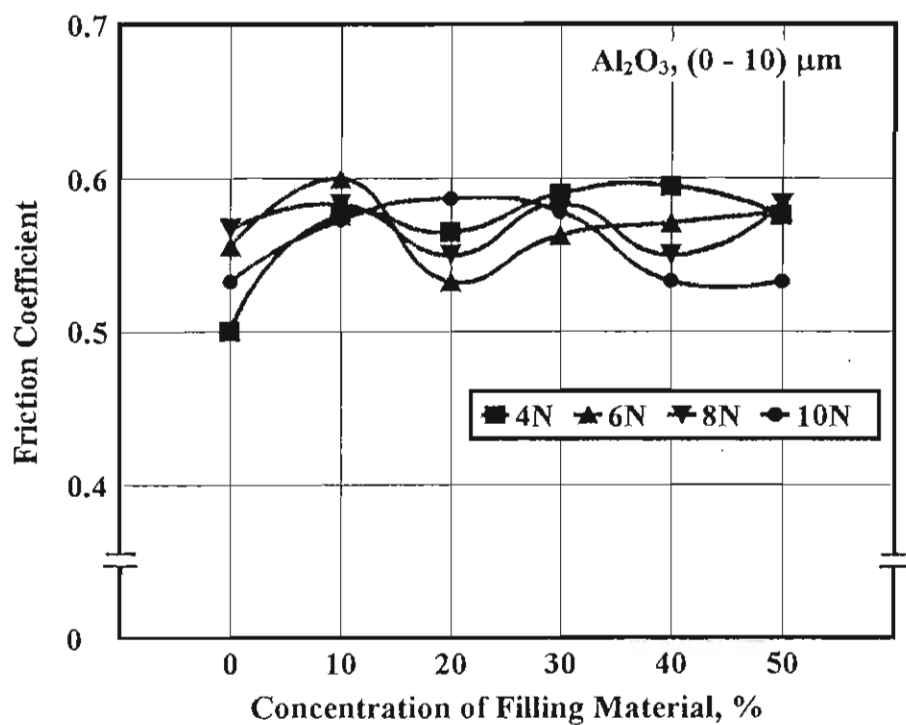


Fig. 10 Friction coefficient of epoxy resin filled by aluminium oxide of $(0 - 10) \mu\text{m}$ particle size.

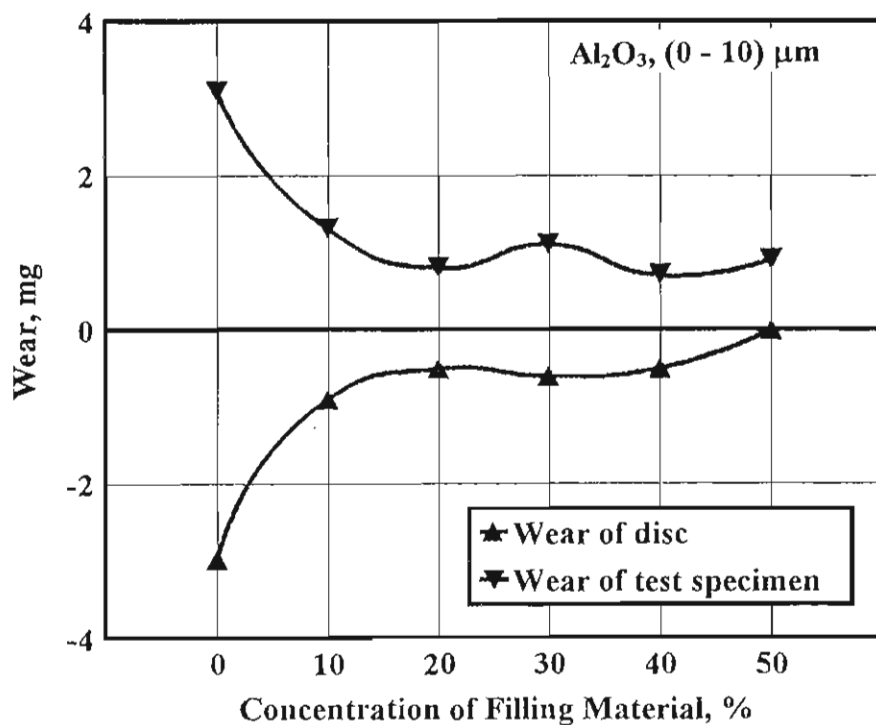


Fig. 11 Wear of epoxy resin test specimens filled by aluminium oxide of $(0 - 10) \mu\text{m}$ particle size at applied load of 6.0 N.

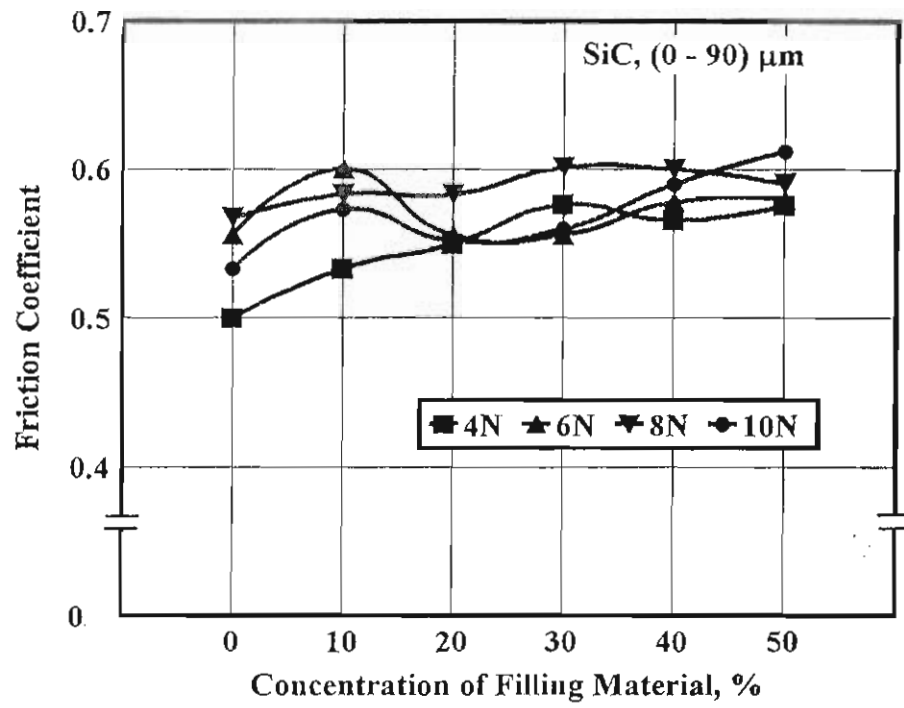


Fig. 12 Friction coefficient of epoxy resin filled by silicon carbide of (0 - 90) μm particle size.

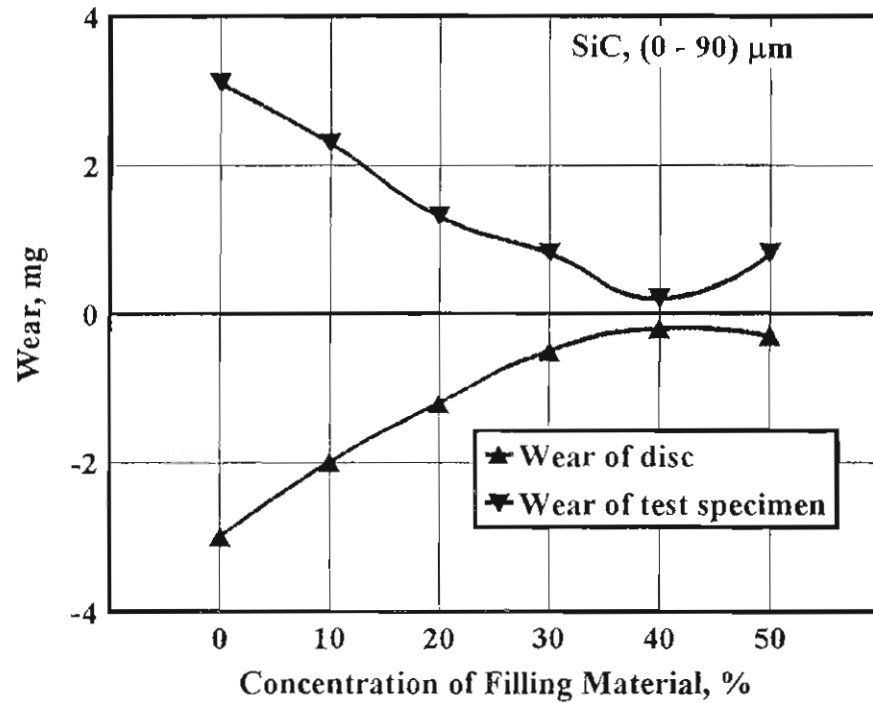


Fig. 13 Wear of epoxy resin test specimens filled by silicon carbide of (0 - 90) μm particle size at applied load of 6.0 N.

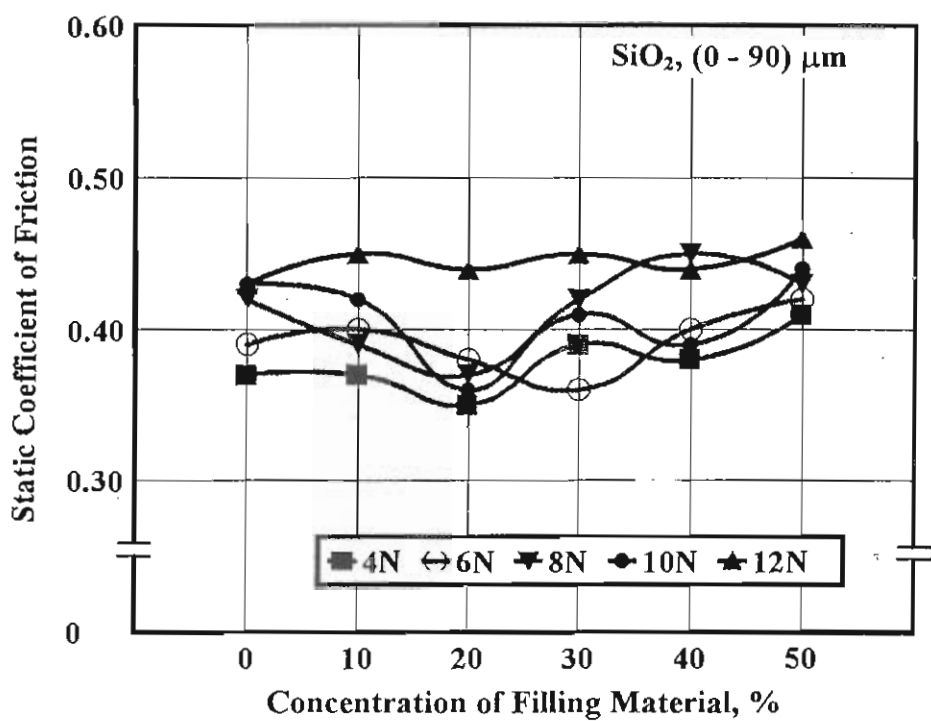


Fig. 14 Static coefficient of friction versus concentration of silicon oxide of (0 - 90) μm particle size.

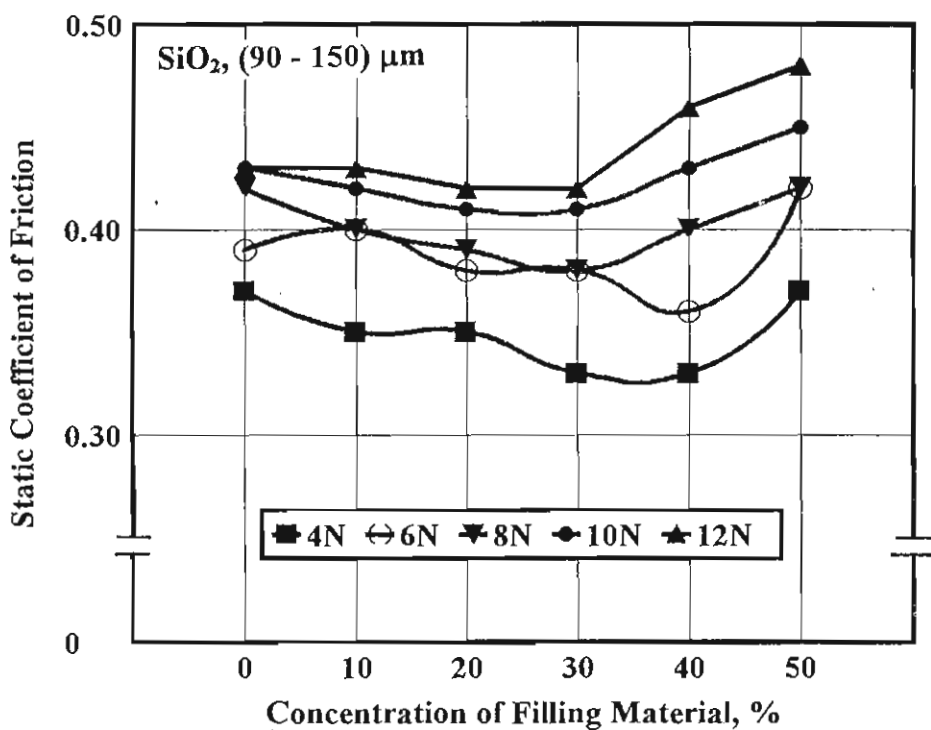


Fig. 15 Static coefficient of friction versus concentration of silicon oxide of (90 - 150) μm particle size.

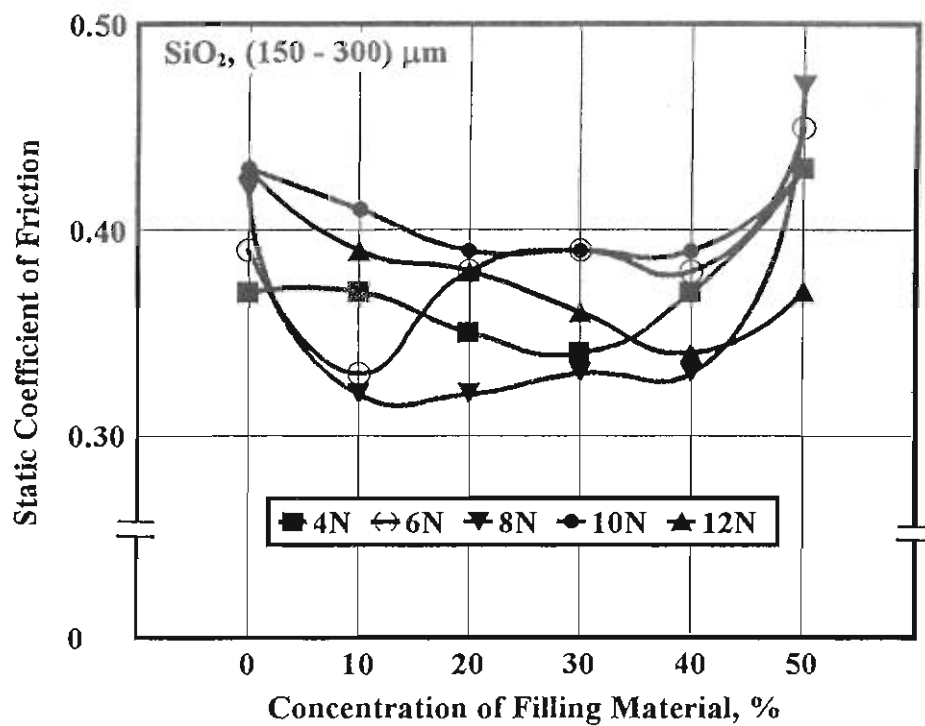


Fig. 16 Static coefficient of friction versus concentration of silicon oxide of (150 - 300) μm particle size.

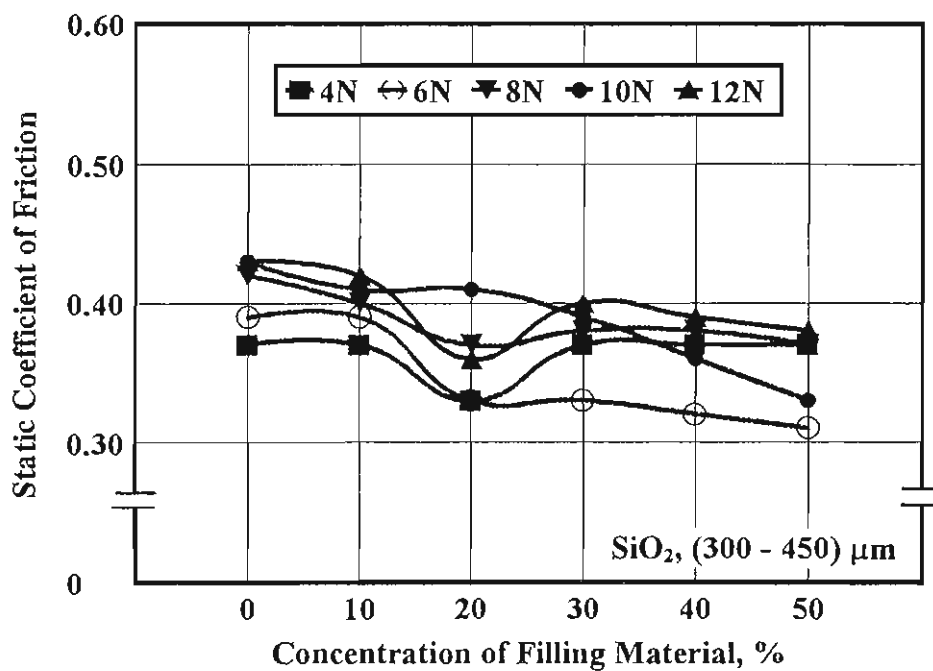


Fig. 17 Static coefficient of friction versus concentration of silicon oxide of (300 - 450) μm particle size.

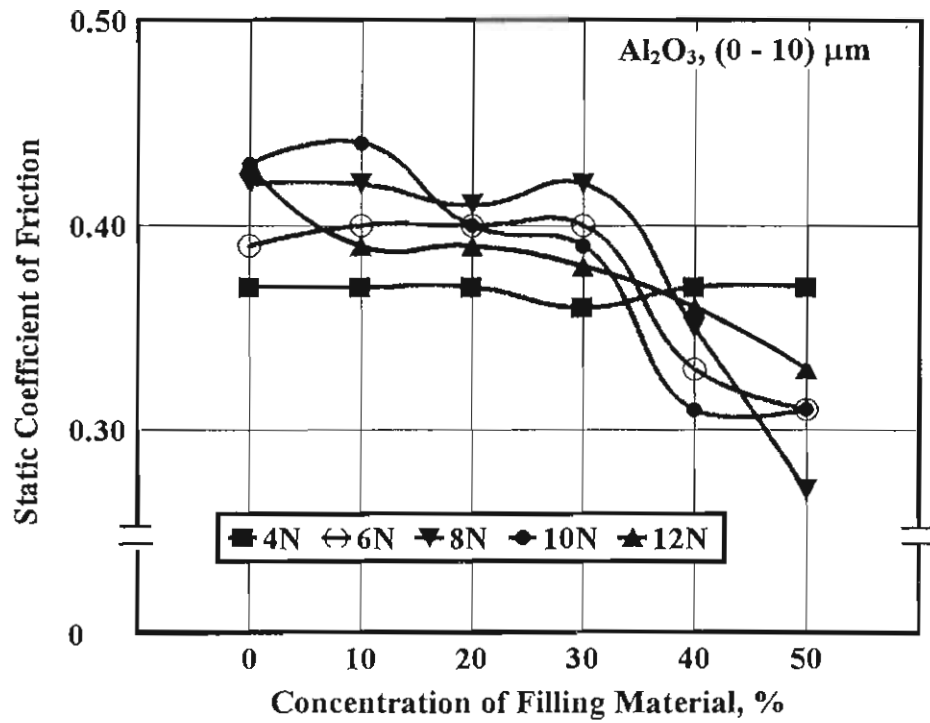


Fig. 18 Static coefficient of friction versus concentration of aluminium oxide of (0 - 10) μm particle size.

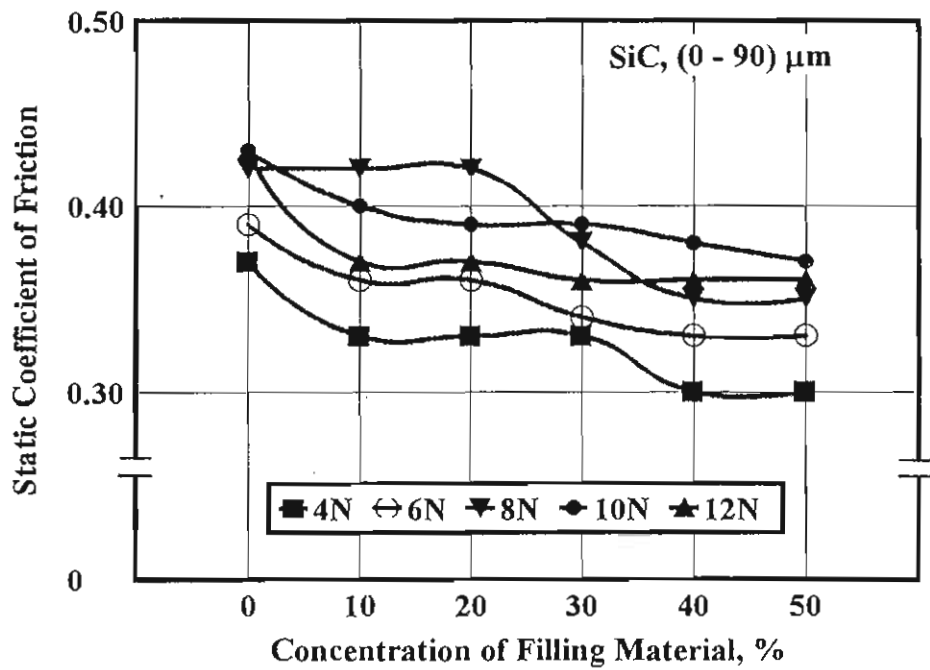


Fig. 19 Static coefficient of friction versus concentration of silicon carbide of (0 - 90) μm particle size.

COMPARATIVE STRESS ANALYSIS IN A THIN-WALLED AXISYMMETRIC MACHINE PART

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ABSTRACT

The paper deals with the stress distribution in a axisymmetric machine part that consists of circular plate and conical shell. The experimental photoelastic frozen stress method has been used. The experimental results have been compared with the results of performed analytical solution based on the theory of thin plates and the moment theory of thin shells of revolution. Experimental method, as a contrast to the analytical ones, gives very clear picture of stresses.

1. INTRODUCTION

Axisymmetric thin-walled bodies that consist of plates and shells are widely used as parts of machine constructions of different purpose. The analysis of stresses and strains of these parts is one of the most important basis of construction and plays the decisive step in their design.

There are three approaches in stress analysis: analytical, numerical and experimental. When talking about machine parts of complicated geometry, analytical and numerical approach can make analysis not reliable enough, because of the problems of recognition and complete satisfaction of boundary conditions. Experimental stress analysis usually knows no problem with boundary conditions and searching for right solution depends on appropriate equipment, knowledge and skill.

This work presents comparative stress analysis of a machine part that consists of circular plate and conical shell. Performed analytical solution, which can be considered approximate due to some assumptions, is based on the theory of thin plates and the moment theory of thin shells of revolution. The experimental solution has been founded by using photoelastic frozen stress method especially made supported and loaded model. On comparing the results their analysis has been made which confirms that it is possible to use the frozen stress method efficiently in the stress analysis of these and similar constructions in mechanical engineering practice.

2. ANALYTICAL SOLUTION

Calculating model of the machine part that consists of circular plate and conical shell is supported and loaded as shown in Fig.1.

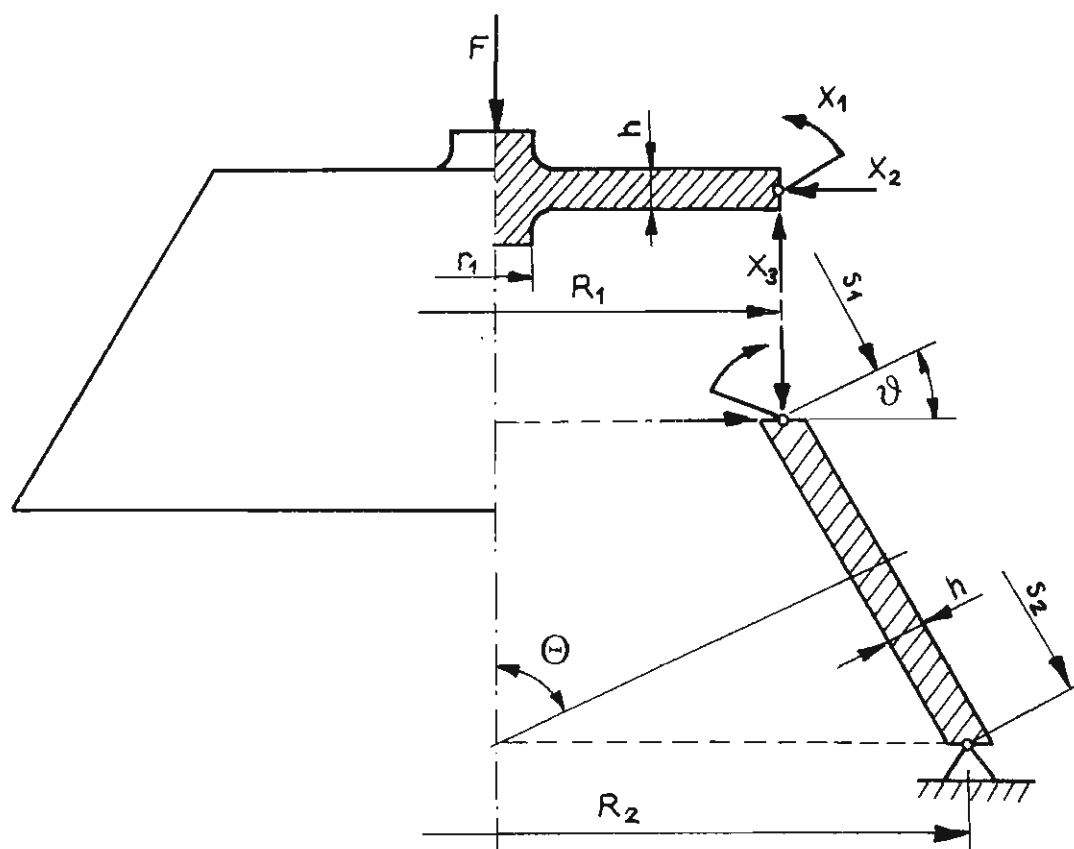


Fig. 1

Both parts are analysed separately and put together provided that boundary conditions on the point of connection are the same, when talking about forces and displacements. Differential equations of thin circular plate [1], is:

$$\frac{d^2\vartheta}{dr^2} + \frac{1}{r} \frac{d\vartheta}{dr} + \frac{1}{r^2} \vartheta = \frac{Q}{D} \quad (1)$$

where:

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

Its general solution has the following form:

$$\vartheta = K_1 r + \frac{K_2}{r} + \frac{1}{Dr} \int (r) Q dr dr \quad (2)$$

In this case is: $Q = -F/(2r\pi)$. Boundary conditions on the basis of which K_1 and K_2 are defined, are: for $r=r_1$, $\vartheta=0$; for $r=R_1$, $X_1=D(d\vartheta/dr+\mu\vartheta/r)$.

The plate is in state of plane stress and the stresses in radial and circular directions are:

$$\sigma_r = \frac{Ez}{1-\mu^2} \left(\frac{d\vartheta}{dr} + \mu \frac{\vartheta}{r} \right) \quad (3)$$

$$\sigma_\varphi = \frac{Ez}{1-\mu^2} \left(\frac{\vartheta}{r} + \mu \frac{d\vartheta}{dr} \right) \quad (4)$$

On the basis of theory of thin shells of revolution [2], differential equations of conical shell can be presented in the following way [6]:

$$L(V) = Eh\vartheta \operatorname{tg} \Theta + \phi(s)$$

$$L(\vartheta) = - \frac{\operatorname{tg} \Theta}{D} V \quad (5)$$

where:

$$L(\vartheta) = s \frac{d^2(\vartheta)}{ds^2} + \frac{d(\vartheta)}{ds} - \frac{1}{s} (\vartheta) \\ \phi(s) = - \frac{F(s)}{s \sin \Theta \cos \Theta}$$

The solutions of the equation system (5), are represented as the sum of general solutions of belonging homogeneous equations and particular solutions of given equations. In this work the general solutions of homogeneous equations are expressed by the second-order Thomson functions with real variable. The particular solutions were performed by the way of loading the shell. The solutions are:

$$V = \frac{Eh^2}{\sqrt{12(1-\mu^2)}} (C_1 \varphi_1 + C_2 \varphi_2 + C_3 \varphi_3 + C_4 \varphi_4) \quad (6)$$

$$\vartheta = -C_1 \varphi_2 + C_2 \varphi_1 - C_3 \varphi_4 + C_4 \varphi_3 - \frac{X_3 R_1}{Ehs \sin^2 \Theta}$$

where: $\varphi_1 = \text{ber}_2(x)$, $\varphi_2 = \text{bei}_2(x)$, $\varphi_3 = \text{ker}_2(x)$, $\varphi_4 = \text{kei}_2(x)$,

$$x = 2\sqrt{\left(\frac{\text{stg}\Theta}{h}\right)^4 (12(1-\mu^2))}$$

In the given case, boundary conditions which determine C_1 , C_2 , C_3 and C_4 are: for $x=x_1$ ($C_1=C_2=0$),

$$X_1 = -D\left(\frac{d\vartheta}{ds} + \mu \frac{\vartheta}{s}\right), \quad X_2 = \frac{V}{s(\sin^2\Theta/\cos\Theta + 1)} - \frac{F(s)}{s \sin\Theta}$$

for $x=x_2$ ($C_3=C_4=0$),

$$\frac{d\vartheta}{ds} + \mu \frac{\vartheta}{s} = 0, \quad \frac{dV}{ds} - \mu \left(\frac{V}{s} - \phi(s)\right) = 0$$

The shell is in the state of plane stress, and the stresses in meridional and circular directions are:

$$\sigma_s = -\frac{1}{h} \left(\frac{V}{s} - \phi(s) \right) - \frac{Ez}{1-\mu^2} \left(\frac{d\vartheta}{ds} + \mu \frac{\vartheta}{s} \right) \quad (7)$$

$$\sigma_\varphi = -\frac{1}{h} \frac{dV}{ds} - \frac{Ez}{1-\mu^2} \left(\frac{\vartheta}{s} + \mu \frac{d\vartheta}{ds} \right) \quad (8)$$

On the basis of expressions (3), (4), (7) and (8), the maximum radial, meridional and circular stresses on the inner surface of model were determined. The following values of basic quantities were taken: $r_1=5\text{mm}$, $R_1=35\text{mm}$, $R_2=60\text{mm}$, $s_1=70\text{mm}$, $s_2=120\text{mm}$, $h=5\text{mm}$, $\Theta=60^\circ$, $\mu=0,48$, $F=14\text{N}$, $E=18\text{MPa}$, $X_3=F/(2R_1\pi)$, $F(s)=F/(2\pi)$, $z=h/2$.

3. EXPERIMENTAL SOLUTION

Experimental model was made by casting from Araldite B (Ciba-Geigy). Basic resin has been heated in electric oven up to 195°C . After being melted, hardener was added at the proportion 100:30. After that, the mixture was stirred outside the oven (exothermic nature of the polymerisation). Temperature was being controlled. Casting in previously heated steel mould greased with silicon was performed outside the oven. Afterwards, the mould was cooled down by compressed air at 60°C . The cycle was continued in the oven by increasing and maintaining the temperature at 80°C for the following 48 hours. By gradual increase of temperature of 5°C per hour, the temperature of final polymerisation was reached. It was 150°C and kept constant for 5 hours. By reducing the temperature at a rate of 5°C per hour, the room temperature was

reached. After this the model was mechanically handled to approximate dimensions and left at the temperature of 150°C to remove residual stresses. After gradual cooling, the final handling was done by turning and gnawing according to regular standards [3,4,5], so that the model got its final dimensions.

In especially constructed and precisely made device [6], the model was supported, loaded and heated for 2 hours to the stress freezing temperature 150°C . After 2 hours the temperature was brought down to room temperature at a rate of 5°C . In this way, the strains inside the model were permanently conserved and hereby photoelastic effect as well. Since the form and loading the model was symmetrical, maximum stresses do appear in the plane of symmetry. Because that the model was cut into symmetrical slices and subslices, so that in their planes there are two principal stresses that are determined by isocromatic picture in polariscope. The slices were kept at the temperature of 60°C for 48 hours to eliminate edge effect, and they were photographed in a bright and dark field of circularly polarised light. The photography of radial-meridional slice is given on Fig. 2.

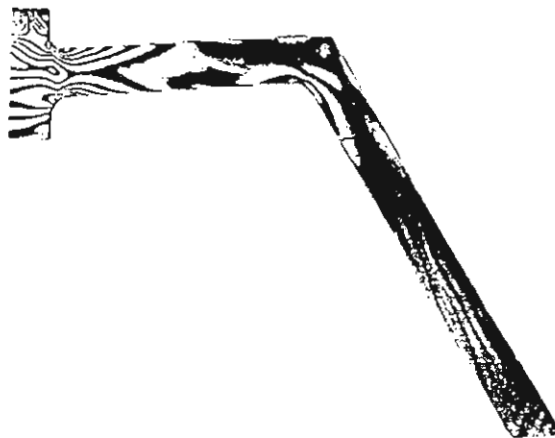


Fig. 2

In radial and meridional slices, the equation of stress-optic law is:

$$\sigma_1 - \sigma_3 = \frac{f}{b} N \quad (9)$$

In circular slices, its form is:

$$\sigma_2 - \sigma_3 = \frac{f}{b} N \quad (10)$$

The maximum values of radial and circular stresses of the plate-like part, and meridional and circular stresses of shell-like part were the result of equations (9) and (10), respectively.

The following values were taken: $\sigma_1 = \sigma_r$ (plate), $\sigma_1 = \sigma_\theta$ (shell), $\sigma_2 = \sigma_\varphi$, $\sigma_3 = 0$ (perpendicular stress on the load-free boundary), $b = 3,5\text{mm}$ (radial and meridional slices), $f = 0,28\text{kN/m}$ (gauging of specimen loaded on pure bending that was exposed to same temperatures as the model itself).

4. ANALYSIS OF RESULTS AND CONCLUSION

On the basis of analytical and experimental solution, the curves of stresses were comparatively drawn on the inner surface of the model, Fig. 3. The stresses are presented in non-dimensional form dividing by $\sigma = F/5h^2$.

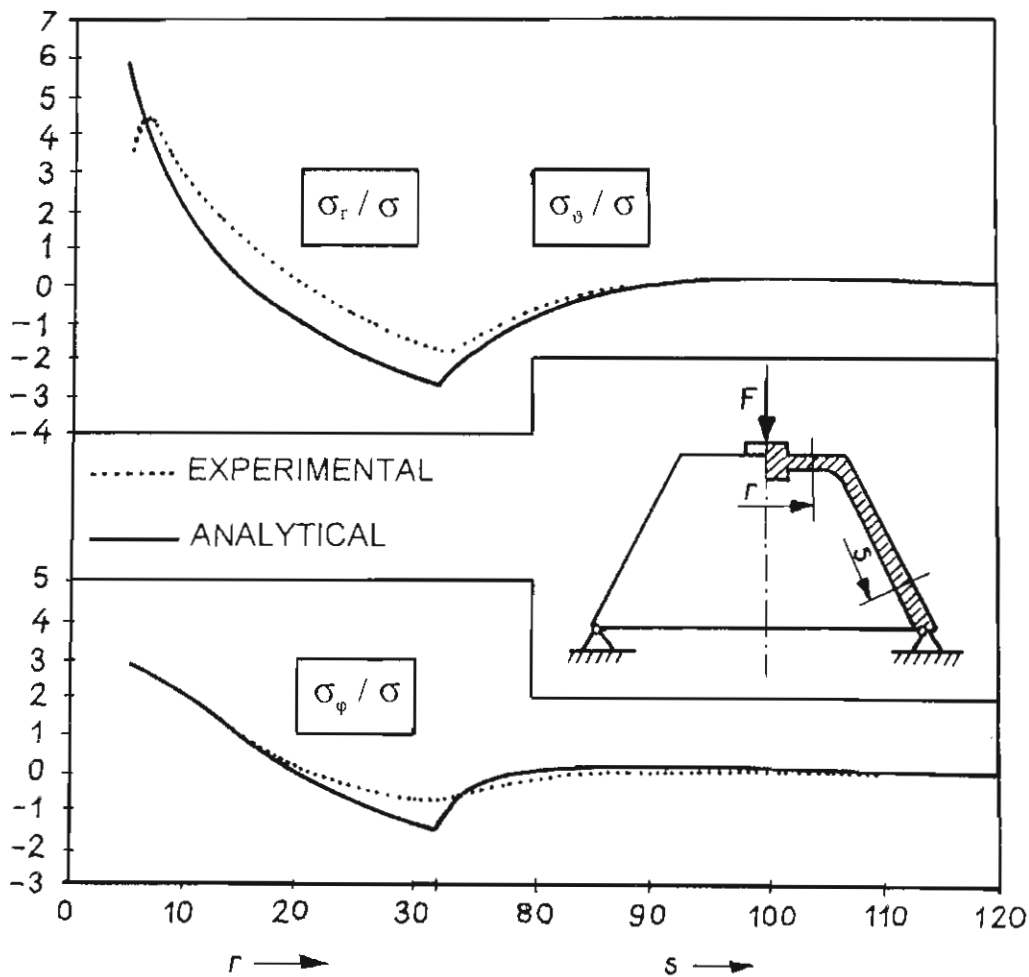


Fig. 3

The similarities of the given curves are evident except for the point of connection between the plate and the shell. Analytical curves make a break here while experimental curves are continuous. At this point the analytical values of stress are higher than the experimental ones, for 34% at radial and meridional stress, and for 59% at circular stress. The extreme value of experimental radial stress is 24% lower than the analytical value, while the extreme analytical and experimental values of circular stress are the same.

Different results at the point of connection of plate and shell can be explained by assumption of approximate analytical solution. It is that the contact of these parts is realised in the point that lies on their central surfaces. It is not true of course, because the contact is realised on the surface with help of radius of curvature, so the connecting conditions are not satisfied in expected point.

Besides the stress concentration appear on the place of connection, and it is not taken into account in this analytical solution. The practice shows that at h to $2h$ distance from the point of connection, this approximate solution gives almost exact values, which is confirmed by the experimental results in this case. Finally the work shows that the experimental frozen stress method gives clear picture of stress and that reliable stress analysis of machine parts of various forms can be performed by using this method.

SYMBOLS:

b - thicknees of slice; C - constant; D - flexural rigidity; E - modulus of elasticity; F - force; f - stress optic coefficient; h - thicknees of model; L - differential operator; N - fringe order; R - radius of shell; r - radius of plate; s - length of meridian of shell; V - Meissner's variable; X - reaction; x - axial co-ordinate, real variable; z - distance from the central surface; Θ - angle between normal on the central surface and axis of symmetry of shell; ϑ - angle of twist of normal; μ - Poisson's ratio; σ - normal stress; ϕ - function depending on stress; φ - Thomson's function.

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APPLICATION FINITE ELEMENTS METHODS IN PLATE BENDING IN THE AREA OF SMALL PLASTIC DEFORMATION

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ABSTRACT

The applied approximate formulas do not guarantee accurate results in determination of bending force for long and thick plates, in relation to the size of the bending radius, especially for plates with great wall thickness and the large bending radius. When bending plates in press tool, it happens very often that stamp and the matrix are of the same size. At one moment the bending force at $\Delta L=0$ represents infinitesimally great bending force. The paper is concerned with the numeric procedure in plate bending. For numeric procedure the finite element method is used. The plate bending analysis is restricted to the data used so far in the manufacture of tanks i.e. to deformations up to 2%.

Key words: bending force, finite element method

1. INTRODUCTION

Bending process has been studied in numerous scientific research papers. Primary subject of interest in these papers has been mainly focused on distribution of strains and bending force.

By free bending (three point bending) the flat plate is being positioned in the press tool, and the force F is being exerted in the middle of the plate bending it as shown in Figure 1.

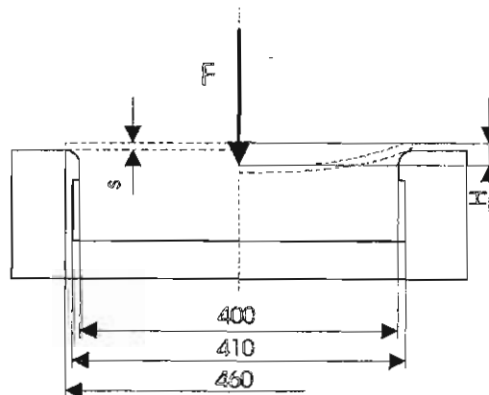


Figure 1. Bending force in free bending (Model for three-point bending)

Wolter [1] has shown that the achieved bend curvature of the plate depends only on the width of the tool $L_k (=460)$, in case that sufficiently small radius of the stamp has been selected. The bending force is obtained by application of unit moment force.

$$\Sigma = \frac{M}{b \cdot s^2} \text{ , N / mm}^2$$

$$F = \frac{4 \Sigma \cdot b \cdot s_o^2}{L_k} \quad (1)$$

Where s_o = wall thickness of the unbended plate in mm, and b stands for length of the plate in mm.

In practice it happens very often that the bending force calculation must be obtained by using approximate formulas. There are several reasons for that: all unit moment forces for all materials are not always accessible, and the bending and elongation yielding curvatures are not identical. Besides that the applied approximate formulas also take into account other operational conditions, such as deviations from plate tolerances, plate friction at margins of the tool and the cold tightening due to bending itself. It has been proved that the following simple formulas when compared with the measured values give reliable results:

$$F = \frac{1,64 \cdot \sigma_B \cdot b \cdot s^2}{L_k} \quad (2)$$

When carrying out plate bending with large bending radius in relation to the width of the tool, this formula gives lower values. The measured values of force F are approximately 2,3 times greater than those calculated by the formula (2). Such a large calculation deviations in relation to the measured values have been established first of all when bending the steel plates with yield strength $>700 \text{ N/mm}^2$ and the tensile strength $>800 \text{ N/mm}^2$.

The bending geometry during the process of bending is altered depending on the pathway of the stamp (Fig. 2), where we have the same lengths of stamp and matrix, being different from the free bending.

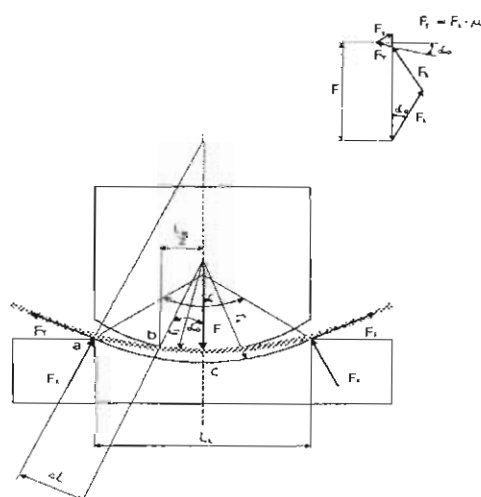


Figure 2. Bending Force

Recently, the finite-element technique for elastoplastic, large deformation has been relatively well established, and theories of contact analysis have been extensively studied. For a unified theory of variational inequalities with a considerable literature review, Panagiotopoulos is cited.

2. MODEL FOR THREE - POINT BENDING

A schematic model for a three-point bending test is shown in Fig 1. Contact occurs between the punch, the specimen and the supports. For simplicity, the punch and supports are assumed rigid and the contact surface unbonded and frictionless. Vertical displacement of the punch and the specimen are taken as the rigid-body degrees of freedom.

The finite-element method is utilized as a numerical approximation with u denoting the vector of the nodal displacement increment. The functional I to be minimized contains polynomial functions of up to the fourth order with respect to u . To solve this nonlinear problem efficiently, a sequence of quadratic problems [3] are solved by using simplifying assumptions with respect to the strain increment. To iteration for the sequence $u^{(i)} = u^{(i-1)} + \Delta u$, $u^{(0)} = 0$ is continued until the nodal displacement change, Δu , at the i -th iteration becomes sufficiently small.

The final problem to be solved is the dual problem whose dimension is just the number of nodal displacement increments on the potential contact region. An efficient numerical method is then used to solve this reduced quadratic programming. For the detailed form of discretized problems, refer to Joo and Kwak [4].

A computer program was developed to solve threepoint bending problems based on the above formulation. The constitutive coefficients in the elastoplastic range were obtained from the Prandtl-Reuss equation, the von Mises yield criterion and the isotropic hardening rule [5].

The specimen material used was steel, StE 500, which has a chemical composition of C=0.20 percent, Si=0.44 percent, P=0.012 percent, S=0.002 percent. The material properties were obtained in a tensile test. From the test results, the effective stress-strain relation, expressed by $\bar{\sigma} = \sigma_y + \sigma_0 \bar{\epsilon}_p^n$, is obtained by curve fitting. The material properties thus obtained are shown in Tabele 1.

Specimens used for bending test were taken from a 29 mm and 13 mm thickness steel plate and manufactured along the rolling direction. 2 types of specimens were prepared, which were the same in length and width but different in thickness. The length and width of the specimens were 460 mm and 2000 mm respectively. The thickens selected were 29 and 13 mm.

Teflon sheet and grease was used to prevent friction. Using machine, the load was increased at a punch velocity of 20 mm/min until the punch displacement reached 10 mm. Then it was removed at the same velocity.

Four experiments were carried out for each type of specimen.

The finite-element mesh and the dimensions of the specimen are shown in Fig. 3. Because of the symmetry, only one half of the body was considered. An eight-node quadratic isoparametric element was used for the analysis and the present problem was solved for both planestrain and plane-stress conditions. Material properties used for the analysis were obtained in a tensile test and given in Table 1. The applied load was increased stepwise until the punch displacement became 10 mm, and then the external forces were removed completely by decreasing the load (Figure 2).

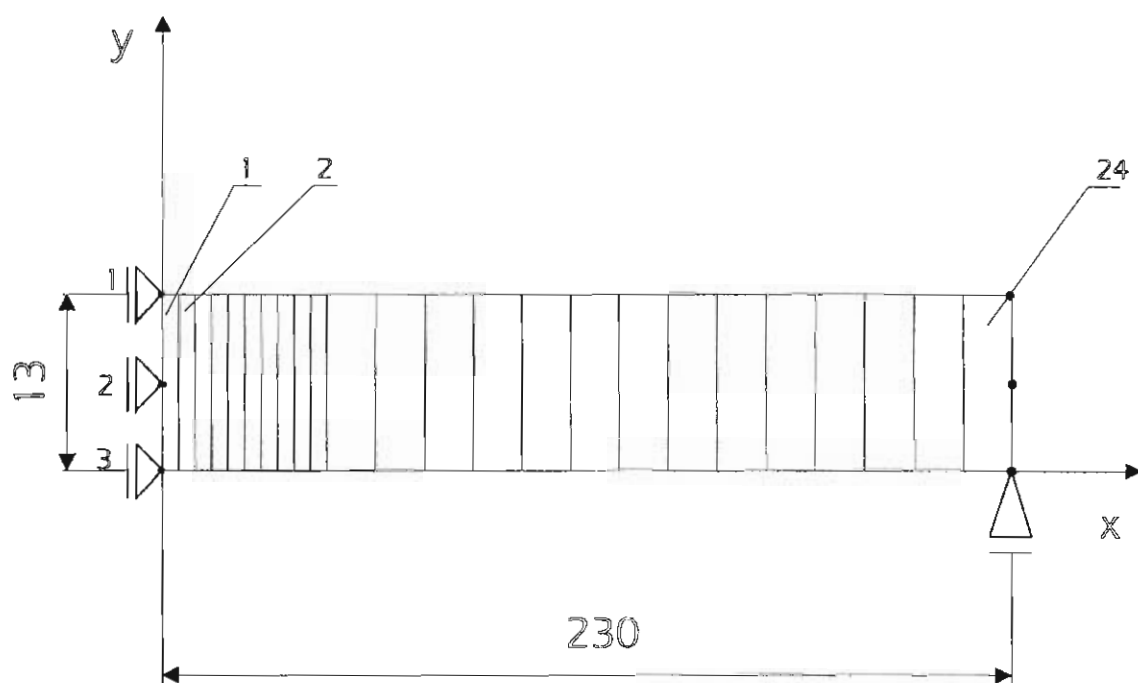


Figure 3. Finite-element mesh for bending problem. Dimension in mm

Table 1 - Material properties used for three point bending tests

E	ν	σ_y	σ_o	n
206 GPa	0.32	588 MPa	700 MPa	0.4

The load-punch displacement curves for numerical and experimental results are shown in Fig. 4.

In actual bending tests, there is a contact region between the bodies, and slipping occurs between the specimen and the support surfaces.

The distributions of the bending stress across the symmetric axis are shown in Fig. 5. In the elastic range, the bending stress varies linearly. But, as the plastic range is enlarged, it is seen that the bending stress is distributed like an S shape and the neutral axis goes up slightly.

3. BENDING SHAPE GEOMETRY

We may precisely speak about free bending only in cases when the plate bending curvature is obtained by force exerted in the middle of tool [1]. By relatively large internal radius the plate is forced to come into contact with the radius of the tool stamp. Figure 2 shows the stamp in one phase of bending process. Fig. 4 FEM and exp. Materials StE 500. Curve (1) 13x460x1mm; curve (2) 29x460x1mm.

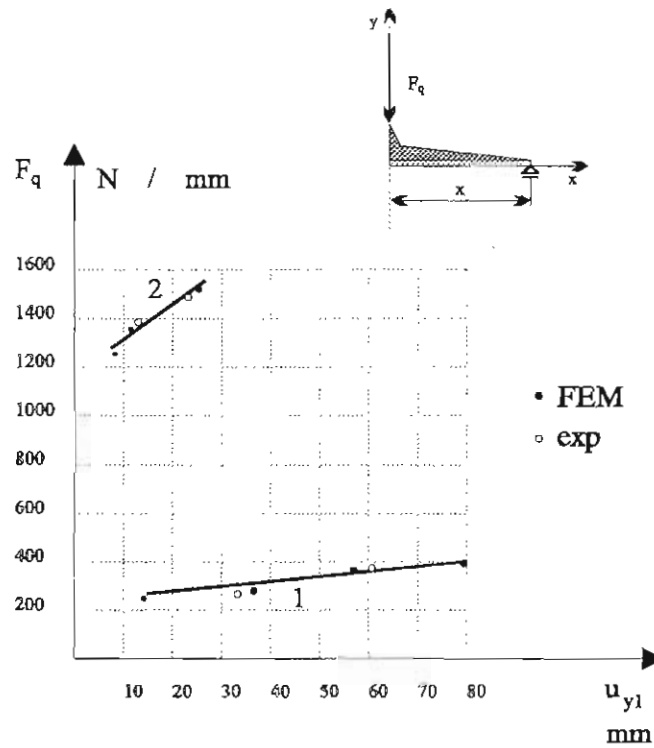


Figure 4. FEM and exp.($H=u_{y1}$)

FEM results are in compliance with experimental data.

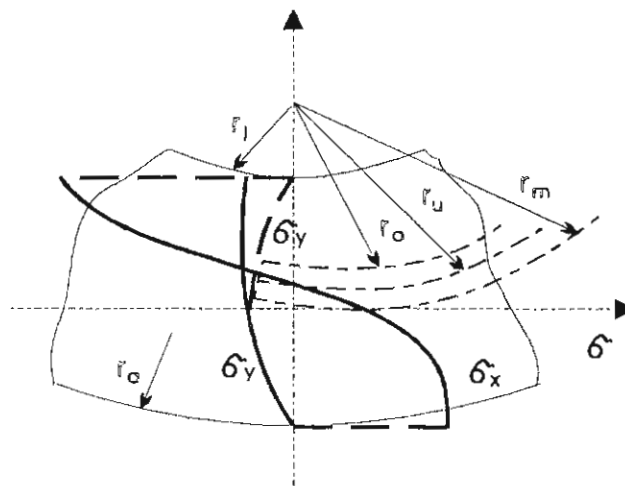


Figure 5. Distribution of bending stress across symmetric axis

4. STAMPING FORCE

The bending force (stamping force) is calculated by formula

$$F = F_q \cdot bN \quad (3)$$

Where:

for steel plates 13x460x1 mm (curve 1 Figure 4)

$$F_q = 1,054u_{y1} + 243, \text{ N/mm} \quad (4)$$

and for steel die plates 29x460x1 mm (curve 2 Figure 4)

$$F_q = 15,638u_{y1} + 1160, \text{ N/mm} \quad (5)$$

$b[\text{mm}]$ stands for length of the plate. The results apply for deformations up to 2%.

5. CONCLUSIONS

It occurs frequently in bending of the plate in the press tool that we have the same diameter of the stamp and matrix. For numerical procedure the finite element method is used. The results are valid for deformations up to 2%.

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